Lagrangian mapping class groups from a group homological point of view

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We focus on two kinds of infinite index subgroups of the mapping class group of a surface associated with a Lagrangian submodule of the first homology of a surface. These subgroups, called Lagrangian mapping class groups, are known to play important roles in the interaction between the mapping class group and finite-type invariants of 3–manifolds. In this paper, we discuss these groups from a group (co)homological point of view. The results include the determination of their abelianizations, lower bounds of the second homology and remarks on the (co)homology of higher degrees. As a byproduct of this investigation, we determine the second homology of the mapping class group of a surface of genus 3.

55R40; 32G15, 57R20

1 Introduction

Let Σ_g be a closed oriented connected surface of genus g and let H_g be an oriented handlebody of the same genus. As shown in Figure 1, put H_g in the standard position in $\mathbb{R}^3$ and consider Σ_g to be the boundary of H_g. Fix a basis $\{x_1, x_2, \dots, x_g, y_1, y_2, \dots, y_g\}$ of $H := H_1(\Sigma_g)$ as in the figure so that $\mathrm{Ker}(H_1(\Sigma_g) \to H_1(H_g))$ coincides with the submodule L of H generated by $\{x_1, x_2, \dots, x_g\}$.

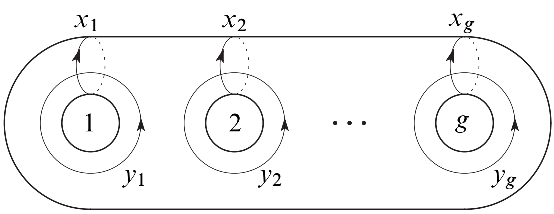

Figure 1. A symplectic basis of $H_1(\Sigma_g)$

The module H has a natural nondegenerate antisymmetric bilinear form $\mu\colon H \otimes H \to \mathbb{Z}$ called the intersection pairing. It is easy to see that L is a maximal direct summand of H on which μ restricts to 0. Such a submodule is said to be *Lagrangian*. By using

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the pairing μ, we can naturally identify the quotient module H/L, the dual module $L^* := \mathrm{Hom}(L, \mathbb{Z})$ and the submodule L_y of H generated by $\{y_1, y_2, \ldots, y_g\}$.

The *mapping class group* $\mathcal{M}_g$ of Σ_g is the group of isotopy classes of orientation preserving self-diffeomorphisms of Σ_g. In this paper, we focus on subgroups of $\mathcal{M}_g$ associated with the above fixed Lagrangian submodule L of H. More precisely, two subgroups

$$\begin{aligned} \mathcal{L}_g &:= \{f \in \mathcal{M}_g \mid f_*(L) = L\}, \\ \mathcal{IL}_g &:= \{f \in \mathcal{M}_g \mid f_*|_L = \mathrm{id}_L\} \end{aligned}$$

are studied through their group (co)homology, where f_* denotes the induced automorphism of H for $f \in \mathcal{M}_g$. We have $\mathcal{IL}_g \subset \mathcal{L}_g$ by definition and call them *Lagrangian mapping class groups* or *Lagrangian subgroups*. The *Torelli group* $\mathcal{I}_g$ is defined by

$$\mathcal{I}_g := \{f \in \mathcal{M}_g \mid f_* = \mathrm{id}_H\}.$$

One motivation by which the author started to study the groups $\mathcal{L}_g$ and $\mathcal{IL}_g$ is the fact that they are *infinite* index subgroups of $\mathcal{M}_g$ *including* $\mathcal{I}_g$. The importance to study this kind of subgroups will be explained in Section 7.2 with the relationship to the (non)triviality problem of *even Miller–Morita–Mumford classes* $e_{2i} \in H^{4i}(\mathcal{M}_g; \mathbb{Q})$ pulled back to $H^{4i}(\mathcal{I}_g; \mathbb{Q})$.

Lagrangian subgroups have been studied by several researchers. Hirose studied a generating system of $\mathcal{L}_g$ in [16], where $\mathcal{L}_g$ is called the *homological handlebody group*. In fact, the group $\mathcal{L}_g$ can be seen as a homological extension of the handlebody mapping class group $\mathcal{H}_g$. Recall that the group $\mathcal{H}_g$ is the subgroup of $\mathcal{M}_g$ consisting of isotopy classes of orientation preserving self-diffeomorphisms of $\Sigma_g = \partial H_g$ that can be extended to self-diffeomorphisms of the handlebody H_g. We can easily check that $\mathcal{L}_g = \mathcal{H}_g \mathcal{I}_g$. Prior to Hirose's work, Birman gave a generating set of $\mathcal{L}_g/\mathcal{I}_g \cong \mathcal{H}_g/(\mathcal{H}_g \cap \mathcal{I}_g)$ in [4] and we can give a generating set of $\mathcal{L}_g$ by combining her result with Johnson's finite generating set of $\mathcal{I}_g$ [19].

As for $\mathcal{IL}_g$, Levine conducted a series of investigations in [23; 24; 25]. He defined a filtration of $\mathcal{IL}_g$ called the *Lagrangian filtration*, which is analogous to the Johnson filtration of $\mathcal{I}_g$, by modifying the theory of Johnson homomorphisms so that it conforms well to $\mathcal{IL}_g$. Then he gave an application of this filtration to the theory of homology 3–spheres.

Recently, the groups $\mathcal{L}_g$ and $\mathcal{IL}_g$ appear and play important roles in the theory of finite-type invariants of 3–manifolds. See Andersen, Bene, Meilhan and Penner [1], Cheptea, Habiro and Massuyeau [10], Cheptea and Le [11] (with a slightly different definition) and Garoufalidis and Levine [13], for example. However, it seems that the groups $\mathcal{L}_g$ and $\mathcal{IL}_g$ have been studied separately. In this paper, we put $\mathcal{L}_g$ on the top of the Lagrangian filtration of $\mathcal{IL}_g$ and study them simultaneously as in the case of $\mathcal{M}_g$ and $\mathcal{I}_g$.

We first summarize the notation and fundamental facts on $\mathcal{L}_g$ and $\mathcal{IL}_g$ in Section 2. Then we will discuss the following in order.

- Section 3: Computation of $H_1(\mathcal{IL}_g)$
- Section 4: Computations of $H_1(\mathcal{L}_g/\mathcal{I}_g)$ and $H_2(\mathcal{L}_g/\mathcal{I}_g)$
- Section 5: Computation of $H_1(\mathcal{L}_g)$ and a lower bound of $H_2(\mathcal{L}_g)$
- Section 7: Remarks on higher (co)homology of $\mathcal{L}_g$ and $\mathcal{IL}_g$

Precisely speaking, we study in Sections 3 and 5 the Lagrangian mapping class groups of a surface with one boundary component and then derive the statements for those of a closed surface in Section 6.

As a byproduct, we will give a remark that the second homology of the full mapping class group of genus 3 has $\mathbb{Z}_2$ as a direct summand (Theorem 4.9 and Corollary 4.10). This homology group has been almost determined by Korkmaz and Stipsicz [22] up to this $\mathbb{Z}_2$ summand.

In this paper, we use the same notation $H_*(\,\cdot\,)$ for the homology of both topological spaces and groups unless otherwise stated. We refer to Brown's book [9] for generalities of group (co)homology.

2 Lagrangian mapping class groups

By using the ordered basis $\{x_1, x_2, \dots, x_g, y_1, y_2, \dots, y_g\}$ of H, we fix an isomorphism between $\mathbb{Z}^{2g}$ and H, which enables us to identify the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ with the group of automorphisms of H preserving the intersection pairing μ. Then the action of $\mathcal{M}_g$ on H gives the exact sequence

$$1 \longrightarrow \mathcal{I}_g \longrightarrow \mathcal{M}_g \xrightarrow{\sigma} \mathrm{Sp}(2g,\mathbb{Z}) \longrightarrow 1 \tag{1}$$

with $\operatorname{Ker}\sigma = \mathcal{I}_g$, the Torelli group. The symplecticity condition for a $(2g)\times(2g)$ matrix

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $g\times g$ matrices A, B, C, D is given by

$${}^tX \begin{pmatrix} O & I_g \\ -I_g & O \end{pmatrix} X = \begin{pmatrix} O & I_g \\ -I_g & O \end{pmatrix},$$

where we denote by I_g the identity matrix of size g. The left hand side is equal to

$$\begin{pmatrix} -{}^tCA + {}^tAC & -{}^tCB + {}^tAD \\ -{}^tDA + {}^tBC & -{}^tDB + {}^tBD \end{pmatrix}.$$

From this we see that if $C = O$, then $D = {}^tA^{-1}$ holds and $A^{-1}B$ is symmetric. This case corresponds to $\sigma(\mathcal{L}_g)$. That is, if we put

$$\mathrm{urSp}(2g) := \left\{ \begin{pmatrix} A & B \\ O & {}^tA^{-1} \end{pmatrix} \;\middle|\; A^{-1}B\colon \text{symmetric} \right\},$$

then it is a subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$ and the equality $\mathcal{L}_g = \sigma^{-1}(\mathrm{urSp}(2g))$ follows by definition. The notation $\mathrm{urSp}(2g)$ meaning "upper right" was introduced by Hirose [16]. We have the exact sequence

$$1 \longrightarrow \mathcal{I}_g \longrightarrow \mathcal{L}_g \xrightarrow{\sigma|_{\mathcal{L}_g}} \mathrm{urSp}(2g) \longrightarrow 1. \tag{2}$$

Moreover, if $C = O$ and $A = D = I_g$, then the matrix B itself is symmetric. In this case, the subgroup

$$\left\{ \begin{pmatrix} I_g & B \\ O & I_g \end{pmatrix} \;\middle|\; B\colon \text{symmetric} \right\}$$

is naturally isomorphic to the second symmetric power S^2L of L because

$$\mathrm{Hom}(L_y, L) \cong \mathrm{Hom}(L^*, L) \cong L \otimes L$$

and B is symmetric. By definition, the equality $\mathcal{IL}_g = \sigma^{-1}(S^2L)$ holds and we have the exact sequence

$$1 \longrightarrow \mathcal{I}_g \longrightarrow \mathcal{IL}_g \xrightarrow{\sigma|_{\mathcal{IL}_g}} S^2L \longrightarrow 1. \tag{3}$$

Note that S^2L is a free abelian group. The groups S^2L and $\mathrm{urSp}(2g)$ are related by the exact sequence

$$1 \longrightarrow S^2L \longrightarrow \mathrm{urSp}(2g) \xrightarrow{\mathrm{ul}} \mathrm{GL}(g,\mathbb{Z}) \longrightarrow 1, \tag{4}$$

where the map ul assigns to each matrix its upper left block of size $g \times g$. Note that this group extension has a splitting defined by

$$\mathrm{GL}(g,\mathbb{Z}) \longrightarrow \mathrm{urSp}(2g) \qquad \left(A \longmapsto \begin{pmatrix} A & O \\ O & {}^tA^{-1} \end{pmatrix} \right).$$

Using (4), we obtain the exact sequence

$$1 \longrightarrow \mathcal{IL}_g \longrightarrow \mathcal{L}_g \xrightarrow{\mathrm{ul}\,\circ\,\sigma|_{\mathcal{L}_g}} \mathrm{GL}(g,\mathbb{Z}) \longrightarrow 1. \tag{5}$$

In the subsequent sections, we will use the above exact sequences to discuss the homology of $\mathcal{L}_g$ and $\mathcal{IL}_g$. By a technical reason, however, we first consider the mapping class group $\mathcal{M}_{g,1}$ of the surface $\Sigma_{g,1}$ obtained from Σ_g by removing an open disk, where each mapping class is supposed to fix the boundary of $\Sigma_{g,1}$ pointwise. The subgroups $\mathcal{L}_{g,1}$, $\mathcal{IL}_{g,1}$ and $\mathcal{I}_{g,1}$ are defined similarly. Exact sequences similar to the above hold for these groups. We naturally identify H with $H_1(\Sigma_{g,1})$. Also, we assume that $g \geq 3$ to avoid the complexity of $\mathcal{I}_{2,1}$, which is not covered by Johnson's work (see the next section).

