Unstable Adams operations on p–local compact groups

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A p–local compact group is an algebraic object modelled on the p–local homotopy theory of classifying spaces of compact Lie groups and p–compact groups. In the study of these objects unstable Adams operations are of fundamental importance. In this paper we define unstable Adams operations within the theory of p–local compact groups and show that such operations exist under rather mild conditions. More precisely, we prove that for a given p–local compact group $\mathcal{G}$ and a sufficiently large positive integer m, there exists an injective group homomorphism from the group of p–adic units which are congruent to 1 modulo p^m to the group of unstable Adams operations on $\mathcal{G}$.

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Let p be a prime number. Broto, the second author and Oliver [9] developed the theory of p–local compact groups. The theory is modelled on the p–local homotopy theory of classifying spaces of compact Lie groups and p–compact groups and generalises the earlier concept of p–local finite groups, also due to Broto, the second author and Oliver [8]. It provides a coherent context in which classifying spaces of compact Lie groups and p–compact groups (see Dwyer and Wilkerson [12]) can be studied and also gives rise to many exotic examples. Roughly speaking, a p–local compact group is a triple $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$, where S is a discrete p–toral group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. More specifically, the group S is an extension of a finite p–group by a group of the form $(\mathbb{Z}/p^\infty)^r$, where $\mathbb{Z}/p^\infty = \bigcup_k \mathbb{Z}/p^k$. One can think of S as a Sylow p–subgroup of $\mathcal{G}$ in an appropriate sense. The saturated fusion system $\mathcal{F}$ is a category whose set of objects consists of all subgroups of S, and whose morphisms model conjugacy relations among subgroups of S. The centric linking system $\mathcal{L}$ is again a category which is an enrichment of $\mathcal{F}$ with just about enough structure to allow one to associated with $\mathcal{F}$ a homotopy theoretically meaningful "classifying space" given by the space $|\mathcal{L}|^\wedge_p$ which we shall denote by $B\mathcal{G}$. Here $(-)^\wedge_p$ denotes the Bousfield–Kan [5] p–completion functor, and $|-|$ stands for the functor which associates with a small category its nerve. Compact Lie groups and p–compact groups give rise to p–local compact groups,

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whose classifying spaces coincide up to homotopy with the p–completed classifying space of the object from which they originate.

Unstable Adams operations are certain self-equivalences of classifying spaces of compact Lie groups, which were also defined and studied for p–compact groups (see for instance Jackowski, McClure and Oliver [13] and Andersen et al [2]). For a compact connected Lie group G, one defines an unstable Adams operation of degree k on G, for some integer k, to be any self-map of the classifying space BG which induces multiplication by k^i on $H^{2i}(BG, \mathbb{Q})$. There is an analogous definition for connected p–compact groups, where the degree is allowed to be any p–adic unit and rational cohomology is replaced by $H^*_{\mathbb{Q}_p}(-) = H^*(-, \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$.

Unstable Adams operations are of fundamental importance in the study of classifying spaces of compact Lie and p–compact groups, which motivates the study undertaken in this paper, the aim of which is to prove the existence of unstable Adams operations on p–local compact groups.

For a compact Lie group G, the rational cohomology $H^*(BG, \mathbb{Q})$ is given by the invariants of the action of the Weyl group of G on $H^*(BT, \mathbb{Q})$, where $T \leq G$ is a maximal torus by Borel [4]. Thus it is not hard to see, as we will observe in Section 2, that the definition of an unstable Adams operation of degree k on BG is equivalent to the existence of a self-map of BG, which extends the map induced by the k–th power map on BT. The analogous description of cohomology holds for connected p–compact groups by Dwyer and Wilkerson [12, Theorem 9.7], and so here as well one can define an unstable Adams operation as a self-map extending the map induced by the appropriate power map on the maximal torus.

A precise algebraic definition of what unstable Adams operations actually mean for p–local compact groups will be given in Section 3. Roughly and more geometrically speaking, given a p–local compact group $\mathcal{G}$ and a p–adic unit ζ, an unstable Adams operation of $\mathcal{G}$ of degree ζ is a self-equivalence of $B\mathcal{G}$, which restricts up to homotopy to the map induced by the ζ power map on the classifying space of the maximal torus of $\mathcal{G}$ (see Definition 1.1). A comparison of this notion with the algebraic concept in Definition 3.3 will be carried out in Section 3 as well. The collection of all ("algebraic") unstable Adams operation on $\mathcal{G}$ forms a group under composition called the group of unstable Adams operations on $\mathcal{G}$, which we denote by $\mathrm{Ad}(\mathcal{G})$.

For a positive integer k, let $\Gamma_k(p)$ denote the subgroup of all p–adic units $\zeta \in \mathbb{Z}_p^\times$ such that $\zeta - 1$ is divisible by p^k. Thus $\Gamma_0(p) = \mathbb{Z}_p^\times$, and for each $k > 0$, elements of $\Gamma_k(p)$ have the form $1 + a_k p^k + \cdots$. We are now ready to state our main theorem.

Theorem A *Let $\mathcal{G}$ be a p–local compact group. Then for a sufficiently large positive integer k there exists a group homomorphism*

$$\alpha_{\mathcal{G}}\colon \Gamma_k(p) \to \mathrm{Ad}(\mathcal{G}),$$

such that for any $\zeta \in \Gamma_k(p)$, $\alpha_{\mathcal{G}}(\zeta)$ is an unstable Adams operation of $\mathcal{G}$ of degree ζ.

The theorem will be restated as Theorem 4.1 in terms both more precise and more suitable to the theory of p–local compact groups. Several remarks are in place. Adams operations of degree ζ as defined in this paper induce the ζ^i–power map on $H^{2i}_{\mathbb{Q}_p}(B\mathcal{G})$, where $H^*_{\mathbb{Q}_p}$ is the functor $H^*(-,\mathbb{Z}_p)\otimes\mathbb{Q}$. The proof requires a result of Broto, Levi and Oliver [7], which describes $H^*_{\mathbb{Q}_p}(B\mathcal{G})$ as the invariants of the action of the Weyl group on $H^*_{\mathbb{Q}_p}(BT)$, where T is the maximal torus of $\mathcal{G}$, in analogy with the classical result for compact Lie groups and p–compact groups. A discussion of the precise relationship between unstable Adams operations as defined here, and the cohomological concept will be discussed in [7].

As we make no assumption at all on the p–local compact group in Theorem A, one can not expect the degree of an unstable Adams operation to determine the operation, even up to homotopy. Also, in general, not every degree $\zeta \in \mathbb{Z}_p^\times$ can occur as the degree of an unstable Adams operation. See Example 3.6 and Example 3.7.

Given a p–local compact group $\mathcal{G}$, we do not have a way to determine which degrees can be realised by an unstable Adams operation. However, the Adams operations constructed in this paper satisfy a certain strong condition. We do not know whether or not this condition is satisfied by an arbitrary unstable Adams operation. In a subsequent paper by the second and third authors, operations which do satisfy this extra condition will be studied. We show there in particular that for such operations the possible degrees are exactly the subring $\Gamma_k(p)$ where k is determined in terms of certain cohomology groups.

We point out that in our construction no connectivity assumption has to be made, although one would expect that some conditions, analogous to connectivity in compact Lie groups, will have to be assumed in dealing with uniqueness questions.

This paper is an elaboration of the PhD dissertation of the first author. It is organised as follows. Section 1 collects some basic facts on p–local compact groups. In Section 2 we define and study Adams automorphisms of p–toral groups, and in Section 3 we define and discuss unstable Adams operations on p–local compact groups. Finally, in Section 4 we prove Theorem A. More precisely, for a p–local compact group $\mathcal{G}$, we construct in Theorem 4.1 families of subgroups of the group $\mathrm{Ad}(\mathcal{G})$ of unstable Adams operations of $\mathcal{G}$.

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1 Preliminaries on p–local compact groups

We recall the definition and some basic properties of p–local compact groups. The reader is referred to [9] for a comprehensive account of these objects.

The fundamental objects in this theory are discrete p–toral groups. By $\mathbb{Z}/p^\infty$ we mean the union of all $\mathbb{Z}/p^r$ with respect to the natural inclusions.

Definition 1.1 A group of the form $(\mathbb{Z}/p^\infty)^r$ for some positive integer r will be referred to as a *discrete p–torus*. A *discrete p–toral group* is an extension S of a finite p–group Γ by a discrete p–torus S_0. The normal subgroup S_0 will be referred to as the *maximal torus* or the *identity component* of S, and the quotient group $\Gamma \cong S/S_0$ will be called the *group of components* of S.

The identity component P_0 of a discrete p–toral group P can be characterised as the subset of all infinitely p–divisible elements in P, and also as the minimal subgroup of finite index in P. Thus, P_0 is a characteristic subgroup. The rank of P is the number $r = \mathrm{rk}(P)$ such that $P_0 \cong (\mathbb{Z}/p^\infty)^r$.

Recall that given $P, Q \leq S$ the elements $g \in S$ such that $gPg^{-1} \leq Q$ form the transporter set $N_S(P, Q)$. The set $\mathrm{Hom}_S(P, Q)$ of all homomorphisms $c_g\colon P \to Q$, which are restrictions of an inner automorphism of S is obtained by identifying two elements in $N_S(P, Q)$ if they differ by an element of the centraliser $C_S(P)$. Let $\mathrm{Inj}(P, Q)$ denote the set of all the injective homomorphisms $P \to Q$. We are now ready to recall the definition of fusion systems over discrete p–toral groups.

Definition 1.2 A *fusion system* $\mathcal{F}$ over a discrete p–toral group S is a category whose objects are the subgroups of S, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ satisfy the following conditions:

(i) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ for all $P, Q \leq S$.

(ii) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

Two subgroups $P, P' \leq S$ are called *$\mathcal{F}$–conjugate* if $\mathrm{Iso}_{\mathcal{F}}(P, P') \neq \varnothing$. A subgroup $P \leq S$ is said to be *$\mathcal{F}$–centric* if for every subgroup $P' \leq S$ which is $\mathcal{F}$–conjugate to P, $C_S(P') = Z(P')$.

The theory of p–local compact groups makes a very essential use of an extra set of axioms one requires fusion systems to satisfy. Fusion systems that satisfy these axioms are called *saturated*. In this paper we do not directly need to use the saturation axioms, and we will therefore not spell them out. It should be emphasised, however, that throughout the paper only saturated fusion systems will be considered and we will heavily rely upon some of their properties. The interested reader is referred to [9, Definition 2.2].

The next fundamental concept in the theory is that of a centric linking system.

Definition 1.3 Let $\mathcal{F}$ be a fusion system over a discrete p–toral group S. A *centric linking system associated to* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$–centric subgroups of S, together with a functor

$$\pi\colon \mathcal{L} \to \mathcal{F}^c$$

and "distinguished" monomorphisms $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$–centric subgroup $P \leq S$, which satisfy the following conditions.

(A) π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \mathrm{Aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

(B) For each $\mathcal{F}$–centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ {\scriptstyle \delta_P(g)}\downarrow & & \downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow{f} & Q. \end{array}$$

Next we recall the definition of our fundamental object.

Definition 1.4 A *p–local compact group* is a triple $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$, where S is a discrete p–toral group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The *classifying space* of $\mathcal{G}$ is the p–completed nerve $|\mathcal{L}|^\wedge_p$, which we will generally denote by $B\mathcal{G}$.

In [9] the authors show that compact Lie groups and p–compact groups give rise to particular examples of p–local compact groups. Another large family of examples arises from linear torsion groups. In all cases the respective classifying space coincides up to homotopy (after p–completion in the case of genuine groups) with the classifying space of the p–local compact group it gives rise to.

Before moving on, we will need to record several key properties of linking systems that will be used later.

