Some bounds for the knot Floer τ–invariant of satellite knots

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This paper uses four dimensional handlebody theory to compute upper and lower bounds for the Heegaard Floer τ–invariant of almost all satellite knots in terms of the τ–invariants of the pattern and the companion.

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1 Introduction

An oriented knot $K \subset S^3$ induces a filtration, $\mathcal{F}_m(K)$, on the Heegaard Floer chain complex $\widehat{CF}(S^3)$ (see Oszváth and Szabó [8]), whose homology, over the rationals, is $\widehat{HF}(S^3) \cong \mathbb{Q}_0$. For each m there is an inclusion $I_m\colon \mathcal{F}_m(K) \hookrightarrow \widehat{CF}(S^3)$ of chain complexes which induces a map $I_{m,*}\colon H_*(\mathcal{F}_m(K)) \to \widehat{HF}(S^3)$.

Definition 1.1 Oszváth–Szabó [7] Let $K \subset S^3$ be a knot. Define

$$\tau(K) = \min\left\{ m \in \mathbb{Z} \,\middle|\, I_{m,*} \text{ is nontrivial}\right\}.$$

P Ozsváth and Z Szabó [7] proved that

(1) $\tau(K)$ is an invariant of the concordance class of K,

(2) $\tau(K)$ does not depend on the orientation of K,

(3) $|\tau(K)| \leq g_4(K)$,

(4) $\tau(\bar{K}) = -\tau(K)$,

(5) $\tau(K \# J) = \tau(K) + \tau(J)$,

(6) $\tau(T_{p,q}) = (p-1)(q-1)/2$ where $p, q > 0$ and $T_{p,q}$ is the (p,q)–torus knot.

In this paper, we use four dimensional surgery techniques to compute the τ–invariant of satellite knots up to a bounded error. This approach is provided by [7, Proposition 3.1], where properties of surgeries on a knot are shown to compute the τ–invariant up to a bounded error. The resulting bounds on τ for a satellite are not particularly strong (they recover the connect sum formula for τ only up to an error of ± 2, for example), but are quite general. Furthermore, the technique could be useful in other situations, such as string link infections. The simplest form of these inequalities is:

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Theorem 1.2 *Let $S_r(C,P)$ be the r–twisted satellite knot formed from a companion C in S^3 and a pattern P in $S^1\times D^2$ (with framing fixed below). Let l be the intersection number of P with D^2, with P oriented so that $l>0$. Let n_+ be the minimal number of positive intersections contributing to l. Defining*

$$T(S_r(C,P))=\left(\tau(P)+l\,\tau(C)+\frac{l(l-1)}{2}\,r\right),$$

then

$$-n_+(P)-l\le\tau(S_r(C,P))-T(S_r(C,P))\le n_+(P)+l$$

whereas, if $l=0$,

$$-n_+(P)-1\le\tau(S_r(C,P))-T(S_r(C,P))\le n_+(P)+1.$$

Note that P will be embedded in $S^1\times D^2$, with a prescribed framing of $S^1\times D^2$, making the r–twisted satellite well-defined.

There are stronger inequalities for certain restricted ranges of r.

Theorem 1.3 *In the situation of Theorem 1.2, for $r\neq 0$, we have for $l>0$*

$$\begin{aligned}-(1+l)\le\tau(S_r(C,P))-T(S_r(C,P))&\le n_+(P)+l &&\text{when } r<2\tau(C)-1,\\ -n_+(P)-l\le\tau(S_r(C,P))-T(S_r(C,P))&\le 1+l &&\text{when } r>2\tau(C)+1,\end{aligned}$$

but for $l=0$,

$$\begin{aligned}-1\le\tau(S_r(C,P))-T(S_r(C,P))&\le n_+(P)+1 &&\text{when } r<2\tau(C)-1,\\ -n_+(P)-1\le\tau(S_r(C,P))-T(S_r(C,P))&\le 1 &&\text{when } r>2\tau(C)+1.\end{aligned}$$

Previous results on $\tau(K)$ for satellites revolve around two cases: Whitehead doubles and cables. For cables, M Hedden proved the following estimates, based on an analysis of specific Heegaard diagrams.

Theorem (Hedden [2]) *Let $K_{l,lr+1}$ be the $(l,lr+1)$–cable of K. Then*

$$l\,\tau(K)+\frac{lr(l-1)}{2}\le\tau(K_{l,lr+1})\le l\,\tau(K)+\frac{lr(l-1)}{2}+l-1.$$

Furthermore, he gave some cases in which on or other inequality is actually an equality. Additionally, I Petkova has used bordered Heegaard Floer homology to compute the τ–invariant explicitly for cables of a knot Floer homologically thin companion C [11].

M Hedden [1] also completely described the τ–invariant for the twisted Whitehead doubles of K. This completed the work of several authors including C Livingston and S Naik [5] and M Hedden and P Ording [3].

Using a different technique, C Van Cott also considered cables and discovered, in a slightly different form:

Theorem (Van Cott [13]) *Let $h(n) = \tau(K_{l,n}) - (l-1)(n-1)/2$. Then for $n > r$, n, r relatively prime to l,*

$$-(l-1) \le h(n) - h(r) \le 0.$$

In Section 4 we will make use of a similar result that we proved in [12].

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2 Requisite Heegaard Floer Results

Let $K \subset S^3$ be a knot, and let W_n be the four dimensional manifold found by attaching a 2–handle along K with framing n. Denote the n–framed Dehn surgery on K by $S^3_n(K)$; we will regard W_n as an oriented cobordism from S^3 to $S^3_n(K)$. Let Σ be a surface in B^4 with boundary K and $\widehat{\Sigma}_K$ be the surface in W_n found by capping off Σ. We can relate $\tau(K)$ to W_n through:

Lemma 2.1 [7, Proposition 3.1] *For each k, when $n > 0$ is sufficiently large relative to k, the cobordism map $\widehat{F}_{n,k}\colon \widehat{HF}(S^3) \to \widehat{HF}(S^3_{-n}(K), [k])$ is nontrivial if $k < \tau(K)$ and is trivial if $k > \tau(K)$, where the map is for the $Spin^c$ structure on W_{-n} with $\langle c_1(\mathfrak{s}_k), [\widehat{\Sigma}_K]\rangle - n = 2k$.*

Thus the triviality/nontriviality of certain cobordism maps characterizes $\tau(K)$ to within 1. However, the lemma excludes the case when $k = \tau(K)$: the cobordism map may or may not be trivial, depending upon K. To account for this ambiguity we define

Definition 2.2 For a knot K, the τ–correction, $\mathcal{C}(K)$, equals 0 if the cobordism map $\widehat{F}_{n,k}$ is nontrivial for $k = \tau(K)$ and equals 1 if it is not (assuming $n > 0$ is sufficiently large).

The property in this definition may, a priori, depend upon the value of n, but for n sufficiently large, we will see that it does not.

Using the integer surgeries techniques in P Ozsváth and Z Szabó [10], we can make the previous lemma more precise.

Lemma 2.3 *Let $\widehat{F}_{\mathfrak{u}_k}\colon \widehat{HF}(S^3) \to \widehat{HF}(S^3_r(K),[k])$ be the cobordism map induced from W_r and the Spin^c structure, $\mathfrak{u}_k$ with $\langle c_1(\mathfrak{u}_k), [\widehat{\Sigma}_K]\rangle + r = 2k$. Then for $r \neq 0$,*

(1) *$\widehat{F}_{\mathfrak{u}_k}$ is nontrivial for any k in the range $-\tau(K)+r < k < \tau(K)$, if such exists,*

(2) *$\widehat{F}_{\mathfrak{u}_k} = 0$ if $k < -|\tau(K)| + r$ or $k > |\tau(K)|$, and*

(3) *if $r > 0$, $\widehat{F}_{\mathfrak{u}_k} = 0$ for all k when $\tau(K) \leq 0$.*

Note that there is no contradiction between items (1) and (3) since $\tau(K) < 0$ and $r > 0$ implies $-\tau(K) + r > 0 > \tau(K)$, so there are no k in the requisite range of item (1).