3 The first homology of $\mathcal{IL}_{g,1}$

We begin our investigation by determining the first homology, namely the abelianization, of $\mathcal{IL}_{g,1}$. For that, we use the five-term exact sequence

$$H_2(\mathcal{IL}_{g,1}) \to H_2(S^2L) \to H_1(\mathcal{I}_{g,1})_{S^2L} \to H_1(\mathcal{IL}_{g,1}) \to H_1(S^2L) \to 0 \tag{6}$$

associated with the group extension (3). Put

$$X_i^2 := x_i \otimes x_i, \quad X_{ij} = X_{ji} := x_i \otimes x_j + x_j \otimes x_i.$$

The set

$$\{X_i^2 \mid 1 \leq i \leq g\} \cup \{X_{ij} \mid 1 \leq i < j \leq g\}$$

forms a basis of S^2L in $L \otimes L$. As a subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$, the group S^2L acts on H by

$$X_i^2 \colon \begin{cases} x_k \mapsto x_k, \\ y_k \mapsto \delta_{ik}x_i + y_k, \end{cases} \qquad X_{ij} \colon \begin{cases} x_k \mapsto x_k, \\ y_k \mapsto \delta_{jk}x_i + \delta_{ik}x_j + y_k, \end{cases} \tag{7}$$

where δ_{ij} is the Kronecker delta.

Lemma 3.1 *The homomorphism $(\sigma|_{\mathcal{IL}_{g,1}})_*\colon H_2(\mathcal{IL}_{g,1}) \to H_2(S^2L) \cong \wedge^2(S^2L)$ is surjective.*

Proof We use the technique of *abelian cycles* to construct homology classes in $\mathrm{Im}(\sigma|_{\mathcal{IL}_{g,1}})_*$. That is, for each homomorphism $\varphi\colon \mathbb{Z}^2 \to \mathcal{IL}_{g,1}$, we have a homology class $\varphi_*(1) \in H_2(\mathcal{IL}_{g,1})$ by sending the fundamental class $1 \in H_2(\mathbb{Z}^2) \cong \mathbb{Z}$ to $H_2(\mathcal{IL}_{g,1})$. Such a class $\varphi_*(1)$, which is in fact defined on cycle level, is called an *abelian cycle associated with φ*. Moreover, we can see that

$$(\sigma|_{\mathcal{IL}_{g,1}} \circ \varphi)_*(1) = (\sigma|_{\mathcal{IL}_{g,1}} \circ \varphi)((1,0)) \wedge (\sigma|_{\mathcal{IL}_{g,1}} \circ \varphi)((0,1)) \in \wedge^2(S^2L) \cong H_2(S^2L),$$

where $(1,0), (0,1) \in \mathbb{Z}^2$ (see [32, Lemma 2.2] for details).

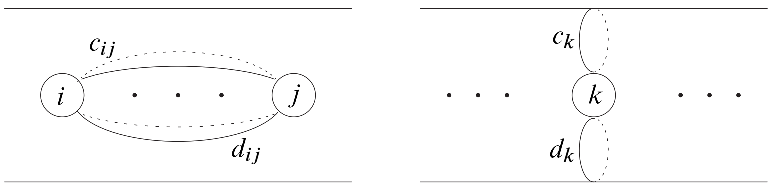

Figure 2

Define simple closed curves c_{ij}, d_{ij}, c_k, d_k $(1 \leq i < j \leq g, 1 \leq k \leq g)$ on $\Sigma_{g,1}$ as in Figure 2. Let G_1 (resp. G_2) be the subgroup of $\mathcal{IL}_g$ generated by $\{T_{c_{ij}}\}_{i,j} \cup \{T_{c_k}\}_k$ (resp. $\{T_{d_{ij}}\}_{i,j} \cup \{T_{d_k}\}_k$), where T_c denotes the right-handed Dehn twist along a simple closed curve c. Since

$$\sigma|_{\mathcal{IL}_{g,1}}(T_{c_{ij}}) = \sigma|_{\mathcal{IL}_{g,1}}(T_{d_{ij}}) = X_i^2 - X_{ij} + X_j^2, \quad \sigma|_{\mathcal{IL}_{g,1}}(T_{c_k}) = \sigma|_{\mathcal{IL}_{g,1}}(T_{d_k}) = X_k^2,$$

each of $\sigma|_{\mathcal{IL}_{g,1}}(G_1)$ and $\sigma|_{\mathcal{IL}_{g,1}}(G_2)$ generates S^2L. Clearly $fg = gf \in \mathcal{IL}_g$ holds for any $f \in G_1$ and $g \in G_2$. Hence, for each element of the form $a \wedge b$ in $\wedge^2(S^2L)$, we can take $f_1 \in G_1$ and $f_2 \in G_2$ satisfying

$$(\sigma|_{\mathcal{IL}_{g,1}})(f_1) = a, \quad (\sigma|_{\mathcal{IL}_{g,1}})(f_2) = b, \quad f_1 f_2 = f_2 f_1.$$

They give a homomorphism $\varphi\colon \mathbb{Z}^2 \to \mathcal{IL}_{g,1}$ with $(\sigma|_{\mathcal{IL}_{g,1}} \circ \varphi)_*(1) = a \wedge b$, which implies the surjectivity of $(\sigma|_{\mathcal{IL}_{g,1}})_*$. $\square$

Lemma 3.1 shows that $H_2(\mathcal{IL}_{g,1})$ is nontrivial (see also Theorem 7.1). In particular, its rank, which may be infinite, gets bigger and bigger when g grows.

Before going further, here we recall some results on the Torelli group $\mathcal{I}_{g,1}$ obtained by Johnson in [17; 18; 19; 20; 21]. First, he showed in [19] that $\mathcal{I}_{g,1}$ is finitely generated for $g \geq 3$. This fact together with the sequences (2), (3) imply that $\mathcal{L}_{g,1}$ and $\mathcal{IL}_{g,1}$ are also finitely generated. At present, it is not known whether they are finitely presentable or not, where the same question for $\mathcal{I}_{g,1}$ is a well-known open problem. Second, he showed that $\mathcal{I}_{g,1}$ is normally generated by only one element $T_{c_2}T_{d_2}^{-1}$ (see Figure 2). Finally, in [21], he determined the abelianization of $\mathcal{I}_{g,1}$ written as follows. Let B be a commutative $\mathbb{Z}_2$–algebra with unit 1 generated by formal elements $\overline{x}$ for $x \in H \otimes \mathbb{Z}_2$ and having relations

$$\overline{x}^2 = \overline{x}, \quad \overline{x+y} = \overline{x} + \overline{y} + \overline{\mu}(x,y)$$

for $x, y \in H \otimes \mathbb{Z}_2$, where $\overline{\mu}(x,y) := \mu(x,y) \bmod 2$. The algebra B can be graded by supposing that each $\overline{x}$ has degree 1 (after replacing $\overline{x}^2$ by $\overline{x}$). Let B^i be the submodule of B generated by elements of degree at most i. This endows B with a filtration

$$B^3 \supset B^2 \supset B^1 \supset B^0 = \{0,1\}.$$

We have a natural action of $\mathcal{M}_{g,1}$ on B^3 defined by $f\,\overline{x} := \overline{f_*(x)}$. It is easily checked that there exists a natural $\mathcal{M}_{g,1}$–equivariant isomorphism

$$B^3/B^2 \cong \wedge^3(H \otimes \mathbb{Z}_2).$$

Therefore we can take the fiber product $\wedge^3 H \times_{\wedge^3(H\otimes\mathbb{Z}_2)} B^3$ of the natural projections $B^3 \to B^3/B^2 \cong \wedge^3(H\otimes\mathbb{Z}_2)$ and $\wedge^3 H \to \wedge^3(H\otimes\mathbb{Z}_2)$. Then Johnson gave an $\mathcal{M}_{g,1}$–equivariant isomorphism

$$(\tau,\beta)\colon H_1(\mathcal{I}_{g,1}) \xrightarrow{\cong} \wedge^3 H \times_{\wedge^3(H\otimes\mathbb{Z}_2)} B^3,$$

where $\mathcal{M}_{g,1}$ acts on $\mathcal{I}_{g,1}$ and $H_1(\mathcal{I}_{g,1})$ by conjugation and on $\wedge^3 H \times_{\wedge^3(H\otimes\mathbb{Z}_2)} B^3$ diagonally. The homomorphism τ is now called the *Johnson homomorphism* [17; 20] and β is called the *Birman–Craggs–Johnson homomorphism* (see Johnson [18] and Birman and Craggs [6]). Explicitly, the isomorphism is given by

$$T_{c_2}T_{d_2}^{-1} \longmapsto (x_1\wedge y_1 \wedge y_2, \overline{x}_1\overline{y}_1(\overline{y}_2+1)),$$

which characterizes an $\mathcal{M}_{g,1}$–equivariant homomorphism uniquely because $\mathcal{I}_{g,1}$ is normally generated by $T_{c_2}T_{d_2}^{-1}$.

