More on the anti-automorphism of the Steenrod algebra

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The relations of Barratt and Miller are shown to include all relations among the elements $P^i \chi P^{n-i}$ in the mod p Steenrod algebra, and a minimal set of relations is given.

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1 Introduction

Milnor [4] observed that the mod 2 Steenrod algebra $\mathcal{A}$ forms a Hopf algebra with commutative diagonal determined by

$$\Delta \operatorname{Sq}^n = \sum_i \operatorname{Sq}^i \otimes \operatorname{Sq}^{n-i} \ . \tag{1}$$

This allowed him to interpret the Cartan formula as the assertion that the cohomology of a space forms a module-algebra over $\mathcal{A}$. The anti-automorphism χ in the Hopf algebra structure, defined inductively by

$$\chi \operatorname{Sq}^0 = \operatorname{Sq}^0 , \quad \sum_i \operatorname{Sq}^i \chi \operatorname{Sq}^{n-i} = 0 \quad \text{for } n > 0 , \tag{2}$$

has a topological interpretation too: If K is a finite complex then the homology of the Spanier–Whitehead dual DK_+ of K_+ is canonically isomorphic to the cohomology of K. Under this isomorphism the left action by $\theta \in \mathcal{A}$ on $H^*(K)$ corresponds to the right action of $\chi\theta \in \mathcal{A}$ on $H_*(DK_+)$.

In 1974 Davis [3] proved that sometimes much more efficient ways exist to compute $\chi \operatorname{Sq}^n$; for example

$$\chi \operatorname{Sq}^{2^r-1} = \operatorname{Sq}^{2^{r-1}} \chi \operatorname{Sq}^{2^{r-1}-1} , \tag{3}$$

$$\chi \operatorname{Sq}^{2^r-r-1} = \operatorname{Sq}^{2^{r-1}-1} \chi \operatorname{Sq}^{2^{r-1}-r} + \operatorname{Sq}^{2^{r-1}} \chi \operatorname{Sq}^{2^{r-1}-r-1} \ . \tag{4}$$

Similarly, Straffin [6] proved that if $r \geq 0$ and $b \geq 2$ then

$$\sum_i \operatorname{Sq}^{2^r i} \chi \operatorname{Sq}^{2^r (b-i)} = 0 \, . \tag{5}$$

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Both authors give analogous identities among reduced powers and their images under χ at an odd prime as well. Further relations among the Steenrod squares and their conjugates appear in these articles and elsewhere (eg Silverman [5]).

Barratt and Miller [1] found a general family of identities which includes (3), (4) and (5), and their odd-prime analogues, as special cases. We state it for the general prime. When $p=2$, P^n denotes Sq^n. Let $\alpha(n)$ denote the sum of the p–adic digits of n.

Theorem 1.1 [1; 2] *For any integer k and any integer $l \geq 0$ such that $pl-\alpha(l) < (p-1)n$,*

$$\sum_i \binom{k-i}{l} P^i \chi P^{n-i} = 0\,. \tag{6}$$

The relations defining χ occur with $l=0$. Davis' formulas (for $p=2$) are the cases in which $(n,l,k) = (2^r-1, 2^{r-1}-1, 2^r-1)$ or $(n,l,k) = (2^r-r-1, 2^{r-1}-2, 2^r-2)$. Straffin's identities (for $p=2$) occur as $(n,l,k) = (2^r b, 2^r-1, -1)$.

Since $\binom{(k+1)-i}{l} - \binom{k-i}{l} = \binom{k-i}{l-1}$, the cases $(l, k+1)$ and (l,k) of (6) imply it for $(l-1,k)$. Thus the relations for $l = \phi(n)-1$, where

$$\phi(n) = 1 + \max\{j : pj - \alpha(j) < (p-1)n\}\,, \tag{7}$$

imply all the rest. Here we have adopted the notation $\phi(n)$ used in [2]; we note that it is not the Euler function $\varphi(n)$.

