More on the anti-automorphism of the Steenrod algebra

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The relations of Barratt and Miller are shown to include all relations among the elements $P^i \chi P^{n-i}$ in the mod $p$ Steenrod algebra, and a minimal set of relations is given.

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1 Introduction

Milnor [4] observed that the mod 2 Steenrod algebra $\mathcal{A}$ forms a Hopf algebra with commutative diagonal determined by

$$\Delta \operatorname{Sq}^n = \sum_i \operatorname{Sq}^i \otimes \operatorname{Sq}^{n-i}.$$ (1)

This allowed him to interpret the Cartan formula as the assertion that the cohomology of a space forms a module-algebra over $\mathcal{A}$. The anti-automorphism $\chi$ in the Hopf algebra structure, defined inductively by

$$\chi \operatorname{Sq}^0 = \operatorname{Sq}^0, \quad \sum_i \operatorname{Sq}^i \chi \operatorname{Sq}^{n-i} = 0 \quad \text{for } n > 0.$$ (2)

has a topological interpretation too: If $K$ is a finite complex then the homology of the Spanier–Whitehead dual $DK_+$ of $K_+$ is canonically isomorphic to the cohomology of $K$. Under this isomorphism the left action by $\theta \in \mathcal{A}$ on $H^*(K)$ corresponds to the right action of $\chi \theta \in \mathcal{A}$ on $H_*(DK_+)$. In 1974 Davis [3] proved that sometimes much more efficient ways exist to compute $\chi \operatorname{Sq}^n$; for example

$$\chi \operatorname{Sq}^{2^r-1} = \operatorname{Sq}^{2^r-1} \chi \operatorname{Sq}^{2^r-1-1},$$ (3)

$$\chi \operatorname{Sq}^{2^r-r-1} = \operatorname{Sq}^{2^r-1-1} \chi \operatorname{Sq}^{2^r-1-r} + \operatorname{Sq}^{2^r-1} \chi \operatorname{Sq}^{2^r-1-r-1}.$$ (4)

Similarly, Straffin [6] proved that if $r \geq 0$ and $b \geq 2$ then

$$\sum_i \operatorname{Sq}^{2^r i} \chi \operatorname{Sq}^{2^r (b-i)} = 0.$$ (5)
Both authors give analogous identities among reduced powers and their images under \( \chi \) at an odd prime as well. Further relations among the Steenrod squares and their conjugates appear in these articles and elsewhere (eg Silverman [5]).

Barratt and Miller [1] found a general family of identities which includes (3), (4) and (5), and their odd-prime analogues, as special cases. We state it for the general prime. When \( p = 2 \), \( P^n \) denotes \( \text{Sq}^n \). Let \( \alpha(n) \) denote the sum of the \( p \)-adic digits of \( n \).

**Theorem 1.1** [1; 2] For any integer \( k \) and any integer \( l \geq 0 \) such that \( pl - \alpha(l) < (p - 1)n \),

\[
\sum_i \binom{k - i}{l} P^i \chi P^n - i = 0 .
\]

The relations defining \( \chi \) occur with \( l = 0 \). Davis’ formulas (for \( p = 2 \)) are the cases in which \((n, l, k) = (2^r - 1, 2^r - 1, 2^r - 1) \) or \((n, l, k) = (2^r - r - 1, 2^r - 1 - 2, 2^r - 2) \). Straffin’s identities (for \( p = 2 \)) occur as \((n, l, k) = (2^r b, 2^r - 1, 1) \). Since \( \binom{(k+1) - i}{l} - \binom{k - i}{l} = (k - i) \binom{k - i}{l - 1} \), the cases \((l, k + 1)\) and \((l, k)\) of (6) imply it for \((l - 1, k)\). Thus the relations for \( l = \phi(n) - 1 \), where

\[
\phi(n) = 1 + \max \{ j : pj - \alpha(j) < (p - 1)n \},
\]

imply all the rest. Here we have adopted the notation \( \phi(n) \) used in [2]; we note that it is not the Euler function \( \varphi(n) \).

When \( p = 2 \), \( \phi(2^r - 1) = 2^r - 1 \) and \( \phi(2^r - r - 1) = 2^r - 1 - 2 \), so Davis’s relations are among these basic relations.

Two questions now arise. To express them uniformly in the prime, let \( \mathcal{P} \) denote the algebra of Steenrod reduced powers (which is the full Steenrod algebra when \( p = 2 \)), but assign \( P^n \) degree \( n \). Write

\[
V_n = \text{Span}\{ P^i \chi P^{n-i} : 0 \leq i \leq n \} \subseteq \mathcal{P}^n .
\]

It is natural to ask:

- Are there yet other linear relations among the \( n + 1 \) elements \( P^i \chi P^{n-i} \) in \( \mathcal{P}^n \)?
- What is a basis for \( V_n \)?

We answer these questions in Theorem 1.4 below.