Lemma 3.2 *We have the following isomorphism:*

$$H_1(\mathcal{I}_{g,1})_{S^2L} \cong \begin{cases} \wedge^3 L^* \oplus L^* \oplus \wedge^2(L^*\otimes\mathbb{Z}_2) & g=3, \\ \wedge^3 L^* \oplus L^* & g \geq 4. \end{cases}$$

Proof By definition, the coinvariant part $H_1(\mathcal{I}_{g,1})_{S^2L}$ is the quotient of $H_1(\mathcal{I}_{g,1})$ by the submodule Q_0 generated by $\{\sigma x - x \mid \sigma \in S^2L, x \in H_1(\mathcal{I}_{g,1})\}$. We now list a generating set of Q_0 explicitly. Assuming that the indices $i,j,k,l \in \{1,2,\dots,g\}$ are distinct from each other, we have

$$\begin{aligned} X_j^2&(x_i\wedge x_j\wedge y_j, \overline{x}_i\overline{x}_j\overline{y}_j) - (x_i\wedge x_j\wedge y_j, \overline{x}_i\overline{x}_j\overline{y}_j) \\ &= (x_i\wedge x_j\wedge (x_j+y_j), \overline{x}_i\overline{x}_j\overline{x_j+y_j}) - (x_i\wedge x_j\wedge y_j, \overline{x}_i\overline{x}_j\overline{y}_j) \\ &= (x_i\wedge x_j\wedge y_j, \overline{x}_i\overline{x}_j(\overline{x}_j+\overline{y}_j+1)) - (x_i\wedge x_j\wedge y_j, \overline{x}_i\overline{x}_j\overline{y}_j) \\ &= (0, \overline{x}_i\overline{x}_j^2 + \overline{x}_i\overline{x}_j) = (0,0), \end{aligned}$$

where we used the relations $\overline{x}_j^2 = \overline{x}_j$ and $2\overline{x}_i\overline{x}_j = 0$ in B^3. We denote this result by

(1a) $[X_j^2; (x_i\wedge x_j\wedge y_j, \overline{x}_i\overline{x}_j\overline{y}_j)] := (0,0)$

for short. Similar calculations show that

(1b) $[X_{kj};(x_i \wedge x_j \wedge y_j, \overline{x}_i\overline{x}_j\overline{y}_j)] = (x_i \wedge x_j \wedge x_k, \overline{x}_i\overline{x}_j\overline{x}_k)$,

(1c) $[X_{ij};(x_i \wedge x_j \wedge y_j, \overline{x}_i\overline{x}_j\overline{y}_j)] = (0, \overline{x}_i\overline{x}_j)$,

(2a) $[X_k^2;(x_i \wedge x_j \wedge y_k, \overline{x}_i\overline{x}_j\overline{y}_k)] = (x_i \wedge x_j \wedge x_k, \overline{x}_i\overline{x}_j\overline{x}_k + \overline{x}_i\overline{x}_j)$,

(2b) $[X_{jk};(x_i \wedge x_j \wedge y_k, \overline{x}_i\overline{x}_j\overline{y}_k)] = (0, \overline{x}_i\overline{x}_j)$,

(2c)* $[X_{lk};(x_i \wedge x_j \wedge y_k, \overline{x}_i\overline{x}_j\overline{y}_k)] = (x_i \wedge x_j \wedge x_l, \overline{x}_i\overline{x}_j\overline{x}_l)$,

(3a) $[X_j^2;(x_i \wedge y_i \wedge y_j, \overline{x}_i\overline{y}_i\overline{y}_j)] = (-x_i \wedge x_j \wedge y_i, \overline{x}_i\overline{x}_j\overline{y}_i + \overline{x}_i\overline{y}_i)$,

(3b) $[X_{ik};(x_i \wedge y_i \wedge y_j, \overline{x}_i\overline{y}_i\overline{y}_j)] = (x_i \wedge x_k \wedge y_j, \overline{x}_i\overline{x}_k\overline{y}_j)$,

(3c) $[X_{jk};(x_i \wedge y_i \wedge y_j, \overline{x}_i\overline{y}_i\overline{y}_j)] = (-x_i \wedge x_k \wedge y_i, \overline{x}_i\overline{x}_k\overline{y}_i)$,

(3d) $[X_{ij};(x_i \wedge y_i \wedge y_j, \overline{x}_i\overline{y}_i\overline{y}_j)] = (x_i \wedge x_j \wedge y_j, \overline{x}_i\overline{x}_j\overline{y}_j + \overline{x}_i\overline{x}_j + \overline{x}_i\overline{y}_i)$,

(4a) $[X_j^2;(x_i \wedge y_j \wedge y_k, \overline{x}_i\overline{y}_j\overline{y}_k)] = (x_i \wedge x_j \wedge y_k, \overline{x}_i\overline{x}_j\overline{y}_k + \overline{x}_i\overline{y}_k)$,

(4b) $[X_{ij};(x_i \wedge y_j \wedge y_k, \overline{x}_i\overline{y}_j\overline{y}_k)] = (0, \overline{x}_i\overline{y}_k)$,

(4c)* $[X_{jl};(x_i \wedge y_j \wedge y_k, \overline{x}_i\overline{y}_j\overline{y}_k)] = (x_i \wedge x_l \wedge y_k, \overline{x}_i\overline{x}_l\overline{y}_k)$,

(4d) $[X_{jk};(x_i \wedge y_j \wedge y_k, \overline{x}_i\overline{y}_j\overline{y}_k)] = (x_i \wedge x_k \wedge x_j + x_i \wedge x_k \wedge y_k - x_i \wedge x_j \wedge y_j,$
$\overline{x}_i\overline{x}_k\overline{x}_j + \overline{x}_i\overline{x}_k\overline{y}_k + \overline{x}_i\overline{x}_j\overline{y}_j)$,

(5a) $[X_i^2;(y_i \wedge y_j \wedge y_k, \overline{y}_i\overline{y}_j\overline{y}_k)] = (x_i \wedge y_j \wedge y_k, \overline{x}_i\overline{y}_j\overline{y}_k + \overline{y}_j\overline{y}_k)$,

(5b)* $[X_{il};(y_i \wedge y_j \wedge y_k, \overline{y}_i\overline{y}_j\overline{y}_k)] = (x_l \wedge y_j \wedge y_k, \overline{x}_l\overline{y}_j\overline{y}_k)$,

(5c) $[X_{ij};(y_i \wedge y_j \wedge y_k, \overline{y}_i\overline{y}_j\overline{y}_k)] = (x_j \wedge x_i \wedge y_k + x_j \wedge y_j \wedge y_k - x_i \wedge y_i \wedge y_k,$
$\overline{x}_j\overline{x}_i\overline{y}_k + \overline{x}_j\overline{y}_j\overline{y}_k + \overline{x}_i\overline{y}_i\overline{y}_k)$,

(6) $[X_{ij};(0, \overline{x}_i\overline{y}_i)] = (0, \overline{x}_i\overline{x}_j)$,

(7a) $[X_j^2;(0, \overline{x}_i\overline{y}_j)] = (0, \overline{x}_i\overline{x}_j + \overline{x}_i)$,

(7b) $[X_{ij};(0, \overline{x}_i\overline{y}_j)] = (0, \overline{x}_i)$,

(7c) $[X_{jk};(0, \overline{x}_i\overline{y}_j)] = (0, \overline{x}_i\overline{x}_k)$,

(8a) $[X_i^2;(0, \overline{y}_i\overline{y}_j)] = (0, \overline{x}_i\overline{y}_j + \overline{y}_j)$,

(8b) $[X_{ik};(0, \overline{y}_i\overline{y}_j)] = (0, \overline{x}_k\overline{y}_j)$,

(8c) $[X_{ij};(0, \overline{y}_i\overline{y}_j)] = (0, \overline{x}_i\overline{x}_j + \overline{x}_i\overline{y}_i + \overline{x}_j\overline{y}_j)$,

(9a) $[X_i^2;(0, \overline{y}_i)] = (0, \overline{x}_i + 1)$,

(9b) $[X_{ij};(0, \overline{y}_i)] = (0, \overline{x}_j)$, where $(\cdot)^*$ means that it is valid for $g \geq 4$. The actions not listed above are all trivial, namely $\sigma x - x = (0,0)$, so that they do not contribute to Q_0. In particular, there are no contribution from the elements

$$(x_i \wedge x_j \wedge x_k, \bar{x}_i \bar{x}_j \bar{x}_k), \quad (0, \bar{x}_i \bar{x}_j), \quad (0, \bar{x}_i), \quad (0, 1).$$

From (7b), (9a), (1c), (1b), (4b), (8a), (3b), (3c), (3a), (5b), (5a), we see that, for $g \geq 4$, Q_0 contains $(0, \bar{x}_j)$, $(0,1)$, $(0, \bar{x}_i \bar{x}_j)$, $(x_i \wedge x_j \wedge x_k, \bar{x}_i \bar{x}_j \bar{x}_k)$, $(0, \bar{x}_i \bar{y}_k)$, $(0, \bar{y}_j)$, $(x_i \wedge x_k \wedge y_j, \bar{x}_i \bar{x}_k \bar{y}_j)$, $(x_i \wedge x_k \wedge y_i, \bar{x}_i \bar{x}_k \bar{y}_i)$, $(0, \bar{x}_i \bar{y}_i)$, $(x_l \wedge y_j \wedge y_k, \bar{x}_l \bar{y}_j \bar{y}_k)$, $(0, \bar{y}_j \bar{y}_k)$ in order, and combinations of these elements express all the generators listed above except (5c). Finally (5c) shows $(x_j \wedge y_j \wedge y_k - x_i \wedge y_i \wedge y_k, \bar{x}_j \bar{y}_j \bar{y}_k + \bar{x}_i \bar{y}_i \bar{y}_k)$ are in Q_0. Our claim for $g \geq 4$ follows from this, where we assign $y_k \in L^*$ to $(y_k \wedge x_i \wedge y_i, \bar{y}_k \bar{x}_i \bar{y}_i) \in H_1(\mathcal{I}_{g,1})_{S^2 L}$, which does not depend on i.

When $g = 3$, differently from the above, we cannot remove $(x_l \wedge y_j \wedge y_k, \bar{x}_l \bar{y}_j \bar{y}_k)$ and $(0, \bar{y}_j \bar{y}_k)$ simultaneously. In this case, we use (5a) to eliminate $(x_l \wedge y_j \wedge y_k, \bar{x}_l \bar{y}_j \bar{y}_k)$ and conclude that $(0, \bar{y}_j \bar{y}_k)$ survive in $H_1(\mathcal{I}_{g,1})_{S^2 L}$ and form $\wedge^2(L^* \otimes \mathbb{Z}_2)$. □

By the exact sequence (6) together with Lemmas 3.1, 3.2, we conclude the following.

