Algebraic & Geometric Topology 12 (2012) 601–641 **601**

Characteristic classes of proalgebraic varieties and motivic measures

SHOJI YOKURA

Gromov initiated what he calls "symbolic algebraic geometry", in which he studied proalgebraic varieties. In this paper we formulate a general theory of characteristic classes of proalgebraic varieties as a natural transformation, which is a natural extension of the well-studied theories of characteristic classes of singular varieties. Fulton–MacPherson bivariant theory is a key tool for our formulation and our approach naturally leads us to the notion of motivic measure and also its generalization.

14C17, 18F99; 55N99, 14E18, 18A99, 55N35

Dedicated to Clint McCrory on the occasion of his 65th birthday

1 Introduction

This work was originally motivated by Gromov's papers [38; 39] and partly by our paper [78].

A pro-category was introduced by Grothendieck [41] and used to develop the Etale Homotopy Theory by Artin and Mazur [3] and Shape Theory by Borsuk [8], Edwards [28], Mardešić and Segal [53], etc. A *pro-algebraic variety* is a projective system of complex algebraic varieties and a *proalgebraic variety* is the projective limit of a pro-algebraic variety. In [38], Gromov investigated *surjunctivity* (see Gottschalk [37]), ie, *being either surjective or noninjective*, in the category of proalgebraic varieties. The original surjunctivity theorem is *Ax' Theorem* [5], saying that every regular self-mapping of a complex algebraic variety is surjunctive.

In this paper we do not deal with Ax-type theorems, but we consider characteristic classes of proalgebraic varieties. In [38] Gromov uses *proconstructible set* or *proconstructible space* at several places, but he does not seem to give precise definitions for these terms. So a naïve question is how one should define "proconstructible set" or equivalently "proconstructible function" on a proalgebraic variety, and therefore how one should define the Chern–Schwartz–MacPherson class of a proalgebraic variety. This is surely hinted by MacPherson's Chern class transformation $c_*\colon F \to H_*$ from

Published: 8 April 2012 DOI: 10.2140/agt.2012.12.601

the covariant constructible function functor F to the homology theory [51] (also see Brasselet and Schwarz [15], and Schwarz [67; 68]). A very simple example of a proalgebraic variety is the Cartesian product $X^{\mathbb{N}}$ of an infinite countable copies of a complex algebraic variety X, which is one of the main objects treated in [38]. What would be the Chern–Schwartz–MacPherson class of $X^{\mathbb{N}}$? In particular, what would be the Euler–Poincaré characteristic of $X^{\mathbb{N}}$? Our answers are that they are respectively the Chern–Schwartz–MacPherson class $c_*(X)$ and the Euler–Poincaré characteristic $\chi(X)$ in a sense which will be clarified later. It is this very simple observation that led us to this work.

We will give a general theory of characteristic classes of proalgebraic varieties, which is formulated as a natural transformation just like the now well-known theories of characteristic classes of singular varieties. In Section 2 we quickly recall the theory of characteristic classes of singular varieties, which is the base of the present work. We discuss whether one should consider the inductive limit or the projective limit in order to consider a reasonable notion of "proconstructible function" on the proalgebraic variety. As reasonable models for characteristic classes of proalgebraic varieties we consider some simple but instructive situations. In Section 3, suggested by the results in Section 2, from a bifunctor $\mathcal{F}$ we introduce the notion of *$\chi_{\mathcal{F}}$–stable objects* and obtain *$\mathcal{F}$–characteristics* of proalgebraic varieties or more generally projective systems, as a simple or natural pro-analogue of the classical characteristics such as Euler–Poincaré characteristic, arithmetic genus, signature and Hirzebruch χ_y–genus. We show that our approach naturally leads us to *motivic measure* (see, eg, Craw [23], Denef and Loeser [24; 25], Kontsevich [45], Looijenga [50] and Veys [70]) and its generalization. In Section 4 we give a class version of the above $\mathcal{F}$–characteristics. For that we need more requirements on the bifunctor $\mathcal{F}$ and also pro-morphisms between pro-algebraic varieties; in particular, we need the base change formula for the bifunctor (pre-Mackey functor). Furthermore in a general context we formulate a general theory of characteristic classes of pro-objects as a natural transformation, using Fulton–MacPherson bivariant theory [34]. In Section 5 we consider Green functors and Grothendieck–Green functors, which are (pre-)Mackey functors with additional structure, and show a uniqueness theorem concerning the constructible function functor $F(X)$ and the relative Grothendieck group $K_0(\mathcal{V}/X)$.

2 Preliminaries

2.1 Pro-category

Let $\mathcal{C}$ be a given category and I a directed set. A projective system is a system $\{X_i, \pi_{ii'}\colon X_{i'} \to X_i \ (i < i')\}$ consisting of objects $X_i \in \mathrm{Obj}(\mathcal{C})$, morphisms

$\pi_{ii'}\colon X_{i'} \to X_i \in \mathrm{Mor}(\mathcal{C})$ for each $i < i'$ and the index set I. The morphism $\pi_{ii'}\colon X_{i'} \to X_i$ is called a *structure morphism* or a *bonding morphism*; see Mardešić and Segal [53]. The projective system $\{X_i, \pi_{ii'}\colon X_{i'} \to X_i \ (i < i')\}$ is sometimes simply denoted by $\{X_i\}_{i \in I}$.

Given a category $\mathcal{C}$, pro–$\mathcal{C}$ is the category whose objects are projective systems $X = \{X_i\}_{i \in I}$ in $\mathcal{C}$ and whose set of morphisms from $X = \{X_i\}_{i \in I}$ to $Y = \{Y_j\}_{j \in J}$ is pro–$\mathcal{C}(X, Y) := \varprojlim_J (\varinjlim_I \mathcal{C}(X_i, Y_j))$. An object in pro–$\mathcal{C}$ is called a *pro-object* and a morphism in pro–$\mathcal{C}$ a *pro-morphism*.

The above definition of a pro-morphism is not crystal clear, but a more down-to-earth definition is given by, eg, Fox [31] or Mardešić and Segal [53]. It follows from [53] that for two pro-objects $X = \{X_i\}_{i \in I}$ and $Y = \{Y_j\}_{j \in J}$, a pro-morphism $f = \{f_i\}_{i \in I}\colon X \to Y$ is described as follows: there is an order-preserving map $\xi\colon J \to I$, ie, $\xi(i) < \xi(i')$ for $i < i'$, and for each $i \in I$ there is a morphism $f_i\colon X_{\xi(i)} \to Y_i$ such that for $i < i'$ the following diagram commutes:

$$\begin{array}{ccc} X_{\xi(i')} & \xrightarrow{f_{i'}} & Y_{i'} \\ \Big\downarrow{\scriptstyle \pi_{\xi(i)\xi(i')}} & & \Big\downarrow{\scriptstyle \rho_{ii'}} \\ X_{\xi(i)} & \xrightarrow[f_i]{} & Y_i. \end{array}$$

From now on, to make the presentation simpler, we assume that a pro-morphism (promorphism, resp.) is (the projective limit of, resp.) a projective system of morphisms of objects with the same directed set I and that the order-preserving map $\xi\colon I \to I$ is the identity.

2.1.1 Remark Given a projective system $X = \{X_i\}_{i \in I}$ the projective limit $X_\infty := \varprojlim X_i$ may not belong to the source category $\mathcal{C}$. For a certain sufficient condition for the existence of the projective limit in the category $\mathcal{C}$; see Mardešić and Segal [53] for example.

Let $\mathfrak{F}\colon \mathcal{C} \to \mathcal{D}$ be a covariant functor between two categories $\mathcal{C}, \mathcal{D}$. Obviously the covariant functor $\mathfrak{F}$ extends to a covariant pro-functor pro–$\mathfrak{F}$: pro–$\mathcal{C} \to$ pro–$\mathcal{D}$ defined by pro–$\mathfrak{F}(\{X_i\}_{i \in I}) := \{\mathfrak{F}(X_i)\}_{i \in I}$. Let $\mathfrak{F}_1, \mathfrak{F}_2\colon \mathcal{C} \to \mathcal{D}$ be two covariant functors and $\mathfrak{N}\colon \mathfrak{F}_1 \to \mathfrak{F}_2$ be a natural transformation between the two functors $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Then the natural transformation $\mathfrak{N}\colon \mathfrak{F}_1 \to \mathfrak{F}_2$ extends to a natural pro-transformation pro–$\mathfrak{N}$: pro–$\mathfrak{F}_1 \to$ pro–$\mathfrak{F}_2$, taking the projective limit of which gives rise to the pro-transformation $\varprojlim \mathfrak{N}\colon \varprojlim \mathfrak{F}_1 \to \varprojlim \mathfrak{F}_2$. Here we denote $\mathfrak{F}^{\mathrm{pro}}(X_\infty) := \varprojlim \mathfrak{F}(X_i)$ for a covariant functor $\mathfrak{F}$ and $\mathfrak{N}^{\mathrm{pro}} = \varprojlim \mathfrak{N}$.

2.2 Characteristic classes of singular varieties

In this section we quickly recall well-studied theories of characteristic classes of singular varieties, which are typical models for the natural transformation $\mathfrak{N}\colon \mathfrak{F}_1 \to \mathfrak{F}_2$, where $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are covariant functors from the category of complex algebraic varieties to the category of abelian groups.

The classical theory of characteristic class is a natural transformation from the contravariant functor of vector bundles to the contravariant cohomology theory. When it comes to characteristic classes of singular spaces, they are considered in homology theory, instead of cohomology theory and still formulated as *natural transformations from a covariant functor $\mathcal{F}$ to a (suitable) homology theory H_**. Topologically or geometrically the following are most important and interesting:

(1) MacPherson's Chern class transformation [51]:

$$c_*\colon F(X) \to H_*(X),$$

(2) Baum, Fulton and MacPherson's Todd class or Riemann–Roch [6]:

$$\mathrm{td}_*\colon G_0(X) \to H_*(X) \otimes \mathbb{Q},$$

(3) Goresky and MacPherson's homology L–class [36], which is extended as a natural transformation by Cappell and Shaneson [21] (also see Yokura [73]):

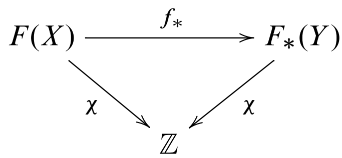
$$L_*\colon \Omega(X) \to H_*(X) \otimes \mathbb{Q}.$$

Here $F(X)$ is the abelian group of constructible functions on X, $G_0(X)$ is the Grothendieck group of coherent sheaves on X, $\Omega(X)$ is the Cappell–Shaneson–Youssin cobordism group of self-dual constructible sheaves (see Cappell and Shaneson [21], Woolf [71] and You [82]) and $H_*(X)$ is the Borel–Moore homology theory (eg, see Fulton [33]).

Since MacPherson's Chern class is the most fundamental one in the development of characteristic classes of singular varieties and we also need some facts on constructible functions in sections below, we quickly recall MacPherson's Chern class.