Proof We use [10, Theorem 4.2] to compute when $\widehat{F}_{\mathfrak{u}_i}$ is nontrivial for $r \neq 0$. During the proof of Theorem 4.2 in [10], several arguments are given which allow us to compute the cobordism maps $\widehat{HF}(S^3) \to \widehat{HF}(S^3_r(K), \mathfrak{u}_k)$. Although the main theorems of [10] are for the HF^+–version of the cobordism maps, the isomorphisms established in [10] apply to a more general set of complexes denoted by HF^δ with $\delta \geq 0$; the results we use are for $\delta = 0$. The interested reader should examine [10, Sections 4.3, 4.7 and 4.9]; the proof in Section 4.9 with $\delta = 0$ provides the necessary result. However, we summarize the procedure here.

The map $\widehat{F}_{\mathfrak{u}_i}$ can be described as an inclusion of chain complexes constructed from the knot Floer homology complex $CFK^\infty(K)$. This complex $CFK^\infty(K)$ is a bi-filtered complex whose chain groups decompose as $\bigoplus_{i,j\in\mathbb{Z}} CFK_{i,j}(K)$ which are supported on diagonals $j = i + C$ for a finite set of C's. The differential is a map $\partial^\infty_K\colon CFK_{i,j}(K) \to \bigoplus_{(i',j')\leq(i,j)} CFK_{i',j'}(K)$. From this general setup we may define several complexes as combinations of sub- and quotient complexes. Namely, $CFK^- = \bigoplus_{i<0,j\in\mathbb{Z}} CFK_{i,j}(K)$ is a subcomplex. Following [10], we will denote this $C\{i<0\}$, where the terms inside the brackets denote the range of i, j–indices, and we include the terms in ∂^∞ which map between summands in that range. The quotient by this subcomplex is $C\{i \geq 0\}$, which has a subcomplex $\widehat{B} = C\{i = 0\}$ that is isomorphic to $\widehat{CF}(S^3)$ (equipped with a filtration by the j–indices). B will figure prominently below; in fact, we will require identical copies of $\widehat{B}$, one for each $s \in \mathbb{Z}$, and will distinguish them with a subscript, $\widehat{B}_s$, $s \in \mathbb{Z}$. In addition, let A^+_s be the quotient of $CFK^\infty(K)$ by the subcomplex $C\{i<0, j<s\}$, then A^+_s has a subcomplex, $\widehat{A}_s$ given by $C\{\max(i, j-s) = 0\}$, whose differential comes from that on A^+_s.

Given r and k, let $\widehat{C}^{r,k}$ be the mapping cone complex

$$\bigoplus_{s\equiv k \bmod |r|} \widehat{A}_s \xrightarrow{\partial_{AB}} \bigoplus_{s\equiv k \bmod |r|} \widehat{B}_s,$$

where ∂_{AB} is defined by a sum $\widehat{v}_s+[r]\widehat{h}_s$ on $\widehat{A}_s$ with $\widehat{v}_s\colon \widehat{A}_s\to\widehat{B}_s$ and $[r]\widehat{h}_s\colon \widehat{A}_s\to\widehat{B}_{s+r}$. Here $\widehat{v}_s$ is the map in the sequence

$$\begin{tikzcd}
0 \arrow[r] & C\{i<0,\, j=s\} \arrow[r] & \widehat{A}_s \arrow[r, hook] \arrow[dr, "\widehat{v}_s"'] & C\{i=0,\, j\le s\}\cong \mathcal{F}(K,s) \arrow[d, hook] \\
 & & & C\{i=0\}\cong \widehat{B}_s
\end{tikzcd}$$

and $[r]\widehat{h}_s$ is the map $\widehat{D}^{-1}\circ U^s\circ \widehat{H}_s\colon \widehat{A}_s\to\widehat{B}_{s+r}$, where

$$\begin{tikzcd}
0 \arrow[r] & C\{i=0,\, j<s\} \arrow[r] & \widehat{A}_s \arrow[r, "{[r]\widehat{h}_s}"] \arrow[d, hook, "\widehat{H}_s"] & C\{i=0\}\cong \widehat{B}_{s+r} \\
 & & C\{i\le 0,\, j=s\} \arrow[r, "U^s"] & C\{j=0\}. \arrow[u, "\widehat{D}^{-1}"]
\end{tikzcd}$$

The map U shifts both i and j down by 1, therefore $U^s\colon C\{i<0, j=s\}\to C\{i<-s, j=0\}\subset C\{j=0\}$. $\widehat{D}$ is an isomorphism $C\{i=0\}\to C\{j=0\}$, which reflects changing the orientation of K, but whose precise form will not be needed in the sequel. We will only need that $\mathcal{F}(K,s)\cong C\{i=0, j<s\}$ considered as a subcomplex of $C\{i=0\}$ is isomorphically mapped by D to $C\{i<s, j=0\}$ in $C\{j=0\}$. A main result of [10] states that there is a quasi-isomorphism $\widehat{CF}(S^3_r(K),[k])\to\widehat{C}^{r,k}$ (see [10, Section 4.3]). Under this quasi-isomorphism, the cobordism map $\widehat{F}_{\mathfrak{u}_s}$ becomes the inclusion $\widehat{B}_s\to\widehat{C}^{r,k}$, where $s\equiv k$ modulo r and $\mathfrak{u}_s$ is determined by $\langle c_1(\mathfrak{u}_s),[\Sigma_K]\rangle+r=2s$.

Lemma 2.4 *When $s<\tau(K)$, $\widehat{v}_{s,*}\equiv 0$. When $s>\tau(K)$, $\widehat{v}_{s,*}$ is onto $H_*(\widehat{B}_s)\cong\mathbb{Z}$. When $s>-\tau(K)$, $\widehat{h}_{s,*}\equiv 0$, but when $s<-\tau(K)$, $\widehat{h}_{s,*}$ is onto.*

Proof When $s<\tau(K)$ the image of $\widehat{v}_{s,*}$ lies in the image of $H_*(\mathcal{F}(K,s))$, which is trivial by the definition of $\tau(K)$. For $s>-\tau(K)$ the image of $\widehat{h}_{s,*}$ lies in the image of $\mathcal{F}(K,-s)$ in homology after applying the U–shift (and D). Since $-s<\tau(K)$, the image of $\widehat{h}_{s,*}$ is trivial if $s>-\tau(K)$. For $s>\tau(K)$, the inclusion $\mathcal{F}(K,\tau(K))\hookrightarrow C\{i=0\}$ can be decomposed as the inclusion $I\colon \mathcal{F}(K,\tau(K))\subset \widehat{A}_s/C\{i<0, j=s\}=Q$ followed by the inclusion $J\colon \widehat{A}_s/C\{i<0, j=s\}\to C\{i=0\}$. On the other hand, I is $\pi\circ I'$ where I' is the inclusion $\mathcal{F}(K,\tau(K))\hookrightarrow \widehat{A}_s\cap C\{i=0\}\subset\widehat{A}_s$ and $\pi\colon \widehat{A}_s\to Q$. I and I' are chain maps when $s>\tau(K)$. The map $\widehat{v}_s$ is then the composition $J \circ \pi$. Therefore, $\widehat{v}_s$ induces the map $J \circ \pi \circ I' = J \circ I$ on $\mathcal{F}(K, \tau(K))$. Since $(J \circ I)_* \neq 0$ on homology, by the definition of τ, so too $\widehat{v}_{s,*} \neq 0$. A similar argument verifies the claim for $\widehat{h}_s$ when $s < -\tau(K)$. $\square$

At $s = \tau(K)$, I' may not be a chain map: there may be a nonzero component in the differential for $\widehat{A}_{\tau(K)}$ mapping $C\{i = 0, j = \tau(K)\} \to C\{i < 0, j = \tau(K)\}$. This component can prevent generators of $\widehat{A}_{\tau(K)}$ in $C\{i = 0, j \leq \tau(K)\}$ which map nontrivially into $\widehat{C}\{i = 0, j \leq \tau(K)\} \subset \widehat{CF}(S^3)$ from being closed in $\widehat{A}_{\tau(K)}$. Thus, $\widehat{v}_{\tau(K),*}$ can be the zero map even though the inclusion $\mathcal{F}(K, \tau(K)) \hookrightarrow \widehat{CF}(S^3)$ is not trivial on homology. A similar problem occurs with $\widehat{h}_s$ at $s = -\tau(K)$.