Theorem 3.3

$$H_1(\mathcal{IL}_{g,1}) \cong \begin{cases} \wedge^3 L^* \oplus L^* \oplus \wedge^2(L^* \otimes \mathbb{Z}_2) \oplus S^2 L & g = 3, \\ \wedge^3 L^* \oplus L^* \oplus S^2 L & g \geq 4. \end{cases}$$

Remark 3.4 In [23, Theorem 1], Levine constructed a surjective homomorphism

$$\mathcal{J} \colon H_1(\mathcal{IL}_{g,1}) \twoheadrightarrow \wedge^3 L^* \oplus L^*$$

by using the Johnson homomorphism τ for $\mathcal{I}_{g,1}$. We can check that $\mathcal{J}$ coincides with the projection to the first two components of the isomorphism in Theorem 3.3. In [8, Section 5.1], Broaddus, Farb and Putman gave another construction of $\mathcal{J}$. In fact, their homomorphisms called *relative Johnson homomorphisms* cover not only $\mathcal{IL}_{g,1}$ but any subgroup of $\mathcal{M}_{g,1}$ fixing a given submodule of H.

4 The first and second homology of urSp($2g$)

In this section, we determine the first and second homology of urSp($2g$) for later use. By a technical reason, we first consider its index 2 subgroup $\mathrm{urSp}^+(2g)$ defined by

(8) $$1 \longrightarrow \mathrm{urSp}^+(2g) \longrightarrow \mathrm{urSp}(2g) \xrightarrow{\det \circ \mathrm{ul}} \mathbb{Z}_2 \longrightarrow 1.$$

By restricting the sequence (4) to $\mathrm{urSp}^+(2g)$, we have a split exact sequence

$$1 \longrightarrow S^2L \longrightarrow \mathrm{urSp}^+(2g) \xrightarrow{\mathrm{ul}} \mathrm{SL}(g,\mathbb{Z}) \longrightarrow 1. \tag{9}$$

Proposition 4.1 (1) *The group* $\mathrm{urSp}^+(2g)$ *is perfect, that is* $H_1(\mathrm{urSp}^+(2g)) = 0$ *for* $g \geq 3$.

(2) $$H_2(\mathrm{urSp}^+(2g)) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & g = 3, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & g = 4, \\ \mathbb{Z}_2 & g \geq 5. \end{cases}$$

We will prove this proposition by using the Lyndon–Hochschild–Serre spectral sequence

$$E^2_{p,q} = H_p(\mathrm{SL}(g,\mathbb{Z}); H_q(S^2L)) \Longrightarrow H_n(\mathrm{urSp}^+(2g)) \tag{10}$$

associated with (9). Before that, we recall the first and second homology of $\mathrm{SL}(g,\mathbb{Z})$. Refer to books of Milnor [26, Sections 5 and 10] and Rosenberg [31, Sections 4.1 and 4.2] for the facts below and generalities of the second homology of groups. The group $\mathrm{SL}(g,\mathbb{Z})$ has a presentation given by

- generators: $\{e_{ij} \mid 1 \leq i \leq g, 1 \leq j \leq g \text{ and } i \neq j\}$,
- relations: $$\begin{array}{ll} [e_{ij}, e_{kl}] = 1 & \text{if } j \neq k \text{ and } i \neq l, \\ [e_{ik}, e_{kj}] = e_{ij} & \text{if } i \neq j \neq k \neq i, \\ (e_{12}e_{21}^{-1}e_{12})^4 = 1, & \end{array}$$

where e_{ij} corresponds to the matrix whose diagonal entries and (i,j)–entry are 1 with the others 0. From this presentation, we immediately see that $\mathrm{SL}(g,\mathbb{Z})$ is perfect for every $g \geq 3$. The second homology, which is also called the *Schur multiplier*, of $\mathrm{SL}(g,\mathbb{Z})$ is also known:

$$H_2(\mathrm{SL}(g,\mathbb{Z})) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & g = 3, 4, \text{(by van der Kallen [35])}, \\ \mathbb{Z}_2 & g \geq 5, \end{cases}$$

where van der Kallen also showed in [35] that one summand of $H_2(\mathrm{SL}(3,\mathbb{Z})) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ survives in the stable homology $\lim_{g\to\infty} H_2(\mathrm{SL}(g,\mathbb{Z})) \cong K_2(\mathbb{Z}) \cong \mathbb{Z}_2$ under stabilization, while the other one vanishes in $H_2(\mathrm{SL}(4,\mathbb{Z}))$.

For computations of the zeroth and first homology of a group G, we can use any connected CW–complex X with $\pi_1 X = G$. Let X_g be a connected CW–complex associated with the above presentation of $\mathrm{SL}(g,\mathbb{Z})$. Namely X_g consists of one vertex, edges $\{\langle e_{ij}\rangle \mid 1 \leq i \leq g, 1 \leq j \leq g \text{ and } i \neq j\}$ and faces

$$\{\langle [e_{ij}, e_{kl}]\rangle \mid j \neq k \text{ and } i \neq l\} \cup \{\langle [e_{ik}, e_{kj}]e_{ij}^{-1}\rangle \mid \text{if } i \neq j \neq k \neq i\} \cup \{\langle (e_{12}e_{21}^{-1}e_{12})^4\rangle\}$$

attached to the 1–skeleton of X_g along the words. We consider S^2L to be a local coefficient system on X_g. The boundary maps

$$\partial_1\colon C_1(X_g;S^2L)\to C_0(X_g;S^2L)\cong S^2L,$$
$$\partial_2\colon C_2(X_g;S^2L)\to C_1(X_g;S^2L)$$

of the complex $C_*(X_g;S^2L)=C_*(X_g)\otimes S^2L$ are given by

$$\partial_1(\langle e_{ij}\rangle\otimes c)=(e_{ij}^{-1}-1)c,$$
$$\partial_2(\langle e_1e_2\cdots e_n\rangle\otimes c)=\langle e_1\rangle\otimes c+\langle e_2\rangle\otimes e_1^{-1}c+\langle e_3\rangle\otimes(e_1e_2)^{-1}c +\cdots+\langle e_n\rangle\otimes(e_1e_2\cdots e_{n-1})^{-1}c$$

for $c\in S^2L$, where $e_1,e_2,\dots,e_n\in\{e_{ij}\}_{i,j}\cup\{e_{ij}^{-1}\}_{i,j}$ and $\langle e_{ij}^{-1}\rangle\otimes c:=-\langle e_{ij}\rangle\otimes e_{ij}c$. The action of $\mathrm{SL}(g,\mathbb{Z})$ on L is given by

$$e_{ij}\colon x_k\mapsto\delta_{jk}x_i+x_k,\quad e_{ij}^{-1}\colon x_k\mapsto-\delta_{jk}x_i+x_k.$$

Lemma 4.2 (1) $H_0(\mathrm{SL}(g,\mathbb{Z});S^2L)\cong(S^2L)_{\mathrm{SL}(g,\mathbb{Z})}=0$ *for* $g\geq 3$.

(2) $H_1(\mathrm{SL}(g,\mathbb{Z});S^2L)=0$ *for* $g\geq 4$.

Proof Here and hereafter, we suppose that the indices i,j,k,l are distinct from each other. We have

$$\partial_1(\langle e_{ij}\rangle\otimes X_j^2)=(e_{ij}^{-1}-1)X_j^2=(-x_i+x_j)^{\otimes 2}-X_j^2=X_i^2-X_{ij},$$
$$\partial_1(\langle e_{ij}\rangle\otimes X_{jk})=(e_{ij}^{-1}-1)X_{jk}=(-X_{ik}+X_{jk})-X_{jk}=-X_{ik}.$$

By running i,j,k in $\{1,2,\dots,g\}$ with $g\geq 3$, we immediately see that ∂_1 is surjective and (1) holds. To show (2), it suffices to check that $\partial_1\colon C_1(\mathrm{SL}(g,\mathbb{Z});S^2L)/\operatorname{Im}\partial_2\to S^2L$ is an isomorphism. Assume that $g\geq 4$. $C_1(\mathrm{SL}(g,\mathbb{Z});S^2L)$ is generated by elements of types

$$\mathrm{I}:\langle e_{ij}\rangle\otimes X_i^2,\quad \mathrm{II}:\langle e_{ij}\rangle\otimes X_j^2,\quad \mathrm{III}:\langle e_{ij}\rangle\otimes X_k^2,$$
$$\mathrm{IV}:\langle e_{ij}\rangle\otimes X_{ij},\quad \mathrm{V}:\langle e_{ij}\rangle\otimes X_{jk},\quad \mathrm{VI}:\langle e_{ij}\rangle\otimes X_{il},\quad \mathrm{VII}:\langle e_{ij}\rangle\otimes X_{kl}.$$

For $c\in S^2L$, we have

$$\partial_2(\langle[e_{ik},e_{kj}]e_{ij}^{-1}\rangle\otimes c)=\langle e_{ik}\rangle\otimes(1-e_{kj}^{-1}e_{ij}^{-1})c+\langle e_{kj}\rangle\otimes(e_{ik}^{-1}-e_{ij}^{-1})c-\langle e_{ij}\rangle\otimes c.$$

By putting $c=X_i^2,X_j^2,X_k^2,X_l^2,X_{jk}$ and X_{jl}, we see that

(i) $-\langle e_{ij}\rangle\otimes X_i^2$ (type I),

(ii) $\langle e_{ik}\rangle\otimes(X_{ij}+X_{jk}-X_{ik}-X_i^2-X_k^2)+\langle e_{kj}\rangle\otimes(-X_i^2+X_{ij})-\langle e_{ij}\rangle\otimes X_j^2$,

(iii) $\langle e_{kj}\rangle \otimes (X_i^2 - X_{ik} + X_k^2) - \langle e_{ij}\rangle \otimes X_k^2$,

(iv) $-\langle e_{ij}\rangle \otimes X_l^2$ (type III),

(v) $\langle e_{ik}\rangle \otimes (X_{ik} + 2X_k^2) + \langle e_{kj}\rangle \otimes (X_{ik} - X_{ij}) - \langle e_{ij}\rangle \otimes X_{jk}$,

(vi) $\langle e_{ik}\rangle \otimes (X_{il} + X_{kl}) + \langle e_{kj}\rangle \otimes X_{il} - \langle e_{ij}\rangle \otimes X_{jl}$

are in $\operatorname{Im}\partial_2$. From (i), (iii) and (iv), $-\langle e_{kj}\rangle \otimes X_{ik}$ (type VI) is in $\operatorname{Im}\partial_2$. Also

$$\text{(vii)}\quad \partial_2(\langle [e_{ij}, e_{kl}]\rangle \otimes X_l^2) = \langle e_{ij}\rangle \otimes (1 - e_{kl}^{-1})X_l^2 + \langle e_{kl}\rangle \otimes (e_{ij}^{-1} - 1)X_l^2 , \\ = \langle e_{ij}\rangle \otimes (X_{kl} - X_k^2)$$

$$\text{(viii)}\quad \partial_2(\langle [e_{ij}, e_{kj}]\rangle \otimes X_j^2) = \langle e_{ij}\rangle \otimes (-X_k^2 + X_{kj}) + \langle e_{kj}\rangle \otimes (X_i^2 - X_{ij})$$

are in $\operatorname{Im}\partial_2$. From (iv) and (vii), $\langle e_{ij}\rangle \otimes X_{kl}$ (type VII) is in $\operatorname{Im}\partial_2$. Then we can derive from (vi) that

(ix) $\langle e_{ik}\rangle \otimes X_{kl} - \langle e_{ij}\rangle \otimes X_{jl} \in \operatorname{Im}\partial_2$.

