Noninjectivity of the “hair” map

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Kricker constructed a knot invariant $Z^{\text{rat}}$ valued in a space of Feynman diagrams with beads. When composed with the “hair” map $H$, it gives the Kontsevich integral of the knot. We introduce a new grading on diagrams with beads and use it to show that a nontrivial element constructed from Vogel’s zero divisor in the algebra $\Lambda$ is in the kernel of $H$. This shows that $H$ is not injective.

57M25, 57M27

Introduction

The Kontsevich integral $Z$ is a universal rational finite type invariant for knots (see the Bar-Natan survey [1]). For a knot $K$, $Z(K)$ lives in the space of Chinese diagrams isomorphic to $\overline{\mathcal{B}}(*)$ (see Section 1.1). Rozansky [5] conjectured and Kricker [3] proved that $Z$ can be organized into a series of “lines” called $Z^{\text{rat}}$. They can be represented by finite $\mathbb{Q}$–linear combinations of diagrams whose edges are labelled, in an appropriate way, with rational functions. Garoufalidis and Kricker [2] directly proved that the map $Z^{\text{rat}}$ with values in a space of diagrams with beads is an isotopy invariant and that $Z$ factors through $Z^{\text{rat}}$. For a knot $K$ with trivial Alexander polynomial, $Z(K) = H \circ Z^{\text{rat}}(K)$ where $H$ is the hair map (see Section 1.3). Rozansky, Garoufalidis and Kricker conjectured (see Ohtsuki [4, Conjecture 3.18]) that $H$ could be injective. Theorem 4 gives a counterexample to this conjecture.

1 The hair map

1.1 Classical diagrams

Let $X$ be a finite set. A $X$–diagram is an isomorphism class of finite unitrivalent graphs $K$ with the following data:

- At each trivalent vertex $x$ of $K$, we have a cyclic ordering on the three oriented edges starting from $x$.
- A bijection between the set of univalent vertices of $K$ and the set $X$. 

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We define $A(X)$ to be the quotient of the $\mathbb{Q}$–vector space generated by $X$–diagrams by the relations:

1. The (AS) relations for “antisymmetry”: 
   \[
   \begin{array}{c}
   \begin{array}{cc}
   \includegraphics[width=0.2\textwidth]{as.pdf}
   \\
   +
   \\
   \includegraphics[width=0.2\textwidth]{as.pdf}
   \\
   \end{array}
   \end{array}
   = 0
   \]

2. The (IHX) relations for three diagrams which differ only in a neighborhood of an edge:
   \[
   \begin{array}{c}
   \begin{array}{cc}
   \includegraphics[width=0.2\textwidth]{ihx.pdf}
   \\
   =
   \\
   \includegraphics[width=0.2\textwidth]{ihx.pdf}
   \\
   \end{array}
   \end{array}
   \]

These spaces are graded. The degree of an $X$–diagram is given by half the total number of vertices.

Let $[n] = \{1, 2, \ldots, n\}$ and define $F_n$ to be the subspace of $A([n])$ generated by connected diagrams with at least one trivalent vertex. The permutation group $\mathfrak{S}(X)$ acts on $A(X)$. Let $B(*)$ be the coinvariant space for this action:

\[ B(*) = \bigoplus_{n \in \mathbb{N}} A([n]) \otimes \mathfrak{S}_n \mathbb{Q} \]

and let $\hat{B}(*)$ be the completion of $B(*)$ for the grading.

Finally let $\Lambda$ be Vogel’s algebra generated by totally antisymmetric elements of $F_3$ (for the action of $\mathfrak{S}_3$).

We recall (see [6]) that $\Lambda$ acts on the modules $F_n$ and that for this action, $F_0$ and $F_2$ are free $\Lambda$–modules of rank one. Furthermore, the following elements are in $\Lambda$:

\[ t = \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{t.pdf}
\end{array}
\end{array} = \frac12 \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{xt.pdf}
\end{array}
\end{array}, \quad x_n = \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{x.pdf}
\end{array}
\end{array}_{n-2} \]

**Theorem 1** (Vogel [6, Section 8 and Proposition 8.5]) The element $t$ is a divisor of zero in $\Lambda$.

**Corollary 2** There exists an element $r \in \Lambda \setminus \{0\}$ such that $t \cdot r = 0$. So one has 

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{coro.pdf}
\end{array}
\end{array} \neq 0 \in F_0 \quad \text{but} \quad \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{coro.pdf}
\end{array}
\end{array} = 0 \in F_3.
\]

