More on the anti-automorphism of the Steenrod algebra

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The relations of Barratt and Miller are shown to include all relations among the elements $P^i \chi P^{n-i}$ in the mod $p$ Steenrod algebra, and a minimal set of relations is given.

1 Introduction

Milnor [4] observed that the mod 2 Steenrod algebra $A$ forms a Hopf algebra with commutative diagonal determined by

$$\Delta \text{Sq}^n = \sum_i \text{Sq}^i \otimes \text{Sq}^{n-i}.$$  

This allowed him to interpret the Cartan formula as the assertion that the cohomology of a space forms a module-algebra over $A$. The anti-automorphism $\chi$ in the Hopf algebra structure, defined inductively by

$$\chi \text{Sq}^0 = \text{Sq}^0, \quad \sum_i \text{Sq}^i \chi \text{Sq}^{n-i} = 0 \quad \text{for } n > 0,$$

has a topological interpretation too: If $K$ is a finite complex then the homology of the Spanier–Whitehead dual $DK_+$ of $K_+$ is canonically isomorphic to the cohomology of $K$. Under this isomorphism the left action by $\theta \in A$ on $H^*(K)$ corresponds to the right action of $\chi^\theta \in A$ on $H_*(DK_+)$.  

In 1974 Davis [3] proved that sometimes much more efficient ways exist to compute $\chi \text{Sq}^n$; for example

$$\chi \text{Sq}^{2^r-1} = \text{Sq}^{2^r-1} \chi \text{Sq}^{2^r-1-1},$$

$$\chi \text{Sq}^{2^r-r-1} = \text{Sq}^{2^r-1-1} \chi \text{Sq}^{2^r-1-r} + \text{Sq}^{2^r-1} \chi \text{Sq}^{2^r-1-r-1}.$$  

Similarly, Straffin [6] proved that if $r \geq 0$ and $b \geq 2$ then

$$\sum_i \text{Sq}^{2^r i} \chi \text{Sq}^{2^r (b-i)} = 0.$$
Both authors give analogous identities among reduced powers and their images under $\chi$ at an odd prime as well. Further relations among the Steenrod squares and their conjugates appear in these articles and elsewhere (e.g. Silverman [5]).

Barratt and Miller [1] found a general family of identities which includes (3), (4) and (5), and their odd-prime analogues, as special cases. We state it for the general prime. When $p = 2$, $P^n$ denotes $Sq^n$. Let $\alpha(n)$ denote the sum of the $p$–adic digits of $n$.

**Theorem 1.1** [1; 2] For any integer $k$ and any integer $l \geq 0$ such that $pl - \alpha(l) < (p - 1)n$,

$$\sum_i \binom{k-i}{l} P^i \chi P^{n-i} = 0.$$  

The relations defining $\chi$ occur with $l = 0$. Davis’ formulas (for $p = 2$) are the cases in which $(n, l, k) = (2^r - 1, 2^{r-1} - 1, 2^r - 1)$ or $(n, l, k) = (2^r - r - 1, 2^{r-1} - 2, 2^r - 2)$. Straffin’s identities (for $p = 2$) occur as $(n, l, k) = (2^r b, 2^r - 1, -1)$.

Since $\binom{k+1}{l-i} - \binom{k-i}{l} = \binom{k-i}{l-1}$, the cases $(l, k + 1)$ and $(l, k)$ of (6) imply it for $(l - 1, k)$. Thus the relations for $l = \phi(n) - 1$, where

$$\phi(n) = 1 + \max\{ j : pj - \alpha(j) < (p - 1)n \},$$

imply all the rest. Here we have adopted the notation $\phi(n)$ used in [2]; we note that it is not the Euler function $\varphi(n)$.

When $p = 2$, $\phi(2^r - 1) = 2^{r-1}$ and $\phi(2^r - r - 1) = 2^{r-1} - 1$, so Davis’s relations are among these basic relations.

Two questions now arise. To express them uniformly in the prime, let $\mathcal{P}$ denote the algebra of Steenrod reduced powers (which is the full Steenrod algebra when $p = 2$), but assign $P^n$ degree $n$. Write

$$V_n = \text{Span}\{ P^i \chi P^{n-i} : 0 \leq i \leq n \} \subseteq \mathcal{P}^n.$$  

It is natural to ask:

- Are there yet other linear relations among the $n + 1$ elements $P^i \chi P^{n-i}$ in $\mathcal{P}^n$?
- What is a basis for $V_n$?

We answer these questions in **Theorem 1.4** below.

Write $e_i$, $0 \leq i \leq n$, for the $i$–th standard basis vector in $\mathbb{F}_p^{n+1}$. 