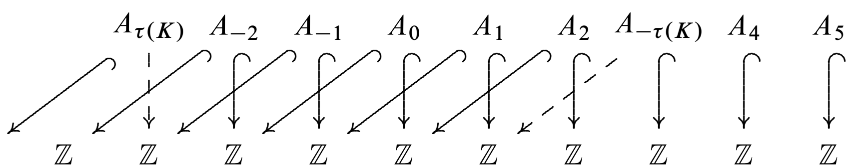
To finish the proof, we now consider what happens in three cases:

Case i $\tau(K) \geq 0$. When $\tau(K) \geq 0$, $\widehat{v}_s = 0$ when $s < \tau(K)$, and $\widehat{h}_s = 0$ when $s > -\tau(K)$. We may think of $\widehat{C}^{r,k}$ as the mapping cone of ∂_{AB}, with all the $\widehat{A}_s$ in one level and the $\widehat{B}_s$ in the other. In the first step of the induced spectral sequence, the nontriviality of including $\widehat{B}_s$ can be determined by whether its homology includes into the remaining steps. But $H_*(\widehat{B}_k) \cong \mathbb{Z}$ is not in the image of any nontrivial map, h_* or v_*, for $-\tau(K) + r < k < \tau(K)$. To be in the image of $\widehat{v}_s$ we would need $k = s$, and we know that $\widehat{v}_s \equiv 0$ on homology in this range. Likewise, for $H_*(\widehat{B}_k)$ to intersect the image of $[r]\widehat{h}_s$ we would need, $k = s + r$ or $s = k - r > -\tau(K)$, but $\widehat{h}_s \equiv 0$ when $s > -\tau(K)$. Therefore, $H_*(\widehat{B}_k) \cong \mathbb{Z}$ is not in the image of ∂_{AB}, and it nontrivially includes into the homology of the mapping cone. The map with $\langle c_1(\mathfrak{u}_k), \Sigma_K \rangle + r = 2k$ is thus nontrivial. In particular, we note for the arguments in the following sections that $\widehat{F}_{\mathfrak{u}_k} \neq 0$ for $k = \tau(K) - 1$ when $r < 2\tau(K) - 1$. We give a figurative example of this case, where the ambiguity at $s = \tau(K)$ is depicted as a dashed arrow (where we have taken $\tau(K), r = 3$ for concreteness), and we have replaced each $\widehat{A}_s$ and $\widehat{B}_s$ with their homologies (indicated by dropping the hats):

$$\begin{tikzcd}
A_{-5} \arrow[drrr, hook] & A_{-4} \arrow[drrr, hook] & A_{-\tau(K)} \arrow[drrr, dashed] & A_{-2} & A_{-1} & A_0 & A_1 & A_2 & A_{\tau(K)} \arrow[d, dashed] & A_4 \arrow[d, hook] & A_5 \arrow[d, hook] \\
 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}
\end{tikzcd}$$

The solid arrows indicate maps which are definitely nonzero. As can be seen in the diagram, there is a swath of B_s's into which no arrow points. This occurs until $-\tau(K)+r = \tau(K)$, and we are outside the range considered in the lemma. Furthermore, if $k > \tau(K)$ then $H_*(B_k)$ is in the image of $\widehat{v}_{k,*}$, but $\widehat{h}_{k,*} \equiv 0$ since $k > -\tau(K)$. Therefore, the image of $H_*(B_k)$ into the mapping cone will be trivial. For $k < -\tau(K) + r$ then $H_*(B_k)$ is in the image of $[r]\widehat{h}_{k-r,*}$ but $\widehat{v}_{k,*}$ is trivial, so again the inclusion will be trivial when considered into $H_*(\widehat{C}^{r,k})$. This verifies items (1) and (2) of Lemma 2.3 for these values of $\tau(K)$ and r.

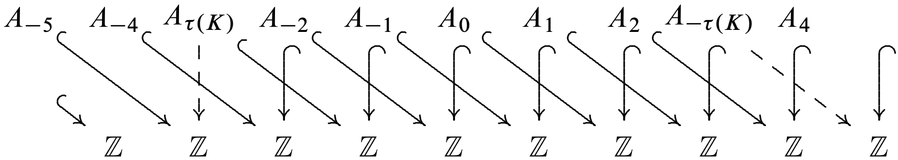
Case ii $\tau(K) < 0, r < 0$. For $s > -\tau(K) > \tau(K)$, the map $\hat{v}_{s,*}$ is onto the image of $H_*(\widehat{B}_s)$, but $[r]\widehat{h}_{s-r,*}$ is trivial since $s - r > -\tau(K)$. Thus, the inclusion of $H_*(\widehat{B}_s)$ into the homology of the mapping cone has trivial image so $\widehat{F}_{\mathfrak{u}_k} = 0$. Similarly, when $s < \tau(K) + r$, $\hat{v}_{s,*} = 0$ since $s < \tau(K)$, but $[r]\widehat{h}_{s-r,*}$ is nontrivial since $\tau(K) + r < -\tau(K) + r < -\tau(K)$. Thus, $H_*(\widehat{B}_s)$ will be trivial in the mapping cone, so $\widehat{F}_{\mathfrak{u}_k} = 0$. This verifies item (2) in Lemma 2.3 for this range (the remaining range for item (2) is subsumed by case iii, below, where we verify item (3)). Assume then, that $-\tau(K) + r < k < \tau(K)$. Thus, for there to be any such k we must have $r < 2\tau(K) - 1$. The nontrivial images of maps $\hat{v}_{s,*}$ occur only for $s \geq \tau(K)$, so for any k in $-\tau(K) + r < k < \tau(K)$, $H_*(\widehat{B}_k)$ is not in the image of $\hat{v}_{k,*}$. The nontrivial maps $[r]\widehat{h}_{s-r,*}$ only occur for $s - r \leq -\tau(K)$ or $s \leq -\tau(K) + r$, but since $r < 2\tau(K) + 1$ we have that $[r]\widehat{h}_{s-r,*}\colon H_*(\widehat{A}_s) \to H_*(\widehat{B}_{s+r})$ has its image in $H_*(\widehat{B}_{s'})$ for $s' \leq \tau(K) + r$. Consequently, for $-\tau(K) + r < k < \tau(K)$, $H_*(\widehat{B}_k)$ is not in the image of $\partial_{AB,*}$, and thus $\widehat{F}_{\mathfrak{u}_k}$ is nontrivial, verifying item (1) of Lemma 2.3 for this case. Our figurative depiction for this case (with $\tau(K) = -3$ and $r = -2$):