**Proof** $F_0$ is a free $\Lambda$–module of rank one generated by the diagram $\Theta$ and the previous diagram of $F_0$ is $r \cdot \Theta \neq 0$. The diagram of $F_3$ of the corollary is the product

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{coro.pdf}
\end{array}
\end{array} = 2tr \in \Lambda.
\]
Remark Vogel shows that $r$ can be chosen with degree fifteen in $\Lambda$ (the degree in $\Lambda$ is the degree in $F_3$ minus two), and in the algebra generated by the $x_n$. This element is killed by all the weight systems coming from Lie algebras (but $r$ is not killed by the Lie superalgebras $\mathfrak{D}_{2,1,\alpha}$).

1.2 Diagrams with beads

Diagrams with beads were introduced by Kricker and Garoufalidis [3; 2]. A presentation of $B$ which uses the first cohomology classes of diagrams is already present in [5]. Vogel explained me this point of view for diagrams with beads.

Let $G$ be the multiplicative group $\{b^n, n \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$ and consider its group algebra $R = \mathbb{Q}G = \mathbb{Q}[b, b^{-1}]$. Let $a \mapsto \overline{a}$ be the involution of the $\mathbb{Q}$–algebra $R$ that maps $b$ to $b^{-1}$.

A diagram with beads in $R$ is an $\emptyset$–diagram with the following supplementary data: The beads form a map $f : E \to R$ from the set of oriented edges of $K$ such that if $-e$ denotes the same edge than $e$ with opposite orientation, one has $f(-e) = \overline{f(e)}$.

We will represent the beads by some arrows on the edges with label in $R$. The value of the bead $f$ on $e$ is given by the product of these labels and we will not represent the beads with value 1. So with graphical notation, we have:

$$f(b) = \overline{f(b)} \quad \text{and} \quad \overline{f(b)} \cdot g(b) = \overline{f(b)g(b)}$$

The loop degree of a diagram with beads is the first Betti number of the underlying graph.

Let $\mathcal{A}^R(\emptyset)$ be the quotient of the $\mathbb{Q}$–vector space generated by diagrams with beads in $R$ by the following relations:

1. (AS)
2. The (IHX) relations should only be considered near an edge with bead 1.
3. PUSH:

$$\begin{array}{c}
\begin{array}{c}
\overline{b} \\
\downarrow \\
\overline{b}
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
\overline{b} \\
\downarrow \\
\overline{b}
\end{array}
\end{array}
$$

4. Multilinearity:

$$\alpha f(b) + \beta g(b) = \alpha \overline{f(b)} + \beta \overline{g(b)}$$
$\mathcal{A}^R(\emptyset)$ is graded by the loop degree:

$$\mathcal{A}^R(\emptyset) = \bigoplus_{n \in \mathbb{N}} \mathcal{A}^R_n(\emptyset)$$

We will prefer another presentation of $\mathcal{A}^R(\emptyset)$:

- Note that it is enough to consider diagrams with beads in $G$ and the multilinear relation can be viewed as a notation.
- Next note that for a diagram with beads in $G$, the map $f$ defines a 1–cochain $\tilde{f}$ with values in $\mathbb{Z} \simeq G$ on the underlying simplicial set of $K$. The elements $\tilde{f}$ are in fact 1–cocycles because of the condition $f(-e) = \overline{f(e)}$ which implies $\tilde{f}(-e) = -\tilde{f}(e)$.
- The “PUSH” relation at a vertex $v$ implies that $\tilde{f}$ is only given up to the coboundary of the 0–cochain with value 1 on $v$ and 0 on the other vertices. Hence $\mathcal{A}^R(\emptyset)$ is also the $\mathbb{Q}$–vector space generated by the pairs $(3$–valent graph $D, x \in H^1(D, \mathbb{Z}))$ quotiented by the relations (AS) and (IHX). With these notation one can describe the (IHX) relations in the following way:

Let $K_I, K_H$ and $K_X$ be three graphs which appear in a (IHX) relation on an edge $e$. Let $K_\bullet$ be the graph obtained by collapsing the edge $e$. The maps $p_i: K_I \to K_\bullet$ induce three cohomology isomorphisms. If $x \in H^1(K_\bullet, \mathbb{Z})$ then the (IHX) relation at $e$ says that

$$(K_I, p_I^*x) = (K_H, p_H^*x) - (K_X, p_X^*x)$$

holds in $\mathcal{A}^R(\emptyset)$.