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Proposition 1.2  For any integers \(l, m, n\), with \(0 \leq l \leq n\),
\[
\left\{ \sum_i \binom{k-i}{l} e_i : m \leq k \leq m+l \right\}
\]
is linear independent in \(\mathbb{F}_p^{n+1}\).

Proposition 1.3  The set
\[
\{ P^i \chi P^{n-i} : \phi(n) \leq i \leq n \}
\]
is linearly independent in \(\mathcal{P}^n\).

Define a linear map
\[
\mu: \mathbb{F}_p^{n+1} \to \mathcal{P}^n, \quad \mu e_i = P^i \chi P^{n-i}.
\]

Theorem 1.1 implies that if \(l = \phi(n) - 1\) the elements in (8) lie in \(\ker \mu\), so Propositions 1.2 and 1.3 imply that (8) with \(l = \phi(n) - 1\) is a basis for \(\ker \mu\) and that (9) is a basis for \(V_n \subseteq \mathcal{P}^n\). Thus:

Theorem 1.4  Any \(\phi(n)\) consecutive relations from the set (6) with \(l = \phi(n) - 1\) form a basis of relations among the elements of \(\{ P^i \chi P^{n-i} : 0 \leq i \leq n \}\). The set \(\{ P^i \chi P^{n-i} : \phi(n) \leq i \leq n \}\) is a basis for \(V_n\).

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2 Independence of the relations

We wish to show that (8) is a linearly independent set. Regard elements of \(\mathbb{F}_p^{n+1}\) as column vectors, and arrange the \(l+1\) vectors in (8) as columns in a matrix, which we claim is of rank \(l+1\). The top square portion is the mod \(p\) reduction of the \((l+1) \times (l+1)\) integral Toeplitz matrix \(A_l(m)\) with \((i, j)\)-th entry
\[
\binom{m+j-i}{l}, \quad 0 \leq i, j \leq l.
\]

Lemma 2.1  \(\det A_l(m) = 1\).
Proof By induction on $m$. Since $(-1)^l = (-1)^l$ and $(-1)^{l+j} = 0$ for $0 < j \leq l$, $A_l(1)$ is lower triangular with determinant $((-1)^l)^{l+1} = 1$. Now we note the identity

$$BA_l(m) = A_l(m + 1)$$

where

$$B = \begin{pmatrix}
\binom{l+1}{1} & -(l+1) & \cdots & -1 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix}.$$ 

The matrix identity is an expression of the binomial identity

$$(11) \quad \sum_k (-1)^k \binom{l+1}{k} \binom{n-k}{l} = 0$$

(taking $n = m + 1 - j$ and $k = j + 1$). Since $\det B = 1$, the result follows for all $m \in \mathbb{Z}$. \hfill \square

For completeness, we note that (11) is the case $m = l + 1$ of the equation

$$(12) \quad \sum_k (-1)^k \binom{m}{k} \binom{n-k}{l} = \binom{n-m}{l-m}.$$ 

To prove this formula, note that the defining identity for binomial coefficients implies the case $m = 1$, and also that both sides satisfy the recursion $C(l, m, n) - C(l, m, n - 1) = C(l, m + 1, n)$.

3 Independence of the operations

We will prove Proposition 1.3 by studying how $P^i \chi P^{n-i}$ pairs against elements in $\mathcal{P}_*$, the dual of the Hopf algebra of Steenrod reduced powers. According to Milnor [4], with our grading conventions

$$\mathcal{P}_* = \mathbb{F}_p[\xi_1, \xi_2, \ldots], \quad |\xi_j| = \frac{p^j - 1}{p - 1},$$

$$(13) \quad \Delta \xi_k = \sum_{i + j = k} \xi_i^{p^j} \otimes \xi_j.$$
For a finitely nonzero sequence of nonnegative integers \( R = (r_1, r_2, \ldots) \) write \( \xi^R = \xi_1^{r_1} \xi_2^{r_2} \cdots \) and let \( ||R|| = r_1 + p r_2 + p^2 r_3 + \cdots \) and

\[
|R| = |\xi^R| = r_1 + \left( \frac{p^2 - 1}{p - 1} \right) r_2 + \left( \frac{p^3 - 1}{p - 1} \right) r_3 + \cdots.
\]

The following clearly implies Proposition 1.3.

**Proposition 3.1** For any integer \( n > 0 \) there exist sequences \( R_{n,j}, 0 \leq j \leq n - \phi(n), \) such that \( |R_{n,j}| = n \) and

\[
\langle P_i \chi^{P^n - i}, \xi^{R_{n,j}} \rangle = \begin{cases} 
\pm 1 & \text{for } i = n - j, \\
0 & \text{for } i > n - j.
\end{cases}
\]

The starting point in proving this is the following result of Milnor.