Proposition 1.5 *There is a choice of functions $\delta_{P,Q}\colon N_S(P,Q) \to \mathrm{Mor}_{\mathcal{L}}(P,Q)$ for all $\mathcal{F}$–centric $P, Q \leq S$ such that the following holds.*

- *For all $x \in N_S(P,Q)$, $\pi(\delta_{P,Q}(x)) = c_x$.*
- *The restriction of $\delta_{P,P}\colon N_S(P) \to \mathrm{Aut}_{\mathcal{L}}(P)$ to P is equal to the distinguished monomorphism $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$.*
- *$\delta_{Q,R}(x) \circ \delta_{P,Q}(y) = \delta_{P,R}(xy)$ for all $x \in N_S(Q,R)$ and all $y \in N_S(P,Q)$.*

Proof For every $P \in \mathcal{F}^c$ choose once and for all morphisms $\iota_P \in \mathrm{Mor}_{\mathcal{L}}(P,S)$ which project to the inclusion $P \leq S$ in $\mathcal{F}$. In particular let ι_S be the identity on S. The proof is identical to that in [8, Proposition 1.11]. One uses [9, Lemma 4.3(a)] instead of [8, Proposition 1.10(a)]. □

Remark 1.6 We will write $[\varphi]$ for the image in $\mathcal{F}$ of a morphism $\varphi \in \mathcal{L}$ under the projection π. Throughout this paper we will always assume that p–local compact groups are equipped with a choice of functions $\delta_{P,Q}$ as in Proposition 1.5. The image of $g \in N_S(P,Q)$ under $\delta_{P,Q}$ will be denoted $\widehat{g}$, and with this notation $\widehat{g} \circ \widehat{h} = \widehat{gh}$. For $P \leq Q$, the image of the identity element $e \in N_S(P,Q)$ will be denoted ι_P^Q. Note that $\iota_P^P = \mathrm{Id}_P$, and that $[\widehat{g}] = c_g$.

Lemma 1.7 *Let $(S, \mathcal{F}, \mathcal{L})$ be a p–local compact group.*

(i) *Let $P \xrightarrow{a} Q \xrightarrow{b} R$ be morphisms in $\mathcal{F}^c$. If $\beta \in \mathrm{Mor}_{\mathcal{L}}(Q,R)$ and $\gamma \in \mathrm{Mor}_{\mathcal{L}}(P,R)$ satisfy $[\beta] = b$ and $[\gamma] = b \circ a$ then there exists a unique $\alpha \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ such that $[\alpha] = a$ and $\beta \circ \alpha = \gamma$.*

(ii) *If P, Q, R and a, b and γ as above and if $\alpha \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ satisfies $[\alpha] = a$ then there exists a unique $\beta \in \mathrm{Mor}_{\mathcal{L}}(Q,R)$ such that $\beta \circ \alpha = \gamma$, (but β is not generally a lift of b).*

(iii) *For any $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ there exists a unique isomorphism $\varphi' \in \mathrm{Iso}_{\mathcal{L}}(P,P')$ such that $\varphi = \iota_{P'}^Q \circ \varphi'$.*

Proof Point (i) is proven in [9, Proposition 4.3(a)], and point (iii) is an immediate consequence of (i), since one has a factorization $[\varphi] = \mathrm{incl}_{P'}^{Q} \circ f$, for some $f \in \mathrm{Iso}_{\mathcal{F}}(P, P')$.

It remains to prove point (ii). We may assume, in light of point (iii), that $P \leq Q$ and $\alpha = \iota_P^Q$. Choose an arbitrary $\beta' \in \mathrm{Mor}_{\mathcal{L}}(Q, R)$ such that $[\beta'] = b$. Then $[\gamma] = b \circ a = [\beta' \circ \iota_P^Q]$. By axiom (A) in Definition 1.3 there is some $z \in Z(P)$ such that $\gamma = \beta' \circ \iota_P^Q \circ \hat{z} = \beta' \circ \hat{z} \circ \iota_P^Q$. Note that $z \in Z(P) \leq Q$ and therefore $\beta := \beta' \circ \hat{z} \in \mathrm{Mor}_{\mathcal{L}}(Q, R)$ satisfies $\gamma = \beta \circ \iota_P^Q$.

To prove uniqueness, assume that $\beta_1 \circ \iota_P^Q = \beta_2 \circ \iota_P^Q$ for some $\beta_1, \beta_2 \in \mathrm{Mor}_{\mathcal{L}}(Q, R)$. In particular $[\beta_1]|_P = [\beta_2]|_P$, and [9, Proposition 2.8] implies that there exists $g \in Z(P)$ such that $[\beta_2] = [\beta_1] \circ c_g$. By axioms (A) and (B) of Definition 1.3 and by Remark 1.6, there is some $z \in Z(Q)$ such that $\beta_2 = \beta_1 \circ \widehat{gz}$. Note that $Z(Q) = C_S(Q) \leq C_S(P) = Z(P)$ so $gz \in Z(P)$. By the hypothesis on β_1 and β_2 we now get $\beta_1 \circ \iota_P^Q = \beta_2 \circ \iota_P^Q = \beta_1 \circ \widehat{gz} \circ \iota_P^Q = \beta_1 \circ \iota_P^Q \circ \widehat{gz}$. Since $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, R)$ we deduce that $gz = 1$ and therefore $\beta_1 = \beta_2$, whence the morphism β is unique. □

Corollary 1.8 (cf [6, Proposition 3.10]) *All morphisms in a centric linking system $\mathcal{L}$ associated to a saturated fusion system are both monomorphisms and epimorphisms in the categorical sense.*

Proof If $\beta \circ \alpha_1 = \beta \circ \alpha_2$ in $\mathcal{L}$ then $[\alpha_1] = [\alpha_2]$, since $[\beta]$ is group monomorphism. Therefore $\alpha_1 = \alpha_2$, by point (i) of Lemma 1.7. Hence all the morphisms in $\mathcal{L}$ are monomorphisms. Similarly, by point (ii) of the same Lemma, if $\beta_1 \circ \alpha = \beta_2 \circ \alpha$ in $\mathcal{L}$, then $\beta_1 = \beta_2$, hence every morphism in $\mathcal{L}$ is an epimorphism. □

An important tool in the study of p–local compact groups, which will be useful in this paper as well, is the reduction of the object set of $\mathcal{F}$ to a subset of subgroups of S which contains only a finite number of S conjugacy classes. The precise details of this construction are only relevant in the proof of Proposition 1.13 and are recalled there. It suffices to say that with any $P \leq S$ one associates a subgroup $P^\bullet$ containing P. The family of objects which results from this construction has some remarkably useful properties, as we state below. Set

$$\mathcal{H}^\bullet(\mathcal{F}) = \{P^\bullet \mid P \leq S\},$$

and let $\mathcal{F}^\bullet \subseteq \mathcal{F}$ be the full subcategory with object set $\mathcal{H}^\bullet(\mathcal{F})$.

The following is a summary of some important properties of this construction. The statements and proofs are contained in [9, Lemma 3.2, Proposition 3.3, Corollaries 3.4–3.5].

Proposition 1.9 *The following hold for every saturated fusion system $\mathcal{F}$ over a discrete p–toral group S.*

(i) *The set $\mathcal{H}^\bullet(\mathcal{F})$ contains finitely many S–conjugacy classes of subgroups of S.*

(ii) *For subgroups $P, Q \leq S$, any morphism $f \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ extends to a unique morphism $f^\bullet \in \mathrm{Hom}_{\mathcal{F}}(P^\bullet, Q^\bullet)$. Thus $(-)^\bullet\colon \mathcal{F} \to \mathcal{F}^\bullet$ is a functor, which is left adjoint to the natural inclusion $\mathcal{F}^\bullet \subseteq \mathcal{F}$ and whose unit of adjunction is the inclusion $P \leq P^\bullet$. Moreover, $(-)^\bullet$ is an idempotent functor, namely $(P^\bullet)^\bullet = P^\bullet$, and it carries inclusions to inclusions, ie, if $P \leq Q$ then $P^\bullet \leq Q^\bullet$.*

Definition 1.10 Let $\mathcal{F}$ be a fusion system over S. An automorphism $\phi\colon S \to S$ is called *fusion preserving* if for every morphism $f \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ there exists a morphism $f' \in \mathrm{Hom}_{\mathcal{F}}(\phi(P), \phi(Q))$ such that $\phi \circ f = f' \circ \phi$.

If $\mathcal{F}$ is a fusion system over S, and ϕ is a fusion preserving automorphism of S, then ϕ induces an automorphism $\phi_*\colon \mathcal{F} \to \mathcal{F}$ where $\phi_*(P) = \phi(P)$ for all $P \leq S$ and $\phi_*(f) = \phi \circ f \circ \phi^{-1}$ for all morphisms $f \in \mathcal{F}$. Such automorphisms of $\mathcal{F}$ are referred to in the literature as "isotypical".

Definition 1.11 Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p–local compact group. Let $\phi\colon S \to S$ be a fusion preserving automorphism. We say that a functor $\Phi\colon \mathcal{L}' \to \mathcal{L}''$ where $\mathcal{L}'$ and $\mathcal{L}''$ are subcategories of $\mathcal{L}$, *covers* ϕ if:

(i) $\pi|_{\mathcal{L}''} \circ \Phi = \phi_* \circ \pi|_{\mathcal{L}'}$ where $\pi\colon \mathcal{L} \to \mathcal{F}$ is the projection functor, and ϕ_* is the isotypical automorphism of $\mathcal{F}$ induced by ϕ.

(ii) For each $P, Q \in \mathcal{L}'$ and $g \in N_S(P, Q)$, $\Phi(\widehat{g}) = \widehat{\phi(g)}$.

For a p–local compact group $(S, \mathcal{F}, \mathcal{L})$ let $\mathcal{F}^{c\bullet}$ denote the full subcategory of $\mathcal{F}^\bullet$ consisting of the $\mathcal{F}$–centric objects $P \leq S$. Let $\mathcal{L}^\bullet$ be the full subcategory of $\mathcal{L}$ whose objects are those of $\mathcal{F}^{c\bullet}$.

Proposition 1.12 *There is a unique functor $\mathcal{L} \xrightarrow{(-)^\bullet} \mathcal{L}^\bullet \subseteq \mathcal{L}$ which lifts $\mathcal{F} \xrightarrow{(-)^\bullet} \mathcal{F}^\bullet$ and which satisfies $\iota_Q^{Q^\bullet} \circ \varphi = \varphi^\bullet \circ \iota_P^{P^\bullet}$ for every morphism $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$. Moreover, $(\widehat{g})^\bullet = \widehat{g}$ for all $g \in N_S(P, Q)$.*

Proof On objects define $\mathcal{L} \xrightarrow{(-)^\bullet} \mathcal{L}^\bullet$ by $P \mapsto P^\bullet$. Let $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ be any morphism. By Proposition 1.9(ii), $\mathrm{incl}_Q^{Q^\bullet} \circ [\varphi] = [\varphi]^\bullet \circ \mathrm{incl}_P^{P^\bullet}$, and Lemma 1.7(ii) implies that there exists a unique morphism $\varphi^\bullet \in \mathrm{Mor}_{\mathcal{L}}(P^\bullet, Q^\bullet)$ such that

$$\varphi^\bullet \circ \iota_P^{P^\bullet} = \iota_Q^{Q^\bullet} \circ \varphi. \tag{1}$$

Define $(-)^\bullet$ on φ in this way. The fact that $[\varphi^\bullet] = [\varphi]^\bullet$ follows from the uniqueness statement in Proposition 1.9(ii) and shows in particular that $(-)^\bullet$, as defined on $\mathcal{L}$, lifts $(-)^\bullet\colon \mathcal{F} \to \mathcal{F}^\bullet$, provided we show it is a functor. The uniqueness of $\varphi^\bullet$ solving (1) easily implies that $(1_P)^\bullet = 1_{P^\bullet}$, and that $(-)^\bullet$ respects compositions. This shows that $(-)^\bullet$ is a functor on $\mathcal{L}$. The uniqueness of $\varphi^\bullet$ in Equation (prop-lift-bullet-to-l:eq-def-bullet) also shows that $(-)^\bullet$ with these properties is unique.