A constructible function on a variety is an integer-valued function for which the variety has a finite stratification into constructible sets such that the function is constant on each constructible set. Another simpler description of $F(X)$ is the following. For a subvariety $W \subset X$, let $\mathbb{1}_W$ be the characteristic function supported on W, ie, $\mathbb{1}_W(x) = 1$ for $x \in W$ and $\mathbb{1}_W(x) = 0$ for $x \notin W$. Then $F(X)$ consists of all finite linear combinations of such characteristic functions supported on subvarieties with integer coefficients. With the pullback $f^*\colon F(Y) \to F(X)$ defined by $f^*\alpha :=$

$\alpha \circ f$, the assignment $X \longmapsto F(X)$ is a *contravariant functor*. With the pushforward $f_*\colon F(X) \to F(Y)$ defined by $f_*(\mathbb{1}_W)(p) = \chi(f^{-1}(p) \cap W)$, the assignment $X \longmapsto F(X)$ is a *covariant functor* by MacPherson [51, Proposition 1]. The *Euler–Poincaré characteristic* homomorphism $\chi\colon F(X) \to \mathbb{Z}$ is defined by $\chi(\alpha) := \sum_{n\in\mathbb{Z}} n\chi(\alpha^{-1}(n))$. Then for a morphism $\pi_X\colon X \to \mathrm{pt}$ to a point pt, the pushforward $(\pi_X)_*\colon F(X) \to F(\mathrm{pt}) = \mathbb{Z}$ is nothing but $\chi\colon F(X) \to \mathbb{Z}$. So, for a morphism $f\colon X \to Y$, from the covariance of F we get the following commutative diagram:

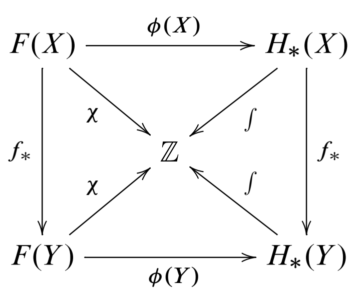


In fact, the commutativity of this diagram follows from the definition of the pushforward $f_*\colon F(X) \to F(Y)$ and the stratification theory (see [51]).

Deligne and Grothendieck conjectured and MacPherson affirmatively solved the following:

2.2.1 Theorem [51, Theorem 1] *There exists a unique natural transformation from the covariant constructible function functor to the Borel–Moore homology covariant functor $c_*\colon F(-) \to H_*(-)$ such that for a nonsingular variety X the value of the characteristic function $\mathbb{1}_X$ is the Poincaré dual of the total Chern cohomology class: $c_*(\mathbb{1}_X) = c(TX) \cap [X]$ where TX is the tangent bundle of X.*

2.2.2 Remark The above theorem is an answer for the question of if there exists (uniquely) a homomorphism $\phi(X)\colon F(X) \to H_*(X)$ such that the following diagram commutes:



Here $\int\colon H_*(X) \to \mathbb{Z}$ is the integration or equal to $(\pi_X)_*\colon H_*(X) \to H_*(\mathrm{pt}) = \mathbb{Z}$. It is obviously a Grothendieck–Riemann–Roch type question for Chern classes just like Grothendieck extended Hirzebruch–Riemann–Roch theorem to the Grothendieck–Riemann–Roch theorem (cf Grothendieck [40, Part II, note(87_1), pages 361 ff]).

The constructible function functor F is a bifunctor, ie, a functor which is both covariant and contravariant. Another bifunctor which we need in later sections is the *relative Grothendieck group* $K_0(\mathcal{V}/X)$. This was introduced by Looijenga [50] and further studied by Bittner [7].

2.2.3 Definition $K_0(\mathcal{V}/X)$ is the quotient of the free abelian group of isomorphism classes of morphisms to X (denoted by $[Y \to X]$, $[Y \xrightarrow{h} X]$ or $[h\colon Y \to X]$), modulo the relation

$$[Y \xrightarrow{h} X] = [Z \hookrightarrow Y \xrightarrow{h} X] + [Y \setminus Z \hookrightarrow Y \xrightarrow{h} X]$$

for a closed subvariety $Z \subset Y$. The isomorphism class $[h\colon Y \to X]$ shall be called the *Grothendieck class* of $h\colon Y \to X$.

- The ring structure is given for for $[f\colon Y \to X], [g\colon W \to X] \in K_0(\mathcal{V}/X)$ by
$$[Y \xrightarrow{f} X] \cdot [W \xrightarrow{g} X] := [Y \times_X W \xrightarrow{f \times_X g} X].$$
- For a morphism $f\colon X' \to X$, the pushforward $f_*\colon K_0(\mathcal{V}/X') \to K_0(\mathcal{V}/X)$ is defined by $f_*[h\colon Y \to X'] := [f \circ h\colon Y \to X]$.
- For a morphism $f\colon X' \to X$, the pullback $f^*\colon K_0(\mathcal{V}/X) \to K_0(\mathcal{V}/X')$ is defined by $f^*[g\colon Y \to X] := [g'\colon Y' \to X']$, using the fiber square
$$\begin{array}{ccc} Y' & \xrightarrow{g'} & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & X. \end{array}$$
- The exterior product $\times\colon K_0(\mathcal{V}/X) \times K_0(\mathcal{V}/Y) \to K_0(\mathcal{V}/X \times Y)$ is defined by
$$[V \xrightarrow{h} X] \times [W \xrightarrow{k} Y] := [V \times W \xrightarrow{h \times k} X \times Y].$$

2.2.4 Remark When $X = \mathrm{pt}$ is a point, the relative Grothendieck ring $K_0(\mathcal{V}/\mathrm{pt})$ is the usual Grothendieck ring $K_0(\mathcal{V})$. By considering $\pi_X\colon X \to \mathrm{pt}$, we get the homomorphism $\pi_{X*}\colon K_0(\mathcal{V}/X) \to K_0(\mathcal{V})$, which shall be denoted by χ_{Gro}. Then we have the following commutative diagram:

$$\begin{array}{ccc} K_0(\mathcal{V}/X) & \xrightarrow{\quad f_* \quad} & K_0(\mathcal{V}/Y) \\ & {\scriptstyle \chi_{\mathrm{Gro}}} \searrow \quad \swarrow {\scriptstyle \chi_{\mathrm{Gro}}} & \\ & K_0(\mathcal{V}) & \end{array}$$

In Brasselet, Schürmann and Yokura [14] (cf work of Brasselet, Schürmann and Yokura [11; 64; 66; 81]) we showed the following theorem (originally, using Saito's theory of mixed Hodge modules [60]):

2.2.5 Theorem (Motivic Hirzebruch class of singular varieties) *There exists a unique natural transformation* $T_{y*}\colon K_0(\mathcal{V}/-) \to H_*(-)\otimes \mathbb{Q}[y]$ *satisfying the normalization condition that for a smooth variety* X, $T_{y*}([\mathrm{id}_X\colon X\to X]) = \mathrm{td}_{(y)}(TX)\cap [X]$.

Here the Hirzebruch class $\mathrm{td}_{(y)}(E)$ of the complex vector bundle E [42; 43] is

$$\mathrm{td}_{(y)}(E) := \prod_{i=1}^{\operatorname{rank} E}\left(\frac{\alpha_i(1+y)}{1-e^{-\alpha_i(1+y)}} - \alpha_i y\right) \in H^*(X)\otimes \mathbb{Q}[y],$$

where α_i is the Chern root of E, ie, $c(E)=\prod_{i=1}^{\operatorname{rank} E}(1+\alpha_i)$. Note that

- $\mathrm{td}_{(-1)}(E) = c(E)$, the Chern class,
- $\mathrm{td}_{(0)}(E) = \mathrm{td}(E)$, the Todd class,
- $\mathrm{td}_{(1)}(E) = L(E)$, the Thom–Hirzebruch L–class.

2.2.6 Definition $T_{y*}(X) := T_{y*}([\mathrm{id}_X\colon X\to X])$ is called the *Hirzebruch class of* X and $\chi_y(X) := \int T_{y*}(X) = T_{y*}([X\to \mathrm{pt}])$ is called the *Hirzebruch* χ_y*–characteristic of* X.

2.2.7 Remark [14] Here we should note that for a smooth variety X we have $T_{-1*}(X) = c_*(X)$, $T_{0*}(X) = \mathrm{td}_*(X)$, $T_{1*}(X) = L_*(X)$. If X is singular, in general we have $T_{0*}(X) \neq \mathrm{td}_*(X)$, $T_{1*}(X)\neq L_*(X)$, $\chi_0(X)\neq \chi_a(X)$, $\chi_1(X)\neq \sigma(X)$, although $T_{-1*}(X) = c_*(X)$ always holds whether X is singular or not. Here $\chi_a(X) := \chi(X, \mathcal{O}_X)$ is the arithmetic genus and $\sigma(X)$ is the signature defined via the intersection homology of Goresky and MacPherson [36]. Thus $T_{0*}(X)$ and $T_{1*}(X)$ are respectively called *Hodge–Todd class* and *Hodge–*L *class*, and $\chi_0(X)$ and $\chi_1(X)$ are respectively called *Hodge–arithmetic genus* and *Hodge signature*.

This motivic Hirzebruch class $T_{y*}\colon K_0(\mathcal{V}/-)\to H_*(-)\otimes\mathbb{Q}[y]$ "unifies" the aforementioned three characteristic classes of singular varieties (cf MacPherson [52] and Yokura [74]). For more details, see Brasselet, Schürmann and Yokura [14]. For further and related works on the motivic Hirzebruch class, eg, see the works by Cappell, Libgober, Maxim, Saito, Schürmann, Shaneson and Yokura [20; 16; 18; 17; 19; 54; 55; 63; 65; 80].

2.3 Proconstructible functions, indconstructible functions and cylinder functions

From MacPherson's Chern class transformation $c_*\colon F \to H_*$, we get

$$c_*^{\mathrm{pro}}\colon F^{\mathrm{pro}}(X_\infty) \to H_*^{\mathrm{pro}}(X_\infty).$$

What would be the value of an element $\alpha_\infty = (\alpha_i) \in F^{\mathrm{pro}}(X_\infty)$ at a point $(x_i) \in X_\infty$? A very naïve definition could be $\alpha_\infty((x_i)) = \alpha_i(x_i)$ for all i. The equality $\pi_{ij\,*}(\alpha_j) = \alpha_i$ implies that $\alpha_i(x_i) = \pi_{ij\,*}(\alpha_j)(x_i) = \chi(\pi_{ij}^{-1}(x_i); \alpha_j)$. Here, $\chi(A;\alpha) := \chi(\alpha|_A)$. Thus, in general, $\pi_{ij}(x_j) = x_i$ and $\pi_{ij\,*}(\alpha_j) = \alpha_i$ do not imply that $\alpha_i(x_i) = \alpha_j(x_j)$. Thus $\alpha_\infty((x_i)) = \alpha_i(x_i)$ is not well-defined. Hence, an element of the pro-group $F^{\mathrm{pro}}(X_\infty)$ would not be a good candidate to be considered as a function on the proalgebraic variety X_∞. However, the equalities $\pi_{ij}(x_j) = x_i$ and $\pi_{ij}^*(\alpha_i) = \alpha_j$ imply that $\alpha_\infty((x_i)) = \alpha_i(x_i)(\forall i)$ is well-defined, since we have $\alpha_j(x_j) = (\pi_{ij}^*(\alpha_i))(x_j) = \alpha_i(\pi_{ij}(x_j)) = \alpha_i(x_i)$. Namely, to define a reasonable notion of *proconstructible function* on a proalgebraic variety we need to take the *inductive limit* instead of the projective limit, using the *contravariant nature* of the covariant functor F. So the following definition is reasonable.

2.3.1 Definition For a proalgebraic variety $X_\infty = \varprojlim\{X_i, \pi_{ij}\colon X_j \to X_i\ (i < j)\}$, the inductive limit $\varinjlim\{F(X_i), \pi_{ij}^*\colon F(X_i) \to F(X_j)\ (i < j)\}$ is denoted by $F^{\mathrm{ind}}(X_\infty)$ and the equivalence class of α_j is denoted by $[\alpha_j]$. An element of the indgroup $F^{\mathrm{ind}}(X_\infty)$ is called an *indconstructible function* on the proalgebraic variety X_∞.