Case iii $\tau(K) < 0, r > 0$. Now all the maps $\widehat{F}_{\mathfrak{u}_s}$ will be trivial on homology. For $s < \tau(K)$, $H_*(\widehat{B}_s)$ is the image of $[r]\widehat{h}_{s-r}$ since $s - r < -\tau(K)$ by Lemma 2.4, but not in the image of any $\hat{v}_s$. If the map $\hat{v}_s$ at $s = \tau(K)$ is trivial then the same reasoning applies to see that $\widehat{F}_{\mathfrak{u}_s}$ is trivial. For $s > -\tau(K) + r$ we see that $\hat{v}_s$ is onto since $s > \tau(K)$, but $[r]\widehat{h}_{s-r}$ does not map onto $H_*(\widehat{B}_s)$ since $s - r > -\tau(K)$. If $[r]\widehat{h}_{s-r}$ is trivial when $s = -\tau(K) + r$ then the same reasoning applies to see that $\widehat{F}_{\mathfrak{u}_s}$ is trivial. We now consider the range $\tau(K) \leq s \leq -\tau(K) + r$. We still assume that $\hat{v}_s$ is nontrivial at $s = \tau(K)$ and $[r]\widehat{h}_{s-r}$ is nontrivial at $s = -\tau(K) + r$ since we have already dealt with the cases when they are not. For such a $\tau(K) \leq s \leq -\tau(K) + r$ consider the inclusion of $H_*(\widehat{B}_s)$. The map $\hat{v}_{s,*}$ is onto, so there is a cycle $\xi_s \in \widehat{A}_s$ which maps onto a generator of $H_*(\widehat{B}_s)$ under $\hat{v}_s$. If $[r]\widehat{h}_{s,*}$ is trivial on $[\xi]$ then $H_*(\widehat{B}_s)$ is in the image of $\partial_{AB,*}$ and the inclusion is trivial, so $\widehat{F}_{\mathfrak{u}_s}$ is trivial. If not, let $\xi_{s+r} \in \widehat{A}_{s+r}$ be a cycle mapping under $\hat{v}_{s+r,*}$ to the image of $[r]\widehat{h}_{s,*}([\xi_s])$. We repeat the analysis, with $[r]\widehat{h}_{s+r}$ and $[\xi_{s+r}]$. With each step, ξ_* has its index incremented by r. However, $\hat{v}_{s,*}$ is nontrivial for $s \geq \tau(K)$ (including our assumption on endpoints), and $[r]h_s$ is trivial once $s > -\tau(K)$, so the process eventually stops with a cycle $\xi_{s+kr} \in \widehat{A}_{s+kr}$

which maps nontrivially onto a generator of $\widehat{B}_{s+kr}$, but upon which $[r]h_{s+kr}$ is trivial in homology. The cycle in $\widehat{C}^{r,k}$ given by $\xi_s + \xi_{s+r} + \cdots + \xi_{s+kr}$ maps under $\partial_{AB,*}$ to a generator of $H_*(\widehat{B}_s)$, yielding the conclusions that $H_*(\widehat{B}_s)$ includes trivially into the mapping cone and, consequently, $\widehat{F}_{\mathfrak{u}_s}$ is trivial. This verifies item (3) of Lemma 2.3.

Again, here is a figurative depiction of this situation with $r = 2, \tau(K) = -3$:



As an example of the argument, for $s = 0$, there is an element in A_0 mapping onto the generator of the $\mathbb{Z}$ term. If this element does not also map nontrivially to B_2 then B_0 is in the image of A_0. If, however, it does map to B_2 then there is an element in A_2 with the same image. We repeat the process with this element. Due to the termination of the slanted arrows at $A_{-\tau(K)+1}$, the process stops. This can be performed for each B_s and shows that the inclusion of each B_s will be trivial in the homology of the mapping cone. Of course, the number of arrows involved will depend upon r, s and $\tau(K)$. □

We note that in each case in the lemma, the behavior at the ends of the intervals depend upon the knot in question. We will thus have correction terms for each endpoint, and for each framing, but will only need the one at $\tau(K)$ below. We now record some results on these corrections.

Let $\mathcal{C}_r(K)$ be the correction at $\tau(K)$ for r surgery on K. Recall that $\mathcal{C}(K)$ is the correction for sufficiently negative surgeries.

Lemma 2.5 *If $r \leq 2\tau(K) - 1$ then $\mathcal{C}_r(K) = \mathcal{C}(K)$.*

Proof We refer to the previous diagrams. First, if $\tau(K) > 0$ then $\mathcal{C}_r(K)$ is determined by whether or not $\hat{v}\colon H_*(A_{\tau(K)}) \to \mathbb{Z}_0$ is surjective. If it is then $\mathcal{C}_r(K) = 1$ and 0 otherwise. When $r \leq 2\tau(K) - 1$, this map is uninfluenced by the maps $\widehat{h}$ as they will map into factors to the left of $i = \tau(K)$. (The first nonzero such map is at $-\tau(K)$ and maps to the factor in position $-\tau(K) + r \leq \tau(K) - 1$). Thus for $r \leq 2\tau(K) - 1$ and $\tau(K) > 0$ we have $\mathcal{C}_r(K) = \mathcal{C}(K)$. In fact, if $r \leq 2\tau(K) - 1$ and $\tau(K) \leq 0$ then $r < 0$ and the map from $H_*(A_{-\tau(K)})$ to $B_{r-\tau(K)}$ also maps into a factor to the left of $i = \tau(K)$. It again follows, taking into account the possibilities for $\widehat{h}$ that $\mathcal{C}_r(K) = \mathcal{C}(K)$. □

Lemma 2.6 *If $\tau(K) = g(K)$ then $\mathcal{C}(K) = 1$. If $\tau(K) = -g(K)$ then $\mathcal{C}(K) = 0$.*

Proof The reader should consult the diagrams in the proof of Lemma 2.3. If $\tau(K) = g(K)$ then $H_*(\widehat{A}_{g(K)}) = \mathbb{Z}$, and $\widehat{v}_{g(K),*}$ is an isomorphism onto $H_*(\widehat{B}_{g(K)})$ since $\widehat{A}_{g(K)} \cong \widehat{B}_{g(K)}$ (there cannot be any horizontal component to the differential in $\widehat{A}_{g(K)}$!). However, $g(K) > -\tau(K)$, hence $[r]h_{g(K)} \equiv 0$. Therefore inclusion of $B_{g(K)}$ is trivial in homology, thus the requisite cobordism map is trivial. If $\tau(K) = -g(K)$, let ξ be a class in $\widehat{CF}(K, -g(K)) \cong C\{i = 0, j = -g(K)\}$ which maps isomorphically to $\widehat{HF}(S^3)$ under the inclusion $C\{i = 0, j = -g(K)\} \hookrightarrow C\{i = 0\}$. Let ∂_h: $C\{i = 0, j = -g(K)\} \to C\{i < 0, j = -g(K)\}$ be the map induced from ∂^∞, and suppose $\partial_h \xi = 0$. Then $\xi' = U^{-g(K)}\xi$ is closed in $C\{j = 0\} \cong \widehat{CF}(S^3)$. Suppose $\xi' = y + \partial z$ with $y \in C\{i < g(K), j = 0\}$, and $z = \sum z_i$ with $z_i \neq 0$ in $C\{i, j = 0\}$. There is an $m \geq g(K)$ with $z_m \neq 0$. Take the largest such m, and call it M. Then $\xi' = y + \partial z$ implies $\partial z_M = 0$ in $C\{i = M, j = 0\}$. If $M > g(K)$, there is a u_M with $\partial u_M = z_M$ in $C\{i = M, j = 0\}$. By taking $z + \partial u_M$ in $C\{j = 0\}$ we obtain a new z' with $\partial z' = \xi'$ for which the corresponding largest filtration index is less than M. Consequently, we may assume $M = g(K)$. Then $\partial z_{g(K)} = \xi'$ in $C\{i = g(K), j = 0\}$ since the highest filtration index for any term in y is $< g(K)$. Applying $U^{g(K)}$ produces $\partial U^{g(K)} z_{g(K)} = \xi$ which contradicts that ξ maps nontrivially to a generator of $H_*(C\{i = 0\})$. Therefore, ξ' is closed in $C\{j = 0\}$ but is not exact and is not homologous to any element with $i < g(K)$. Since $\tau(K) = -g(K)$. this is a contradiction, as $H_*(C\{j = 0\}) \cong \mathbb{Z}$ is generated by elements in $C\{i = -g(K), j = 0\}$. Thus, $\partial_h \xi \neq 0$ in $\widehat{A}_{-g(K)}$. Consequently $v_{-g(K),*} = 0$. □

This last lemma applies, for instance, to strongly quasi-positive knots by Livingston [4].