We see from (iv) and (viii) that

(x) $\langle e_{ij}\rangle \otimes X_{jk} - \langle e_{kj}\rangle \otimes X_{ji} \in \operatorname{Im}\partial_2$.

Finally, we can derive from (ii) and (v) that

(xi) $\langle e_{ik}\rangle \otimes (X_{jk} - X_{ik} - X_k^2) + \langle e_{kj}\rangle \otimes X_{ij} - \langle e_{ij}\rangle \otimes X_j^2$,

(xii) $\langle e_{ik}\rangle \otimes (X_{ik} + 2X_k^2) - \langle e_{kj}\rangle \otimes X_{ij} - \langle e_{ij}\rangle \otimes X_{jk}$

are in $\operatorname{Im}\partial_2$.

We have so far shown that $C_1(\mathrm{SL}(g,\mathbb{Z}); S^2L)/\operatorname{Im}\partial_2$ is a quotient of the module M generated by the elements of types (II), (IV) and (V) with the relations (ix), (x), (xi) and (xii). We can use (xii) to remove $\langle e_{ik}\rangle \otimes X_{ik}$ (type IV) and to produce a relation

(xiii) $\langle e_{ik}\rangle \otimes (X_{jk} + X_k^2) - \langle e_{ij}\rangle \otimes (X_j^2 + X_{jk})$

in M from (xi). Therefore M is generated by the elements of types (II) and (V) with the relations (ix), (x), (xiii). The relation (ix) enables us to put $Y_{il} := -\langle e_{ij}\rangle \otimes X_{jl} \in M$, which does not depend on j, and the relation (x) shows that $Y_{il} = Y_{li}$. On the other hand, if we put $Y_i(j,k) := \langle e_{ij}\rangle \otimes X_j^2 - \langle e_{ik}\rangle \otimes X_{kj}$, it follows from (ix) and (xiii) that $Y_i(j,l) = Y_i(j,k) = Y_i(k,j) \in M$. This implies that $Y_i := Y_i(j,l) \in M$ is independent of j and l. Consequently, M is a free module with a basis $\{Y_i \mid 1 \le i \le g\} \cup \{Y_{jk} \mid 1 \le j < k \le g\}$. It is easy to see that the homomorphism

$$\widetilde{\partial}_1\colon M \to C_0(\mathrm{SL}(g,\mathbb{Z}); S^2 L) \cong S^2 L$$

induced from the surjection

$$\partial_1\colon C_1(\mathrm{SL}(g,\mathbb{Z}); S^2 L)/\operatorname{Im}\partial_2 \twoheadrightarrow C_0(\mathrm{SL}(g,\mathbb{Z}); S^2 L)$$

is an isomorphism since $\widetilde{\partial}_1(Y_i) = X_i^2$ and $\widetilde{\partial}_1(Y_{jk}) = X_{jk}$. Therefore

$$\partial_1\colon C_1(\mathrm{SL}(g,\mathbb{Z}); S^2 L)/\operatorname{Im}\partial_2 \to S^2 L$$

is an isomorphism and (2) is proved. □

Lemma 4.3 *$H_0(\mathrm{SL}(g,\mathbb{Z}); H_2(S^2 L)) \cong (\wedge^2(S^2 L))_{\mathrm{SL}(g,\mathbb{Z})} = 0$ for $g \ge 4$.*

Proof By definition, the coinvariant part $(\wedge^2(S^2 L))_{\mathrm{SL}(g,\mathbb{Z})}$ is the quotient of $\wedge^2(S^2 L)$ by the submodule Q_1 generated by $\{[e;x] \mid e \in \mathrm{SL}(g,\mathbb{Z}),\ x \in \wedge^2(S^2 L)\}$, where we put $[e;x] := ex - x$. Direct computations show that

(i) $[e_{ij}; X_i^2 \wedge X_j^2] = X_i^2 \wedge X_{ij}$,

(ii) $[e_{kl}; X_i^2 \wedge X_{jl}] = X_i^2 \wedge X_{jk}$,

(iii) $[e_{kj}; X_i^2 \wedge X_j^2] = X_i^2 \wedge (X_k^2 + X_{jk})$,

(iv) $[e_{ji}; X_i^2 \wedge X_{jk}] = (X_j^2 + X_{ij}) \wedge X_{jk}$,

(v) $[e_{ij}; X_j^2 \wedge X_{kl}] = X_i^2 \wedge X_{kl} + X_{ij} \wedge X_{kl}$

are in Q_1 and that they generate $\wedge^2(S^2 L)$ by running i, j, k, l in $\{1, 2, \dots, g\}$ with $g \ge 4$. This completes the proof. □

Proof of Proposition 4.1(1) for $g \ge 3$ and (2) for $g \ge 4$ When $g \ge 3$, we have $E^2_{1,0} = E^2_{0,1} = 0$ in the spectral sequence (10) by Lemma 4.2(1) and the fact that $H_1(\mathrm{SL}(g,\mathbb{Z})) = 0$. This proves (1).

Assume further that $g \ge 4$. By Lemma 4.2(2) and Lemma 4.3, we have $E^2_{1,1} = E^2_{0,2} = 0$ in the spectral sequence (10). It follows that $H_2(\mathrm{urSp}^+(2g)) \cong H_2(\mathrm{SL}(g,\mathbb{Z}))$. We finish the proof of (2) for $g \ge 4$ by using the explicit description of $H_2(\mathrm{SL}(g,\mathbb{Z}))$. □

Corollary 4.4 (1) *$H_1(\mathrm{urSp}(2g)) \cong H_1(\mathrm{GL}(g,\mathbb{Z})) \cong \mathbb{Z}_2$ for $g \ge 3$.*

(2) *$H_2(\mathrm{urSp}(2g)) \cong H_2(\mathrm{urSp}^+(2g))$ for $g \ge 3$.*

Proof By using the Lyndon–Hochschild–Serre spectral sequence associated with the split extension (8) and the fact that $H_1(\mathrm{urSp}^+(2g)) = 0$, we have $H_1(\mathrm{urSp}(2g)) \cong \mathbb{Z}_2$ and $H_2(\mathrm{urSp}(2g)) \cong H_2(\mathrm{urSp}^+(2g))_{\mathbb{Z}_2}$. For $g \geq 4$, we have that $H_2(\mathrm{urSp}^+(2g))_{\mathbb{Z}_2} \cong H_2(\mathrm{SL}(g,\mathbb{Z}))_{\mathbb{Z}_2}$. The action of $\mathbb{Z}_2$ on $H_2(\mathrm{urSp}^+(2g))$ is compatible with that on $H_2(\mathrm{SL}(g,\mathbb{Z}))$ and the latter one is known to be trivial. Hence $H_2(\mathrm{urSp}(2g)) \cong H_2(\mathrm{urSp}^+(2g))$ follows. When $g = 3$, the action of $\mathbb{Z}_2$ on $H_2(\mathrm{urSp}^+(2g))$ is also trivial, since we can take the minus of the identity matrix as a lift of the generator of $\mathbb{Z}_2$ and it is central. Therefore $H_2(\mathrm{urSp}(2g)) \cong H_2(\mathrm{urSp}^+(2g))$ holds also for $g = 3$. □

It remains to compute $H_2(\mathrm{urSp}^+(2g))$ for $g = 3$.

Lemma 4.5 $H_1(\mathrm{SL}(3,\mathbb{Z}); S^2L) \cong \mathbb{Z}_2$ *and it is generated by* $\langle e_{12}\rangle \otimes X_3^2$.

Sketch of Proof Now $\mathrm{SL}(3,\mathbb{Z})$ has a presentation consisting of 6 generators and 13 relations. Also we have $S^2L \cong \mathbb{Z}^6$. Hence the complex

$$C_2(\mathrm{SL}(3,\mathbb{Z}); S^2L) \xrightarrow{\partial_2} C_1(\mathrm{SL}(3,\mathbb{Z}); S^2L) \xrightarrow{\partial_1} C_0(\mathrm{SL}(3,\mathbb{Z}); S^2L)$$

can be explicitly written as

$$\mathbb{Z}^{78} \xrightarrow{D_2\cdot} \mathbb{Z}^{36} \xrightarrow{D_1\cdot} \mathbb{Z}^6$$

with some matrices D_1 and D_2. The author with an aid of a computer calculated the homology by using the Smith normal form. We omit the details. □

Lemma 4.6 $H_0(\mathrm{SL}(3,\mathbb{Z}); H_2(S^2L)) \cong H_2(S^2L)_{\mathrm{SL}(3,\mathbb{Z})} \cong (\wedge^2(S^2L))_{\mathrm{SL}(3,\mathbb{Z})} \cong \mathbb{Z}_2$ *and it is generated by* $X_3^2 \wedge X_2^2$. *Moreover this generator is mapped nontrivially to* $H_2(\mathrm{Sp}(6,\mathbb{Z}))$ *by the composition* $H_2(S^2L)_{\mathrm{SL}(3,\mathbb{Z})} \to H_2(\mathrm{urSp}^+(6)) \to H_2(\mathrm{Sp}(6,\mathbb{Z}))$ *induced from the inclusions* $S^2L \hookrightarrow \mathrm{urSp}^+(6) \hookrightarrow \mathrm{Sp}(6,\mathbb{Z})$.

In the proof of this lemma, the following theorem by Stein plays a key role.

Theorem 4.7 (Stein [34, Theorem 2.2]) $H_2(\mathrm{Sp}(6,\mathbb{Z})) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ *and the abelian cycle associated with the homomorphism* $\varphi\colon \mathbb{Z}^2 \to \mathrm{Sp}(6,\mathbb{Z})$ *defined by*

$$\varphi((1,0)) = X_3^2, \quad \varphi((0,1)) = X_2^2$$

gives the element of order 2, *where* X_3^2 *and* X_2^2 *are in* $S^2L \subset \mathrm{urSp}^+(6) \subset \mathrm{Sp}(6,\mathbb{Z})$.