2.3.2 Remark (1) In [2] (cf [1]), Aluffi considered the above projective limit F^{pro} for a certain special projective system of morphisms called *modification system*, which is more precisely a projective system of birational morphisms.

(2) Since the above indconstructible function is a function defined on a proalgebraic variety, we could still call it a "proconstructible" function as in Gromov [38], but we want to emphasize the fact that it is defined via the inductive limit and call it so.

(3) The indconstructible function $[\mathbb{1}_{X_i}]$ shall be called the *indcharacteristic function* on X_∞ and denoted by $\mathbb{1}_{X_\infty}$.

For a later reference we take a closer look at the above fact that an element of $F^{\mathrm{ind}}(X_\infty)$ can be considered as a function on the proalgebraic variety X_∞. We denote the above correspondence as $\Psi\colon F^{\mathrm{ind}}(X_\infty) \to \mathrm{Fun}(X_\infty, \mathbb{Z})$ defined by $\Psi([\alpha_i])((x_j)) := \alpha_i(x_i)$. One can describe this in a fancier way as follows. Let $\pi_i\colon X_\infty \to X_i$ denote the canonical projection. Consider the following commutative diagram (which follows from $\pi_i = \pi_{ij} \circ \pi_j$ $(i < j)$):

$$\begin{array}{ccc} F(X_i) & \xrightarrow{\pi_{ij}^*} & F(X_j) \\ & {\scriptstyle \pi_i^*} \searrow \quad \swarrow {\scriptstyle \pi_j^*} & \\ & \mathrm{Fun}(X_\infty, \mathbb{Z}) & \end{array}$$

Then it follows from a standard fact on inductive limits that the homomorphism $\Psi\colon F^{\mathrm{ind}}(X_\infty) \to \mathrm{Fun}(X_\infty, \mathbb{Z})$ is nothing but the unique homomorphism such that the following diagram commutes:

$$\begin{array}{ccc} & F(X_i) & \\ & {\scriptstyle \rho^i} \swarrow \quad \searrow {\scriptstyle \pi_i^*} & \\ F^{\mathrm{ind}}(X_\infty) & \xrightarrow[\Psi]{} & \mathrm{Fun}(X_\infty, \mathbb{Z}). \end{array}$$

To avoid possible confusion, the image $\Psi([\alpha_i]) = \pi_i^* \alpha_i$ shall be denoted by $[\alpha_i]_\infty$. For a constructible set $W_i \in X_i$, by the definition we have $[\mathbb{1}_{W_i}]_\infty = \mathbb{1}_{\pi_i^{-1}(W_i)}$. ${\pi_i}^{-1}(W_i)$ is called a *cylinder set (of level i)*, mimicking Craw [23]. And the characteristic function supported on a cylinder set (of level i) is called a *cylinder-characteristic function (of level i)* and a finite linear combination of cylinder-characteristic functions is called a *cylinder function*. Let $F^{\mathrm{cyl}}(X_\infty)$ denote the abelian group of all cylinder functions on the proalgebraic variety X_∞. Thus we have the following:

2.3.3 Proposition *For a proalgebraic variety $X_\infty = \varprojlim \{X_i, \pi_{ij}\colon X_j \to X_i\}$,*

$$F^{\mathrm{cyl}}(X_\infty) = \mathrm{Image}\, \Psi\colon F^{\mathrm{ind}}(X_\infty) \to \mathrm{Fun}(X_\infty, \mathbb{Z})) = \bigcup_i \pi_i^*(F(X_i)).$$

2.3.4 Proposition *If the structure morphisms $\pi_{ij}\colon X_j \to X_i$ $(i < j)$ are all surjective, then for the proalgebraic variety $X_\infty = \varprojlim \{X_i, \pi_{ij}\colon X_j \to X_i\}$ we have*

$$F^{\mathrm{ind}}(X_\infty) \cong F^{\mathrm{cyl}}(X_\infty).$$

Proof That all the structure morphisms $\pi_{ij}\colon X_j \to X_j$ $(i < j)$ are surjective implies that all the projections $\pi_i\colon X_\infty \to X_i$ are surjective. Which implies in turn that all the homomorphism $\pi_i^*\colon F(X_i) \to \mathrm{Fun}(X_\infty, \mathbb{Z})$ are injective. Since the inductive limit is an exact functor, it follows that the homomorphism $\Psi\colon \varinjlim F(X_\lambda) \to \mathrm{Fun}(X_\infty, \mathbb{Z})$ is also injective. Thus we get the above isomorphism. □

2.3.5 Question Is $\Psi\colon F^{\mathrm{ind}}(X_\infty) \to \mathrm{Fun}(X_\infty, \mathbb{Z})$ always injective?

2.3.6 Remark Suppose that $\Psi([\alpha_j]) = 0$, ie, $\Psi([\alpha_j])((x_i)) = \alpha_j(x_j) = 0$ for any $(x_i) \in X_\infty$. Hence we have $\alpha_j(\pi_j(X_\infty)) = 0$. At the moment we do not know whether we can conclude $[\alpha_j] = 0$ from this condition. A key point of the above proof is the fact that all the projections $\pi_i \colon X_\infty \to X_i$ are surjective, thus one might guess that the existence of a nonsurjective projection $\pi_i \colon X_\infty \to X_i$ might lead to a negative answer to the above question. But that is not the case; here is a very simple example such that $\alpha_j(\pi_j(X_\infty)) = 0, \pi_j(X_\infty) \neq X_j$ and $\alpha_j \neq 0$, but $[\alpha_j] = 0$: Let $X_1 = \{a, b\}$ be a space of two different points, and let $X_n = \{a\}$ for any $n > 1$. Let $\pi_{12} \colon X_2 \to X_1$ be the injection map sending a to a and the other structure morphism $\pi_{n(n+1)} \colon X_{n+1} \to X_n$ is the identity for $n > 1$. Then the projective limit $X_\infty = \{(a)\}$ consists of one point $(a, a, a, \ldots)$. Let $\alpha_1 = p \cdot \mathbb{1}_b \in F(X_1)$. Then we have $\alpha_1(\pi_1(X_\infty)) = 0, \pi_1(X_\infty) \neq X_1$ and $\alpha_1 \neq 0$, but $[\alpha_1] = 0$. We suspect that in general Ψ might be not necessarily injective, but we have been unable to find such an example.

2.4 Characteristic "indhomology" classes

First of all, let us consider a projective system $\{M_i, \pi_{ij} \colon M_j \to M_i \ (i < j)\}$ of compact complex manifolds M_i's. From this we get the inductive system of homology groups $\{H_*(M_i), (\pi_{ij})^! \colon H_*(M_i) \to H_*(M_j) \ (i < j)\}$. Here $(\pi_{ij})^! \colon H_*(M_i) \to H_*(M_j)$ is the Gysin homomorphism, ie, for a morphism $f \colon M \to N$, $f^! := \mathrm{PD}_M \circ f^* \circ \mathrm{PD}_M^{-1}$, where $\mathrm{PD}_W \colon H^*(W) \to H_*(W)$ is the Poincaré duality isomorphism via capping with the fundamental class; $\mathrm{PD}_W(x) = x \cap [W]$.

Let $c\ell \colon K^0(-) \to H^*(-)$ be a multiplicative characteristic class of complex vector bundles, ie, a multiplicative sequence of Chern classes and let $c\ell_*(M) := c\ell(TM) \cap [M]$. Then the family $\{c\ell_*(M_i)\}_{i \in I}$ is not compatible with the above inductive system $\{H_*(M_i)\}$, but it is compatible with the inductive system of the twisted pullback homomorphism $\{H_*(M_i), (\pi_{ij})^{!!} \colon H_*(M_i) \to H_*(M_j) \ (i < j)\}$, where $f^{!!} := c\ell(T_f) \cap f^!$ with $T_f := TM - f^*TN \in K^0(M)$. Namely, for each $i < j$ we have $c\ell_*(M_j) = (\pi_{ij})^{!!}(c\ell_*(M_i))$. Therefore any $c\ell_*(M_i)$ determines the unique "indhomology" class $[c\ell_*(M_i)]$ in the inductive limit: $[c\ell_*(M_i)] \in H_{**}^{\mathrm{ind}}(M_\infty)$. Here $H_{**}^{\mathrm{ind}}(M_\infty)$ denotes the inductive limit of the above inductive system. So, this "indhomology" class $[c\ell_*(M_i)]$ can be considered as the characteristic "indhomology" class $c\ell_*(\{M_i\})$ of the pro-manifold $\{M_i\}$ or the characteristic "indhomology" class $c\ell_*(M_\infty)$ of the pro-manifold M_∞.

2.4.1 Remark (1) If we consider the cohomology group $H^*(M)$ instead of the homology group $H_*(M)$, then the twisted pullback $f^{!!} \colon H^*(N) \to H^*(M)$ is $f^{!!} = c\ell(T_f) \cup f^*$ and we get the characteristic "indcohomology" class $c\ell^{\mathrm{ind}}(\{M_i\})$ or $c\ell^{\mathrm{ind}}(M_\infty)$.

(2) Here it should be warned that for *any* projective system of compact complex manifolds M_i we get the same "indhomology" class $[c\ell_*(M_i)]$, *independent of the structure of the projective limit* M_∞.

(3) For *any* projective system of compact complex manifolds $\{M_i\}$ the "indcharacteristic function" $\mathbb{1}_{M_\infty}$ and the Chern "indhomology" class $[c_*(M_i)]$ are both available. However, we do not necessarily have a homomorphism $c_*^{\mathrm{ind}}\colon F^{\mathrm{ind}}(X_\infty) \to H_*^{\mathrm{ind}}(X_\infty)$ such that $c_*^{\mathrm{ind}}(\mathbb{1}_{M_\infty}) = [c_*(M_i)]$. For example, consider the case when at least one map $\pi_{ij}\colon X_j \to X_i$ is constant. If $f\colon M \to N$ is a constant map with $f(M) = x_0$, then the following diagram is not commutative:

$$\begin{array}{ccc} F(N) & \xrightarrow{c_*} & H_*(N) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle c(T_f)\cap f^!} \\ F(M) & \xrightarrow[c_*]{} & H_*(M). \end{array}$$

Indeed, let us suppose that the above diagram commutes. Take another point x from N such that $x \neq x_0$ and x is in the same component of x_0. Then $c_*(\mathbb{1}_x) = [x] = [x_0] = c_*(\mathbb{1}_{x_0}) \in H_*(N)$. Thus we have $c_*(M) = c_*(\mathbb{1}_M) = c_*(f^*\mathbb{1}_{x_0}) = c(T_f) \cap f^!(c_*(\mathbb{1}_{x_0})) = c(T_f) \cap f^!(c_*(\mathbb{1}_x)) = c_*(f^*\mathbb{1}_x) = c_*(0) = 0$. This is a contradiction.