3 Analyzing satellites knots using the lemma

Our goal in this section will be to use the previous Heegaard Floer results to prove the following proposition:

Proposition 3.1 *Let $S_r(C, P)$ be the r–twisted satellite knot formed from a companion, C, in S^3 and a pattern, P, in $S^1 \times D^2$. Let l be the intersection number of P with D^2 and orient P so that $l \geq 0$. Let*

$$D(S_r(C, P)) = \tau(S_r) - \left(\tau(P) + l\,\tau(C) + \frac{l(l-1)}{2}\, r\right).$$

Then when $r \neq 0$,

$$\text{(1)}\quad \begin{aligned} -\big(\mathcal{C}(P) + l\,\mathcal{C}(C)\big) \leq D(S_r) \leq D(S_{\Delta(r)}(U, P)) + 1 + l\,\mathcal{C}(\overline{C}),& \\ \textit{when } r < 2\tau(C) - 1,& \\ D(S_{\Delta'(r)}(U, P)) - 1 - l\,\mathcal{C}(C) \leq D(S_r) \leq \big(\mathcal{C}(\overline{P}) + l\,\mathcal{C}(\overline{C})\big)& \\ \textit{when } r > 2\tau(C) + 1,& \end{aligned}$$

where

(a) U *is the unknot,*

(b) $\Delta(r) = r - 2\tau(C) - 1 - \mathcal{C}(\overline{C})$, *and*

(c) $\Delta'(r) = r - 2\tau(C) + 1 + \mathcal{C}(C)$.

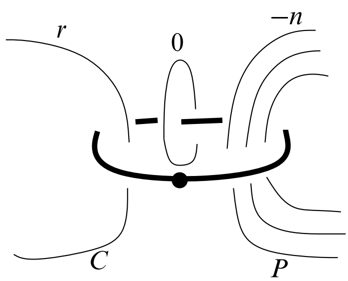
The first inequality applies to $r = 2\tau(C) - 1$ *when* $\mathcal{C}(C) = 0$ *and the second set of inequalities applies to* $r = 2\tau(C) + 1$ *when* $\mathcal{C}(\overline{C}) = 0$. *Furthermore, in all instances,*

$$D(S_{\Delta'(r)}(U, P)) - 1 - l\,\mathcal{C}(C) \leq D(S_r) \leq D(S_{\Delta(r)}(U, P)) + 1 + l\,\mathcal{C}(\overline{C}).$$

In the next section we address replacing $D(S_{\Delta(r)}(U, P))$ and $D(S_{\Delta'(r)}(U, P))$ by more congenial representations.

Proof The proof breaks into six steps.

Step I We consider the four manifold, $W_{r,n}$, with $r \neq 0$, depicted in



where C and P are geometrically isolated from each other. We will consider this four manifold as a cobordism from S^3 to its boundary, by removing a small ball. There are two different descriptions of this four dimensional cobordism, up to diffeomorphism:

(1) We may slide P and C over the 0–framed two handle and then cancel the one handle with the 0–framed two handle. $W_{r,n}$ is then seen to be diffeomorphic to r surgery on C and $-n$ surgery on P, where C and P are geometrically isolated knots in S^3. Thus $W_{r,n} \cong W_r(C) \# W_{-n}(P)$, where we have taken a sum along unknotted arcs in the two four manifolds.

(2) Alternately, we may slide all the strands of P over C, leaving C linking the one handle. Sliding the 0–framed two handle over C as well, we may cancel C with the one handle. This results in a pair of two handles, with the one attached along the image of P under the cancellation, being attached in S^3 along $S_r(C, P)$. Note that this diagram specifies the framing used on $S^1 \times D^2$ in constructing $S_r(C, P)$. Thus $W_{r,n}$ can be thought of as the composition

of two cobordisms: the attachment of a two handle with some framing along $S_r(C,P)$, which results in a four dimensional cobordism we will call W_S below, and another four dimensional cobordism H which occurs when we add the other two handle to the boundary of W_S.

In the second case, we can compute the framing on $S_r(C,P)$ using standard Kirby calculus. $W_{r,n}$ decomposes as $-n+l^2r$ surgery on the r–twisted satellite, $S_r(C,P)$, and a cobordism resulting from the attachment of a two handle . The framing on the satellite is determined by row/column operations in the linking matrix for a diagram derived from the one above by removing the 0–framed meridional two handle and changing the one handle into a 0–framed two handle

$$\begin{bmatrix} r & 0 & 1 \\ 0 & -n & -l \\ 1 & -l & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} r & lr & 1 \\ lr & -n+l^2r & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

For convenience, we will also use the notation P^r for $S_r(U,P)$ where U is the unknot; P^r is just P with r full twists added to the strands passing over the 1–handle. For instance, we may find 0–twisted satellites by the identity $S_0(C,P) = S_i(C,P^{-i})$.

Step II For the first way of decomposing the manifold $W_{r,n}$ as $W_r(C) \# W_{-n}(P)$, we can decompose spin^c structures along the sum into a spin^c structure, $\mathfrak{s}_C$ for $W_r(C)$ and one $\mathfrak{s}_P$ for $W_{-n}(P)$, and these are uniquely determined. The map $\widehat{F}_{W_{r,n},\mathfrak{s}_C\#\mathfrak{s}_P}$ is equivalent to $\widehat{F}_{W_r(C),\mathfrak{s}_C} \otimes \widehat{F}_{W_{-n}(P),\mathfrak{s}_P}$ under the connect sum isomorphisms of [9]. We now explain this fact.

In [9, Section 6], an isomorphism map

$$\widehat{HF}_{\mathbb{Q}}(Y_1,\mathfrak{s}_1) \otimes \widehat{HF}_{\mathbb{Q}}(Y_2,\mathfrak{s}_2) \to \widehat{HF}_{\mathbb{Q}}(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2)$$

is constructed. In [9], the map is constructed on chain complexes with $\mathbb{Z}$–coefficients and results only in $\widehat{HF}(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) \cong H_*(\widehat{CF}(Y_1,\mathfrak{s}_1) \otimes \widehat{CF}(Y_2,\mathfrak{s}_2))$. Over $\mathbb{Q}$, however, this isomorphism extends to the tensor product of the homologies. We call this isomorphism $\widehat{R}_{Y_1\#Y_2,\mathfrak{s}_1\#\mathfrak{s}_2}$.