Write \( e_i, 0 \leq i \leq n \), for the \( i \)-th standard basis vector in \( \mathbb{F}_p^{n+1} \).
Proposition 1.2  For any integers \( l, m, n \), with \( 0 \leq l \leq n \),

\[
\sum_i \binom{k-i}{l} e_i : m \leq k \leq m+l
\]

is linear independent in \( \mathbb{F}_p^{n+1} \).

Proposition 1.3  The set

\[
\{ P^i \chi P^{n-i} : \phi(n) \leq i \leq n \}
\]

is linearly independent in \( \mathcal{P}^n \).

Define a linear map

\[
\mu : \mathbb{F}_p^{n+1} \to \mathcal{P}^n, \quad \mu e_i = P^i \chi P^{n-i}.
\]

Theorem 1.1 implies that if \( l = \phi(n) - 1 \) the elements in (8) lie in \( \ker \mu \), so Propositions 1.2 and 1.3 imply that (8) with \( l = \phi(n) - 1 \) is a basis for \( \ker \mu \) and that (9) is a basis for \( V_n \subseteq \mathcal{P}^n \). Thus:

**Theorem 1.4** Any \( \phi(n) \) consecutive relations from the set (6) with \( l = \phi(n) - 1 \) form a basis of relations among the elements of \( \{ P^i \chi P^{n-i} : 0 \leq i \leq n \} \). The set \( \{ P^i \chi P^{n-i} : \phi(n) \leq i \leq n \} \) is a basis for \( V_n \).

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## 2 Independence of the relations

We wish to show that (8) is a linearly independent set. Regard elements of \( \mathbb{F}_p^{n+1} \) as column vectors, and arrange the \( l+1 \) vectors in (8) as columns in a matrix, which we claim is of rank \( l+1 \). The top square portion is the mod \( p \) reduction of the \( (l+1) \times (l+1) \) integral Toeplitz matrix \( A_l(m) \) with \((i, j)\)-th entry

\[
\binom{m+j-i}{l}, \quad 0 \leq i, j \leq l.
\]

**Lemma 2.1** \( \det A_l(m) = 1 \).
Proof. By induction on $m$. Since $\binom{-1}{i} = (-1)^i$ and $\binom{-1+j}{i} = 0$ for $0 < j \leq l$, $A_l(-1)$ is lower triangular with determinant $((-1)^l)^{l+1} = 1$. Now we note the identity

$$BA_l(m) = A_l(m + 1)$$

where

$$B = \begin{bmatrix}
\binom{l+1}{1} & -\binom{l+1}{2} & \cdots & -\binom{l-1}{l-1} & (l+1)l \binom{l}{l+1} \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}.$$ 

The matrix identity is an expression of the binomial identity

$$(11) \quad \sum_k (-1)^k \binom{l+1}{k} \binom{n-k}{l} = 0$$

(taking $n = m + 1 - j$ and $k = j + 1$). Since $\det B = 1$, the result follows for all $m \in \mathbb{Z}$. \hfill \Box

For completeness, we note that (11) is the case $m = l + 1$ of the equation

$$(12) \quad \sum_k (-1)^k \binom{m}{k} \binom{n-k}{l} = \binom{n-m}{l-m}.$$ 

To prove this formula, note that the defining identity for binomial coefficients implies the case $m = 1$, and also that both sides satisfy the recursion $C(l, m, n) - C(l, m, n - 1) = C(l, m + 1, n)$.

3 Independence of the operations

We will prove Proposition 1.3 by studying how $P^i \chi P^{n-i}$ pairs against elements in $\mathcal{P}_*$, the dual of the Hopf algebra of Steenrod reduced powers. According to Milnor [4], with our grading conventions

$$\mathcal{P}_* = \mathbb{F}_p[\xi_1, \xi_2, \ldots], \quad |\xi_j| = \frac{p^j - 1}{p - 1},$$

$$(13) \quad \Delta \xi_k = \sum_{i+j=k} \xi_i^{p^j} \otimes \xi_j.$$
For a finitely nonzero sequence of nonnegative integers $R = (r_1, r_2, \ldots)$ write $\xi^R = \xi_1^{r_1} \xi_2^{r_2} \cdots$ and let $\|R\| = r_1 + pr_2 + p^2r_3 + \cdots$ and

$$|R| = |\xi^R| = r_1 + \left(\frac{p^2 - 1}{p - 1}\right)r_2 + \left(\frac{p^3 - 1}{p - 1}\right)r_3 + \cdots.$$ 

The following clearly implies Proposition 1.3.

**Proposition 3.1** For any integer $n > 0$ there exist sequences $R_{n,j}, 0 \leq j \leq n - \phi(n)$, such that $|R_{n,j}| = n$ and

$$\langle P^i \chi P^{n-i}, \xi_{R_{n,j}} \rangle = \begin{cases} 1 & \text{for } i = n - j, \\ 0 & \text{for } i > n - j. \end{cases}$$

The starting point in proving this is the following result of Milnor.