So, to construct $c_*^{\mathrm{ind}}\colon F^{\mathrm{ind}}(X_\infty) \to H_{**}^{\mathrm{ind}}(X_\infty)$ we need to consider the commutativity of the following diagram with some suitable contravariantly functorial pullback homomorphism $\pi_{ij}^{\P}\colon H_*(X_i) \to H_*(X_j)$:

$$\begin{array}{ccc} F(X_i) & \xrightarrow{c_*} & H_*(X_i) \\ {\scriptstyle \pi_{ij}^*}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{ij}^{\P}} \\ F(X_j) & \xrightarrow{c_*} & H_*(X_j). \end{array}$$

Hence, to get an inductive version $\mathfrak{N}^{\mathrm{ind}}\colon \mathfrak{F}^{\mathrm{ind}}(-) \to H_*^{\mathrm{ind}}(-) \otimes R$ from a theory $\mathfrak{N}\colon \mathfrak{F} \to H_*(-) \otimes R$ of characteristic classes of singular varieties, which is any theory recalled in Section 2.2, we need to consider such a commutative diagram as above. One immediate answer for such a commutative diagram is the following Verdier-type Riemann–Roch theorem (see Schürmann [62] and Yokura [75]):

2.4.2 Theorem *Let X, Y be complex algebraic varieties and $f\colon X \to Y$ be a smooth morphism between them. Then the following diagram commutes:*

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle c(T_f)\cap f^!} \\ F(X) & \xrightarrow[c_*]{} & H_*(X). \end{array}$$

Here T_f is the relative tangent bundle of f and $f^!$ is the Gysin homomorphism.

The same formulas hold for Todd class td_* (which is the original Verdier–Riemann–Roch theorem [6]), for Hirzebruch class T_{y*} [14] and also for Whitney class w_* [34, Proposition 6B] (cf Fu and McCrory [32] and Sullivan [69]).

2.4.3 Remark For a more generalized Verdier-type Riemann–Roch theorem for Chern class, see Schürmann [62]. For the above Verdier–Riemann–Roch formula for Todd class, smooth morphism can be replaced by local complete intersection morphism [6]. We would speculate that for a smooth morphism $f\colon X \to Y$ the following Verdier-type Riemann–Roch formula for the Cappell–Shaneson homology L–class $L_*\colon \Omega(-) \to H_*(-)\otimes\mathbb{Q}$ also holds, but we have been unable to prove or disprove it (cf [14]):

$$\begin{array}{ccc} \Omega(Y) & \xrightarrow{L_*} & H_*(Y)\otimes\mathbb{Q} \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle L(T_f)\cap f^!} \\ \Omega(X) & \xrightarrow[L_*]{} & H_*(X)\otimes\mathbb{Q}. \end{array}$$

2.4.4 Corollary *For a projective system $\{X_i, \pi_{ij}\colon X_j \to X_i\ (i<j)\}$ of smooth morphisms π_{ij} we have*

(1) *the homomorphism $c_*^{\mathrm{ind}}\colon F^{\mathrm{ind}}(X_\infty) \to H_{**}^{\mathrm{ind}}(X_\infty)$ and the Chern– Schwartz–MacPherson "indhomology" class $c_*^{\mathrm{ind}}(X_\infty) = c_*^{\mathrm{ind}}(\mathbb{1}_{X_\infty}) = [c_*(X_i)]$.*

(2) *the homomorphism $\mathrm{td}_*^{\mathrm{ind}}\colon G_0^{\mathrm{ind}}(X_\infty) \to H_{**}^{\mathrm{ind}}(X_\infty)\otimes\mathbb{Q}$ and the Baum–Fulton–MacPherson Todd "indhomology" class $\mathrm{td}_*^{\mathrm{ind}}(X_\infty) = \mathrm{td}_*^{\mathrm{ind}}(\mathcal{O}_{X_\infty}) = [\mathrm{td}_*(X_i)]$.*

(3) *the homomorphism ${T_y}_*^{\mathrm{ind}}\colon K_0^{\mathrm{ind}}(\mathcal{V}/X_\infty) \to H_{**}^{\mathrm{ind}}(X_\infty)\otimes\mathbb{Q}[y]$ and Hirzebruch "indhomology" class ${T_y}_*^{\mathrm{ind}}(X_\infty) = {T_y}_*^{\mathrm{ind}}([\mathrm{id}\colon X_\infty \to X_\infty]) = [T_{y*}(X_i)]$.*

2.4.5 Remark (1) At the moment $c_*^{\mathrm{ind}}, \mathrm{td}_*^{\mathrm{ind}}$ and ${T_y}_*^{\mathrm{ind}}$ are just transformations. In fact, to make $c_*^{\mathrm{ind}}, \mathrm{td}_*^{\mathrm{ind}}$ and ${T_y}_*^{\mathrm{ind}}$ natural transformations, we need more requirements and we treat that in Section 4 in a more general context.

(2) The above homomorphism $c_*^{\mathrm{ind}}\colon F^{\mathrm{ind}}(X_\infty) \to H_{**}^{\mathrm{ind}}(X_\infty)$ is closely related to the construction of equivariant Chern class due to Ohmoto [56] (cf [57]).

2.4.6 Example Let us take a look at another example which deals with nonsmooth maps. Consider an infinite sequence $\{N_1, N_2, \dots, N_n, \dots\}$ of compact complex algebraic varieties N_i's and let us set $X_n := N_1 \times N_2 \times \cdots \times N_n$ $(n \in \mathbb{N})$. For each $i < j$ we let $\pi_{ij}\colon X_j \to X_i$ be the canonical projection $\pi_{ij}(x_1, x_2, \dots, x_j) = (x_1, x_2, \dots, x_i)$. Then consider the projective system $\{X_i, \pi_{ij}\colon X_j \to X_i \ (i < j)\}$, the projective limit of which is actually by definition to be the infinite product $N_1 \times N_2 \times \cdots \times N_n \times \cdots$. Then we have the commutative diagram

$$\begin{array}{ccc} F(X_n) & \xrightarrow{c_*} & H_*(X_n) \\ {\scriptstyle \pi^*_{n,n+1}}\downarrow & & \downarrow{\scriptstyle \times c_*(N_{n+1})} \\ F(X_{n+1}) & \xrightarrow[c_*]{} & H_*(X_{n+1}). \end{array}$$

This commutativity follows from the (exterior) product formula of MacPherson's Chern class [46] (cf [47]): $c_*(\alpha \times \beta) = c_*(\alpha) \times c_*(\beta)$ for $\alpha \in F(X)$ and $\beta \in F(Y)$. Thus $\{H_*(X_i), \times c_*(N_{i+1}) \times \cdots \times c_*(N_j)\colon H_*(X_i) \to H_*(X_j)\ (i < j)\}$ is clearly an inductive system, the inductive limit of which is again denoted by $H^{\mathrm{ind}}_{**}(X_\infty)$, and thus we get the homomorphism $c^{\mathrm{ind}}_*\colon F^{\mathrm{ind}}(X_\infty) \to H^{\mathrm{ind}}_{**}(X_\infty)$ and we have $c^{\mathrm{ind}}_*(X_\infty) = [c_*(N_1)]$.

In fact, the other three characteristic classes td_*, L_* and T_{y*} also commute with the exterior products (see Baum, Fulton and MacPherson [6], Woolf [71] and Brasselet, Schürmann and Yokura [14], respectively), thus we get the following:

2.4.7 Corollary *Let the situation be as above. Let $c\ell = c, \mathrm{td}, L, \mathrm{td}_{(y)}$ and let $H^{\mathrm{ind}}_{**}(X_\infty) \otimes R$ be the inductive limit of the inductive system of homology classes $\{H_*(X_i) \otimes R, \times c\ell_*(N_{i+1}) \times \cdots \times c\ell_*(N_j)\colon H_*(X_i) \otimes R \to H_*(X_j) \otimes R\ (i < j)\}$, where $R = \mathbb{Z}$ when $c\ell = c$, $R = \mathbb{Q}$ when $c\ell = \mathrm{td}, L$ and $R = \mathbb{Q}[y]$ when $c\ell = \mathrm{td}_{(y)}$. Then we have the homomorphism $c\ell^{\mathrm{ind}}_*\colon F^{\mathrm{ind}}(X_\infty) \to H^{\mathrm{ind}}_{**}(X_\infty) \otimes R$ and $c\ell^{\mathrm{ind}}_*(X_\infty) = [c\ell_*(N_1)]$.*

2.4.8 Corollary *Let $c\ell = c, \mathrm{td}, L, \mathrm{td}_{(y)}$ and let X be a compact complex algebraic variety. The $c\ell$ "indhomology" class of the infinite product $X^{\mathbb{N}}$ is $c\ell^{\mathrm{ind}}_*(X^{\mathbb{N}}) = [c\ell_*(X)]$.*

2.5 Characteristic "indnumbers"

Let $c\ell$ be a multiplicative characteristic class as before and let us consider the corresponding characteristic "number" of a compact complex manifold M:

$$\sharp_{c\ell}(M) := \int_M c\ell_*(M) \in H_0(M) = R.$$

2.5.1 Example (i) If $c\ell = c$ is Chern class, $\sharp_c(M) = \chi(M)$ is the Euler–Poincaré characteristic of M.

(ii) If $c\ell = \mathrm{td}$ is Todd class, $\sharp_{\mathrm{td}}(M) = \chi_a(M)$ is the arithmetic genus of M.

(iii) If $c\ell = L$ is Thom–Hirzebruch's L–class, $\sharp_L(M) = \sigma(M)$ is the signature of M.

For a morphism $f\colon M \to N$ of compact complex manifolds we do have $f^{!!}(c\ell_*(N)) = c\ell_*(M)$, but clearly in general $\sharp_{c\ell}(M) \neq \sharp_{c\ell}(N)$. So, for a projective system $\{M_i\} := \{M_i, \pi_{ij}\colon M_j \to M_i \ (i < j)\}$ of compact complex manifolds, if there is an inductive system $\{R_i = R, \phi_{ij}\colon R_i \to R_j \ (i < j)\}$ such that $\phi_{ij}(\sharp_{c\ell}(M_i)) = \sharp_{c\ell}(M_j)$, then we have the "indcharacteristic" of the pro-manifold M_∞: $\sharp^{\mathrm{ind}}_{c\ell}(M_\infty) = [\sharp_{c\ell}(M_i)]$.

2.5.2 Example As before, let us consider an infinite sequence $\{N_1, N_2, \dots, N_n, \dots\}$ of compact complex manifolds N_i's, $M_n := N_1 \times N_2 \times \cdots \times N_n \quad (n \in \mathbb{N})$ and $\pi_{ij}\colon M_j \to M_i$. Then we have $\sharp_{c\ell}(M_n) = \sharp_{c\ell}(N_1) \cdot \sharp_{c\ell}(N_2) \cdot \cdots \cdot \sharp_{c\ell}(N_n)$ and $\sharp_{c\ell}(M_j) = p_{ij} \cdot \sharp_{c\ell}(M_i)$. Here $p_{ij} := \sharp_{c\ell}(M_{i+1}) \cdots \sharp_{c\ell}(M_j) \in R$ and $p_{ii} := 1$. In this case the above homomorphism $\phi_{ij}\colon R \to R$ is the multiplication by p_{ij}, ie, $\phi_{ij}(a) = p_{ij} \cdot a$. Hence the characteristic "indnumber" of the infinite product of complex manifolds $N_1 \times N_2 \times \cdots \times N_n \times \cdots$ is equal to $[\sharp_{c\ell} N_1] \in \varinjlim \{R_i = R; p_{ij} \times \colon R \to R\}$. Furthermore, we let $p_1 := 1$, $p_n := \sharp_{c\ell}(N_2) \cdot \cdots \cdot \sharp_{c\ell}(N_n)$ and let P be the multiplicatively closed subset of R generated by all p_n's (ie, consisting of $(p_{i_1})^{m_1} \cdot (p_{i_2})^{m_2} \cdots (p_{i_k})^{m_k}$ with $m_j \geqq 0$) such that $0 \notin P$ (see Remark 2.5.3 below). Let us consider the localization R_P of R with respect to P and the renormalization $\phi_n\colon R_n = R \to R_P$ defined by $\phi_n(a) := a/p_n$. Then it follows from the standard facts of the inductive limits that there exists a unique homomorphism $\Phi\colon \varinjlim R_n \to R_P$ such that the following diagram commutes:

$$\begin{array}{ccc} & R_n = R & \\ {\scriptstyle \rho^n} \swarrow & & \searrow {\scriptstyle \phi_n} \\ \varinjlim R_n & \xrightarrow[\Phi]{} & R_P. \end{array}$$

This homomorphism $\Phi\colon \varinjlim R_n \to R_P$ is a kind of "realization homomorphism" of the abstract ring $\varinjlim R_n$. Thus we get the inductive characteristic

$$\sharp^{\mathrm{ind}}_{c\ell}(N_1 \times N_2 \times \cdots \times N_n \cdots) = [\sharp_{c\ell}(N_1)] \in R_P.$$

For example, if we consider the Chern class c for the infinite product $N^\infty = N \times N \times \cdots$ of a compact complex manifold N, then we have the inductive Euler–Poincaré characteristic $\chi^{\mathrm{ind}}(N^\infty) = [\chi(N)] \in \mathbb{Z}_{(\chi(N))}$. As we see above, the manifold N can be replaced by a complex algebraic variety X and we have the same answer, ie, $\chi^{\mathrm{ind}}(X^\infty) = [\chi(X)] \in \mathbb{Z}_{(\chi(X))}$.