Using this isomorphism, and the associativity of Heegaard triple maps, P Ozsváth and Z Szabó prove the following proposition:

Lemma 3.2 [6, Proposition 4.4] *The map $\widehat{R}_{Y\#Z,\mathfrak{s}\#\mathfrak{t}}$ is independent of the Heegaard diagrams used for Y and Z. Moreover, if W is cobordism from Y to Y', equipped with a* spin^c *structure $\mathfrak{u}$, restricting to the ends appropriately, then the following* *diagram is commutative:*

$$\begin{tikzcd} \widehat{HF}(Y,\mathfrak{s}) \otimes \widehat{HF}(Z,\mathfrak{t}) \arrow[r, "\widehat{R}_{Y\#Z,\mathfrak{s}\#\mathfrak{t}}"] \arrow[d, "\widehat{F}_{W,\mathfrak{u}} \otimes \mathrm{Id}"] & \widehat{HF}(Y\#Z,\mathfrak{s}\#\mathfrak{t}) \arrow[d, "\widehat{F}_{W\#(Z\times I),\mathfrak{u}\#\mathfrak{t}}"] \\ \widehat{HF}(Y',\mathfrak{s}') \otimes \widehat{HF}(Z,\mathfrak{t}) \arrow[r, "\widehat{R}_{Y'\#Z,\mathfrak{s}'\#\mathfrak{t}}"] & \widehat{HF}(Y'\#Z,\mathfrak{s}'\#\mathfrak{t}) \end{tikzcd}$$

Proposition 4.4 is stated for HF^+–cobordism maps, but is easily altered to apply to $\widehat{HF}$. The order of the factors does not matter, and we can apply the lemma also to a nontrivial cobordism $Z \to Z'$. To complete step II we now apply this lemma twice: once to

$$\widehat{F}_{W_r(C),\mathfrak{s}_C} \otimes \mathrm{Id}\colon S^3 \# S^3 \xrightarrow{W_r(C)\#(S^3\times I)} S^3_r(C)\# S^3$$

and the second time to

$$\mathrm{Id} \otimes \widehat{F}_{W_{-n}(P),\mathfrak{s}_P}\colon S^3_r(C)\# S^3 \xrightarrow{S^3_r(C)\times I\# W_{-n}(P).} S^3_r(C)\# S^3_{-n}(P)$$

Composing these maps yields $\widehat{F}_{W_r(C),\mathfrak{s}_C} \otimes \widehat{F}_{W_{-n}(P),\mathfrak{s}_P}$, which is the left side of a commutative square found from two squares as in the proposition. Composing the cobordism maps from the right of the corresponding commutative squares yields $\widehat{F}_{W_{r,n},\mathfrak{s}_C\#\mathfrak{s}_P}$ (since $S^3_r(C)$ is a rational homology sphere) for the spin^c structures chosen.

Step III On the other hand ∂W_S is a rational homology sphere for sufficiently large n. Recall that $W_{r,n}$ is the cobordism resulting from surgery on both C and P as in Step I. Thus, we can decompose $\widehat{F}_{W_{r,n}} = \widehat{F}_H \circ \widehat{F}_{W_S}$ where H is a particular cobordism $\partial W_S \longrightarrow \partial W_{r,n}$ (which will play no further role). The Spin^c structures on $W_{r,n}$ likewise decompose, and are determined by how they restrict to the two parts. We thus have that $\widehat{F}_{W_{r,n}} \neq 0$ implies that $\widehat{F}_S \neq 0$. Using part II we can find cases when $\widehat{F}_{W_{r,n}} \neq 0$. We now undertake to determine which Spin^c structures on W_S occur for the resulting nontrivial $\widehat{F}_S$ maps.

Step IV From Lemma 2.3 we know that $\widehat{F}^C_{r,k}$ with $\langle c_1(\mathfrak{s}_C), [\widehat{\Sigma}_C]\rangle + r = 2k$ is nontrivial for $-\tau(C)+r < k < \tau(C)$. By the definition on $\mathcal{C}(C)$ and Lemma 2.5, if $\mathcal{C}(C) = 0$ we may also use $r = 2\tau(C)-1$ and $k = \tau(C)$. So for $r < 2\tau(C)-1$, we may take $k = \tau(C) - \mathcal{C}(C)$ and have a nonzero map. Suppose further that $\langle c_1(\mathfrak{s}_P), [\widehat{\Sigma}_P]\rangle - n = 2\,(\tau(P) - \mathcal{C}(P))$. The spin^c structure $\mathfrak{s}_C\#\mathfrak{s}_P$ on $W_{r,-n}$ restricts to W_S according to the value of

$$\langle c_1(\mathfrak{s}), [\widehat{\Sigma}_{S_r}]\rangle + (-n + l^2\, r),$$

but due to the handleslides, $[\widehat{\Sigma}_{S_r}] = [\widehat{\Sigma}_P] + l[\widehat{\Sigma}_C]$, and

$$\begin{aligned}\langle c_1(\mathfrak{s}), [\widehat{\Sigma}_P] + l[\widehat{\Sigma}_C]\rangle &= (2\,(\tau(P) - \mathcal{C}(P)) + n) + l\,(2(\tau(C) - \mathcal{C}(C)) - r) \\ &= 2((\tau(P) - \mathcal{C}(P)) + l\,(\tau(C) - \mathcal{C}(C))) + n - lr + l^2 r - l^2 r.\end{aligned}$$

Consequently,

$$\langle c_1(\mathfrak{s}), [\widehat{\Sigma}_{S_r}]\rangle + (-n + l^2\, r) = 2((\tau(P) - \mathcal{C}(P)) + l\,(\tau(C) - \mathcal{C}(C))) + l(l-1)\,r.$$

We have the freedom to make n as large as we wish. We choose so that $-n + l^2\, r$ is sufficiently negative for Lemma 2.1 to apply to both P and S_r. Therefore, the map $\widehat{F}_{\mathfrak{s}_P}$ will also be nontrivial, so $\widehat{F}_W$ will be nontrivial. Thus, $\widehat{F}_{W_S}$ is nontrivial by Step III. By Lemma 2.1

$$(\tau(P) - \mathcal{C}(P)) + l\,(\tau(C) - \mathcal{C}(C))) + \frac{l(l-1)}{2}\,r \le \tau(S_r)$$

which simplifies to

$$\tau(P) + l\,\tau(C) + \frac{l(l-1)}{2}\,r \le \tau(S_r) + \mathcal{C}(P) + l\,\mathcal{C}(C) \quad \text{when } r < 2\tau(C) - 1.$$

If $\mathcal{C}(C) = 0$ we may include $r = 2\tau(C) - 1$ as well, so the range on r may be replaced by $r \le 2\tau(C) - 1 - \mathcal{C}(C)$.

Let
$$T(S_r) = \tau(P) + l\,\tau(C) + \frac{l(l-1)}{2}\,r,$$

which is thought of as an analog of $\tau(S_r)$. Rearranging terms establishes the one-sided bound

$$-\big(\mathcal{C}(P) + l\,\mathcal{C}(C)\big) \le \tau(S_r) - T(S_r) \quad \text{when } r \le 2\tau(C) - 1 - \mathcal{C}(C). \tag{2}$$

Step V The crucial observation in finding bounds for the other side is

$$\tau(S_m(\overline{C}, S_r(C, P))) = \tau(S_{r+m}(U, P)) = \tau(P^{r+m}).$$

This follows from:

Lemma 3.3 *The knot $S_m(\overline{C}, S_r(C, P))$ is concordant to $S_{m+r}(U, P) = P^{m+r}$.*

Proof Using the specific representations of the satellites in the diagrams above, we can conclude that $S_m(\overline{C}, S_r(C, P))$ is $S_{m+r}(\overline{C} \# C, P)$. Since $\overline{C} \# C$ is slice, this lemma merely restates the well-known fact that the satellite of a slice knot, with a given framing, is concordant to the satellite of the unknot with the same pattern and framing. □

Applying inequality (2) to the satellite $S_m(\bar{C}, S_r(C,P))$ gives

$$-\big(\mathcal{C}(S_r) + l\,\mathcal{C}(\bar{C})\big) \le \tau(S_m(\bar{C}, S_r(C,P))) - T(S_m(\bar{C}, S_r(C,P))) \qquad \text{when } m \le 2\tau(\bar{C}) - 1 - \mathcal{C}(\bar{C})$$

but
$$T(S_m(\bar{C}, S_r(C,P))) = \tau(S_r) + l\,\tau(\bar{C}) + \frac{l(l-1)}{2}\,m.$$

Therefore since $S_m(\bar{C}, S_r(C,P))$ is concordant to P^{r+m},

$$\tau(S_r) \le \tau(P^{r+m}) - \left(l\,\tau(\bar{C}) + \frac{l(l-1)}{2}\,m\right) + \mathcal{C}(S_r) + l\,\mathcal{C}(\bar{C}) \qquad \text{when } m \le -2\tau(C) - 1 - \mathcal{C}(\bar{C}).$$