**Lemma 3.2** [4, Corollary 6] $\langle \chi P^n, \xi^R \rangle = \pm 1$ for all sequences $R$ with $|R| = n$.

In the basis of $\mathcal{P}$ dual to the monomial basis of $\mathcal{P}_*$, the element corresponding to $\xi_1^i$ is $P^i$. Since the diagonal in $\mathcal{P}_*$ is dual to the product in $\mathcal{P}$, it follows from (13) and Lemma 3.2 that

$$\langle P^i \chi P^{n-i}, \xi^R \rangle = \begin{cases} 1 & \text{for } i = \|R\|, \\ 0 & \text{for } i > \|R\|. \end{cases}$$

So we wish to construct sequences $R_{n,j}$, for $\phi(n) \leq j \leq n$, such that $|R_{n,j}| = n$ and $\|R_{n,j}\| = j$. We deal first with the case $j = \phi(n)$.

**Proposition 3.3** For any $n \geq 0$ there is a sequence $M = (m_1, m_2, \ldots)$ such that

1. $|M| = n$,
2. $0 \leq m_i \leq p$ for all $i$,
3. if $m_j = p$ then $m_i = 0$ for all $i < j$.

For any such sequence, $\|M\| = \phi(n)$.

**Proof** Give the set of sequences of dimension $n$ the right-lexicographic order. We claim that the maximal sequence satisfies the hypotheses.

Suppose that $R = (r_1, r_2, \ldots)$ does not satisfy the hypotheses. If $r_1 > p$ then the sequence $(r_1 - (p + 1), r_2 + 1, r_3, \ldots)$ is larger. If $r_j > p$, with $j > 1$, then the sequence $(r_1, \ldots, r_{j-2}, r_{j-1} + p, r_j - (p + 1), r_{j+1} + 1, r_{k+2}, \ldots)$ is larger. This proves (2). To prove (3), suppose that $r_j = p$ with $j > 1$, and suppose that some earlier entry is nonzero. Let $i = \min\{k : r_k > 0\}$. If $i = 1$, then the sequence

\[ \cdots, m_{i-1}, m_i - p, m_{i+1}, \ldots \]
we must show that

This makes sense by monotonicity of \( \alpha \).

Then inductively define \( M \)

To see that (15) holds for \( l = \| M \| - 1 \). To see that \( l = \phi(n) - 1 \)

we must show that

\[
\begin{align*}
(14) & \quad p(l + 1) - \alpha(l + 1) \geq (p - 1)n, \\
(15) & \quad pl - \alpha(l) < (p - 1)n.
\end{align*}
\]

The excess \( e(R) \) is the sum of the entries in \( R \), so that \( p\|R\| - e(R) = (p - 1)\|R\| \).

The \( p \)-adic representation of a number minimizes excess, so for any sequence \( R \) we have \( e(R) \geq \alpha(\|R\|) \) and hence \( p\|R\| - \alpha(\|R\|) \geq (p - 1)\|R\| \) so (14) holds for any sequence.

To see that (15) holds for \( M \), let \( j = \min\{i : m_i > 0\} \), so that \( (p - 1)n = (p^j - 1)m_j + (p^{j+1} - 1)m_{j+1} + \cdots \) and \( l + 1 = p^{j-1}m_j + p^j m_{j+1} + \cdots \). The hypotheses imply that \( l \) has \( p \)-adic expansion

\[
(1 + \cdots + p^{j-2})(p-1) + p^{j-1}(m_j - 1) + p^j m_{j+1} + \cdots,
\]

so

\[
\alpha(l) = (j - 1)(p - 1) + (m_j - 1) + m_{j+1} + \cdots
\]

from which we deduce

\[
pl - \alpha(l) = (p - 1)(n - j) < (p - 1)n.
\]

This completes the proof of Proposition 3.3. \( \square \)

**Corollary 3.4** The function \( \phi(n) \) is weakly increasing.

**Proof** Let \( M \) be a sequence satisfying the conditions of Proposition 3.3, and note that the sequence \( R = (1, 0, 0, \ldots) + M \) has \( |R| = n + 1 \) and \( \|R\| = \|M\| + 1 = \phi(n) + 1 \).

If \( p \) does not occur in \( M \), then \( R \) satisfies the hypotheses of the proposition (in degree \( n + 1 \)) and hence \( \phi(n) \leq \phi(n + 1) \). If \( p \) does occur in \( M \), then the moves described above will lead to a sequence \( M' \) satisfying the hypotheses. None of the moves decrease \( \| - \| \), so \( \phi(n) \leq \phi(n + 1) \). \( \square \)

**Remark 3.5** Properties (1)–(3) of Proposition 3.3 in fact determine \( M \) uniquely.

**Proof of Proposition 3.1** Define \( R_{n, \phi(n)} \) to be a sequence \( M \) as in Proposition 3.3. Then inductively define

\[
R_{n,j} = (1, 0, 0, \ldots) + R_{n-1,j-1} \quad \text{for } \phi(n) < j \leq n.
\]

This makes sense by monotonicity of \( \phi(n) \), and the elements clearly satisfy \( |R_{n,j}| = n \) and \( \|R_{n,j}\| = j \). This completes the proof. \( \square \)
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