Proof of Lemma 4.6 We use the same notation as in the proof of Lemma 4.3. The computational results (i), (iii) and (iv) are valid also for $g = 3$. In particular, the elements $X_i^2 \wedge X_{ij}$, $X_{ij} \wedge X_{jk}$ and $X_i^2 \wedge X_k^2 + X_i^2 \wedge X_{jk}$ are in Q_1. We also see that

$$[e_{ki}; X_i^2 \wedge X_{ij}] = X_k^2 \wedge X_{kj} + X_k^2 \wedge X_{ij} + X_{ik} \wedge X_{kj} + X_{ik} \wedge X_{ij} + X_i^2 \wedge X_{kj},$$
$$[e_{ji}; X_i^2 \wedge X_{ij}] = 2X_i^2 \wedge X_j^2 + X_{ij} \wedge X_j^2.$$

are in Q_1, from which $X_k^2 \wedge X_{ij} + X_i^2 \wedge X_{kj}$ and $2X_i^2 \wedge X_j^2$ are in Q_1. Then there remains only two possibilities: $(\wedge^2(S^2L))_{\mathrm{SL}(3,\mathbb{Z})} = 0$ or $\mathbb{Z}_2$ generated by

$$X_1^2 \wedge X_{23} = X_3^2 \wedge X_{12} = X_2^2 \wedge X_{13} = X_1^2 \wedge X_2^2 = X_1^2 \wedge X_3^2 = X_2^2 \wedge X_3^2.$$

By using Theorem 4.7, we see that the latter is true. Indeed, the element $X_2^2 \wedge X_3^2$ just maps to the element of order 2 in $H_2(\mathrm{Sp}(6,\mathbb{Z}))$ by the map $(\wedge^2(S^2L))_{\mathrm{SL}(3,\mathbb{Z})} = H_2(S^2L)_{\mathrm{SL}(3,\mathbb{Z})} \to H_2(\mathrm{Sp}(6,\mathbb{Z}))$. □

Proof of Proposition 4.1 for $g = 3$ The E^2–term of the Lyndon–Hochschild–Serre spectral sequence associated with the split extension (8) is given as follows:

$(\wedge^2(S^2L))_{\mathrm{SL}(3,\mathbb{Z})} \cong \mathbb{Z}_2$			
$(S^2L)_{\mathrm{SL}(3,\mathbb{Z})} = 0$	$H_1(\mathrm{SL}(3,\mathbb{Z}); S^2L) \cong \mathbb{Z}_2$	$H_2(\mathrm{SL}(3,\mathbb{Z}); S^2L)$	
$\mathbb{Z}$	$H_1(\mathrm{SL}(3,\mathbb{Z})) = 0$	$H_2(\mathrm{SL}(3,\mathbb{Z})) \cong \mathbb{Z}_2^2$	$H_3(\mathrm{SL}(3,\mathbb{Z}))$

By Lemma 4.6, the generator of $(\wedge^2(S^2L))_{\mathrm{SL}(3,\mathbb{Z})} = \mathbb{Z}_2$ survives in $H_2(\mathrm{urSp}^+(6))$. Therefore $d_2\colon H_2(\mathrm{SL}(3,\mathbb{Z}); S^2L) \to (\wedge^2(S^2L))_{\mathrm{SL}(3,\mathbb{Z})}$ is a trivial map. The existence of the splitting of the extension (8) shows that $d_2\colon H_3(\mathrm{SL}(3,\mathbb{Z})) \to H_1(\mathrm{SL}(3,\mathbb{Z}); S^2L)$ and $d_3\colon H_3(\mathrm{SL}(3,\mathbb{Z})) \to (\wedge^2(S^2L))_{\mathrm{SL}(3,\mathbb{Z})}$ are also trivial. Hence $E^2_{p,q} = E^\infty_{p,q}$ for $p + q \le 2$. The E^∞–term says that there exists a filtration

$$H_2(\mathrm{urSp}^+(6)) \supset F_0 \supset F_1 = E^\infty_{0,2}$$

with $H_2(\mathrm{urSp}^+(6))/F_0 \cong E^\infty_{2,0}$ and $F_0/F_1 \cong E^\infty_{1,1}$. Again the existence of the splitting of the extension (8) shows that $H_2(\mathrm{urSp}^+(6)) \cong F_0 \oplus E^\infty_{2,0} \cong F_0 \oplus H_2(\mathrm{SL}(3,\mathbb{Z}))$. Finally we consider the extension

$$0 \longrightarrow (\wedge^2(S^2L))_{\mathrm{SL}(3,\mathbb{Z})} \cong \mathbb{Z}_2 \longrightarrow F_0 \longrightarrow H_1(\mathrm{SL}(3,\mathbb{Z}); S^2L) \cong \mathbb{Z}_2 \longrightarrow 0.$$

Suppose $F_0 \cong \mathbb{Z}_4$. Then the second map $\mathbb{Z}_2 \to \mathbb{Z}_4$ should send $1 \in \mathbb{Z}_2$ to $2 \in \mathbb{Z}_4$. This contradicts to the fact that the generator of $(\wedge^2(S^2L))_{\mathrm{SL}(3,\mathbb{Z})}$ maps to the element of order 2 in $H_2(\mathrm{Sp}(6,\mathbb{Z})) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Therefore $F_0 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, finishing the proof. □

Remark 4.8 The homology of $\mathrm{SL}(3,\mathbb{Z})$ was completely determined by Soulé [33]. In particular, $H_3(\mathrm{SL}(3,\mathbb{Z})) \cong \mathbb{Z}_3^2 \oplus \mathbb{Z}_4^2$.

We finish this section by pointing out a byproduct of our argument (see also Remark 5.2). Consider the second homology of the full mapping class group $\mathcal{M}_{3,1}$ of genus 3. Korkmaz and Stipsicz [22] showed that $H_2(\mathcal{M}_3)$ is $\mathbb{Z}$ or $\mathbb{Z}\oplus\mathbb{Z}_2$. Now we can use Lemma 3.1 and the fact that the generator of $(\wedge^2(S^2L))_{\mathrm{SL}(3,\mathbb{Z})} \cong \mathbb{Z}_2$ maps to the element of order 2 in $H_2(\mathrm{Sp}(6,\mathbb{Z})) \cong \mathbb{Z}\oplus\mathbb{Z}_2$ to show that there exists an element of $H_2(\mathcal{M}_{3,1})$ which comes from $H_2(\mathcal{IL}_{g,1})$ and maps to the element of order 2 in $H_2(\mathrm{Sp}(6,\mathbb{Z}))$. Consequently, we have:

Theorem 4.9 $H_2(\mathcal{M}_{3,1}) \cong \mathbb{Z}\oplus\mathbb{Z}_2$.

By using an argument of Korkmaz and Stipsicz in [22], we can derive the following.

Corollary 4.10 $H_2(\mathcal{M}_3) \cong \mathbb{Z}\oplus\mathbb{Z}_2$ *and* $H_2(\mathcal{M}_{3,*}) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2$, *where* $\mathcal{M}_{g,*}$ *denotes the mapping class group of a surface of genus* g *with one puncture.*

5 The first and second homology of $\mathcal{L}_{g,1}$

We use our results in the previous sections to determine $H_1(\mathcal{L}_{g,1})$ and give a lower bound of $H_2(\mathcal{L}_{g,1})$.

Theorem 5.1

(1) $H_1(\mathcal{L}_{g,1}) \cong \begin{cases} \mathbb{Z}_2\oplus\mathbb{Z}_2 & g=3, \\ \mathbb{Z}_2 & g\geq 4. \end{cases}$

(2) *The map* $(\sigma|_{\mathcal{L}_{g,1}})_*\colon H_2(\mathcal{L}_{g,1}) \to H_2(\mathrm{urSp}(2g))$ *is surjective for* $g\geq 3$.

Proof Consider the five-term exact sequence

$$(11)\quad H_2(\mathcal{L}_{g,1}) \to H_2(\mathrm{urSp}(2g)) \\ \to H_1(\mathcal{I}_{g,1})_{\mathrm{urSp}(2g)} \to H_1(\mathcal{L}_{g,1}) \to H_1(\mathrm{urSp}(2g)) \to 0$$

associated with the group extension (2). We have seen that $H_1(\mathrm{urSp}(2g)) \cong \mathbb{Z}_2$. We now show that

$$H_1(\mathcal{I}_{g,1})_{\mathrm{urSp}(2g)} \cong \begin{cases} \mathbb{Z}_2 & g=3, \\ 0 & g\geq 4, \end{cases}$$

which proves the theorem for $g\geq 4$.

Put $H_1(\mathcal{I}_{g,1})_{\mathrm{urSp}(2g)} = H_1(\mathcal{I}_{g,1})/Q_2$ with Q_2 generated by

$$\{[\sigma; x] \mid \sigma \in \mathrm{urSp}(2g), x \in H_1(\mathcal{I}_{g,1})\}.$$

Note that Q_2 includes Q_0 in the proof of Lemma 3.2 since $S^2 L \subset \mathrm{urSp}(2g)$.

We have

$$[e_{kl}^{-1} \oplus e_{lk}; (y_i \wedge y_j \wedge y_k, \bar{y}_i \bar{y}_j \bar{y}_k)] = (y_i \wedge y_j \wedge y_l, \bar{y}_i \bar{y}_j \bar{y}_l)$$

for $g \geq 4$ and also have

$$[e_{ik}^{-1} \oplus e_{ki}; (y_i \wedge x_j \wedge y_j, \bar{y}_i \bar{x}_j \bar{y}_j)] = (y_k \wedge x_j \wedge y_j, \bar{y}_k \bar{x}_j \bar{y}_j)$$

for $g \geq 3$. So $H_1(\mathcal{I}_{g,1})/Q_2 = 0$ holds for $g \geq 4$. In the case where $g = 3$, we have

$$\left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; (y_1 \wedge y_2 \wedge y_3, \bar{y}_1 \bar{y}_2 \bar{y}_3)\right] = -2(y_1 \wedge y_2 \wedge y_3, \bar{y}_1 \bar{y}_2 \bar{y}_3),$$

$$[e_{ik}^{-1} \oplus e_{ki}; (0, \bar{y}_i \bar{y}_j)] = (0, \bar{y}_k \bar{y}_j).$$

Therefore $H_1(\mathcal{I}_{3,1})/Q_2$ is at most $\mathbb{Z}_2$ generated by $(y_1 \wedge y_2 \wedge y_3, \bar{y}_1 \bar{y}_2 \bar{y}_3)$. To see that $H_1(\mathcal{I}_{3,1})/Q_2 \cong \mathbb{Z}_2$, which proves (1) and (2) for $g = 3$ simultaneously, we now show that there exists a splitting $H_1(\mathcal{L}_{3,1}) \to H_1(\mathcal{I}_{3,1})/Q_2$ by constructing a homomorphism $H_1(\mathcal{L}_{3,1}) \to \mathbb{Z}_2$ whose precomposition by $H_1(\mathcal{I}_{3,1}) \to H_1(\mathcal{L}_{3,1})$ is nontrivial. Indeed if such a homomorphism exists, $H_1(\mathcal{I}_{3,1})/Q_2 \cong \mathbb{Z}_2$ immediately follows and the composition $H_1(\mathcal{I}_{3,1})/Q_2 \to H_1(\mathcal{L}_{3,1}) \to \mathbb{Z}_2 \cong H_1(\mathcal{I}_{3,1})/Q_2$ becomes the identity map.