### 1.3 The hair map

The hair map $H: \mathcal{A}^R(\emptyset) \to \hat{\mathcal{B}}(*)$ replaces beads by legs (or hair): Just replace a bead $b^n$ by the exponential of $n$ times a leg.

$$b^n \mapsto \exp\left(\sum_{k=0}^{n} \frac{n^k}{k!} \right) = n + \frac{n^2}{2!} + \cdots$$

$H$ is well defined (see [2]).
2 Grading on diagrams with beads

Note that for a 3-valent graph $K$, $H^1(K, \mathbb{Z})$ is a free $\mathbb{Z}$-module. The beads $x \in H^1(K, \mathbb{Z})$ which occur in an (AS) or (IHX) relation are the same up to isomorphisms. We will call $p \in \mathbb{N}$ the bead degree of $(K, x)$ if $x$ is $p$ times an indivisible element of $H^1(K, \mathbb{Z})$.

**Theorem 3** The bead degree is well defined in $\mathcal{A}^R_n(\emptyset)$. Thus we have a grading

$$\mathcal{A}^R_n(\emptyset) = \bigoplus_{p \in \mathbb{N}} \mathcal{A}^R_{n,p}(\emptyset),$$

where $\mathcal{A}^R_{n,p}(\emptyset)$ is the subspace of $\mathcal{A}^R_n(\emptyset)$ generated by diagrams with bead degree $p$.

Furthermore, $\mathcal{A}^R_{n,0}(\emptyset) \cong \mathcal{A}_n(\emptyset)$ and for $p > 0$, $\mathcal{A}^R_{n,p}(\emptyset) \cong \mathcal{A}^R_{n,1}(\emptyset)$.

**Proof** The second presentation we have given for $\mathcal{A}^R_n(\emptyset)$ implies that this degree is well defined. Indeed, the elements in a IHX relation have the same degree because the set of indivisible elements of the cohomology is preserved by isomorphisms.

Now, the map $\psi: R \rightarrow \mathbb{Q}$ that sends $b$ to 1 induces the isomorphism $\mathcal{A}^R_{n,0}(\emptyset) \cong \mathcal{A}_n(\emptyset)$ and the group morphism $\phi_p: G \rightarrow G$ that sends $b$ to $b^p$ (or the multiplication by $p$ in $H^1(\cdot, \mathbb{Z})$) induces the isomorphism $\mathcal{A}^R_{n,1}(\emptyset) \cong \mathcal{A}^R_{n,p}(\emptyset)$. These maps are isomorphisms because they have obvious inverses.

3 A nontrivial element in the kernel of $H$

**Theorem 4** This nontrivial element of $\mathcal{A}^R(\emptyset)$ is in the kernel of $H$:

$$\begin{align*}
\begin{array}{c}
\hline
\hline
\end{array}
\end{align*}\begin{array}{c}
\hline
\hline
\end{array}\begin{array}{c}
\hline
\hline
\end{array}$$

Thus $H$ is not injective.

**Proof** This element is not zero because its bead degree zero part is the opposite of the element $r \cdot \emptyset$ of Corollary 2. Then, one has

$$\begin{align*}
\begin{array}{c}
\hline
\hline
\end{array}
\end{align*}\begin{array}{c}
\hline
\hline
\end{array}\begin{array}{c}
\hline
\hline
\end{array} H \begin{array}{c}
\hline
\hline
\end{array}\begin{array}{c}
\hline
\hline
\end{array} + \frac{1}{2!} \begin{array}{c}
\hline
\hline
\end{array} + \frac{1}{3!} \begin{array}{c}
\hline
\hline
\end{array} + \cdots
\end{align*}$$

but all these diagrams are zero in $B(\ast)$ because they contain, as a subdiagram, the element of $F_3$ of Corollary 2. 

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Remark  The element of Theorem 4 has a loop degree seventeen.

The hair map is obviously injective on the space of diagrams with bead degree zero. I don’t know if the same is true in other degrees.

References


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