**Lemma 3.2** [4, Corollary 6] \( \langle \chi^{P^n}, \xi^R \rangle = \pm 1 \) for all sequences \( R \) with \( |R| = n \).

In the basis of \( \mathcal{P} \) dual to the monomial basis of \( \mathcal{P}_* \), the element corresponding to \( \xi_1^i \) is \( P^i \). Since the diagonal in \( \mathcal{P}_* \) is dual to the product in \( \mathcal{P} \), it follows from (13) and Lemma 3.2 that

\[
\langle P^i \chi^{P^n - i}, \xi^R \rangle = \begin{cases} 
\pm 1 & \text{for } i = ||R||, \\
0 & \text{for } i > ||R||.
\end{cases}
\]

So we wish to construct sequences \( R_{n,j}, \) for \( \phi(n) \leq j \leq n, \) such that \( |R_{n,j}| = n \) and \( ||R_{n,j}|| = j \). We deal first with the case \( j = \phi(n) \).

**Proposition 3.3** For any \( n \geq 0 \) there is a sequence \( M = (m_1, m_2, \ldots) \) such that

1. \( |M| = n, \)
2. \( 0 \leq m_i \leq p \) for all \( i, \)
3. if \( m_j = p \) then \( m_i = 0 \) for all \( i < j. \)

For any such sequence, \( ||M|| = \phi(n). \)

**Proof** Give the set of sequences of dimension \( n \) the right-lexicographic order. We claim that the maximal sequence satisfies the hypotheses.

Suppose that \( R = (r_1, r_2, \ldots) \) does not satisfy the hypotheses. If \( r_1 > p \) then the sequence \( (r_1 - (p + 1), r_2 + 1, r_3, \ldots) \) is larger. If \( r_j > p, \) with \( j > 1, \) then the sequence \( (r_1, \ldots, r_{j-2}, r_{j-1} + 1, r_j - (p + 1), r_{j+1} + 1, r_{k+2}, \ldots) \) is larger. This proves (2). To prove (3), suppose that \( r_j = p \) with \( j > 1, \) and suppose that some earlier entry is nonzero. Let \( i = \min\{k : r_k > 0\}. \) If \( i = 1, \) then the sequence
Let $M$ be a sequence satisfying (1)–(3), and write $l = \|M\| - 1$. To see that (15) holds for $M$, let $j = \min\{i : m_i > 0\}$, so that $(p-1)n = (p^j - 1)m_j + (p^{j+1} - 1)m_{j+1} + \cdots$ and $l + 1 = p^{j-1}m_j + p^jm_{j+1} + \cdots$. The hypotheses imply that $l$ has $p$–adic expansion
\[
1 + \cdots + p^{j-2}(p-1) + p^{j-1}(m_j - 1) + p^jm_{j+1} + \cdots,
\]
so
\[
\alpha(l) = (j - 1)(p-1) + (m_j - 1) + m_{j+1} + \cdots
\]
from which we deduce
\[
pl - \alpha(l) = (p-1)(n-j) < (p-1)n.
\]
This completes the proof of Proposition 3.3.

\[\square\]

Corollary 3.4 The function $\phi(n)$ is weakly increasing.

Proof Let $M$ be a sequence satisfying the conditions of Proposition 3.3, and note that the sequence $R = (1, 0, 0, \ldots) + M$ has $|R| = n + 1$ and $\|R\| = \|M\| + 1 = \phi(n) + 1$. If $p$ does not occur in $M$, then $R$ satisfies the hypotheses of the proposition (in degree $n+1$) and hence $\phi(n) \leq \phi(n+1)$. If $p$ does occur in $M$, then the moves described above will lead to a sequence $M'$ satisfying the hypotheses. None of the moves decrease $\|\cdot\|$, so $\phi(n) \leq \phi(n+1)$.

\[\square\]

Remark 3.5 Properties (1)–(3) of Proposition 3.3 in fact determine $M$ uniquely.

Proof of Proposition 3.1 Define $R_{n,\phi(n)}$ to be a sequence $M$ as in Proposition 3.3. Then inductively define
\[
R_{n,j} = (1, 0, 0, \ldots) + R_{n-1,j-1} \text{ for } \phi(n) < j \leq n.
\]
This makes sense by monotonicity of $\phi(n)$, and the elements clearly satisfy $|R_{n,j}| = n$ and $\|R_{n,j}\| = j$. This completes the proof.

\[\square\]
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