2.5.3 Remark The requirement $0 \notin P$ for the multiplicatively closed subset P of R generated by all p_n's is necessary when we consider the quotient ring R_P, ie, when we consider the realization homomorphism $\Phi\colon \varinjlim R_n \to \mathcal{R}_P$. $0 \notin P$ implies that each p_i is nonnilpotent. If R is an integral domain, then $0 \notin P$ if and only if each p_i is nonnilpotent. However, if R is not an integral domain, that each p_i is nonnilpotent is not necessarily sufficient. Later we deal with the Grothendieck ring $K_0(\mathcal{V})$, which is not an integral domain; see Poonen [58].

2.5.4 Example Let $f\colon M \to N$ be a submersion of compact complex manifolds and assume that the fundamental group of the base manifold acts trivially on the cohomology $H^*(F)$ of its fiber F, then we have that $\chi(M) = \chi(F)\chi(N)$ [22]. Provisionally such a submersion shall be called a *perfect submersion* for short. (Note that Atiyah [4] and Kodaira [44] showed that if a submersion is not perfect, then the above multiplicative formula does not necessarily hold.) In general we have $\chi_y(M) = \chi_y(F)\chi_y(N)$. So, for a projective system $\{M_i\} := \{M_i, \pi_{ij}\colon M_j \to M_i \ (i < j)\}$ of perfect submersions of compact complex manifolds with $p_{ij} := \chi_y(F_{ij})$ for a fiber F_{ij} of π_{ij}, we have $\chi_y(M_i) = p_{ij} \cdot \chi_y(M_j)$. Hence we can define the inductive Hirzebruch χ_y–characteristic $\chi_y^{\mathrm{ind}}(M_\infty) = [\chi_y(M_i)]$.

3 Characteristics of proalgebraic varieties

3.1 Bifunctors

In the previous section we saw that a bifunctor plays a key role in the consideration of proalgebraic analogues of characteristics and characteristic classes. So, from now on we consider a *bifunctor* $\mathcal{F}\colon \mathcal{C} \to \mathcal{AB}$ from a category $\mathcal{C}$ to the category $\mathcal{AB}$ of abelian groups, ie, $\mathcal{F}$ is a pair $(\mathcal{F}_*, \mathcal{F}^*)$ of a covariant functor $\mathcal{F}_*$ and a contravariant functor $\mathcal{F}^*$ such that $\mathcal{F}_*(X) = \mathcal{F}^*(X)$ for any object X. Unless some confusion occurs, we just denote $\mathcal{F}(X)$ for $\mathcal{F}_*(X) = \mathcal{F}^*(X)$. Furthermore we assume that for a final object $\mathrm{pt} \in \mathrm{Obj}(\mathcal{C})$, $\mathcal{F}(\mathrm{pt})$ is a commutative ring $\mathcal{R}$ with a unit. Then the covariance of the bifunctor $\mathcal{F}$ induces the homomorphism $\mathcal{F}(\pi_X)\colon \mathcal{F}(X) \to \mathcal{F}(\mathrm{pt}) = \mathcal{R}$, which shall be denoted by $\chi_{\mathcal{F}}\colon \mathcal{F}(X) \to \mathcal{R}$ and called the *$\mathcal{F}$–characteristic*, just mimicking the Euler– Poincaré characteristic $\chi\colon F(X) \to \mathbb{Z}$ in the case when $\mathcal{F} = F$. Then furthermore the covariance of $\mathcal{F}$ implies that for a morphism $f\colon X \to Y$ we get the commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{f_*} & \mathcal{F}(Y) \\ & {\scriptstyle \chi_{\mathcal{F}}} \searrow \quad \swarrow {\scriptstyle \chi_{\mathcal{F}}} & \\ & \mathcal{R} & \end{array}$$

3.1.1 Remark In the above definition we require the contravariance of $\mathcal{F}$ for all morphisms in the category $\mathcal{C}$. But, as seen before, when it comes to dealing with pullbacks, the contravariance is not required on all morphisms but only on projective systems which you consider. From now on our bifunctors are understood to be such ones. Sloppily we say that $\mathcal{F}$ is a bifunctor on projective systems.

3.2 $\chi_{\mathcal{F}}$–Stable objects and inductive characteristics

3.2.1 Definition Let $\{X_i, \pi_{ij}\colon X_j \to X_i\}$ be a projective system and let $P = \{p_{ij}\}$ be a projective system of elements of $\mathcal{R}$ by the directed set I, ie, a set such that $p_{ii} = 1$ and $p_{ij} \cdot p_{jk} = p_{ik}$ $(i < j < k)$. Let $\mathcal{F}$ be a bifunctor on the projective system $\{X_i\}$.

(1) For each $i \in I$ we define the following subobject of $\mathcal{F}(X_i)$:

$$\mathcal{F}^{\mathrm{st}}_P(X_i) := \left\{\alpha_i \in \mathcal{F}(X_i) \mid \chi_{\mathcal{F}}({\pi_{ij}}^*\alpha_i) = p_{ij} \cdot \chi_{\mathcal{F}}(\alpha_i) \text{ for all } j > i\right\}.$$

(2) For each $i \in I$, an element of $\mathcal{F}^{\mathrm{st}}_P(X_i)$ is called a *$\chi_{\mathcal{F}}$–stable object (of level i) with respect to the projective system P.*

3.2.2 Lemma *For each structure morphism $\pi_{ij}\colon X_j \to X_i$ the pullback homomorphism $\pi^*_{ij}\colon \mathcal{F}(X_i) \to \mathcal{F}(X_j)$ preserves $\chi_{\mathcal{F}}$–stable objects with respect to the projective system $P = \{p_{ij}\}$, namely it induces the homomorphism (using the same symbol)*

$$\pi^*_{ij}\colon \mathcal{F}^{\mathrm{st}}_P(X_i) \to \mathcal{F}^{\mathrm{st}}_P(X_j).$$

3.2.3 Definition The inductive limit of the inductive system

$$\left\{\mathcal{F}^{\mathrm{st}}_P(X_i),\ \pi^*_{ij}\colon \mathcal{F}^{\mathrm{st}}_P(X_i) \to \mathcal{F}^{\mathrm{st}}_P(X_j)\ (i < j)\right\}$$

is denoted by $\mathcal{F}^{\mathrm{st.ind}}_P(X_\infty)$ and an element of this inductive limit shall be called a *$\chi_{\mathcal{F}}$–stable indobject of $\mathcal{AB}$ on the proalgebraic variety X_∞ with respect to the projective system P.*

3.2.4 Remark We see that this can be also directly defined as

$$\left\{[\alpha_i] \in \mathcal{F}^{\mathrm{ind}}(X_\infty) \mid \chi_{\mathcal{F}}(\pi^*_{ij}\alpha_\lambda) = p_{ij} \cdot \chi_{\mathcal{F}}(\alpha_i)\ (i < j)\right\}.$$

The following is an application of standard facts on inductive systems and inductive limits, but nevertheless it is a key and important observation for the rest of the paper, in particular in connection to motivic measures, so it is stated as a theorem.

3.2.5 Theorem (1) *For a projective system $\{X_i, \pi_{ij}\colon X_j \to X_i\}$ and a projective system $P = \{p_{ij}\}$ of nonzero elements of $\mathcal{R}$, we have the homomorphism*

$$\chi_{\mathcal{F}}^{\mathrm{ind}}\colon \mathcal{F}_{\boldsymbol{P}}^{\mathrm{st.ind}}(X_\infty) \to \varinjlim\{\times p_{ij}\colon \mathcal{R} \to \mathcal{R}\},$$

which is called the inductive $\mathcal{F}$–characteristic (or "$\mathcal{F}$–indcharacteristic") homomorphism.

(2) *In the case when $\Lambda = \mathbb{N}$, for a projective system $X_\infty = \varprojlim\{X_n, \pi_{nm}\colon X_m \to X_n\}$ and a projective system $P = \{p_{nm}\}$ of nonzero elements of $\mathcal{R}$ such that the multiplicatively closed set S generated by P does not contain the zero, the inductive $\mathcal{F}$–characteristic homomorphism*

$$\chi_{\mathcal{F}}^{\mathrm{ind}}\colon \mathcal{F}_{\boldsymbol{P}}^{\mathrm{st.ind}}(X_\infty) \to \varinjlim\{\times p_{nm}\colon \mathcal{R} \to \mathcal{R}\}$$

is realized as the homomorphism $\widetilde{\chi_{\mathcal{F}}^{\mathrm{ind}}}\colon \mathcal{F}_{\boldsymbol{P}}^{\mathrm{st.ind}}(X_\infty) \to \mathcal{R}_{\boldsymbol{P}}$ defined by

$$\widetilde{\chi_{\mathcal{F}}^{\mathrm{ind}}}([\alpha_n]) := \frac{\chi_{\mathcal{F}}(\alpha_n)}{p_{01} \cdot p_{12} \cdot p_{23} \cdots p_{(n-1)n}}.$$

Here $p_{01} := 1$ and $\mathcal{R}_{\boldsymbol{P}}$ is the ring $\mathcal{R}_{\boldsymbol{S}}$ of fractions of $\mathcal{R}$ with respect to S.