To show that $(\widehat{g})^\bullet = \widehat{g}$ for all $g \in N_S(P, Q)$, first note that by Equation (1) $(-)^\bullet$ induces the identity on $\mathrm{Aut}_{\mathcal{L}}(S)$. Then apply $(-)^\bullet$ to the equation $\iota^S_Q \circ \widehat{g} = \widehat{g} \circ \iota^S_P$, where $\widehat{g}$ on the right hand side is the corresponding element of $\mathrm{Aut}_{\mathcal{L}}(S)$, and use the fact that $\iota^S_{Q^\bullet}$ is a monomorphism by Corollary 1.8. □

Proposition 1.13 *If $\phi\colon S \to S$ is a fusion preserving automorphism then $\phi(P^\bullet) = \phi(P)^\bullet$ for any $P \le S$.*

Proof Recall the definition of $P^\bullet$ [9, Definition 3.1], $P^\bullet := P \cdot I(P^{[m]})$, where $P^{[m]} = \{g^{p^m} \mid g \in P\} \le S_0$, with $p^m = |S/S_0|$, and for any $Q \le S_0$, $I(Q) := S_0^{C_W(Q)}$ is again a subgroup of S_0. Clearly $\phi(P^{[m]}) = \phi(P)^{[m]}$ for any $P \le S$, and for $Q \le S_0$,

$$I(\phi(Q)) = S_0^{C_W(\phi(Q))} = S_0^{\phi C_W(Q))\phi^{-1}} = \phi(S_0^{C_W(Q)}) = \phi(I(Q)).$$

Therefore, $\phi(P)^\bullet = \phi(P) \cdot I(\phi(P^{[m]})) = \phi(P \cdot I(P^{[m]})) = \phi(P^\bullet)$. □

Proposition 1.14 *Fix a p–local compact group $(S, \mathcal{F}, \mathcal{L})$ and let $\phi\colon S \to S$ be a fusion preserving isomorphism. Let $\Phi'\colon \mathcal{L}^\bullet \to \mathcal{L}^\bullet$ be a functor which covers ϕ. Then Φ' extends to a unique functor $\Phi\colon \mathcal{L} \to \mathcal{L}$ which covers ϕ.*

Proof Define Φ on objects by $\Phi(P) = \phi(P)$. To define Φ on morphisms, observe first that

$$[\Phi'(\varphi^\bullet)] = \phi_*([\varphi^\bullet]) = \phi_*([\varphi]^\bullet) \quad \text{and} \quad [\varphi]^\bullet \circ \mathrm{incl}_P^{P^\bullet} = \mathrm{incl}_Q^{Q^\bullet} \circ [\varphi].$$

Applying ϕ_* to the second equation, and using the first, we obtain

$$[\Phi'(\varphi^\bullet)] \circ \mathrm{incl}_{\phi(P)}^{\phi(P^\bullet)} = \mathrm{incl}_{\phi(Q)}^{\phi(Q^\bullet)} \circ \phi_*([\varphi]).$$

Lemma 1.7(i) now implies that there is a unique morphism $\Phi(\varphi) \in \mathrm{Mor}_{\mathcal{L}}(\phi(P), \phi(Q))$ such that

$$\Phi'(\varphi^\bullet) \circ \iota_{\phi(P)}^{\phi(P^\bullet)} = \iota_{\phi(Q)}^{\phi(Q^\bullet)} \circ \Phi(\varphi) \quad \text{and} \quad [\Phi(\varphi)] = \phi_*([\varphi]).$$

Functoriality of Φ follows from the uniqueness statement of Lemma 1.7(i). By construction, and since $(-)^\bullet$ is the identity on $\mathcal{L}^\bullet$, the functor Φ extends Φ'. From Remark 1.6 and the last statement of Proposition 1.12 one deduces that $\Phi(\widehat{g}) = \widehat{\phi(g)}$ for all $g \in N_S(P, Q)$, and so Φ covers ϕ as well. □

2 Adams automorphisms of discrete p–toral groups

In this section we define a certain class of automorphisms of discrete p–toral groups, which we call *Adams automorphisms*. These automorphisms are the starting point of our study of unstable Adams operations on p–local compact groups.

Throughout this section, for any discrete group Γ, $\mathbb{Z}[\Gamma]$–modules will be referred to as Γ–modules. For a group G which is an extension $0 \to M \to G \to \Gamma \to 1$, of a group Γ by a Γ–module M, let $[G] \in H^2(\Gamma, M)$ denote the corresponding extension class (see eg [10, Chapter IV.3]).

Definition 2.1 Let G be a fixed extension of a group Γ by a Γ–module M. For $\lambda \in \mathrm{Aut}(M)$ and $\varphi \in \mathrm{Aut}(\Gamma)$, let $\mathrm{Aut}(G; \lambda, \varphi)$ denote the set of all automorphisms $\psi \in \mathrm{Aut}(G)$ which render the following diagram commutative.

(2)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & \Gamma & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & \Gamma & \longrightarrow & 1 \end{array}$$

For any group G and subgroups $H, K \leq G$, let $\mathrm{Aut}_H(K) \leq \mathrm{Aut}(K)$ denote the subgroup of all automorphisms of K which are induced by conjugation by elements of H. In particular, if $G = H$ one has $\mathrm{Aut}_G(K) = N_G(K)/C_G(K)$. On the other hand, if $G = K$, then $\mathrm{Aut}_H(G)$ is the image of H in $\mathrm{Inn}(G)$. For an extension as above, notice that $\mathrm{Aut}_M(G) \leq \mathrm{Aut}(G; 1_M, 1_\Gamma)$.

Let M be a Γ–module, and let $\varphi \in \mathrm{Aut}(\Gamma)$. Let $\varphi^* M$ denote the Γ–module M where the action of Γ is twisted by φ, namely $x \in \Gamma$ and $m \in M$, $x \cdot m := \varphi(x) \cdot m$. Thus one has an induced homomorphism $\varphi^*\colon H^*(\Gamma, M) \longrightarrow H^*(\Gamma, \varphi^* M)$. If $\lambda \in \mathrm{Aut}(M)$ is an automorphism, we say that λ is compatible with φ if for each $x \in \Gamma$ and $m \in M$, $\lambda(x \cdot m) = \varphi(x) \cdot \lambda(m)$. In that case one has an induced homomorphism $\lambda_*\colon H^*(\Gamma, M) \to H^*(\Gamma, \varphi^* M)$.

Lemma 2.2 *Let G be an extension of a group Γ by a Γ module M. Fix $\varphi \in \mathrm{Aut}(\Gamma)$ and $\lambda \in \mathrm{Aut}(M)$.*

(i) *$\mathrm{Aut}(G; \lambda, \varphi)$ is not empty if and only if λ is compatible with φ, and $\varphi^*([G]) = \lambda_*([G])$.*

(ii) *There is an isomorphism $H^1(\Gamma, M) \cong \mathrm{Aut}(G; 1_M, 1_\Gamma)/\mathrm{Aut}_M(G)$, and thus the group $H^1(\Gamma, M)$ acts freely and transitively on $\mathrm{Aut}(G; \lambda, \varphi)/\mathrm{Aut}_M(G)$.*

(iii) *Assume the action of Γ on M is faithful (ie, $C_G(M) = M$), and that λ is central in $\mathrm{Aut}(M)$. Then a necessary condition for $\mathrm{Aut}(G; \lambda, \varphi)$ to be nonempty is $\varphi = 1_\Gamma$.*

Proof (i) The classes $\varphi^*([G])$ and $\lambda_*([G])$ correspond to the extensions described in [3, Definitions 1.3 and 1.5], namely the "backward" and "forward" induced extensions. The result follows from [3, Theorem 1.10].

(ii) Clearly $\mathrm{Aut}(G; 1_M; 1_\Gamma)$ acts freely and transitively on $\mathrm{Aut}(G; \lambda, \varphi)$ via composition. By [3, Theorem 1.10] there is an isomorphism $\mathrm{Aut}(G; 1_M, 1_\Gamma) \cong \mathrm{Der}(\Gamma, M)$ mapping $\mathrm{Aut}_M(G)$ to $\mathrm{PDer}(\Gamma, M)$. Finally, $H^1(\Gamma, M) \cong \mathrm{Der}(\Gamma, M)/\mathrm{PDer}(\Gamma, M)$; see eg [16, Theorem 6.4.5].

(iii) By point (i), $\mathrm{Aut}(G; \lambda, \varphi) \neq \varnothing$ implies that λ is compatible with φ, and since λ is assume to be a central automorphism of M, it commutes with the action of Γ. Thus, for each $x \in \Gamma$ and $m \in M$,

$$x \cdot \lambda(m) = \lambda(x \cdot m) = \varphi(x) \cdot \lambda(m).$$

Since the action is assumed faithful, this implies at once that $\varphi(x) = x$ for all $x \in \Gamma$. □

In the remainder of this section S always denotes a discrete p–toral group with maximal torus S_0 and group of components Γ (Definition 1.1). Thus, S is an extension of Γ by the Γ–module S_0. Note that the centre of $\mathrm{Aut}(S_0) \cong \mathrm{GL}_r(\mathbb{Z}_p)$, where $r = \mathrm{rk}(S_0)$, is isomorphic to the group of the p–adic units $\mathbb{Z}_p^\times$ which acts on S_0 by power maps, $t \mapsto t^\zeta$, where $\zeta \in \mathbb{Z}_p^\times$ is a fixed unit, and $t \in S_0$. Also note that S_0 is a characteristic subgroup of S, so we obtain homomorphisms res: $\mathrm{Aut}(S) \longrightarrow \mathrm{Aut}(S_0)$ and q: $\mathrm{Aut}(S) \longrightarrow \mathrm{Aut}(\Gamma)$.

Definition 2.3 An *Adams automorphism* on S of degree ζ, where $\zeta \in \mathbb{Z}_p^\times$, is an automorphism of S whose restriction to S_0 is the ζ–power automorphism. Clearly composition of Adams automorphisms of S is an Adams automorphism, where the degree is given by the product of the two individual degrees. Thus the collection of all Adams automorphisms of S forms a group which we denote by $\mathrm{Ad}(S)$. Let $\mathrm{Ad}_\zeta(S)$ denote the subset of all Adams automorphisms of degree ζ.

The obvious map deg: $\mathrm{Ad}(S) \to \mathbb{Z}_p^\times$, which associates with an Adams automorphism its degree, is a group homomorphism. It is the restriction of the natural map res: $\mathrm{Aut}(S) \longrightarrow \mathrm{Aut}(S_0)$ to $\mathrm{Ad}(S)$.

An Adams automorphism is called *normal* if it induces the identity on Γ. Let $\mathrm{Ad}(S; 1_\Gamma)$ denote the subgroup of the normal Adams automorphisms and $\mathrm{Ad}_1(S; 1_\Gamma)$ the subgroup of those of degree 1.

Remark 2.4 For any discrete p–toral group S, the following hold.

(i) $\mathrm{Ad}_1(S) = \mathrm{Ker}(\mathrm{deg})$, and $\mathrm{Ad}(S; 1_\Gamma) = \mathrm{Ker}(\mathrm{Ad}(S) \to \mathrm{Aut}(\Gamma))$, so both subgroups are normal in $\mathrm{Ad}(S)$.

(ii) $\mathrm{Aut}_{S_0}(S) \lhd \mathrm{Ad}_1(S;1_\Gamma)$. In fact $\mathrm{Aut}_{S_0}(S) \lhd \mathrm{Aut}(S)$, because it is the maximal torus, hence a characteristic subgroup of the p–toral group $\mathrm{Inn}(S) = S/Z(S)$, which in turn is normal in $\mathrm{Aut}(S)$.