When $p=2$, $\phi(2^r-1) = 2^{r-1}$ and $\phi(2^r-r-1) = 2^{r-1}-1$, so Davis's relations are among these basic relations.

Two questions now arise. To express them uniformly in the prime, let $\mathcal{P}$ denote the algebra of Steenrod reduced powers (which is the full Steenrod algebra when $p=2$), but assign P^n degree n. Write

$$V_n = \mathrm{Span}\{P^i \chi P^{n-i} : 0 \leq i \leq n\} \subseteq \mathcal{P}^n\,.$$

It is natural to ask:

- Are there yet other linear relations among the $n+1$ elements $P^i \chi P^{n-i}$ in $\mathcal{P}^n$?
- What is a basis for V_n?

We answer these questions in Theorem 1.4 below.

Write $e_i, 0 \leq i \leq n$, for the i–th standard basis vector in $\mathbb{F}_p^{n+1}$.

Proposition 1.2 *For any integers l, m, n, with $0 \leq l \leq n$,*

$$\left\{ \sum_i \binom{k-i}{l} e_i : m \leq k \leq m+l \right\} \tag{8}$$

is linear independent in $\mathbb{F}_p^{n+1}$.

Proposition 1.3 *The set*

$$\left\{ P^i \chi P^{n-i} : \phi(n) \leq i \leq n \right\} \tag{9}$$

is linearly independent in $\mathcal{P}^n$.

Define a linear map

$$\mu \colon \mathbb{F}_p^{n+1} \to \mathcal{P}^n , \quad \mu e_i = P^i \chi P^{n-i} . \tag{10}$$

Theorem 1.1 implies that if $l = \phi(n) - 1$ the elements in (8) lie in $\ker \mu$, so Propositions 1.2 and 1.3 imply that (8) with $l = \phi(n) - 1$ is a basis for $\ker \mu$ and that (9) is a basis for $V_n \subseteq \mathcal{P}^n$. Thus:

Theorem 1.4 *Any $\phi(n)$ consecutive relations from the set* (6) *with $l = \phi(n) - 1$ form a basis of relations among the elements of $\{P^i \chi P^{n-i} : 0 \leq i \leq n\}$. The set $\{P^i \chi P^{n-i} : \phi(n) \leq i \leq n\}$ is a basis for V_n.*

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2 Independence of the relations

We wish to show that (8) is a linearly independent set. Regard elements of $\mathbb{F}_p^{n+1}$ as column vectors, and arrange the $l+1$ vectors in (8) as columns in a matrix, which we claim is of rank $l+1$. The top square portion is the mod p reduction of the $(l+1) \times (l+1)$ integral Toeplitz matrix $A_l(m)$ with (i, j)–th entry

$$\binom{m+j-i}{l}, \quad 0 \leq i, j \leq l .$$

Lemma 2.1 $\det A_l(m) = 1$.

Proof By induction on m. Since $\binom{-1}{l} = (-1)^l$ and $\binom{-1+j}{l} = 0$ for $0 < j \le l$, $A_l(-1)$ is lower triangular with determinant $((-1)^l)^{l+1} = 1$. Now we note the identity

$$BA_l(m) = A_l(m+1)$$

where

$$B = \begin{bmatrix} \binom{l+1}{1} & -\binom{l+1}{2} & \cdots & (-1)^{l-1}\binom{l+1}{l} & (-1)^l\binom{l+1}{l+1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

The matrix identity is an expression of the binomial identity

$$\sum_k (-1)^k \binom{l+1}{k}\binom{n-k}{l} = 0 \tag{11}$$

(taking $n = m+1-j$ and $k = j+1$). Since $\det B = 1$, the result follows for all $m \in \mathbb{Z}$. □

For completeness, we note that (11) is the case $m = l+1$ of the equation

$$\sum_k (-1)^k \binom{m}{k}\binom{n-k}{l} = \binom{n-m}{l-m}. \tag{12}$$

To prove this formula, note that the defining identity for binomial coefficients implies the case $m = 1$, and also that both sides satisfy the recursion $C(l,m,n) - C(l,m,n-1) = C(l,m+1,n)$.