(3) *In particular, if each $p_{nm} = p^{m-n}$ for a nonnilpotent element p, we get the homomorphism*

$$\widetilde{\chi_{\mathcal{F}}^{\mathrm{ind}}}\colon \mathcal{F}_{\boldsymbol{P}}^{\mathrm{st.ind}}(X_\infty) \to \mathcal{R}_{\boldsymbol{P}} \quad \text{defined by} \quad \widetilde{\chi_{\mathcal{F}}^{\mathrm{ind}}}([\alpha_n]) := \frac{\chi_{\mathcal{F}}(\alpha_n)}{p^{n-1}}.$$

Proof (1) follows from taking the inductive limit of the commutative diagram

$$\begin{array}{ccc}
\mathcal{F}_{\boldsymbol{P}}^{\mathrm{st}}(X_i) & \xrightarrow{\chi_{\mathcal{F}}} & \mathcal{R} \\
{\scriptstyle \pi_{ij}{}^*}\downarrow & & \downarrow{\scriptstyle \times p_{ij}} \\
\mathcal{F}_{\boldsymbol{P}}^{\mathrm{st}}(X_j) & \xrightarrow[\chi_{\mathcal{F}}]{} & \mathcal{R}.
\end{array}$$

For a general directed set I, we do not know how to describe the homomorphism $\chi_{\mathcal{F}}^{\mathrm{ind}}$ in a bit more down-to-earth way. However, when it comes to the case when $I = \mathbb{N}$, we can get the above claim as follows:

Let $R_n = \mathcal{R}$ for each n and for $n < m$ let $\rho_{nm}\colon R_n \to R_m$ denote the homomorphism defined by $\rho_{nm}(r_n) = r_n \cdot p_{n(n+1)} \cdot p_{(n+1)(n+2)} \cdots p_{(m-1)m}$. And let $\phi^n\colon R_n \to \mathcal{R}_P$ be the homomorphism defined by

$$\phi^n(r_n) := \frac{r_n}{p_{01} \cdot p_{12} \cdot p_{23} \cdot \cdots \cdot p_{(n-1)n}}.$$

Then for $n < m$ we have $\phi^m \circ \rho_{nm} = \phi^n$. Therefore it follows from the standard facts of the inductive limits that there exists a unique homomorphism $\Phi\colon \varinjlim R_n \to \mathcal{R}_P$ such that the following diagram commutes:

$$\begin{array}{ccc} & R_n & \\ {\scriptstyle \rho^n} \swarrow & & \searrow {\scriptstyle \phi_n} \\ \varinjlim R_n & \xrightarrow[\Phi]{} & \mathcal{R}_P . \end{array}$$

By composing $\chi^{\mathrm{ind}}_{\mathcal{F}}\colon \mathcal{F}^{\mathrm{st.ind}}_P(X_\infty) \to \varinjlim\{\times p_{nm}\colon \mathcal{R} \to \mathcal{R}\}$ with this "realization homomorphism" Φ, we get the above homomorphism

$$\widetilde{\chi^{\mathrm{ind}}_{\mathcal{F}}}\colon \mathcal{F}^{\mathrm{st.ind}}_P(X_\infty) \to \mathcal{R}_P. \qquad \square$$

3.2.6 Remark (1) Let $X_i = \mathrm{pt}$ be a point for any $i \in I$ and let $\pi_{ij} = \mathrm{id}\colon X_i \to X_j$ be the identity. Then $\varprojlim\{X_i, \pi_{ij}\colon X_j \to X_i\}$ is a point and is called a *pro-point* and is denoted by pt_∞. Then for the pro-point pt_∞ we define $\mathcal{F}^{\mathrm{st.ind}}_P(\mathrm{pt}_\infty)$ to be $\varinjlim\{\times p_{ij}\colon \mathcal{F}(\mathrm{pt}) \to \mathcal{F}(\mathrm{pt})\} = \varinjlim\{\times p_{ij}\colon \mathcal{R} \to \mathcal{R}\}$. In this sense, the above inductive $\mathcal{F}$–characteristic homomorphism $\chi^{\mathrm{ind}}_{\mathcal{F}}\colon \mathcal{F}^{\mathrm{st.ind}}_P(X_\infty) \to \varinjlim\{\times p_{ij}\colon \mathcal{R} \to \mathcal{R}\}$ is expressed as $\chi^{\mathrm{ind}}_{\mathcal{F}}\colon \mathcal{F}^{\mathrm{st.ind}}_P(X_\infty) \to \mathcal{F}^{\mathrm{st.ind}}_P(\mathrm{pt}_\infty)$ and it is an inductive limit version of the $\mathcal{F}$–characteristic $\chi_{\mathcal{F}}\colon \mathcal{F}(X) \to \mathcal{F}(\mathrm{pt}) = \mathcal{R}$.

(2) The above realization is a *canonical* one in the sense that there are many other realizations by considering

$$\phi'_n(r_n) = \frac{r_n}{\omega \cdot p_{01} \cdot p_{12} \cdot p_{23} \cdots p_{(n-1)n}}$$

with any nonzero element ω such that the multiplicatively closed set generated by $P \cup \{\omega\}$ does not contain the zero.

3.2.7 Definition Let $\mathcal{F}$ be a bifunctor on a category $\mathcal{C}$ such that $\mathcal{R} = \mathcal{F}(\mathrm{pt})$ is a commutative ring with a unit and let $\chi_{\mathcal{F}}\colon \mathcal{F}(X) \to \mathcal{R}$ be the $\mathcal{F}$–characteristic.

(1) If a morphism $f\colon X \to Y$ satisfies the condition that for an element $\alpha \in \mathcal{F}(Y)$, $\chi_{\mathcal{F}}(f_* f^* \alpha) = c_f \cdot \chi_{\mathcal{F}}(\alpha)$ with some element called "multiplier" $c_f \in \mathcal{R}$, then we

say that f is *$\chi_{\mathcal{F}}$–constant with respect to α with the multiplier c_f*. (c_f could be considered as the "χ_F–characteristic of the fiber of f".)

(2) If f is a *$\chi_{\mathcal{F}}$–constant with respect to any element $\alpha \in \mathcal{F}(Y)$ with the multiplier c_f*, then the morphism $f\colon X \to Y$ is called *$\chi_{\mathcal{F}}$–constant with the multiplier c_f*.

(3) (a bit stronger) Let $\mathcal{F}$ be a bifunctor from a category $\mathcal{C}$ to the category of $\mathcal{R}$–modules such that $\mathcal{F}(\mathrm{pt}) = \mathcal{R}$. If a morphism $f\colon X \to Y$ satisfies the condition that $f_*f^* = c_f \cdot \mathrm{Id}_{\mathcal{F}(Y)}\colon \mathcal{F}(Y) \to \mathcal{F}(Y)$ with some element $c_f \in \mathcal{R}$, where $\mathrm{Id}_{\mathcal{F}(Y)}$ denotes the identity homomorphism, then f is also called *$\chi_{\mathcal{F}}$–constant with the multiplier c_f*. (Note that in this case $f_*f^* = c_f \cdot \mathrm{Id}_{\mathcal{F}(Y)}$ implies that $\chi_{\mathcal{F}}(f_*f^*\alpha) = c_f \cdot \chi_{\mathcal{F}}(\alpha)$ for any $\alpha \in \mathcal{F}(Y)$.)

3.2.8 Remark $\alpha_i \in \mathcal{F}^{\mathrm{st}}_P(X_i)$ means that π_{ij} is $\chi_{\mathcal{F}}$–constant with respect to α_i with the multiplier p_{ij} for any j such that $i < j$. A Zariski locally trivial fiber bundle is a χ_{Gro}–constant morphism with the multiplier being the Grothendieck class $[F]$ of its fiber variety F.

3.2.9 Proposition *Let $\{X_n, \pi_{nm}\colon X_m \to X_n\}$ be a projective system such that each structure morphism $\pi_{n(n+1)}\colon X_{n+1} \to X_n$ is $\chi_{\mathcal{F}}$–constant with the multiplier $c_{n(n+1)} \in \mathcal{R}$. We assume that the multiplicatively closed set S generated by $\{c_{nm}\}$ does not contain the zero. Then we get the inductive $\mathcal{F}$–characteristic homomorphism $\chi^{\mathrm{ind}}_{\mathcal{F}}\colon \mathcal{F}^{\mathrm{ind}}(X_\infty) \to \mathcal{R}_P$ defined by*

$$\chi^{\mathrm{ind}}_{\mathcal{F}}([\alpha_n]) := \frac{\chi_{\mathcal{F}}(\alpha_n)}{c_{01} \cdot c_{12} \cdot c_{23} \cdots c_{(n-1)n}}.$$

Here $c_{01} := 1$ and $\mathcal{R}_P$ is the ring $\mathcal{R}_S$ of fractions of $\mathcal{R}$ with respect to S.

3.2.10 Corollary *Let $\{X_n, \pi_{nm}\colon X_m \to X_n\}$ be a pro-algebraic variety such that for each n the structure morphism $\pi_{n(n+1)}\colon X_{n+1} \to X_n$ satisfies the condition that the Euler–Poincaré characteristics e_n of the fibers of $\pi_{n(n+1)}$ are nonzero. We set $e_0 := 1$. Then we get the inductive Euler–Poincaré characteristic homomorphism*

$$\chi^{\mathrm{ind}}\colon F^{\mathrm{ind}}(X_\infty) \to \mathbb{Q} \quad \textit{described by} \quad \chi^{\mathrm{ind}}([\alpha_n]) = \frac{\chi(\alpha_n)}{e_0 \cdot e_1 \cdot e_2 \cdots e_{n-1}}.$$

Proof Let $f\colon X \to Y$ be a morphism such that its fibers all have the same nonzero Euler–Poincaré characteristic, denoted by e_f. Then we can see that for any characteristic function $\mathbb{1}_W$ we have $f_*f^*\mathbb{1}_W = e_f \cdot \mathbb{1}_W$. Hence f is χ_F–constant with the multiplier e_f. □

3.3 A generalization of motivic measure

In this section we show that our approach automatically leads us to the notion of motivic measure (eg, see Craw [23], Denef and Loeser [24; 25], Kontsevich [45], Looijenga [50] and Veys [70]) and also its generalization.

The canonical homomorphism $e\colon K_0(\mathcal{V}/X) \to F(X)$ (see [14]) defined by

$$e([Y \xrightarrow{f} X]) := f_* \mathbb{1}_Y$$

gives us a natural transformation $e\colon K_0(\mathcal{V}/-) \to F(-)$. It will be explained in Section 5 that this natural transformation is unique in a sense.

There exists a canonical homomorphism $\iota\colon F(X) \to K_0(\mathcal{V}/X)$ defined by $\iota(\mathbb{1}_W) := [i_W\colon W \to X]$, where $i_W\colon W \to X$ is the inclusion map. The composite homomorphism $\Gamma := \chi_{\mathrm{Gro}} \circ \iota\colon F(X) \to K_0(\mathcal{V})$ is more directly and simply defined by $\Gamma(\mathbb{1}_W) := [W]$ or more meaningfully $\Gamma(\alpha) = \sum_{n \in \mathbb{Z}} n[\alpha^{-1}(n)]$. And we have the following commutative diagram:

$$\begin{array}{ccc} F(X) & \xrightarrow{\ \Gamma\ } & K_0(\mathcal{V}) \\ & {}_{\chi}\searrow \quad \swarrow_{e} & \\ & \mathbb{Z} & \end{array}$$

3.3.1 Definition Let R be a commutative ring. A map $\mathfrak{e}\colon \mathrm{Obj}(\mathcal{V}) \to R$ is called a *generalized Euler characteristic with value in* R if the following three conditions hold:

(1) $\mathfrak{e}(X) = \mathfrak{e}(Y)$ if $X \cong Y$.

(2) $\mathfrak{e}(X) = \mathfrak{e}(Y) + \mathfrak{e}(X \setminus Y)$ for $Y \subset X$.

(3) $\mathfrak{e}(X \times Y) = \mathfrak{e}(X) \cdot \mathfrak{e}(Y)$.