As above, if $\mathcal{C}(\bar{C}) = 0$ we can extend to $m = -2\tau(C) - 1$. We choose $m = -2\tau(C) - 1 - \mathcal{C}(\bar{C})$. Rearranging and simplifying we obtain

$$\tau(S_r) \le \tau(P^{r-2\tau(C)-1-\mathcal{C}(\bar{C})}) + l\,\tau(C) + \frac{l(l-1)}{2}\,(2\tau(C) + 1 + \mathcal{C}(\bar{C})) + \mathcal{C}(S_r) + l\,\mathcal{C}(\bar{C}).$$

Let
$$\Delta_C(r) = r - 2\tau(C) - 1 - \mathcal{C}(\bar{C}).$$

C does not vary until the next part so we will drop the subscript until then. With this notation we obtain, upon subtracting $T(S_r)$ and rearranging,

$$\tau(S_r) - T(S_r) \le \tau(P^{\Delta(r)}) - \tau(P) - \frac{l(l-1)}{2}\,\Delta(r) + \mathcal{C}(S_r) + l\,\mathcal{C}(\bar{C}).$$

Since $P^s = S_s(U,P)$ and $\tau(U) = 0$, we can replace $\tau(P) + (l(l-1)/2)\,\Delta(r)$ with $T(P^{\Delta(r)})$ to obtain

$$\tau(S_r) - T(S_r) \le \tau(P^{\Delta(r)}) - T(P^{\Delta(r)}) + 1 + l\,\mathcal{C}(\bar{C}),$$

where we have replaced $\mathcal{C}(S_r)$ with 1. Note that this inequality always applies, regardless of the value of r, since it depended only upon the choice of m, which does not depend on r. This provides the right side of our target bounds. Let

$$D(S_r(C,P)) = \tau(S_r(C,P)) - T(S_r(C,P)).$$

Taken with the inequality (2) we obtain the two sided bounds,

(3) $-\big(\mathcal{C}(P) + l\,\mathcal{C}(C)\big) \le D(S_r) \le D(P^{\Delta(r)}) + 1 + l\,\mathcal{C}(\bar{C})$ when $r \le 2\tau(C) - 1 - \mathcal{C}(C)$,

and the right hand inequality applies regardless of the value of r.

Step VI Now consider the satellite knot $S_{-r}(\overline{C},\overline{P})$ which is the mirror of $S_r(C,P)$. Then $\tau(S_{-r}(\overline{C},\overline{P}))=-\tau(S_r)$ and $T(S_{-r}(\overline{C},\overline{P})=-T(S_r(C,P))$. Thus $D(\overline{S_r})=-D(S_r)$. In addition $\Delta_C(r)=-\Delta'_{\overline{C}}(-r)$ where

$$\Delta'_C(r)=r-2\tau(C)+1+\mathcal{C}(C).$$

Now apply inequality (3) to $\tau(S_{-r}(\overline{C},\overline{P}))$:

$$-\big(\mathcal{C}(\overline{P})+l\,\mathcal{C}(\overline{C})\big)\leq D(\overline{S_r})\leq D(\overline{P}^{\Delta_{\overline{C}}(-r)})+1+l\,\mathcal{C}(C)\ \text{ when } -r\leq 2\tau(\overline{C})-1-\mathcal{C}(\overline{C}).$$

But $-r\leq 2\tau(\overline{C})-1-\mathcal{C}(\overline{C})$ means $r\geq 2\tau(C)+1+\mathcal{C}(\overline{C})$, and

$$\overline{P}^{\Delta_{\overline{C}}(-r)}=\overline{P^{-\Delta_{\overline{C}}(-r)}}=\overline{P^{\Delta'_C(r)}}.$$

Thus, using $D(\overline{S_r})=-D(S_r)$, we have

$$D(P^{\Delta'(r)})-1-l\,\mathcal{C}(C)\leq D(S_r)\leq \big(\mathcal{C}(\overline{P})+l\,\mathcal{C}(\overline{C})\big)\quad\text{when } r\geq 2\tau(C)+1+\mathcal{C}(\overline{C}).$$

Here the left hand inequality holds regardless of the value of r. This concludes the proof of the proposition. □

4 Tidying up the inequalities

We now wish to clean up the results of the previous section. In particular, we would like to compute $D(S_{\Delta(r)}(U,P))$ in some simpler manner. The key will be the following proposition, found in [12], whose proof mimics Van Cott's arguments in [13].

Proposition 4.1 *Let the orientation on P be such that $l\geq 0$ and let*

$$g(r)=\tau(S_r(C,P))-\frac{l(l-1)}{2}r.$$

Let n_+, and n_- be the number of strands of P intersecting the oriented copy of D^2 positively and negatively, respectively. Then if $s>r$ and $n_+>n_-$,

$$-(n_+-1)\leq g(s)-g(r)\leq n_-,$$

while when $n_+=n_-$ we have

$$-n_+\leq g(s)-g(r)\leq (n_--1).$$

With this proposition in hand, we can complete the proof of Theorem 1.2:

Lemma 4.2 *If $n_+ > n_-$, then*

$$-(n_+(P)-1) \leq D(S_{\Delta'(r)}(U,P)) \leq n_-(P) \qquad \text{when } r \geq 2\tau(C)-1,$$
$$-n_-(P) \leq D(S_{\Delta(r)}(U,P) \leq n_+(P)-1 \qquad \text{when } r \leq 2\tau(C)+1,$$

while if $n_+ = n_-$, then

$$-n_+(P) \leq D(S_{\Delta'(r)}(U,P)) \leq (n_-(P)-1) \qquad \text{when } r \geq 2\tau(C)-1,$$
$$-(n_-(P)-1) \leq D(S_{\Delta(r)}(U,P) \leq n_+(P) \qquad \text{when } r \leq 2\tau(C)+1.$$

Proof Since C is the unknot we have

$$D(S_t(U,P)) = \tau(S_t(U,P)) - \tau(P) - \frac{l(l-1)}{2}t,$$

which in turn becomes $g(t)-g(0)$, using the notation in the previous proposition. Now $\Delta'(r) \geq 0$ since we only use it when $r \geq 2\tau(C)-1$. Thus, $-(n_+-1) \leq g(\Delta'(r))-g(0) \leq n_-$ when $r \geq 2\tau(C)-1$ and $n_+ > n_-$ ($-n_+ \leq g(\Delta'(r))-g(0) \leq (n_--1)$ when $n_+ = n_-$). Furthermore, when $r \leq 2\tau(C)+1$ we have that $\Delta(r) \leq 0$, so $-(n_+-1) \leq g(0)-g(\Delta(r)) \leq n_-$ in this case (when $n_+ = n_-$ this becomes $-n_+ \leq g(0)-g(\Delta(r)) \leq (n_--1)$). Multiplying by -1, we can reverse the inequalities. The case when $l=0$ is identical, except that the bounds change as in the previous proposition. □

The inequality (1) then becomes:

Proposition 4.3 *When $r \neq 0, l > 0$, we have*

$$-\big(\mathcal{C}(P) + l\,\mathcal{C}(C)\big) \leq D(S_r) \leq n_+(P) + l\,\mathcal{C}(\bar{C}) \qquad \text{when } r < 2\tau(C)-1,$$
$$-n_+(P) - l\,\mathcal{C}(C) \leq D(S_r) \leq \big(\mathcal{C}(\bar{P}) + l\,\mathcal{C}(\bar{C})\big) \qquad \text{when } r > 2\tau(C)+1.$$

If $\mathcal{C}(C)=0$ then the first inequality also applies for $r = 2\tau(C)-1$, while if $\mathcal{C}(\bar{C})=0$ then the second inequality applies at $r = 2\tau(C)+1$. Furthermore, for all r we have

$$-n_+(P) - l\,\mathcal{C}(C) \leq D(S_r) \leq n_+(P) + l\,\mathcal{C}(\bar{C}).$$