3 Independence of the operations

We will prove Proposition 1.3 by studying how $P^i \chi P^{n-i}$ pairs against elements in $\mathcal{P}_*$, the dual of the Hopf algebra of Steenrod reduced powers. According to Milnor [4], with our grading conventions

$$\mathcal{P}_* = \mathbb{F}_p[\xi_1, \xi_2, \ldots], \quad |\xi_j| = \frac{p^j - 1}{p-1},$$

$$\Delta \xi_k = \sum_{i+j=k} \xi_i^{p^j} \otimes \xi_j\,. \tag{13}$$

For a finitely nonzero sequence of nonnegative integers $R = (r_1, r_2, \dots)$ write $\xi^R = \xi_1^{r_1}\xi_2^{r_2}\cdots$ and let $\|R\| = r_1 + pr_2 + p^2 r_3 + \cdots$ and

$$|R| = |\xi^R| = r_1 + \left(\frac{p^2-1}{p-1}\right) r_2 + \left(\frac{p^3-1}{p-1}\right) r_3 + \cdots .$$

The following clearly implies Proposition 1.3.

Proposition 3.1 *For any integer $n > 0$ there exist sequences $R_{n,j}$, $0 \le j \le n - \phi(n)$, such that $|R_{n,j}| = n$ and*

$$\langle P^i \chi P^{n-i}, \xi^{R_{n,j}} \rangle = \begin{cases} \pm 1 & \text{for } i = n-j\,, \\ 0 & \text{for } i > n-j\,. \end{cases}$$

The starting point in proving this is the following result of Milnor.

Lemma 3.2 [4, Corollary 6] *$\langle \chi P^n, \xi^R \rangle = \pm 1$ for all sequences R with $|R| = n$.*

In the basis of $\mathcal{P}$ dual to the monomial basis of $\mathcal{P}_*$, the element corresponding to ξ_1^i is P^i. Since the diagonal in $\mathcal{P}_*$ is dual to the product in $\mathcal{P}$, it follows from (13) and Lemma 3.2 that

$$\langle P^i \chi P^{n-i}, \xi^R \rangle = \begin{cases} \pm 1 & \text{for } i = \|R\|\,, \\ 0 & \text{for } i > \|R\|\,. \end{cases}$$

So we wish to construct sequences $R_{n,j}$, for $\phi(n) \le j \le n$, such that $|R_{n,j}| = n$ and $\|R_{n,j}\| = j$. We deal first with the case $j = \phi(n)$.

Proposition 3.3 *For any $n \ge 0$ there is a sequence $M = (m_1, m_2, \dots)$ such that*

(1) $|M| = n$,

(2) $0 \le m_i \le p$ *for all i,*

(3) *if $m_j = p$ then $m_i = 0$ for all $i < j$.*

For any such sequence, $\|M\| = \phi(n)$.

Proof Give the set of sequences of dimension n the right-lexicographic order. We claim that the maximal sequence satisfies the hypotheses.

Suppose that $R = (r_1, r_2, \dots)$ does not satisfy the hypotheses. If $r_1 > p$ then the sequence $(r_1 - (p+1), r_2 + 1, r_3, \dots)$ is larger. If $r_j > p$, with $j > 1$, then the sequence $(r_1, \dots, r_{j-2}, r_{j-1} + p, r_j - (p+1), r_{j+1} + 1, r_{k+2}, \dots)$ is larger. This proves (2). To prove (3), suppose that $r_j = p$ with $j > 1$, and suppose that some earlier entry is nonzero. Let $i = \min\{k : r_k > 0\}$. If $i = 1$, then the sequence

$(r_1-1, r_2, \dots, r_{j-1}, 0, r_{j+1}+1, r_{j+2}, \dots)$ is larger. If $i>1$, then S with $s_k=0$ for $k<i-1$ and $i\le k\le j$, $s_{i-1}=p$, $s_{j+1}=r_{j+1}+1$, and $s_k=r_k$ for $k>j+1$, is larger.