Our construction uses the extended Johnson homomorphism

$$\rho = (\tilde{k}, \sigma)\colon \mathcal{M}_{3,1} \longrightarrow \frac{1}{2} \wedge^3 H \rtimes \mathrm{Sp}(6, \mathbb{Z})$$

first defined by Morita [29]. Note that $\tilde{k}\colon \mathcal{M}_{3,1} \to \frac{1}{2} \wedge^3 H$ is a crossed homomorphism which extends the original Johnson homomorphism $\tau\colon \mathcal{I}_{3,1} \to \wedge^3 H$. Precisely speaking, such an extension $\tilde{k}$ is not unique but unique up to certain coboundaries (see [29, Sections 4, 5] for details). Here we use the formulation by Birman, Brendle and Broaddus in [5, Section 2.2] and denote their crossed homomorphism by $\tilde{k}\colon \mathcal{M}_{3,1} \to \frac{1}{2} \wedge^3 H$ again.

Consider the composition

$$\psi\colon \mathcal{L}_{3,1} \xrightarrow{\tilde{k}|_{\mathcal{L}_{g,1}}} \frac{1}{2} \wedge^3 H \xrightarrow{\mathrm{proj}} \frac{1}{2} \wedge^3 L \cong \frac{1}{2} \wedge^3 \mathbb{Z}^3 \cong \frac{1}{2}\mathbb{Z} \longrightarrow \left(\frac{1}{2}\mathbb{Z}\right)/(2\mathbb{Z}),$$

where the second map is induced from the projection $H \twoheadrightarrow L$ (in other word, this map assigns the coefficient of $y_1 \wedge y_2 \wedge y_3$ under our basis of H). We claim the following:

(i) $\operatorname{Im} \psi \subset \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$,

(ii) $\psi\colon \mathcal{L}_{3,1} \to \mathbb{Z}_2$ is a homomorphism,

(iii) the composition $\mathcal{I}_{3,1} \to \mathcal{L}_{3,1} \xrightarrow{\psi} \mathbb{Z}_2$ is nontrivial.

To show (i), we recall that $\mathcal{L}_{3,1} = \mathcal{I}_{3,1}\mathcal{H}_{3,1}$, where $\mathcal{H}_{3,1}$ is the preimage of the handlebody mapping class group $\mathcal{H}_3$ of genus 3 by the natural homomorphism $\mathcal{M}_{3,1} \to \mathcal{M}_3$. Birman, Brendle and Broaddus showed in [5, Section 2.2] that $\widetilde{k}(h)$ does not have the term $n\, y_1 \wedge y_2 \wedge y_3$ with $n \in \frac{1}{2}\mathbb{Z} - \{0\}$ for any $h \in \mathcal{H}_{3,1}$. Since

$$\widetilde{k}(f) = \widetilde{k}(ih) = \widetilde{k}(i) + {}^{\sigma(i)}\widetilde{k}(h) = \widetilde{k}(i) + \widetilde{k}(h)$$

for any element $f = ih \in \mathcal{L}_{3,1}$ with $i \in \mathcal{I}_{3,1}$ and $h \in \mathcal{L}_{3,1}$, and $\widetilde{k}(i) = \tau(i) \in \wedge^3 H$, we see that $\psi(f) = \psi(i) + \psi(h) = \psi(i) \in \mathbb{Z}/2\mathbb{Z}$, which proves (i). Next, (ii) follows from the facts that $\mathcal{L}_{3,1}$ acts on H with keeping L and acts on L through $\mathrm{ul} \circ \sigma|_{\mathcal{L}_g}\colon \mathcal{L}_{3,1} \to \mathrm{GL}(3,\mathbb{Z})$ and that $\mathrm{GL}(3,\mathbb{Z})$ acts on $\wedge^3 L \cong \mathbb{Z}$ through $\det\colon \mathrm{GL}(3,\mathbb{Z}) \to \{1,-1\}$. Finally, (iii) clearly follows from the construction and we finish the proof. □

Remark 5.2 The above computation of $H_1(\mathcal{I}_{3,1})_{\mathrm{urSp}(6)}$ and the equality

$$\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}; (y_1 \wedge y_2 \wedge x_3, \overline{y}_1\overline{y}_2\overline{x}_3)\right] = (y_1 \wedge y_2 \wedge y_3, \overline{y}_1\overline{y}_2\overline{y}_3 + \overline{y}_1\overline{y}_2)$$

show that $H_1(\mathcal{I}_{3,1})_{\mathrm{Sp}(6,\mathbb{Z})} = 0$ (see also Putman [30, Lemma 6.4]). Then by the five term exact sequence associated with (1), the map $H_2(\mathcal{M}_{3,1}) \to H_2(\mathrm{Sp}(6,\mathbb{Z}))$ is onto. Therefore, by using the results of Korkmaz–Stipsicz and Stein mentioned in Section 4, we can obtain another proof of $H_2(\mathcal{M}_{3,1}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

Remark 5.3 We saw that $\lim_{g\to\infty} H_2(\mathrm{urSp}(2g)) \cong \lim_{g\to\infty} H_2(\mathrm{GL}(g,\mathbb{Z}))$ in Section 4. The stable homology $\lim_{g\to\infty} H_2(\mathrm{GL}(g,\mathbb{Z})) \cong \mathbb{Z}_2$ also relates to the second homology of the automorphism group of a free group as shown by Gersten [14].

6 Results for Lagrangian mapping class groups of closed surfaces

We now consider the Lagrangian mapping class groups $\mathcal{L}_g$ and $\mathcal{IL}_g$ for closed surfaces. The relationship of $\mathcal{L}_{g,1}$ and $\mathcal{L}_g$ is given by the exact sequence

$$1 \longrightarrow \pi_1(T_1\Sigma_g) \longrightarrow \mathcal{L}_{g,1} \longrightarrow \mathcal{L}_g \longrightarrow 1,$$

where $T_1\Sigma_g$ is the unit tangent bundle of Σ_g (see Birman [3]), and the relationship of $\mathcal{IL}_{g,1}$ and $\mathcal{IL}_g$ is obtained by replacing $\mathcal{L}_{g,1}$ and $\mathcal{L}_g$ with $\mathcal{IL}_{g,1}$ and $\mathcal{IL}_g$. As a subgroup of $\mathcal{L}_{g,1}$ and $\mathcal{IL}_{g,1}$, the group $\pi_1(T_1\Sigma_g) \subset \mathcal{I}_{g,1}$ is generated by the Dehn twist along the boundary curve of $\Sigma_{g,1}$ and spin-maps (see Birman [3, Theorem 4.3] and Johnson [19, Section 3], for example).

Theorem 6.1 *The homology group $H_1(\mathcal{IL}_g)$ is given by*

$$H_1(\mathcal{IL}_g) \cong \begin{cases} \wedge^3 L^* \oplus \wedge^2(L^* \otimes \mathbb{Z}_2) \oplus S^2 L & g = 3, \\ \wedge^3 L^* \oplus S^2 L & g \geq 4. \end{cases}$$

Proof We have an exact sequence

$$H_1(\pi_1(T_1\Sigma_g)) \longrightarrow H_1(\mathcal{IL}_{g,1}) \longrightarrow H_1(\mathcal{IL}_g) \longrightarrow 0.$$

In [23, Section 3.4], Levine showed that $\pi_1(T_1\Sigma_g)$ projects trivially on $\wedge^3 L^*$ and onto on L^* with respect to the abelianization

$$H_1(\mathcal{IL}_{g,1}) \cong \begin{cases} \wedge^3 L^* \oplus L^* \oplus \wedge^2(L^* \otimes \mathbb{Z}_2) \oplus S^2 L & g = 3, \\ \wedge^3 L^* \oplus L^* \oplus S^2 L & g \geq 4. \end{cases}$$

Since $\pi_1(T_1\Sigma_g)$ is included in $\mathcal{I}_{g,1}$, it projects trivially on $S^2 L$. Hence the theorem for $g \geq 4$ holds. In the case where $g = 3$, we can directly check that all of generators of $\pi_1(T_1\Sigma_3)$ are sent to $0 \in \wedge^2(L^* \otimes \mathbb{Z}_2)$, which completes the proof for $g = 3$. □

Theorem 6.2

(1) $H_1(\mathcal{L}_g) \cong H_1(\mathcal{L}_{g,1}) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & g = 3, \\ \mathbb{Z}_2 & g \geq 4. \end{cases}$

(2) *The map $(\sigma|_{\mathcal{L}_g})_*\colon H_2(\mathcal{L}_g) \to H_2(\mathrm{urSp}(2g))$ is surjective for $g \geq 3$.*

Proof Since $\sigma|_{\mathcal{L}_{g,1}}\colon \mathcal{L}_{g,1} \to \mathrm{urSp}(2g)$ factors through $\mathcal{L}_g$, (1) for $g \geq 4$ immediately holds. (1) for $g = 3$ also holds by explicit computations of the extended Johnson homomorphism for generators of $\pi_1(T_1\Sigma_3)$. The proof of (2) is the same as that of Theorem 5.1. □

7 Remarks on higher (co)homology of $\mathcal{L}_g$ and $\mathcal{IL}_g$

7.1 Relationship to the homology of the pure braid group

In [23], Levine studied various embeddings of the pure braid group P_n of n strands into $\mathcal{M}_{g,1}$ and $\mathcal{M}_g$, where $n = g, 2g$ etc. We now use one of them defined as follows. Let D_g be a disk with g holes. We take an embedding $\iota\colon D_g \hookrightarrow \Sigma_{g,1}$ as in Figure 3, where we consider the surface $\Sigma_{g,1}$ to be a disk with g handles attached and the belt circles of the handles correspond to the loops $x_1, x_2, \dots, x_g$ in Figure 1 after filling the boundary $\partial\Sigma_{g,1}$ by a disk. The mapping class group of D_g, where the self-diffeomorphisms of D_g are supposed to fix the boundary pointwise, is known to be isomorphic to the framed pure braid group of g strands. Here the framing counts how many times one gives Dehn twists along each of the loops parallel to the inner boundary. For any choice of framings, we have a homomorphism from the pure braid group P_g of g strands to $\mathcal{M}_{g,1}$ by extending each mapping class by identity on the outside of $\iota(D_g)$. We can easily check that the image of this map is contained in $\mathcal{IL}_{g,1}$. Therefore we obtain a homomorphism $\Phi\colon P_g \to \mathcal{IL}_{g,1}$. Similarly, we have a homomorphism from P_g to $\mathcal{IL}_g$ also denoted by $\Phi\colon P_g \to \mathcal{IL}_g$.