(iii) $\mathrm{Ad}(S)$ is the union of $\mathrm{Aut}(S;\zeta,\gamma)$, for all $\zeta \in \mathbb{Z}_p^\times$ and all $\gamma \in \mathrm{Aut}(\Gamma)$. Here, by abuse of notation, ζ means the ζ power map on S_0.

We will almost always restrict attention to normal Adams automorphisms. The next lemma shows that this is in fact a reasonable restriction. For a discrete p–toral group S, the maximal torus S_0 is not generally self-centralising, but in most important cases it would be. For instance, if $\mathcal{F}$ is a saturated fusion system over S, and one assumes that every $s \in S$ is $\mathcal{F}$–conjugate to S_0 (which can be taken as a concept analogous to connectivity), then S_0 is automatically self-centralising.

Lemma 2.5 *If S_0 is self-centralising in S, then every Adams automorphism of S is normal, ie $\mathrm{Ad}(S) = \mathrm{Ad}(S;1_\Gamma)$.*

Proof This is immediate from Lemma 2.2(iii) and Remark 2.4(iii). □

Our next goal is to examine which p–adic units ζ can appear as the degree of a normal Adams automorphisms. For a discrete p–toral group S, we call any map of sets, which is a right inverse to the projection $\pi\colon S \to \Gamma$, a *section* of S. Once a section of S is fixed, any element $s \in S$ has a unique expression in the form $s = t\cdot\sigma(x)$, where $x = \pi(s)$ and $t = s\cdot\sigma(x)^{-1}$.

Lemma 2.6 *Let ψ be a normal Adams automorphism of S of degree $\zeta \neq 1$. Then there exists a section σ of S which is fixed by ψ, namely $\psi\circ\sigma = \sigma$. In particular $\psi(t\cdot\sigma(x)) = t^\zeta\cdot\sigma(x)$ for all $t \in S_0$ and $x \in \Gamma$.*

Proof We need to find an element y in any coset $S_0\cdot x$ which is fixed by ψ, namely $\psi(y) = y$. Since ψ is normal, $\psi(s)s^{-1} \in S_0$ for all $s \in S$. Fix some $x \in S$. Since S_0 is divisible and $\zeta \neq 1$, there exists $t \in S_0$ such that $\psi(x)x^{-1} = t^{1-\zeta}$. Set $y = t\cdot x$. Then $y \in S_0\cdot x$, and $\psi(y) = \psi(tx) = \psi(t)\psi(x) = t^\zeta\cdot t^{1-\zeta}x = y$ as needed. □

Definition 2.7 The p–adic valuation of $z \in \mathbb{Z}_p^\times$, denoted $\nu_p(z)$, is the largest integer $m \geq 0$ such that $z \in p^m\mathbb{Z}_p$. For any $m \geq 0$, let $\Gamma_m(p)$ be the subgroup of $\mathbb{Z}_p^\times$ consisting of all ζ such that $\nu_p(\zeta-1) \geq m$. Thus $\Gamma_0(p) = \mathbb{Z}_p^\times$ and for any $m > 0$ any $\zeta \in \Gamma_m(p)$ has the form $\zeta = 1 + ap^m + \cdots$.

We note that since Γ is a finite p–group, and since $S_0 \cong (\mathbb{Z}/p^\infty)^r$, a standard transfer argument shows that $H^i(\Gamma, S_0)$ is a finite abelian p–group of exponent bounded above by the order of Γ for all $i > 0$.

Proposition 2.8 *Let S be a discrete p–toral group, and assume that the order of its extension class $[S] \in H^2(\Gamma, S_0)$ is p^m.*

(i) *If ψ is a normal Adams automorphism, then $\deg(\psi) \in \Gamma_m(p)$.*

(ii) *The homomorphism $\deg\colon \mathrm{Ad}(S; 1_\Gamma) \longrightarrow \Gamma_m(p)$ is split surjective. Moreover, there exists a section σ of S, such that for any $\zeta \in \Gamma_m(p)$ the function ψ_ζ which takes any $s = t \cdot \sigma(x) \in S$ to $t^\zeta \cdot \sigma(x)$ is a normal Adams automorphism of degree ζ on S, and the map taking $\zeta \in \Gamma_m(p)$ to ψ_ζ is a right inverse to the degree map.*

(iii) *The kernel of the degree homomorphism in (ii) is $\mathrm{Ad}_1(S; 1_\Gamma)$ and there is an isomorphism $\mathrm{Ad}_1(S; 1_\Gamma)/\mathrm{Aut}_{S_0}(S) \cong H^1(\Gamma, S_0)$.*

Proof (i) Set $\eta = [S]$. By Lemma 2.2(i), $\mathrm{Aut}(S, \zeta, 1_\Gamma)$ is not empty if and only if $\eta = \eta^\zeta$, namely $\eta^{\zeta-1} = 0 \in H^2(\Gamma, S_0)$. This happens if and only if $\zeta \in 1 + p^m\mathbb{Z}_p$, namely $\zeta \in \Gamma_m(p)$. In addition this shows that the degree map in (ii) is surjective.

(ii) Fix some $\psi \in \mathrm{Aut}(S; 1 + p^m; 1_\Gamma)$ which is not empty by (i). By Lemma 2.6 there exists a section σ of S fixed by ψ. We may assume that $\sigma(1) = 1$. Note that $\delta_{x,y} := \sigma(x)\sigma(y)\sigma(xy)^{-1}$ is fixed by ψ for any $x, y \in \Gamma$. But $\psi(\delta_{x,y}) = (\delta_{x,y})^{1+p^m}$ because $\delta_{x,y} \in S_0$. Thus, $(\delta_{x,y})^{p^m} = 1$ for all $x, y \in \Gamma$. Define for every $\zeta \in \Gamma_m(p)$ a function $\psi_\zeta\colon S \to S$ by

$$\psi_\zeta(t \cdot \sigma(x)) = t^\zeta \cdot \sigma(x), \quad t \in S_0,\ x \in \Gamma.$$

It is easy to check that $\psi(1) = 1$ because $\sigma(1) = 1$, that $\psi_\zeta(t_1\sigma(x) \cdot t_2\sigma(y)) = \psi_\zeta((t_1\sigma(x)) \cdot \psi_\zeta(t_2\sigma(y))$ because $(\delta_{x,y})^{p^m} = 1$, and that ψ_ζ induces the identity on Γ and restricts to the ζ–power map on S_0. In other words ψ_ζ is a normal Adams automorphism of degree ζ. Also $\psi_\zeta \circ \psi_{\zeta'} = \psi_{\zeta\zeta'}$ by inspection. Hence the map which takes $\zeta \in \Gamma_m(p)$ to ψ_ζ is a right inverse to the degree map.

(iii) Apply Lemma 2.2(ii) to $\mathrm{Aut}(S; 1_{S_0}, 1_\Gamma)$. □

3 Unstable Adams operations

In this section we revise the definition and some of the known results for unstable Adams operations on compact Lie groups and p–compact groups, and suggest a suitable definition for p–local compact groups.

For any compact Lie group G, a self-map $f\colon BG \to BG$ is called an unstable Adams operation of degree $k \in \mathbb{N}$, if f induces multiplication by k^i on $H^{2i}(BG, \mathbb{Q})$. Notice that for compact Lie groups this is not a p–local concept.

In the seminal paper by Jackowski, McClure and Oliver [13], the authors prove the following.

Theorem 3.1 [13, Theorems 1, 2] *Let G be a compact connected Lie group.*

(i) *For each natural number $k > 0$ there is at most one unstable Adams operation $\psi^k\colon BG \to BG$ of degree k.*

(ii) *If G is in addition simple, then there is an isomorphism of monoids*

$$[BG, BG] \cong \mathrm{Rep}(G, G) \wedge \{k \geq 0 \mid k = 0 \text{ or } (k, |W|) = 1\},$$

where $W = N_G(T)/T$, and T is the maximal torus of G.

For p–compact groups (X, BX) [12], one defines an unstable Adams operation of degree ζ, where $\zeta \in \mathbb{Z}_p$ is a p–adic unit, to be any self-map f of BX which induces multiplication by ζ^i on $H^{2i}_{\mathbb{Q}_p}(BX) := H^{2i}(BX, \mathbb{Z}_p) \otimes \mathbb{Q}$. In light of [12, Theorem 9.7], this is equivalent to the requirement that the restriction of f to the maximal torus T of X is an Adams automorphism of degree ζ.

Theorem 3.2 [1; 2] *For any connected p–compact group there exists exactly one unstable Adams operation of degree ζ for every p–adic unit ζ.*

Proof This is immediate from [1, Theorem 1.2] which shows $\mathrm{Out}(BX) = \pi_0 \mathrm{Aut}(BX)$ is isomorphic to $\mathrm{Out}(D_X) = \mathrm{Aut}(D_X)/W_X$ where $D_X = (W_X, L_X, \{\mathbb{Z}_p b_\sigma\})$ is a root datum associated to a p–compact group X. The Adams automorphisms form the centre of $\mathrm{Aut}(D_X)$ which is also the centre of $\mathrm{Aut}(L_X) \cong \mathrm{GL}_r(\mathbb{Z}_p)$ where r is the rank of X. We remark that for odd primes the result can be deduced from [2, Theorem 1.1]; for $BDI(4)$ defined in [11] it is contained in [14, Theorem 3.5]. □

While it is possible to define unstable Adams operations of p–local compact groups by referring to their classifying spaces or their rational cohomology, we choose to make the definition using the combinatorial structure of a p–local compact group, in order to enable us to use all the power of the theory in our analysis. It might turn out of course that the definition below is in fact equivalent to the obvious cohomological definition, but this question will not be addressed in this paper.

Definition 3.3 Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p–local compact group. An *Adams operation* of degree $\zeta \in \mathbb{Z}_p^\times$ on $\mathcal{G}$ is a pair (ψ, Ψ) where ψ is a fusion preserving Adams automorphism of S of degree ζ, (see Definitions 2.3 and 1.10), and $\Psi\colon \mathcal{L} \to \mathcal{L}$ is an automorphism which covers ψ (see Definition 1.11).

At this stage it is worth pointing out the reason for including condition (ii) in Definition 1.11. Consider the example of the trivial fusion and linking systems over a discrete torus T. Then $\mathcal{L} = \mathcal{B}(T)$, while in $\mathcal{F}$ the only morphisms are inclusions. Thus any power map on T induces the identity functor on $\mathcal{F}$, and therefore the pair $(\zeta, \mathrm{Id}_{\mathcal{L}})$ satisfies condition (i) of the definition. However the realization of the identity functor clearly induces the identity on BT. The authors are grateful to Alex Gonzalez for this observation.

Let $\mathrm{Ad}(\mathcal{G})$ denote the group of all the Adams operations of $\mathcal{G}$. Let $\mathrm{Ad}(\mathcal{F})$ denote the group of all the fusion preserving Adams automorphism of S. The assignment $(\Psi, \psi) \mapsto \psi$ gives rise to a homomorphism $\alpha\colon \mathrm{Ad}(\mathcal{G}) \to \mathrm{Ad}(\mathcal{F}) \leq \mathrm{Ad}(S)$. Let $\mathrm{Ad}(\mathcal{F}; 1_\Gamma) \leq \mathrm{Ad}(\mathcal{F})$ be the subgroup of fusion preserving normal Adams automorphisms of S. Let $\mathrm{Ad}(\mathcal{G}; 1_\Gamma)$ denote the group of "normal" operations (cf Definition 2.3). Then α restricts to a homomorphism $\alpha\colon \mathrm{Ad}(\mathcal{G}; 1_\Gamma) \to \mathrm{Ad}(\mathcal{F}, 1_\Gamma)$. Finally, let $\mathrm{Ad}_1(\mathcal{G})$ be the subgroup of operations of degree 1.