Let M be a sequence satisfying (1)–(3), and write $l=\|M\|-1$. To see that $l=\phi(n)-1$ we must show that

$$p(l+1)-\alpha(l+1)\ge(p-1)n, \tag{14}$$

$$pl-\alpha(l)<(p-1)n\,. \tag{15}$$

The *excess* $e(R)$ is the sum of the entries in R, so that $p\|R\|-e(R)=(p-1)|R|$. The p–adic representation of a number minimizes excess, so for any sequence R we have $e(R)\ge\alpha(\|R\|)$ and hence $p\|R\|-\alpha(\|R\|)\ge(p-1)|R|$: so (14) holds for any sequence.

To see that (15) holds for M, let $j=\min\{i:m_i>0\}$, so that $(p-1)n=(p^j-1)m_j+(p^{j+1}-1)m_{j+1}+\cdots$ and $l+1=p^{j-1}m_j+p^jm_{j+1}+\cdots$. The hypotheses imply that l has p–adic expansion

$$(1+\cdots+p^{j-2})(p-1)+p^{j-1}(m_j-1)+p^jm_{j+1}+\cdots,$$

so

$$\alpha(l)=(j-1)(p-1)+(m_j-1)+m_{j+1}+\cdots$$

from which we deduce

$$pl-\alpha(l)=(p-1)(n-j)<(p-1)n\,.$$

This completes the proof of Proposition 3.3. □

Corollary 3.4 *The function $\phi(n)$ is weakly increasing.*

Proof Let M be a sequence satisfying the conditions of Proposition 3.3, and note that the sequence $R=(1,0,0,\dots)+M$ has $|R|=n+1$ and $\|R\|=\|M\|+1=\phi(n)+1$. If p does not occur in M, then R satisfies the hypotheses of the proposition (in degree $n+1$) and hence $\phi(n)\le\phi(n+1)$. If p does occur in M, then the moves described above will lead to a sequence M' satisfying the hypotheses. None of the moves decrease $\|-\|$, so $\phi(n)\le\phi(n+1)$. □

Remark 3.5 Properties (1)–(3) of Proposition 3.3 in fact determine M uniquely.

Proof of Proposition 3.1 Define $R_{n,\phi(n)}$ to be a sequence M as in Proposition 3.3. Then inductively define

$$R_{n,j}=(1,0,0,\dots)+R_{n-1,j-1}\quad\text{for } \phi(n)<j\le n\,.$$

This makes sense by monotonicity of $\phi(n)$, and the elements clearly satisfy $|R_{n,j}|=n$ and $\|R_{n,j}\|=j$. This completes the proof. □

References

[1] **M G Barratt**, **H R Miller**, *On the anti-automorphism of the Steenrod algebra*, Contemp. Math. 12 (1982) 47–52

[2] **M C Crabb**, **M D Crossley**, **J R Hubbuck**, *K–theory and the anti-automorphism of the Steenrod algebra*, Proc. Amer. Math. Soc. 124 (1996) 2275–2281 MR1328343

[3] **D M Davis**, *The antiautomorphism of the Steenrod algebra*, Proc. Amer. Math. Soc. 44 (1974) 235–236 MR0328934

[4] **J Milnor**, *The Steenrod algebra and its dual*, Ann. of Math. (2) 67 (1958) 150–171 MR0099653

[5] **J H Silverman**, *Conjugation and excess in the Steenrod algebra*, Proc. Amer. Math. Soc. 119 (1993) 657–661 MR1152292

[6] **P D Straffin, Jr**, *Identities for conjugation in the Steenrod algebra*, Proc. Amer. Math. Soc. 49 (1975) 253–255 MR0380796

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