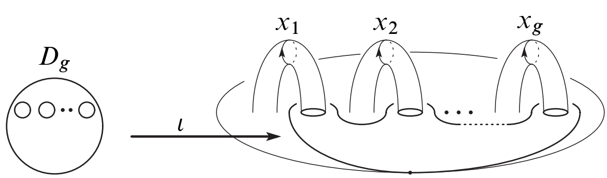

Figure 3. The embedding $\iota\colon D_g \hookrightarrow \Sigma_{g,1}$

Theorem 7.1 *The induced map $\Phi_*\colon H_*(P_g) \to H_*(\mathcal{IL}_g)$ is injective.*

Proof Consider the induced map $H^*(S^2L) \to H^*(P_g)$ of the composition $P_g \xrightarrow{\Phi} \mathcal{IL}_g \to S^2L$ on cohomology. Here the ring structure of $H^*(P_g)$ was completely determined by Arnol'd [2], and in particular, it was shown that $H^*(P_g)$ is a finitely generated free abelian group and is generated by degree 1 elements as a ring. The former shows that $H_*(P_g)$ is also finitely generated free abelian and the latter shows that $H^*(S^2L) \to H^*(P_g)$ is onto since it is clear from a presentation of P_g (see [3] for example) that $H^1(S^2L) \to H^1(P_g)$ is onto. By passing to homology, we see that $H_*(P_g) \to H_*(S^2L)$ is injective. The theorem follows from this. □

7.2 Vanishing of odd Miller–Morita–Mumford classes on $\mathcal{L}_g$

Finally, we discuss the rational cohomology of higher degrees of $\mathcal{L}_g$ with relationships to characteristic classes of oriented Σ_g–bundles called *Miller–Morita–Mumford classes*.

Here we recall the definition of Miller–Morita–Mumford classes following Morita [28]. Let $\pi\colon E \to B$ be an oriented Σ_g–bundle over a closed oriented manifold B. Since Σ_g is 2–dimensional, the relative tangent bundle $\mathrm{Ker}\,\pi_*$ is a vector bundle over E of rank 2. In particular, we can take its Euler class $e \in H^2(E)$. Then i–th Miller–Morita–Mumford class e_i is defined by

$$e_i := \pi_!(e^{i+1}) \in H^{2i}(B),$$

where $\pi_!\colon H^*(E) \to H^{*-2}(B)$ is the Gysin map. This construction is natural with respect to bundle maps, so that we can regard e_i as a cohomology class in the classifying space. Namely $e_i \in H^{2i}(B\mathrm{Diff}_+\Sigma_g)$, where $B\mathrm{Diff}_+\Sigma_g$ is the classifying space of the topological group $\mathrm{Diff}_+\Sigma_g$ of orientation preserving self-diffeomorphisms of Σ_g with C^∞–topology. By a theorem of Earle and Eells [12], we have $B\mathrm{Diff}_+\Sigma_g = K(\mathcal{M}_g, 1)$. Therefore

$$e_i \in H^{2i}(B\mathrm{Diff}_+\Sigma_g) = H^{2i}(K(\mathcal{M}_g, 1)) = H^{2i}(\mathcal{M}_g).$$

Now we ask whether $e_i \in H^{2i}(\mathcal{M}_g;\mathbb{Q})$, regarded as a rational cohomology class, survives in $H^{2i}(\mathcal{I}_g;\mathbb{Q})$ by the pullback of $\mathcal{I}_g \hookrightarrow \mathcal{M}_g$. A partial answer to this question is given as follows (see Morita [28]). It is known that every *odd* class $e_{2i-1} \in H^{4i-2}(\mathcal{M}_g;\mathbb{Q})$ can be obtained as the pullback of some class in $H^{4i-2}(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Q})$, which implies that all the odd classes e_{2i-1} vanish in $H^{4i-2}(\mathcal{I}_g;\mathbb{Q})$. However, this argument says nothing about *even* classes e_{2i} and it has been a long standing problem to determine whether even classes e_{2i} vanish or not in $H^{4i}(\mathcal{I}_g;\mathbb{Q})$.

The author's motivation for the study in this paper is to attack this problem by considering groups locating between $\mathcal{M}_g$ and $\mathcal{I}_g$ and investigating the behavior of e_i on them. As examples of such a kind of groups, finite index subgroups including level L mapping class groups defined as the kernel of the composition

$$\mathcal{M}_g \longrightarrow \mathrm{Sp}(2g,\mathbb{Z}) \longrightarrow \mathrm{Sp}(2g,\mathbb{Z}/L\mathbb{Z})$$

are often studied. However, we cannot solve the above problem by using them since for any finite index subgroup G of $\mathcal{M}_g$ there exists a transfer map

$$\mathrm{tr}\colon H^*(G;\mathbb{Q}) \longrightarrow H^*(\mathcal{M}_g;\mathbb{Q})$$

such that $\mathrm{tr} \circ i^*\colon H^*(\mathcal{M}_g;\mathbb{Q}) \to H^*(\mathcal{M}_g;\mathbb{Q})$ is the multiplication by a positive integer $[\mathcal{M}_g : G]$, where $i\colon G \hookrightarrow \mathcal{M}_g$ denotes the inclusion. In particular, we see that the pullback map on the rational cohomology is always injective for any finite index subgroup. Therefore we shall need infinite index subgroups and we focus on $\mathcal{L}_g$ and $\mathcal{IL}_g$ in this paper. At present, we cannot give the final answer even for $\mathcal{L}_g$, but we now present an observation for odd classes, by which we finish this paper.

Lemma 7.2 *If g is sufficiently larger than q, we have*

$$H^q(\mathrm{urSp}(2g);\mathbb{Q}) \cong H^q(\mathrm{GL}(g,\mathbb{Z});\mathbb{Q}).$$

Proof The E_2–term of the Lyndon–Hochschild–Serre spectral sequence for the group extension (4) is given by

$$E_2^{p,q} = H^p(\mathrm{GL}(g,\mathbb{Z}); H^q(S^2L;\mathbb{Q})).$$

Our claim immediately follows once we show that $H^p(\mathrm{GL}(g,\mathbb{Z}); H^q(S^2L;\mathbb{Q})) = 0$ if $q \geq 1$. Since $H^q(S^2L;\mathbb{Q}) \cong \wedge^q(S^2(L^*\otimes\mathbb{Q}))$ and it is easy to show that the invariant part $\wedge^q(S^2(L^*\otimes\mathbb{Q}))^{\mathrm{GL}(g,\mathbb{Z})}$ is trivial, we can use Borel's vanishing theorem [7] to show that

$$H^p(\mathrm{GL}(g,\mathbb{Z}); H^q(S^2L;\mathbb{Q})) = 0$$

for any $q \geq 1$. $\square$

Theorem 7.3 *For every i, the $(2i-1)$–st Miller–Morita–Mumford class e_{2i-1} vanishes in $H^{4i-2}(\mathcal{L}_g;\mathbb{Q})$ if g is sufficiently larger than i.*

Proof It is known that the group cohomology $H^*(G)$ of a discrete group G can be rewritten as $H^*(BG^\delta)$, where BG denotes the classifying space of G. When G is a Lie group, we write G^{C^∞} for G with C^∞ topology and G^δ for G with discrete topology.

Consider the commutative diagram

$$\begin{tikzcd}
H^*(B\mathrm{Sp}(2g,\mathbb{R})^{C^\infty};\mathbb{Q}) \arrow[rr] \arrow[d, "B(\mathrm{id})^*"] & & H^*(B\mathrm{GL}(g,\mathbb{R})^{C^\infty};\mathbb{Q}) \arrow[d, "B(\mathrm{id})^*"] \\
H^*(B\mathrm{Sp}(2g,\mathbb{R})^{\delta};\mathbb{Q}) \arrow[d] \arrow[rr, phantom, "\circlearrowleft"] & & H^*(B\mathrm{GL}(g,\mathbb{R})^{\delta};\mathbb{Q}) \arrow[d] \\
H^*(B\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Q}) \arrow[r] \arrow[d, "B\sigma^*"] \arrow[dr, phantom, "\circlearrowleft"] & H^*(B\mathrm{urSp}(()2g);\mathbb{Q}) \arrow[r] \arrow[d, "B(\sigma|_{\mathcal{L}_g})^*"] & H^*(B\mathrm{GL}(g,\mathbb{Z});\mathbb{Q}) \\
H^*(B\mathcal{M}_g);\mathbb{Q}) \arrow[r] & H^*(B\mathcal{L}_g;\mathbb{Q}) &
\end{tikzcd}$$

where id denotes the identity map, which always gives a continuous map from G with discrete topology to that with C^∞ topology for a Lie group G, and all the arrows not labeled are pullbacks by the induced maps of inclusions on classifying spaces. We now assume that g is sufficiently large. Since $\mathrm{Sp}(2g,\mathbb{R})^{C^\infty}$ is homotopy equivalent to the unitary group $U(g)^{C^\infty}$, we have $H^*(B\mathrm{Sp}(2g,\mathbb{R})^{C^\infty};\mathbb{Q}) \cong H^*(BU(g)^{C^\infty};\mathbb{Q})$ and the latter is known to be isomorphic to the polynomial algebra $\mathbb{Q}[c_1,c_2,\dots]$ generated by the Chern classes $c_1,c_2,\dots$ independently in the stable range. This polynomial algebra $\mathbb{Q}[c_1,c_2,\dots]$ is mapped onto $\mathbb{Q}[c_1,c_3,c_5,\dots]$ in $H^*(B\mathrm{Sp}(2g,\mathbb{R})^\delta;\mathbb{Q})$, and onto $\mathbb{Q}[e_1,e_3,e_5,\dots]$ in $H^*(B\mathcal{M}_g;\mathbb{Q})$. We refer to Morita [28] again for these arguments. On the other hand, it was shown by Milnor [27, Appendix] that

$$B(\mathrm{id})^*\colon H^*(B\mathrm{GL}(g,\mathbb{R})^{C^\infty};\mathbb{Q}) \longrightarrow H^*(B\mathrm{GL}(g,\mathbb{R})^\delta;\mathbb{Q})$$

is trivial for $* \geq 1$. By combining this fact with Lemma 7.2, the theorem follows. □

Remark 7.4 Recently, Giansiracusa and Tillmann [15] have proved a closely related result that odd Miller–Morita–Mumford classes vanish in the *integral* cohomology of the handlebody subgroup $\mathcal{H}_g$ for $g \geq 2$. In fact, they showed that odd Miller–Morita–Mumford classes are in the kernel of the pullback map on the integral cohomology by $B\mathrm{Diff}_+ M \to B\mathrm{Diff}_+ \Sigma_g$ where M is any compact oriented 3–manifold M with $\partial M = \Sigma_g$.

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