We end this section by introducing a more geometric definition of an unstable Adams operation, and prove that the two definitions are equivalent. For a p–local compact group $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$, let $\iota\colon BS \to B\mathcal{G}$ be the canonical inclusion induced by the distinguished monomorphism $\delta_S\colon S \to \mathrm{Aut}_{\mathcal{L}}(S)$. Recall [9, Theorem 6.3] that for any discrete p–toral group Q there is a bijection

$$[BQ, B\mathcal{G}] \cong \mathrm{Rep}(Q, \mathcal{L}) := \mathrm{Hom}(Q, S)/\!\sim,$$

where for any $\rho, \rho' \in \mathrm{Hom}(Q, S)$, $\rho \sim \rho'$ if there is some $\chi \in \mathrm{Hom}_{\mathcal{F}}(\rho(Q), \rho(Q'))$ such that $\rho' = \chi \circ \rho$. In particular $\mathrm{Rep}(S, \mathcal{L}) = \mathrm{End}(S)/\mathrm{Aut}_{\mathcal{F}}(S)$.

Definition 3.4 Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p–local compact group. A geometric unstable Adams operation on $\mathcal{G}$ is a self-equivalence f of $B\mathcal{G}$ such that the homotopy class of the composite

$$BS \xrightarrow{\iota} B\mathcal{G} \xrightarrow{f} B\mathcal{G}$$

represents the class of an Adams automorphism in $\mathrm{End}(S)$ under the identification above. Equivalently f is a geometric Adams operation on $\mathcal{G}$ if there exists an Adams automorphism $\phi \in \mathrm{Ad}(S)$ such that $\iota \circ B\phi \simeq f \circ \iota$.

Geometric unstable Adams operations on a p–local compact group $\mathcal{G}$ clearly form a topological monoid, which we denote by $\mathrm{Ad}^g(\mathcal{G})$, under composition, where the topology is induced from that of the topological monoid of all self–homotopy equivalences of $B\mathcal{G}$.

Proposition 3.5 *For any p–local compact group $\mathcal{G}$, there is an epimorphism of groups $\gamma\colon \mathrm{Ad}(\mathcal{G}) \to \pi_0(\mathrm{Ad}^g(\mathcal{G}))$ whose kernel is the subgroup of all unstable Adams operations (ψ, Ψ) such that $|\Psi|^\wedge_p \simeq \mathrm{Id}_{B\mathcal{G}}$. In this case $\psi \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $\deg(\psi)$ is a root of unity, whence $\zeta^{p-1} = 1$ if $p \neq 2$, and $\zeta = \pm 1$ if $p = 2$* (cf [15, Section 6.7, Proposition 1,2]).

Proof Fix a p–local compact group $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$. Define $\gamma\colon \mathrm{Ad}(\mathcal{G}) \to \pi_0(\mathrm{Ad}^g(\mathcal{G}))$ by

$$\gamma(\psi, \Psi) = [|\Psi|^\wedge_p]$$

for each unstable Adams operation $(\psi, \Psi) \in \mathrm{Ad}(\mathcal{G})$. By Condition (ii) of Definition 1.11, the following square commutes

$$\begin{array}{ccc} BS & \xrightarrow{B\psi} & BS \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota} \\ B\mathcal{G} & \xrightarrow{|\Psi|^\wedge_p} & B\mathcal{G}. \end{array}$$

Thus $|\Psi|^\wedge_p$ is a geometric unstable Adams operation on $\mathcal{G}$, and so γ is well defined, sends $(1_S, Id_{\mathcal{L}})$ to the identity element and it clearly respects compositions.

Suppose that $\gamma(\psi, \Psi) = 1 \in \pi_0(\mathrm{Ad}^g(\mathcal{G}))$. Then $\iota \circ B\psi \sim \iota$, and so by [9, Theorem 6.3(a)], $\psi \in \mathrm{Aut}_{\mathcal{F}}(S)$. Since $\mathrm{Out}_{\mathcal{F}}(S) := \mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Inn}(S)$ is finite by [9, Definition 2.2, (I)] and since all the elements of S have finite order, this is also the case for all the elements of $\mathrm{Aut}_{\mathcal{F}}(S)$. Therefore $\psi^k = \mathrm{Id}$ for some $k > 0$. But ψ^k induces the ζ^k–power map on the maximal torus S_0 and therefore $\zeta^k = 1$, namely ζ is a root of unity.

It remains to show that γ is surjective. We shall use [9, Theorem 7.1] and various points in its proof. Recall first that a self-equivalence Φ of $\mathcal{L}$ is said to be isotypical if for each $P \in \mathcal{L}$, the isomorphism $\Phi_{P,P}\colon \mathrm{Aut}_{\mathcal{L}}(P) \to \mathrm{Aut}_{\mathcal{L}}(\Phi(P))$ sends $\delta_P(P)$ to $\delta_{\Phi(P)}(\Phi(P))$.

Let $f \in \mathrm{Ad}^g(\mathcal{G})$ be a geometric unstable Adams operation. By [9, Theorem 7.1] there is a group isomorphism

$$\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \cong \mathrm{Out}(B\mathcal{G}),$$

where $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) = \pi_0(\mathrm{Aut}_{\mathrm{typ}}(\mathcal{L}))$ is the group of equivalence classes (under natural isomorphisms) of isotypical self-equivalences of $\mathcal{L}$, and $\mathrm{Out}(B\mathcal{G})$ is the group of homotopy classes of self-equivalences of $B\mathcal{G}$. Let $\Phi' \in \mathrm{Aut}_{\mathrm{typ}}(\mathcal{L})$ be a representative of the homotopy class of f under this isomorphism, so $|\Phi'|^\wedge_p \simeq f$. Let $\phi' \in \mathrm{Aut}(S)$ denote the restriction of Φ' to $S \cong \delta_S(S) \leq \mathrm{Aut}_{\mathcal{L}}(S)$. Without loss of generality we may assume that Φ' sends inclusions to inclusions and that $\Phi'(P) = \phi'(P)$ for every

$P \in \mathcal{L}$ (cf [9, page 368]). This implies that $\Phi'(\widehat{g}) = \widehat{\phi'(g)}$ for all $g \in N_S(P,Q)$ because by applying Φ' to the equality $\iota^S_Q \circ \widehat{g} = \widehat{g} \circ \iota^S_P$ we obtain

$$\iota^S_{\phi'(Q)} \circ \Phi'(\widehat{g}) = \widehat{\phi'(g)} \circ \iota^S_{\phi'(P)},$$

while from axiom (C) we get

$$\iota^S_{\phi'(Q)} \circ \widehat{\phi'(g)} = \widehat{\phi'(g)} \circ \iota^S_{\phi'(P)},$$

so Corollary 1.8 implies that $\Phi'(\widehat{g}) = \widehat{\phi'(g)}$.

The commutative diagram (18) in [9, page 370] shows that ϕ' is fusion preserving and that $\iota \circ B\phi' \simeq f \circ \iota$. Furthermore, if $\alpha \in \mathrm{Iso}_{\mathcal{L}}(P,Q)$, and $g \in P$, then by the functoriality of Φ', and Axiom (C),

$$\widehat{\phi'([\alpha](g))} = \Phi'(\alpha \circ \widehat{g} \circ \alpha^{-1}) = \Phi'(\alpha) \circ \Phi'(\widehat{g}) \circ \Phi'(\alpha)^{-1} = \widehat{[\Phi'(\alpha)](\phi'(g))}.$$

In other words, $\phi'_*([\alpha]) = \phi' \circ [\alpha] \circ (\phi')^{-1} = [\Phi'(\alpha)]$. It follows that Φ' covers ϕ'.

The pair (Φ', ϕ') constructed above satisfies all the requirements from an unstable Adams operation on $\mathcal{G}$, except, ϕ' need not be an Adams automorphism of S. By the assumption on f and the commutative diagram (18) in [9, page 370], there is some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$, such that $\phi' = \phi \circ \alpha^{-1}$, where ϕ is an Adams automorphism of S, as in Definition 3.4. Choose $\widetilde{\alpha} \in \mathrm{Aut}_{\mathcal{L}}(S)$ lifting α. By Lemma 1.7(i) the morphism $\alpha|_P \in \mathrm{Hom}_{\mathcal{F}}(P, \alpha(P))$ has a unique lift to $\widetilde{\alpha}|_P \in \mathrm{Mor}_{\mathcal{L}}(P, \alpha(P))$ such that $\iota^S_{\alpha(P)} \circ \widetilde{\alpha}|_P = \widetilde{\alpha} \circ \iota^S_P$. Define $\Theta\colon \mathcal{L} \to \mathcal{L}$ by $\Theta(P) = \alpha(P)$, and define Θ on morphisms by

$$\Theta(P \xrightarrow{\varphi} Q) = \widetilde{\alpha}|_Q \circ \varphi \circ \widetilde{\alpha}|_P^{-1}.$$

Then Θ is clearly well defined and covers α – condition (i) of Definition 1.11 is immediate and condition (ii) follows from axiom (C). In addition the morphisms $\widetilde{\alpha}|_P$ give rise to a natural isomorphism $\mathrm{Id}_{\mathcal{L}} \to \Theta$, hence $|\Theta| \simeq \mathrm{Id}$. Set $\Phi = \Phi' \circ \Theta$. Then $|\Phi'|^\wedge_p \simeq |\Phi'|^\wedge_p \simeq f$, and $\Phi|_{\delta_S(S)} = \phi$. Hence (ϕ, Φ) is an unstable Adams operation of $\mathcal{G}$ such that $\gamma(\phi, \Phi) = [f]$. □

We end this section with two examples. We note that any p–toral group S gives rise to the "trivial" p–local compact group $\mathcal{G}$ on S whose fusion system is $\mathcal{F}_S(S)$ and whose linking system is the transporter system on the $\mathcal{F}$–centric subgroups of S. In this case, $\mathrm{Ad}(\mathcal{G})$ is merely a subgroup of $\mathrm{Aut}(S)$.

Example 3.6 Set $T = \mathbb{Z}/p^\infty$ and $\Gamma = \mathbb{Z}/p$ and $S = \Gamma \times T$. By Lemma 2.2(ii), the remark above, and since S is abelian, we deduce that $\mathrm{Ad}_1(\mathcal{G}) \supseteq \mathrm{Aut}(S; 1_\Gamma, 1_T) \cong H^1(\Gamma, T) \cong \mathbb{Z}/p$. This shows that degree does not determine the Adams operation in general; not even up to homotopy (ie, its image in $\mathrm{Ad}^g(\mathcal{G})$).

Example 3.7 Set $p = 3$ and let $\Gamma = \mathbb{Z}/3$ act on $S_0 = \mathbb{Z}/3^\infty \oplus \mathbb{Z}/3^\infty$ via the matrix

$$\tau = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}.$$

A direct calculation using the standard periodic $\mathbb{Z}[\Gamma]$–projective resolution of $\mathbb{Z}$ shows that $H^2(\Gamma, S_0) \cong \mathbb{Z}/3$. Let S be a nonsplit extension of Γ by S_0 (ie, one which corresponds to a nontrivial class). Clearly, S_0 is self-centralising in S, so by Lemma 2.5 and Proposition 2.8(i), the degree of any Adams operation on the p–local compact group $\mathcal{G}$ associated to S must be in $\Gamma_1(3)$. In particular not every $\zeta \in \mathbb{Z}_3^\times$ is a degree of an Adams operation on $\mathcal{G}$.