Proof We substitute one side of the inequalities from Lemma 4.2 into the inequalities in Proposition 3.1. All that remains is the inequalities that hold in general. We know that, for all r,

$$D(S_r) \leq D(S_{\Delta(r)}(U,P)) + 1 + l\,\mathcal{C}(\bar{C})$$

and $D(S_{\Delta(r)}(U,P)) + 1 + l\,\mathcal{C}(\bar{C}) \leq n_+(P) + l\,\mathcal{C}(\bar{C})$ for $r \leq 2\tau(C)+1$. But for $r > 2\tau(C)+1$ we also have $D(S_r) \leq \big(\mathcal{C}(\bar{P}) + l\,\mathcal{C}(\bar{C})\big)$. Since $n_+(P) \geq 1 \geq \mathcal{C}(\bar{P})$, the inequality on the right holds for all r. A similar argument establishes the result for the inequality on the left □

To obtain the propositions in the introduction, we set $\mathcal{C}(C), \mathcal{C}(\overline{C}) = 1$, which is the worst case for both sides of the inequalities above. Finally, we address the case when $l = 0$. The argument is an adaptation of that above to the different bounds in Lemma 4.2. The inequality (1) then becomes:

Proposition 4.4 *When $r \neq 0, l = 0$, we have*

$$\begin{aligned} -\mathcal{C}(P) \leq D(S_r) &\leq n_+(P) + 1 \quad \text{when } r < 2\tau(C) - 1, \\ -n_+(P) - 1 \leq D(S_r) &\leq \mathcal{C}(\overline{P}) \quad \text{when } r > 2\tau(C) + 1. \end{aligned}$$

If $\mathcal{C}(C) = 0$ then the first inequality also applies for $r = 2\tau(C) - 1$, while if $\mathcal{C}(\overline{C}) = 0$ then the second inequality applies at $r = 2\tau(C) + 1$. Furthermore, for all r we have

$$-n_+(P) - 1 \leq D(S_r) \leq n_+(P) + 1.$$

5 Special cases

Below, we assume that $r \neq 0$.

5.1 When $l = 0$ and P is an unknot, when considered in S^3

A calculation shows that $\mathcal{C}(P) = 0$ in this case. So we obtain

$$\begin{aligned} 0 \leq D(S_r) &\leq n_+(P) + 1 \quad \text{when } r < 2\tau(C) - 1, \\ -n_+(P) - 1 \leq D(S_r) &\leq 0 \quad \text{when } r > 2\tau(C) + 1, \end{aligned}$$

which conforms to the behavior found for Whitehead doubles in [1; 5].

5.2 When $l = 1$

We obtain the inequalities

$$\begin{aligned} -\big(\mathcal{C}(P) + \mathcal{C}(C)\big) \leq \tau(S_r) - \tau(P) - \tau(C) &\leq n_+(P) + \mathcal{C}(\overline{C}) \quad \text{when } r < 2\tau(C) - 1, \\ -n_+(P) - \mathcal{C}(C) \leq \tau(S_r) - \tau(P) - \tau(C) &\leq \big(\mathcal{C}(\overline{P}) + \mathcal{C}(\overline{C})\big) \quad \text{when } r > 2\tau(C) + 1. \end{aligned}$$

If $l = 1$ both algebraically and geometrically, then $S_r \cong P \# C$ for all r. These inequalities almost give the additivity formula under connect sum – since $n_+(P) = 1$ – but not quite. With some effort, we could replace the correction terms, or simply replace them by 1's. In the latter case, we obtain

$$\begin{aligned} -2 \leq \tau(C \# P) - \tau(P) - \tau(C) \leq 2 \quad \text{when } r < 2\tau(C) - 1, \\ -2 \leq \tau(C \# P) - \tau(P) - \tau(C) \leq 2 \quad \text{when } r > 2\tau(C) + 1. \end{aligned}$$

for the connect sum.

5.3 When P is a specific unknot

Let P be the closure of the braid $\sigma_1\sigma_2\cdots\sigma_{l-1}$. Then $\tau(P)=0$, $\mathcal{C}(P)=0$, and $n_+(P)=l$. Consequently, we have the inequalities

$$\begin{aligned} l\,\tau(C)+\frac{l(l-1)}{2}r-l \le \tau(S_r) &\le l\,\tau(C)+\frac{l(l-1)}{2}r+2l \quad \text{when } r<2\tau(C)-1,\\ l\,\tau(C)+\frac{l(l-1)}{2}r-2l \le \tau(S_r) &\le l\,\tau(C)+\frac{l(l-1)}{2}r+l \quad \text{when } r>2\tau(C)+1. \end{aligned}$$

Up to the final terms on each side, which are different multiples of l, these bounds are similar to those in [2].

5.4 When C is the unknot

We write $P^m = S_m(U,P)$. This is just shorthand for adding full twists to a collection of parallel strands in P. Then

$$\begin{aligned} -\mathcal{C}(P) \le \tau(P^r)-\tau(P)-\frac{l(l-1)}{2}r &\le n_+(P) \quad \text{when } r<-1,\\ -n_+(P) \le \tau(P^r)-\tau(P)-\frac{l(l-1)}{2}r &\le \mathcal{C}(\bar{P}) \quad \text{when } r>+1. \end{aligned}$$

If P is the closure of a l stranded braid then $n_+(P)=l$ and we obtain

$$\begin{aligned} \frac{l(l-1)}{2}r-1 \le \tau(P^r)-\tau(P) &\le \frac{l(l-1)}{2}r+l \quad \text{when } r<-1,\\ \frac{l(l-1)}{2}r-l \le \tau(P^r)-\tau(P) &\le \frac{l(l-1)}{2}r+1 \quad \text{when } r>+1. \end{aligned}$$

These are similar to the results in [13, Section 4].

References

[1] **M Hedden**, *Knot Floer homology of Whitehead doubles*, Geom. Topol. 11 (2007) 2277–2338 MR2372849

[2] **M Hedden**, *On knot Floer homology and cabling: 2*, Int. Math. Res. Not. 2009 (2009) 2248–2274 MR2511910

[3] **M Hedden**, **P Ording**, *The Ozsváth–Szabó and Rasmussen concordance invariants are not equal*, Amer. J. Math. 130 (2008) 441–453 MR2405163

[4] **C Livingston**, *Computations of the Ozsváth–Szabó knot concordance invariant*, Geom. Topol. 8 (2004) 735–742 MR2057779

[5] **C Livingston**, **S Naik**, *Ozsváth–Szabó and Rasmussen invariants of doubled knots*, Algebr. Geom. Topol. 6 (2006) 651–657 MR2240910

[6] **P Ozsváth**, **Z Szabó**, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. 173 (2003) 179–261 MR1957829

[7] **P Ozsváth**, **Z Szabó**, *Knot Floer homology and the four-ball genus*, Geom. Topol. 7 (2003) 615–639 MR2026543

[8] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and knot invariants*, Adv. Math. 186 (2004) 58–116 MR2065507

[9] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. 159 (2004) 1159–1245 MR2113020

[10] **P Ozsváth**, **Z Szabó**, *Knot Floer homology and integer surgeries*, Algebr. Geom. Topol. 8 (2008) 101–153 MR2377279

[11] **I Petkova**, *Cables of thin knots and bordered Heegaard Floer homology* `arXiv:0911.2679v1`

[12] **L Roberts**, *Extending Van Cott's bounds for the τ and s–invariants of a satellite knot*, J. Knot Theory Ramifications 20 (2011) 1237–1245 MR2844805

[13] **C A Van Cott**, *Ozsváth–Szabó and Rasmussen invariants of cable knots*, Algebr. Geom. Topol. 10 (2010) 825–836 MR2629765

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