A typical example of $\mathfrak{e}$ is of course the topological Euler–Poincaré characteristic χ with $R = \mathbb{Z}$ and $\mathfrak{e}$ induces the homomorphism $\mathfrak{e}_F\colon F(X) \to R$ defined simply by $\mathfrak{e}_F(\sum_S a_S \mathbb{1}_S) := \sum_S a_S \mathfrak{e}(S)$. And $\mathfrak{e}_F$ factors through the above "tautological" homomorphism $\Gamma\colon F(X) \to K_0(\mathcal{V})$:

$$\begin{array}{ccc} F(X) & \xrightarrow{\ \Gamma\ } & K_0(\mathcal{V}) \\ & {}_{\mathfrak{e}_F}\searrow \quad \swarrow_{\widetilde{\mathfrak{e}}} & \\ & R & \end{array}$$

where $\widetilde{\mathfrak{e}}\colon K_0(\mathcal{V}) \to \mathcal{R}$ is defined by $\widetilde{\mathfrak{e}}([X]) := \mathfrak{e}(X)$.

So $\Gamma\colon F(X) \to K_0(\mathcal{V})$ is a "motivic" version of the topological Euler–Poincaré characteristic $\chi\colon F(X) \to \mathbb{Z}$ and provisionally called the *Grothendieck class homomorphism*.

3.3.2 Observation Here we emphasize that unlike the Euler–Poincaré characteristic χ, $\Gamma\colon F(X) \to K_0(\mathcal{V})$ *does not* commute with the pushforward $f_*\colon F(X) \to F(Y)$ for a morphism $f\colon X \to Y$, ie, the following diagram is *not* commutative:

$$\begin{array}{ccc} F(X) & \xrightarrow{\quad f_* \quad} & F(Y) \\ & {\scriptstyle\Gamma}\searrow \quad \swarrow{\scriptstyle\Gamma} & \\ & K_0(\mathcal{V}) & \end{array}$$

Let $G = \{\gamma_{ij}\}$ be a projective system of nonnilpotent Grothendieck classes $\gamma_{ij} \in K_0(\mathcal{V})$ indexed by the directed set I. Then in the same way as done before, we can define

$$F^{\mathrm{st}}_G(X_i) := \left\{\alpha_i \in F(X_i) \mid \Gamma({\pi_{ij}}^*\alpha_i) = \gamma_{ij} \cdot \Gamma(\alpha_i) \text{ for any } j > i\right\}.$$

For each $i \in I$, an element of $F^{\mathrm{st}}_G(X_i)$ is called a Γ*–stable constructible function with respect to the projective system G of nonzero Grothendieck classes*. And for a proalgebraic variety $X_\infty = \varprojlim\{X_i, \pi_{ij}\colon X_j \to X_i\}$ we define

$$F^{\mathrm{st.ind}}_G(X_\infty) := \varinjlim \left\{F^{\mathrm{st}}_G(X_i),\ {\pi_{ij}}^*\colon F^{\mathrm{st}}_G(X_i) \to F^{\mathrm{st}}_G(X_j)\ (i < j)\right\}$$

and an element of this group shall be called a Γ*–stable indconstructible function on the proalgebraic variety X_∞ with respect to the projective system G of nonzeros Grothendieck classes*. Then as in Theorem 3.2.5 we get the following:

3.3.3 Corollary (1) *For a proalgebraic variety $X_\infty = \varprojlim\{X_i, \pi_{ij}\colon X_j \to X_i\}$ and a projective system $G = \{\gamma_{ij}\}$ of nonzero Grothendieck classes, we get the proalgebraic Grothendieck class homomorphism*

$$\Gamma^{\mathrm{ind}}\colon F^{\mathrm{st.ind}}_G(X_\infty) \to \varinjlim\{\times\gamma_{ij}\colon K_0(\mathcal{V}) \to K_0(\mathcal{V})\}.$$

(2) *In the case when $\Lambda = \mathbb{N}$, for a pro-algebraic variety $\{X_n, \pi_{nm}\colon X_m \to X_n\}$ and a projective system $G = \{\gamma_{n,m}\}$ of nonzero Grothendieck classes such that the multiplicatively closed set S generated by G does not contain the zero, we have the following canonical proalgebraic Grothendieck class homomorphism*

$$\widetilde{\Gamma^{\mathrm{ind}}}\colon F^{\mathrm{st.ind}}_G(X_\infty) \to K_0(\mathcal{V})_G \quad \text{defined by} \quad \widetilde{\Gamma^{\mathrm{ind}}}([\alpha_n]) := \frac{\Gamma(\alpha_n)}{\gamma_{01} \cdot \gamma_{12} \cdot \gamma_{23} \cdots \gamma_{(n-1)n}}.$$

Here we set $\gamma_{01} := \mathbb{1}$ and $K_0(\mathcal{V})_G$ is the ring of fractions of $K_0(\mathcal{V})$ with respect to S.

(3) *Let* $X_\infty = \varprojlim\{X_n, \pi_{nm}\colon X_m \to X_n\}$ *be a proalgebraic variety such that each structure morphism* $\pi_{n(n+1)}\colon X_{n+1} \to X_n$ *satisfies the condition that for each* n *there exists a nonnilpotent element* $\gamma_n \in K_0(\mathcal{V})$ *such that* $\pi_{n(n+1)}{}^{-1}(S_n) = \gamma_n \cdot [S_n]$ *for any constructible set* $S_n \subset X_n$*; for example,* $\pi_{n(n+1)}\colon X_{n+1} \to X_n$ *is a Zariski locally trivial fiber bundle with fiber variety being* F_n *(in which case* $\gamma_n = [F_n] \in K_0(\mathcal{V})$*). We assume that the multiplicatively closed set* S *generated by* $\{\gamma_m\}$ *does not contain the zero. Then we have the canonical proalgebraic Grothendieck class homomorphism*

$$\Gamma^{\mathrm{ind}}\colon F^{\mathrm{ind}}(X_\infty) \to K_0(\mathcal{V})_G \quad \textit{described by} \quad \Gamma^{\mathrm{ind}}([\alpha_n]) = \frac{\Gamma(\alpha_n)}{\gamma_0 \cdot \gamma_1 \cdot \gamma_2 \cdots \gamma_{n-1}}.$$

Here $\gamma_0 := \mathbb{1}$ *and* $K_0(\mathcal{V})_G$ *is the ring of fractions of* $K_0(\mathcal{V})$ *with respect to* S.

(4) *In particular, if* γ_n *are all the same and nonnilpotent, say* $\gamma_n = \gamma$ *for any* n*, then we have the canonical proalgebraic Grothendieck class homomorphism*

$$\Gamma^{\mathrm{ind}}\colon F^{\mathrm{ind}}(X_\infty) \to K_0(\mathcal{V})_G \quad \textit{described by} \quad \Gamma^{\mathrm{ind}}([\alpha_n]) = \frac{\Gamma(\alpha_n)}{\gamma^{n-1}}.$$

In this special case the quotient ring $K_0(\mathcal{V})_G$ *shall be denoted by* $K_0(\mathcal{V})_\gamma$.

3.3.4 Example (1) The arc space $\mathcal{L}(X)$ of an algebraic variety X is defined to be the projective limit of the projective system consisting of truncated arc varieties $\mathcal{L}_n(X)$ and projections $\pi_{n(n+1)}\colon \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$. Thus the arc space is a nontrivial example of a proalgebraic variety. If X is nonsingular and of complex dimension d, then the projection $\pi_{n(n+1)}\colon \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ is a Zariski locally trivial fiber bundle with fiber being $\mathbb{C}^d$. Thus in this case, in Corollary 3.3.3(4) the Grothendieck class γ is $\mathbb{L}^d$.

(2) In the case of the arc space $\mathcal{L}(X)$ of a nonsingular variety X, each structure morphism $\pi_{n(n+1)}\colon \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ is always surjective, hence it follows from Proposition 2.3.4 we get that for the arc space $\mathcal{L}(X)$ of a nonsingular variety X we have the canonical isomorphism: $F^{\mathrm{ind}}(\mathcal{L}(X)) \cong F^{\mathrm{cyl}}(\mathcal{L}(X))$.

3.3.5 Corollary *When* X *is a nonsingular variety of dimension* d*, we have the following canonical Grothendieck class homomorphism* $\Gamma^{\mathrm{ind}}\colon F^{\mathrm{ind}}(\mathcal{L}(X)) \to K_0(\mathcal{V})_{[\mathbb{L}^d]}$ *described by* $\Gamma^{\mathrm{ind}}([\alpha_n]) = \Gamma(\alpha_n)/[\mathbb{L}]^{nd}$*. In particular,* $\Gamma^{\mathrm{ind}}(\mathbb{1}_{\mathcal{L}(X)}) = \Gamma^{\mathrm{ind}}([\mathbb{1}_X]) = [X]$.

3.3.6 Remark Note that in the case of arc space $\mathcal{L}(X)$, since $\mathcal{L}_0(X) = X$, the indexed set is not $\mathbb{N}$ but $\{0\} \cup \mathbb{N}$. Hence the canonical one is not $\Gamma^{\mathrm{ind}}([\alpha_n]_\infty) = \Gamma(\alpha_n)/[\mathbb{L}]^{(n-1)d}$.

3.3.7 Remark If X is singular, the arc space $\mathcal{L}(X)$ is *not* the projective limit of a projective system of Zariski locally trivial fiber bundles with fiber being $\mathbb{C}^{\dim X}$ any longer and each projection morphism $\pi_{n(n+1)}\colon \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ is complicated and thus as a proalgebraic variety $\mathcal{L}(X)$ is complicated. A crucial ingredient in studying motivic measure or motivic integration is the so-called *stable set* of the arc space $\mathcal{L}(X)$. A subset A of the arc space $\mathcal{L}(X)$ is called a *stable set* if it is a cylinder set, ie, $A = \pi_n^{-1}(C_n)$ for a constructible set C_n in the n–th arc space $\mathcal{L}_n(X)$, such that the restriction of each projection $\pi_{m(m+1)}|_{\pi_{m+1}(A)}\colon \pi_{m+1}(A) \to \pi_m(A)$ for each $m \geq n$ is a Zariski locally trivial fiber bundle with the fiber being $\mathbb{C}^{\dim X}$. So, our Γ–stable indconstructible function is a generalization of the characteristic function supported on this stable set. Therefore we can see that our proalgebraic Grothendieck class homomorphism $\widetilde{\Gamma^{\mathrm{ind}}}\colon F_G^{\mathrm{st.ind}}(X_\infty) \to K_0(\mathcal{V})_G$ given in Corollary 3.3.3 (2) is a generalization of motivic measure.

4 Characteristic classes of proalgebraic varieties

4.1 Pre-Mackey functor

Let $\{f_i\colon X_i \to Y_i\}$ be a pro-morphism of pro-objects $\{X_i, \pi_{ij}\colon X_j \to X_i\}$ and $\{Y_i, \rho_{ij}\colon Y_j \to Y_i\}$. Then it follows from the contravariance of the bifunctor $\mathcal{F}$ that the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(Y_i) & \xrightarrow{f_i^*} & \mathcal{F}(X_i) \\ {\scriptstyle \rho_{ij}{}^*}\downarrow & & \downarrow{\scriptstyle \pi_{ij}{}^*} \\ \mathcal{F}(Y_j) & \xrightarrow{f_j^*} & \mathcal{F}(X_j), \end{array}$$

which in turn implies that the pullback homomorphism $f_\infty^* := \varinjlim\{f_i{}^*\}\colon \mathcal{F}^{\mathrm{ind}}(Y_\infty) \to \mathcal{F}^{\mathrm{ind}}(X_\infty)$ is contravariantly functorial. However, to claim the covariance of $\mathcal{F}^{\mathrm{ind}}$, we need the following requirements; one for the bifunctor $\mathcal{F}$ and one for the pro-morphism $\{f_i\colon Y_i \to X_i\}_{i\in I}$:

4.1.1 Definition If a bifunctor $\mathcal{F}\colon \mathcal{V} \to \mathcal{A}$ satisfies the following two properties (M-1) and (M-2), then it is called a *Mackey functor*:

(M-1) For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

the diagram

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{g'^*} & \mathcal{F}(X') \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f'_*} \\ \mathcal{F}(Y) & \xrightarrow{g^*} & \mathcal{F}(Y'), \end{array}$$

commutes, ie, "pullback and pushforward commute".