4 A construction of Adams operations

We are now ready to construct unstable Adams operations on p–local compact groups. Recall (Definition 2.7) our notation $\Gamma_m(p)$ for the group of p–adic units ζ such that $v_p(\zeta - 1) \geq m$. Throughout this section let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a fixed p–local compact group. Fix a set P of representatives of the S–conjugacy classes of the subgroups in $\mathcal{H}^\bullet(\mathcal{F})$. Then P is a finite set by Proposition 1.9(i). Let $\mathrm{P}^c \subseteq \mathrm{P}$ be the subset consisting of those subgroups which are $\mathcal{F}$–centric.

Theorem 4.1 *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p–local compact group, let $S_0 \leq S$ be the maximal torus, and let $\Gamma = S/S_0$.*

(i) *For a sufficiently large integer $k_\mathcal{F}$ there exists a group homomorphism*

$$\mathfrak{a}_\mathcal{F}\colon \Gamma_{k_\mathcal{F}}(p) \to \mathrm{Ad}(\mathcal{F}; 1_\Gamma)$$

such that $\mathfrak{a}_\mathcal{F}(\zeta)$ is a fusion preserving normal Adams automorphism of degree ζ.

(ii) *For a sufficiently large integer $k_\mathcal{G}$ there exits a group homomorphism*

$$\mathfrak{a}_\mathcal{G}\colon \Gamma_{k_\mathcal{G}}(p) \to \mathrm{Ad}(\mathcal{G}; 1_\Gamma)$$

whose composition with $\alpha\colon \mathrm{Ad}(\mathcal{G}; 1_\Gamma) \to \mathrm{Ad}(\mathcal{F}; 1_\Gamma)$ is equal to $\mathfrak{a}_\mathcal{F}|_{\Gamma_{k_\mathcal{G}}(p)}$.

Proof Let p^{k_S} be the order of the extension class of S in $H^2(\Gamma, S_0)$. By Proposition 2.8(ii), there exists a homomorphism

$$\mathfrak{a}_S\colon \Gamma_{k_S}(p) \to \mathrm{Ad}(S; 1_\Gamma)$$

which is a right inverse of the degree map $\deg\colon \mathrm{Ad}(S; 1_\Gamma) \to \Gamma_{k_S}(p)$. Moreover, there is a section $\sigma\colon \Gamma \to S$ such that $\mathfrak{a}_S(\zeta)(t \cdot \sigma(x)) = t^\zeta \cdot \sigma(x)$ for all $t \in S_0$ and all $x \in \Gamma$.

For each $\zeta \in \Gamma_{k_S}(p)$ set $\mathfrak{a}_S(\zeta) = \psi_\zeta$. Define a function

$$\theta\colon S \to S_0, \quad \theta(x) = x \cdot \sigma(\overline{x})^{-1},$$

where $\overline{x} \in \Gamma$ is the image of $x \in S$. Observe that $x = \theta(x) \cdot \sigma(\overline{x})$ for any $x \in S$ and therefore (see Definition 2.7)

$$\psi_\zeta(x) = x \quad \text{if } \nu_p(\zeta - 1) \geq \log_p(\mathrm{ord}(\theta(x))). \tag{3}$$

For any subset $X \subseteq S$, let $\theta(X) \subseteq S_0$ denote the image of X in S_0, and for $Y \subseteq S_0$, denote by $\mathrm{ord}(Y)$ the supremum of the orders of the elements of Y.

For every $P \in \mathrm{P}$ let r_P be a set of representatives for the cosets of P_0 in P. Given any $P, Q \in \mathrm{P}$ the set $\mathrm{Hom}_{\mathcal{F}}(P, Q)/Q$ is finite by [9, Lemma 2.5], and hence so is the set $\mathrm{Hom}_{\mathcal{F}}(P, Q)/N_S(Q)$. For $P, Q \in \mathrm{P}$, let $\mathrm{M}_{P,Q} \subseteq \mathcal{F}$ be a choice of a set of representatives for $\mathrm{Hom}_{\mathcal{F}}(P, Q)/N_S(Q)$. For convenience, if $P \leq Q$, we will always choose incl_P^Q to represent its image in $\mathrm{Hom}_{\mathcal{F}}(P, Q)/N_S(Q)$. In particular Id_P represents its image in $\mathrm{Aut}_{\mathcal{F}}(P)/N_S(P)$.

Proof of Part (i) For any $P, Q \in \mathrm{P}$ define the following subsets of S:

$$\mathfrak{m}_{P,Q} = \bigcup_{f \in \mathrm{M}_{P,Q}} f(\mathrm{r}_P) \quad \text{and set} \quad \mathfrak{m} = \bigcup_{P,Q \in \mathrm{P}} \mathfrak{m}_{P,Q}.$$

Note that $\mathrm{r}_P \subseteq \mathfrak{m}$ for any $P \in \mathrm{P}$, since $\mathrm{Id}_P \in \mathrm{M}_{P,P}$. Define

$$k_{\mathcal{F}} := \max\{k_S, \log_p \mathrm{ord}(\theta(\mathfrak{m}))\}.$$

Let $\mathfrak{a}_{\mathcal{F}}\colon \Gamma_{k_{\mathcal{F}}}(p) \to \mathrm{Ad}(S; 1_\Gamma)$ be the restriction of $\mathfrak{a}_S$ to $\Gamma_{k_{\mathcal{F}}}(p)$. We claim that its image is contained in $\mathrm{Ad}(\mathcal{F}; 1_\Gamma)$. This will complete the proof of (i).

Fix $\zeta \in \Gamma_{k_{\mathcal{F}}}(p)$, and write $\psi = \psi_\zeta$ for short. It is clear from Proposition 1.9(ii) that it is enough to check that ψ is fusion preserving on $\mathcal{F}^\bullet$. In other words, it suffices to show that $\psi \circ f \circ \psi^{-1} \in \mathcal{F}^\bullet$ for any $P, Q \in \mathcal{F}^\bullet$ and any $f \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$.

Consider first any $P, Q \in \mathrm{P}$ and any $f \in \mathrm{M}_{P,Q}$. Since P_0 is a discrete p–torus, $P_0 \leq S_0$ and also $f(P_0) \subseteq Q_0 \leq S_0$. Since $\psi|_{S_0}$ is the ζ–power map, it follows that $\psi \circ f \circ \psi^{-1}(x) = f(x)$ for any $x \in P_0$. For any $x \in \mathrm{r}_P$, the definition of $k_{\mathcal{F}}$ implies that $\log_p(\mathrm{ord}(\theta(f(x))) \leq k_{\mathcal{F}} \leq \nu_p(\zeta - 1)$ and therefore Equation (3) implies that $\psi(f(x)) = f(x)$. Since P is generated by P_0 and r_P, we deduce that $\psi \circ f \circ \psi^{-1} = f$ for any $f \in \mathrm{M}_{P,Q}$. Since $1_P \in \mathrm{M}_{P,P}$ by construction, it follows that $\psi(P) = P$ for any $P \in \mathrm{P}$.

Let $P, Q \in \mathcal{H}^\bullet(\mathcal{F})$ be any subgroups, and let $f \in \mathcal{F}^\bullet(P, Q)$ be a morphism. There are $P', Q' \in \mathrm{P}$, $g \in N_S(P, P')$, $h \in N_S(Q', Q)$ and $f' \in \mathrm{M}_{P',Q'}$, such that $f = c_h \circ f' \circ c_g$.

Since $\psi \circ f' \circ \psi^{-1} = f'$ one has

$$\psi \circ f \circ \psi^{-1} = c_{\psi(h)} \circ f' \circ c_{\psi(g)},$$

which is clearly a morphism in $\mathcal{F}$. It remains to show only that if $P \in \mathcal{H}^\bullet(\mathcal{F})$, then so is $\psi(P)$. Let $P' \in \mathcal{P}$ be a representative of the S conjugacy class of P, and let $g \in N_S(P', P)$ be any element. Then $\psi(P) = c_{\psi(g)}(P')$, and since conjugation by an element of S is clearly a fusion preserving automorphism of S, the claim follows from Proposition 1.13.

Proof of Part (ii) For any morphism φ in $\mathcal{L}$, denote the corresponding homomorphism in $\mathcal{F}$ by $[\varphi]$ as usual. Let $\mathcal{L}^\bullet$ denote the full subcategory of $\mathcal{L}$ whose object set is $\mathcal{H}^\bullet(\mathcal{F}^c)$. For every $P, Q \in \mathrm{P}^c$ fix a set $\widetilde{\mathrm{M}}_{P,Q}$ of morphisms in $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ which are preimages of the homomorphisms in $\mathrm{M}_{P,Q}$ under $\pi\colon \mathcal{L} \to \mathcal{F}$. In particular take ι_P^Q for the lift of incl_P^Q if $P \le Q$, and note that because of this choice $1_P \in \widetilde{\mathrm{M}}_{P,P}$ for all $P \in \mathrm{P}^c$.

Let $P, Q \in \mathrm{P}^c$ and let $\varphi \in \widetilde{\mathrm{M}}_{P,Q}$ be any morphism. Since $\mathrm{Aut}_{\mathcal{L}}(Q)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ by Corollary 1.8, and since $\widetilde{\mathrm{M}}_{P,Q}$ forms a set of representatives for $\mathrm{Mor}_{\mathcal{L}}(P, Q)/N_S(Q)$, for every $g \in N_S(P)$ there exist unique $\varphi' \in \widetilde{\mathrm{M}}_{P,Q}$ and $\lambda_\varphi(g) \in N_S(Q)$ such that

$$\varphi \circ \hat{g} = \widehat{\lambda_\varphi(g)} \circ \varphi'. \tag{4}$$

We thus obtain a function $\lambda_\varphi\colon N_S(P) \to N_S(Q)$, whose useful properties will be exploited later.

Similarly, if $P, Q, R \in \mathrm{P}^c$ and $\varphi \in \widetilde{\mathrm{M}}_{P,Q}$ and $\gamma \in \widetilde{\mathrm{M}}_{Q,R}$, then there are unique $u_{\gamma,\varphi} \in N_S(R)$, and $\varepsilon \in \widetilde{\mathrm{M}}_{P,R}$ such that

$$\gamma \circ \varphi = \widehat{u_{\gamma,\varphi}} \circ \varepsilon.$$

In particular, if $Q = R$ and $\gamma = 1_Q$, then $\varphi = \hat{x}$ for some $x \in N_S(P, Q)$ if and only if $\varepsilon = \hat{y}$ for some $y \in N_S(P, Q)$. But $\widetilde{\mathrm{M}}_{P,Q}$ contains only one morphism in each left $N_S(Q)$ class, and so $x = y$, and $u_{1_Q,\hat{x}} = 1$. Similarly, if $P = Q$, $\varphi = 1_P$ and $x \in N_S(P, R)$, then $u_{\hat{x},1_P} = 1$.

For any $P \in \mathrm{P}^c$ let $\mathfrak{n}_P$ be a set of representatives for the cosets of P in $N_S(P)$. We choose $1 \in N_S(P)$ to represent the coset $1P$. Consider the following subsets of S.

$$\mathfrak{c} = \bigcup_{P,Q \in \mathrm{P}^c} \bigcup_{\varphi \in \widetilde{\mathrm{M}}_{P,Q}} \lambda_\varphi(\mathfrak{n}_P), \quad \mathfrak{i} = \bigcup_{P,Q \in \mathrm{P}^c} \{x \in S \mid \hat{x} \in \widetilde{\mathrm{M}}_{P,Q}\}$$

$$\mathfrak{d} = \bigcup_{P,Q,R \in \mathrm{P}^c} \{u_{\gamma,\varphi} \mid \varphi \in \widetilde{\mathrm{M}}_{P,Q},\ \gamma \in \widetilde{\mathrm{M}}_{Q,R}\}.$$

Clearly, all these sets are finite. Notice also that $\mathfrak{n}_P \subseteq \mathfrak{c}$ for all $P \in \mathcal{P}^c$, since $1_P \in \widetilde{\mathrm{M}}_{P,P}$, and since $1_P \circ \widehat{g} = \widehat{g} \circ 1_P$ for all $g \in N_S(P)$, so $\lambda_{1_P}(g) = g$. Define

$$k_{\mathcal{G}} = \max\{k_{\mathcal{F}},\ \log_p \mathrm{ord}(\theta(\mathfrak{c})),\ \log_p \mathrm{ord}(\theta(\mathfrak{i})), \log_p \mathrm{ord}(\theta(\mathfrak{d}))\}.$$

Fix some $\zeta \in \Gamma_{k_{\mathcal{G}}}(p)$ and set $\psi = \mathfrak{a}_{\mathcal{F}}(\zeta)$. Equation (3) now shows that

(5) ψ fixes every element in the sets $\mathfrak{c}$, $\mathfrak{i}$ and $\mathfrak{d}$.

Our goal now is to construct $\Psi\colon \mathcal{L} \to \mathcal{L}$ such that $(\psi, \Psi) \in \mathrm{Ad}(\mathcal{G})$. We will then show that the resulting assignment $\zeta \mapsto (\psi, \Psi)$ is a group homomorphism $\Gamma_{k_{\mathcal{G}}}(p) \to \mathrm{Ad}(\mathcal{G})$. The rest of the proof proceeds in five steps. We first study the properties of the function λ_φ essential to our analysis. In Steps 2 and 3 we define Ψ on $\mathcal{L}^\bullet$ and show that it is a functor. Step 4 is the proof that Ψ covers ψ. Finally, in Step 5 we extend the definition of Ψ to $\mathcal{L}$, define the function $\mathfrak{a}_{\mathcal{G}}$ and show that it is a homomorphism, thus completing the proof of (ii).

Step 1: Properties of λ_φ Fix $P, Q \in \mathrm{P}^c$ and $\varphi \in \widetilde{\mathrm{M}}_{P,Q}$.

(a) If $g \in P$ then $\lambda_\varphi(g) = [\varphi](g)$.

(b) Fix some $g \in N_S(P)$ and let $\varphi' \in \widetilde{\mathrm{M}}_{P,Q}$ be the unique morphism such that $\varphi \circ \widehat{g} = \widehat{\lambda_\varphi(g)} \circ \varphi'$. Then $\psi(g) \in N_S(P)$ and $\varphi \circ \widehat{\psi(g)} = \widehat{\psi(\lambda_\varphi(g))} \circ \varphi'$. In particular $\lambda_\varphi(\psi(g)) = \psi(\lambda_\varphi(g))$.

Claim (a) follows immediately from axiom (C) of Definition 1.3 and the definition of λ_φ in (4).

To prove (b) write $g = hg_0$ for some unique $h \in P$ and $g_0 \in \mathfrak{n}_P$. By Equation (4), $\varphi \circ \widehat{g_0} = \widehat{\lambda_\varphi(g_0)} \circ \varphi'$ for a unique $\varphi' \in \widetilde{\mathrm{M}}_{P,Q}$. By Axiom (C) in Definition 1.3,

$$\varphi \circ \widehat{g} = \varphi \circ \widehat{hg_0} = \varphi \circ \widehat{h} \circ \widehat{g_0} = \widehat{[\varphi](h)} \circ \varphi \circ \widehat{g_0} = \widehat{[\varphi](h)} \circ \widehat{\lambda_\varphi(g_0)} \circ \varphi'.$$

In particular $\lambda_\varphi(g) = [\varphi](h) \cdot \lambda_\varphi(g_0)$ by uniqueness. In Part (i) we have shown that $\psi(P) = P$ for all $P \in \mathrm{P}$. Thus in particular $\psi(g) \in N_S(P)$, and $\psi(h) \in P$. Note that ψ fixes g_0 and $\lambda_\varphi(g_0)$ by (5), since $\mathfrak{n}_P \subseteq \mathfrak{c}$. Also $[\varphi] \in \mathrm{M}_{P,Q}$ and by Part (i), $\psi \circ [\varphi] \circ \psi^{-1} = [\varphi]$. By applying Axiom (C) again, one has

$$\begin{aligned}
\varphi \circ \widehat{\psi(g)} &= \varphi \circ \widehat{\psi(h)} \circ \widehat{g_0} \\
&= \widehat{[\varphi](\psi(h))} \circ \varphi \circ \widehat{g_0} \\
&= \widehat{[\varphi](\psi(h))} \circ \widehat{\lambda_\varphi(g_0)} \circ \varphi' = \widehat{\psi([\varphi](h))} \circ \widehat{\psi(\lambda_\varphi(g_0))} \circ \varphi' = \widehat{\psi(\lambda_\varphi(g))} \circ \varphi'.
\end{aligned}$$

The last claim of (b) follows from Equation (4).

Step 2: Definition of $\Psi\colon \mathcal{L}^\bullet \to \mathcal{L}^\bullet$ On objects define $\Psi(P) = \psi(P)$. This makes sense by Proposition 1.13 which states that $\psi(P)$ belongs to $\mathcal{H}^\bullet(\mathcal{F})$ if P does.

Let $P, Q \in \mathcal{H}^\bullet(\mathcal{F}^c)$ be any subgroups and let $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ be a morphism. There are unique $P', Q' \in \mathrm{P}^c$ which are S–conjugate to P and Q respectively. Choose $g \in N_S(P', P)$ and $h \in N_S(Q, Q')$. In particular if $P = P'$, choose $g = 1$, and similarly if $Q = Q'$. Then there are unique $\varphi' \in \widetilde{\mathrm{M}}_{P',Q'}$ and $x \in N_S(Q')$ such that

$$\widehat{h} \circ \varphi \circ \widehat{g} = \widehat{x} \circ \varphi'. \tag{6}$$

Define $\Psi(\varphi)$ to be the composite

$$\Psi(P) \xrightarrow{\widehat{\psi(g^{-1})}} P' \xrightarrow{\varphi'} Q' \xrightarrow{\widehat{\psi(x)}} \Psi(Q') \xrightarrow{\widehat{\psi(h^{-1})}} \Psi(Q).$$

In particular, if $P, Q \in \mathrm{P}^c$ and $\varphi \in \widetilde{\mathrm{M}}_{P,Q}$, then $\Psi(\varphi) = \varphi$.

We need to show that $\Psi(\varphi)$ does not depend on the choice of g and h. If $u \in N_S(P', P)$ and $v \in N_S(Q, Q')$ are any other elements, then there are unique $\varphi'' \in \widetilde{\mathrm{M}}_{P',Q'}$ and $y \in N_S(Q')$ such that $\widehat{v} \circ \varphi \circ \widehat{u} = \widehat{y} \circ \varphi''$. For the two corresponding definitions of $\Psi(\varphi)$ to coincide it suffices to show that the diagram

$$\begin{array}{ccccccccc}
\Psi(P) & \xrightarrow{\widehat{\psi(g^{-1})}} & P' & \xrightarrow{\varphi'} & Q' & \xrightarrow{\widehat{\psi(x)}} & \Psi(Q') & \xrightarrow{\widehat{\psi(h^{-1})}} & \Psi(Q) \\
\| & & \downarrow{\scriptstyle \widehat{\psi(u^{-1}g)}} & & \downarrow{\scriptstyle \widehat{\psi(y^{-1}vh^{-1}x)}} & & \downarrow{\scriptstyle \widehat{\psi(vh^{-1})}} & & \| \\
\Psi(P) & \xrightarrow[\widehat{\psi(u^{-1})}]{} & P' & \xrightarrow[\varphi'']{} & Q' & \xrightarrow[\widehat{\psi(y)}]{} & \Psi(Q') & \xrightarrow[\widehat{\psi(v^{-1})}]{} & \Psi(Q)
\end{array}$$

commutes. By Remark 1.6, the first, third and fourth squares commute. Therefore, the only square whose commutativity is not obvious is the second, namely we must show that

$$\varphi'' \circ \widehat{\psi(u^{-1}g)} = \widehat{\psi(y^{-1}vh^{-1}x)} \circ \varphi'. \tag{7}$$

Write $s = u^{-1}g$ and $t = y^{-1}vh^{-1}x$ for short. Thus we must show that $\varphi'' \circ \widehat{\psi(s)} = \widehat{\psi(t)} \circ \varphi'$. By the choices made, and the definition of $\lambda_{\varphi''}$ we have

$$\widehat{t} \circ \varphi' = \varphi'' \circ \widehat{s} = \widehat{\lambda_{\varphi''}(s)} \circ \varphi'''. \tag{8}$$

Hence $\varphi' = \varphi'''$, and $t = \lambda_{\varphi''}(s)$. Note that $\psi(s) \in N_S(P')$ because $P' \in \mathrm{P}$ so $\psi(P') = P'$. Similarly $\psi(t) \in N_S(Q')$. By Step 1 (b), and since $t = \lambda_{\varphi''}(s)$,

$$\varphi'' \circ \widehat{\psi(s)} = \widehat{\lambda_{\varphi''}(\psi(s))} \circ \varphi_0'' = \widehat{\psi(\lambda_{\varphi''}(s))} \circ \varphi_0'' = \widehat{\psi(t)} \circ \varphi_0'', \tag{9}$$

for a unique $\varphi_0'' \in \widetilde{\mathrm{M}}_{P',Q'}$. Reducing this equation to $\mathcal{F}$ and recalling that ψ_* fixes $\mathrm{M}_{P',Q'}$, we have

$$\psi \circ [\varphi''] \circ c_s \circ \psi^{-1} = [\varphi''] \circ c_{\psi(s)} = c_{\psi(t)} \circ [\varphi_0''] = \psi \circ c_t \circ [\varphi_0''] \circ \psi^{-1}.$$

By applying ψ_* to the reduction of Equation (8) to $\mathcal{F}$, the same equation holds with $[\varphi_0'']$ replaced by $[\varphi']$. It follows that there exists a unique $z \in Z(P')$, such that $\varphi_0'' = \varphi' \circ \hat{z}$. But, $\varphi' \circ \hat{z} = \widehat{[\varphi'](z)} \circ \varphi'$ by Axiom (C), and so $\varphi_0'' = \widehat{[\varphi'](z)} \circ \varphi'$. Since both φ_0'' and φ' are in $\widetilde{\mathrm{M}}_{P',Q'}$ are unique representatives of their class modulo the left action of $N_S(Q')$, we must have $\varphi_0'' = \varphi'$ and $z = 1$. Returning to Equation (9) we deduce that $\varphi'' \circ \widehat{\psi(s)} = \widehat{\psi(t)} \circ \varphi'$ which shows, in turn, that Ψ is well defined on $\mathcal{L}^\bullet$.

Step 3: $\Psi\colon \mathcal{L}^\bullet \to \mathcal{L}^\bullet$ is a functor First notice that if $P \in \mathcal{L}^\bullet$ is any object, and $P' \in \mathrm{P}^c$ represents its S conjugacy class, then for any $g \in N_S(P', P)$, the composite

$$\psi(P) \xrightarrow{\widehat{\psi(g)^{-1}}} P' \xrightarrow{1_{P'}} P' \xrightarrow{\widehat{\psi(g)}} \psi(P)$$

is the identity on $\psi(P)$. Hence $\Psi(1_P) = 1_{\Psi(P)}$.

It remains to prove that Ψ respects compositions. Fix $P, Q, R \in \mathcal{H}^\bullet(\mathcal{F}^c)$ and let $P', Q', R' \in \mathrm{P}^c$ be the unique representatives of their S–conjugacy class. Fix elements $x \in N_S(P', P)$, $y \in N_S(Q', Q)$ and $z \in N_S(R', R)$. Fix morphisms $\alpha \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and $\beta \in \mathcal{L}(Q, R)$. There are unique morphisms $\alpha' \in \widetilde{\mathrm{M}}_{P',Q'}$ and $\beta' \in \widetilde{\mathrm{M}}_{Q',R'}$ and elements $u \in N_S(Q')$ and $v \in N_S(R')$ such that

$$\widehat{y^{-1}} \circ \alpha \circ \hat{x} = \hat{u} \circ \alpha' \qquad \text{and} \qquad \widehat{z^{-1}} \circ \beta \circ \hat{y} = \hat{v} \circ \beta'.$$

There is a unique morphism $\beta'' \in \widetilde{\mathrm{M}}_{Q',R'}$ such that $\beta' \circ \hat{u} = \widehat{\lambda_{\beta'}(u)} \circ \beta''$. Also there is a unique morphism $\gamma' \in \widetilde{\mathrm{M}}_{P',R'}$ and a unique element $u_{\beta'',\alpha'} \in N_S(R')$ such that $\beta'' \circ \alpha' = \widehat{u_{\beta'',\alpha'}} \circ \gamma'$. Now,

$$\widehat{z^{-1}} \circ \beta \circ \alpha \circ \hat{x} = \hat{v} \circ \beta' \circ \hat{u} \circ \alpha' = \hat{v} \circ \widehat{\lambda_{\beta'}(u)} \circ \beta'' \circ \alpha' = \hat{v} \circ \widehat{\lambda_{\beta'}(u)} \circ \widehat{u_{\beta'',\alpha'}} \circ \gamma'.$$

Note that ψ fixes $u_{\beta'',\alpha'}$ by (5). By definition of Ψ on morphisms in $\mathcal{L}^\bullet$ and Step 1(b),

$$\begin{aligned}
\Psi(\alpha \circ \beta) &= \widehat{\psi(z)} \circ \widehat{\psi(v)} \circ \widehat{\psi(\lambda_{\beta'}(u))} \circ \widehat{\psi(u_{\beta'',\alpha'})} \circ \gamma' \circ \widehat{\psi(x)}^{-1} \\
&= \widehat{\psi(z)} \circ \widehat{\psi(v)} \circ \widehat{\lambda_{\beta'}(\psi(u))} \circ \widehat{u_{\beta'',\alpha'}} \circ \gamma' \circ \widehat{\psi(x)}^{-1} \\
&= \widehat{\psi(z)} \circ \widehat{\psi(v)} \circ \widehat{\lambda_{\beta'}(\psi(u))} \circ \beta'' \circ \alpha' \circ \widehat{\psi(x)}^{-1} \\
&= \widehat{\psi(z)} \circ \widehat{\psi(v)} \circ \beta' \circ \widehat{\psi(u)} \circ \alpha' \circ \widehat{\psi(x)}^{-1} \\
&= \widehat{\psi(z)} \circ \widehat{\psi(v)} \circ \beta' \circ \widehat{\psi(y)}^{-1} \circ \widehat{\psi(y)} \circ \widehat{\psi(u)} \circ \alpha' \circ \widehat{\psi(x)}^{-1} \\
&= \Psi(\beta) \circ \Psi(\alpha).
\end{aligned}$$

Step 4: Ψ covers ψ We show next that $\Psi\colon \mathcal{L}^\bullet \to \mathcal{L}^\bullet$ covers $\psi \in \mathrm{Ad}_{\mathcal{F}}(S)$ (see Definition 1.11).

Fix $P, Q \in \mathcal{F}^{\bullet c}$. We show first that $\Psi(\widehat{x}) = \widehat{\psi(x)}$ for any $x \in N_S(P,Q)$. Let $P', Q' \in \mathrm{P}^c$ represent the S conjugacy classes of P and Q respectively, and let $g \in N_S(P', P)$ and $h \in N_S(Q, Q')$ be any elements. Then there exist unique $u \in N_S(Q')$ and $\varphi \in \widetilde{\mathrm{M}}_{P',Q'}$, such that

$$\widehat{h} \circ \widehat{x} \circ \widehat{g} = \widehat{u} \circ \varphi.$$

By Remark 1.6, $\varphi = \widehat{y}$ where $y = u^{-1}hxg \in N_S(P', Q')$. Thus, $y \in \mathfrak{i}$ and by (5), $\psi(y) = y$. Hence, by the definition of Ψ,

$$\Psi(\widehat{x}) = \widehat{\psi(h^{-1}u)} \circ \widehat{y} \circ \widehat{\psi(g^{-1})} = \widehat{\psi(h^{-1}uyg^{-1})} = \widehat{\psi(x)}.$$

It remains to prove that $\pi \circ \Psi = \psi_* \circ \pi$, where $\pi\colon \mathcal{L}^\bullet \to \mathcal{F}^{\bullet c}$ is the projection. First, by definition $\Psi(R) = \psi(R)$ for any $R \in \mathcal{F}^{\bullet c}$. Next, for any morphism $\varphi \in \mathrm{Mor}_{\mathcal{L}^\bullet}(P, Q)$, we have to prove that $\psi_*([\varphi]) = [\Psi(\varphi)]$. Fix g, h, x and φ' such that $\widehat{h} \circ \varphi \circ \widehat{g} = \widehat{x} \circ \varphi'$, as in (6), so that by definition

$$\Psi(\varphi) = \widehat{\psi(h^{-1}x)} \circ \varphi' \circ \widehat{\psi(g^{-1})}.$$

In particular $[\varphi] = c_{h^{-1}x} \circ [\varphi'] \circ c_{g^{-1}}$. Note also that $[\varphi'] \in \mathrm{M}_{P',Q'}$, as it is the reduction to $\mathcal{F}$ of $\varphi' \in \widetilde{\mathrm{M}}_{P',Q'}$, and that by the proof of Part (i), $\psi_*([\varphi']) = [\varphi']$. Therefore

$$\begin{aligned}\psi_*([\varphi]) = \psi \circ (c_{h^{-1}x} \circ [\varphi'] \circ c_{g^{-1}}) \circ \psi^{-1} &= c_{\psi(h^{-1}x)} \circ [\varphi'] \circ c_{\psi(g^{-1})} \\ &= [\widehat{\psi(h^{-1}u)} \circ \varphi' \circ \widehat{\psi(g^{-1})}] = [\Psi(\varphi)].\end{aligned}$$

Step 5: Extension to $\mathcal{L}$ and the homomorphism $\mathfrak{a}_{\mathcal{G}}$ Proposition 1.14 and Step 3 above imply that the functor $\Psi\colon \mathcal{L}^\bullet \to \mathcal{L}^\bullet$ constructed above extends uniquely to a functor $\Psi\colon \mathcal{L} \to \mathcal{L}$ which covers ψ. This defines a function $\mathfrak{a}_{\mathcal{G}}\colon \Gamma_{k_{\mathcal{G}}}(p) \to \mathrm{Ad}(\mathcal{G}; 1_\Gamma)$ whose composition with $\mathrm{Ad}(\mathcal{G}) \to \mathrm{Ad}(\mathcal{F})$ is clearly the restriction of $\mathfrak{a}_{\mathcal{F}}$ to $\Gamma_{k_{\mathcal{G}}}(p)$.

It remains to check that $\mathfrak{a}_{\mathcal{G}}$ is a homomorphism. First, if $\zeta = 1$ then $\psi = \mathfrak{a}_{\mathcal{F}}(\zeta) = 1_S$, and it is easy to check that the definitions in Step 2 implies that the functor Ψ is the identity on $\mathcal{L}^\bullet$ and therefore extends to the identity on $\mathcal{L}$, by Proposition 1.14.

Next fix $\zeta_1, \zeta_2 \in \Gamma_{k_{\mathcal{G}}}(p)$ and set

$$(\psi_{\zeta_i}, \Psi_{\zeta_i}) := \mathfrak{a}_{\mathcal{G}}(\zeta_i),\ i = 1, 2, \quad \text{and} \quad (\psi_{\zeta_1\zeta_2}, \Psi_{\zeta_1\zeta_2}) := \mathfrak{a}_{\mathcal{G}}(\zeta_1\zeta_2).$$

By Part (i), $\psi_{\zeta_1} \circ \psi_{\zeta_2} = \psi_{\zeta_1\zeta_2}$, and it remains to prove that $\Psi_{\zeta_1} \circ \Psi_{\zeta_2} = \Psi_{\zeta_1\zeta_2}$. Alternatively, we need to prove that $\Phi := \Psi_{\zeta_1} \circ \Psi_{\zeta_2} \circ \Psi_{\zeta_1\zeta_2}^{-1}$ is the identity functor on $\mathcal{L}$. On objects $P \in \mathcal{L}$ it is immediate that $\Phi(P) = P$, because $\Psi_\zeta(P) = \psi_\zeta(P)$ for any ζ. Also, it follows from Step 4 applied to $\Psi_{\zeta_1}, \Psi_{\zeta_2}$ and $\Psi_{\zeta_1\zeta_2}$ that $\Phi(\widehat{g}) = \widehat{g}$

for all $P, Q \in \mathcal{L}^\bullet$ and all $g \in N_S(P, Q)$. By Proposition 1.14 it suffices to prove that Φ is the identity on $\mathcal{L}^\bullet$. By the construction for any ζ for which it is defined Ψ_ζ fixes all morphisms in $\widetilde{\mathrm{M}}_{P,Q}$ for any $P, Q \in \mathrm{P}^c$, and therefore so does Φ. Since every morphism $\varphi \in \mathrm{Mor}_{\mathcal{L}^\bullet}(P, Q)$ has the form $\widehat{h} \circ \varphi' \circ \widehat{g}$ for some $\varphi' \in \widetilde{\mathrm{M}}_{P',Q'}$ and some $g \in N_S(P, P')$ and $h \in N_S(Q', Q)$, it follows at once that $\Phi(\varphi) = \Phi(\widehat{h}) \circ \Phi(\varphi') \circ \Phi(\widehat{g}) = \varphi$. This completes the proof that $\alpha_{\mathcal{G}}$ is a homomorphism and with that the proof of the theorem. □

We end with a few corollaries of the proof of Theorem 4.1.

Corollary 4.2 *Fix a p–local compact group $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$, and let $\zeta \in \Gamma_{k_{\mathcal{G}}}(p)$ be any element. Let $(\psi, \Psi) = \alpha_{\mathcal{G}}(\zeta)$. Then there is a set P of representative of all S–conjugacy classes in $\mathcal{H}^\bullet(\mathcal{F})$, and*

- *for each $P, Q \in \mathrm{P}$ there is a set $\mathrm{M}_{P,Q}$ of morphism in $\mathrm{Hom}_{\mathcal{F}}(P, Q)$, which is a complete set of representatives of $\mathrm{Hom}_{\mathcal{F}}(P, Q)/N_S(Q)$, and*
- *for all $\mathcal{F}$–centric subgroups P, Q in P, there are sets $\widetilde{\mathrm{M}}_{P,Q}$ of morphisms in $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ which are representatives for the set $\mathrm{Mor}_{\mathcal{L}}(P, Q)/N_S(Q)$,*

such that the following hold.

(i) $\psi(P) = P$ *for all* $P \in \mathrm{P}$.

(ii) $\psi_*(\varphi) = \psi^{-1} \circ \varphi \circ \psi = \varphi$ *for each* $\varphi \in \mathrm{M}_{P,Q}$.

(iii) $\Psi(\varphi) = \varphi$ *for any* $\varphi \in \widetilde{\mathrm{M}}_{P,Q}$.

Moreover, for any $\mathcal{F}$–centric $P, Q \leq S$ and any $g \in N_S(P, Q)$, $\Psi(\widehat{g}) = \widehat{\psi(g)}$. In particular Ψ is isotypical.

Corollary 4.3 *Let p^m be the order of the extension class of S in $H^2(\Gamma, S_0)$, and let $k_{\mathcal{F}}$ and $k_{\mathcal{G}}$ be as in Theorem 4.1. Then $k_{\mathcal{G}} \geq k_{\mathcal{F}} \geq m$.*

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