(M-2) $\mathcal{F}(X \coprod Y) = \mathcal{F}(X) \oplus \mathcal{F}(Y)$.

4.1.2 Example The constructible function functor $F(X)$ and the relative Grothendieck group functor $K_0(\mathcal{V}/X)$ are both Mackey functors.

4.1.3 Remark The notion of Mackey functor was introduced by Dress [26; 27] (cf Bouc [9]) in the representation theory of finite groups. In what follows, the property we need is just the property (M-1), which is sometimes called the *base change formula*. A bifunctor satisfying only (M-1) is called a *pre-Mackey functor*.

Let $\mathcal{F}, \mathcal{G}\colon \mathcal{V} \to \mathcal{A}$ be two pre-Mackey functors, and let $\mathcal{N}\colon \mathcal{F} \to \mathcal{G}$ be a natural transformation, ie, for a morphism $f\colon X \to Y$ the following diagrams commute:

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\mathcal{N}_X} & \mathcal{G}(X) \\ {\scriptstyle \mathcal{F}_*(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{G}_*(f)} \\ \mathcal{F}(Y) & \xrightarrow[\mathcal{N}_Y]{} & \mathcal{G}(Y) \end{array} \qquad \begin{array}{ccc} \mathcal{F}(Y) & \xrightarrow{\mathcal{N}_Y} & \mathcal{G}(Y) \\ {\scriptstyle \mathcal{F}^*(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{G}^*(f)} \\ \mathcal{F}(X) & \xrightarrow[\mathcal{N}_X]{} & \mathcal{G}(X). \end{array}$$

From now on, unless some confusion is possible, we just denote f_* for both $\mathcal{F}_*(f)$ and $\mathcal{G}_*(f)$, f^* for both $\mathcal{F}^*(f)$ and $\mathcal{G}^*(f)$, and $\mathcal{N}$ for $\mathcal{N}_X, \mathcal{N}_Y$ without subscripts.

4.1.4 Definition Let $\{f_i\colon X_i \to Y_i\}_{i \in I}$ be a pro-morphism of pro-objects

$$\{X_i, \pi_{ij}\colon X_j \to X_i\} \quad \text{and} \quad \{Y_i, \rho_{ij}\colon Y_j \to Y_i\}.$$

If the commutative diagram for $i < j$

$$\begin{array}{ccc} X_j & \xrightarrow{f_j} & Y_j \\ {\scriptstyle \pi_{ij}}\downarrow & & \downarrow{\scriptstyle \rho_{ij}} \\ X_i & \xrightarrow[f_i]{} & Y_i \end{array}$$

is a fiber square, then we call the pro-morphism $\{f_i\colon X_i \to Y_i\}_{i \in I}$ a *fiber-square pro-morphism*, abusing words.

4.1.5 Theorem (1) *Let $\mathcal{F}\colon \mathcal{V} \to \mathcal{A}$ be a pre-Mackey functor and let $\mathcal{F}(\mathrm{pt}) = \mathcal{R}$ be an R–module with a commutative ring R with a unit. Then for a projective system $P = \{p_{ij}\}$ of nonzero elements p_{ij} and for a fiber-square pro-morphism $\{f_i\colon X_i \to Y_i\}_{i\in I}$ of pro-objects $\{X_i, \pi_{ij}\colon X_j \to X_i\}$ and $\{Y_i, \rho_{ij}\colon Y_j \to Y_i\}$, the pushforward homomorphisms*

$$f_{\infty *} := \varinjlim\{f_{i*}\}\colon \mathcal{F}^{\mathrm{ind}}(X_\infty) \to \mathcal{F}^{\mathrm{ind}}(Y_\infty)$$

and

$$f_{\infty *} := \varinjlim\{f_{i*}\}\colon \mathcal{F}_P^{\mathrm{st.ind}}(X_\infty) \to \mathcal{F}_P^{\mathrm{st.ind}}(Y_\infty)$$

are covariantly functorial.

(2) *Let $\mathcal{F}, \mathcal{G}\colon \mathcal{V} \to \mathcal{A}$ be two pre-Mackey functors and $\mathcal{N}\colon \mathcal{F} \to \mathcal{G}$ be a natural transformation. For a projective system $P = \{p_{ij}\}$ of nonzero elements p_{ij} of R and a fiber-square pro-morphism $\{f_i\colon X_i \to Y_i\}$ of pro-objects $\{X_i, \pi_{ij}\colon X_j \to X_i\}$ and $\{Y_i, \rho_{ij}\colon Y_j \to Y_i\}$, then*

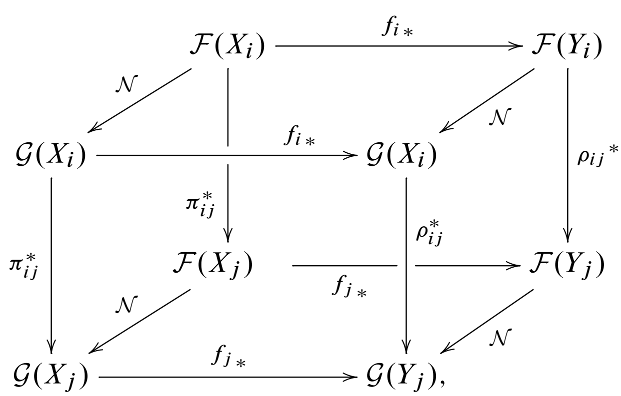
$$\mathcal{N}^{\mathrm{ind}}\colon \mathcal{F}^{\mathrm{ind}}(X_\infty) \to \mathcal{G}^{\mathrm{ind}}(X_\infty) \quad \text{and} \quad \mathcal{N}^{\mathrm{ind}}\colon \mathcal{F}_P^{\mathrm{st.ind}}(X_\infty) \to \mathcal{G}_P^{\mathrm{st.ind}}(X_\infty)$$

are natural transformations.

(3) *Furthermore we suppose that $\nu = \mathcal{N}(\mathrm{pt})\colon \mathcal{R} = \mathcal{F}(\mathrm{pt}) \to \mathcal{R}' = \mathcal{G}(\mathrm{pt})$ is an R–module homomorphism. Then we have the following commutative diagram:*

$$\begin{array}{ccc}
\mathcal{F}^{\mathrm{ind}}(Y_\infty) & \xrightarrow{\ \mathcal{N}^{\mathrm{ind}}\ } & \mathcal{G}^{\mathrm{ind}}(Y_\infty) \\
{\scriptstyle \chi_{\mathcal{F}}^{\mathrm{ind}}}\downarrow & & \downarrow{\scriptstyle \chi_{\mathcal{G}}^{\mathrm{ind}}} \\
\varinjlim\{\times p_{ij}\colon \mathcal{R} \to \mathcal{R}\} & \xrightarrow[\ \nu^{\mathrm{ind}}\ ]{} & \varinjlim\{\times p_{ij}\colon \mathcal{R}' \to \mathcal{R}'\}.
\end{array}$$

Proof It suffices to see that for a fiber-square pro-morphism $\{f_i\colon X_i \to Y_i\}$ of pro-objects $\{X_i, \pi_{ij}\colon X_j \to X_i\}$ and $\{Y_i, \rho_{ij}\colon Y_j \to Y_i\}$, we get the commutative cubic diagram



which completes the proof. □

4.1.6 Remark As seen in the above proof, we do not need to require that $\mathcal{F}$ and $\mathcal{G}$ are pre-Mackey on the whole category $\mathcal{V}$, but we just need to require that $\mathcal{F}$ and $\mathcal{G}$ satisfy the base change formula (M-1) for a fiber-square of each structure morphism of projective systems. To say this property, we simply say that $\mathcal{F}$ and $\mathcal{G}$ are *pre-Mackey functors on projective systems* (cf [79]).

4.2 Fulton–MacPherson bivariant theory

Since we are interested in characteristic homology classes of singular varieties, the target functor is mainly the homology theory H_*. Unfortunately the homology theory is not a pre-Mackey functor on an arbitrary fiber square, but still a pre-Mackey functor on some restricted fiber squares. In order to deal with such pre-Mackey functors in a general setup we use Fulton–MacPherson Bivariant Theory [34] (cf [33]). So, first we quickly recall only necessary ingredients of the Bivariant Theory. For full details, see [34, Section 2].

Let $\mathcal{C}$ be a category with a final object pt, a class of "independent squares" and a class of "confined maps", which is closed under composition and base change in independent squares and contains all identity maps. For example, in the category of topological spaces, a fiber square is an independent square and a proper map is a confined map.

A bivariant theory $\mathbb{B}$ on such a category $\mathcal{C}$ with values in the category of abelian groups is an assignment to each morphism $f\colon X \to Y$ in the category $\mathcal{C}$ an abelian group $\mathbb{B}(f\colon X \to Y)$ which is equipped with the following three basic operations:

Products: For morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, the product operation

$$\bullet\colon \mathbb{B}(X \xrightarrow{f} Y) \otimes \mathbb{B}(Y \xrightarrow{g} Z) \to \mathbb{B}(X \xrightarrow{gf} Z),$$

Pushforwards: For morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ with f *confined*, the pushforward operation $f_\star\colon \mathbb{B}(gf\colon X \to Z) \to \mathbb{B}(g\colon Y \to Z)$,

Pullbacks: For an *independent* square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

the pullback operation $g^\star\colon \mathbb{B}(f\colon X \to Y) \to \mathbb{B}(f'\colon X' \to Y')$.

And these three operations are required to satisfy the seven compatibility axioms [34, Part I, Section 2.2].

A bivariant theory $\mathbb{B}$ is said to *have units* [34, Section 2.2] if there exists an element $1_X \in \mathbb{B}(\mathrm{id}_X\colon X \to X)$ such that

- $\alpha \bullet 1_X = \alpha$ for all maps $W \to X$ and all $\alpha \in \mathbb{B}(W \to X)$,
- $1_X \bullet \beta = \beta$ for all maps $X \to Y$ and all $\beta \in \mathbb{B}(X \to Y)$,
- $g^\star 1_X = 1_{X'}$ for all $g\colon X' \to X$.

Let $\mathbb{B}, \mathbb{B}'$ be two bivariant theories on a category $\mathcal{C}$. Then a *Grothendieck transformation* $\gamma\colon \mathbb{B} \to \mathbb{B}'$ from $\mathbb{B}$ to $\mathbb{B}'$ is a collection of homomorphisms $\mathbb{B}(X \to Y) \to \mathbb{B}'(X \to Y)$ for a morphism $X \to Y$ in the category $\mathcal{C}$, which preserve the above three basic operations:

(1) $\gamma(\alpha \bullet_{\mathbb{B}} \beta) = \gamma(\alpha) \bullet_{\mathbb{B}'} \gamma(\beta)$,

(2) $\gamma(f_\star \alpha) = f_\star \gamma(\alpha)$,

(3) $\gamma(g^\star \alpha) = g^\star \gamma(\alpha)$.

A bivariant theory unifies both a covariant theory and a contravariant theory in the following sense: $\mathbb{B}_*(X) := \mathbb{B}(X \to \mathrm{pt})$ and $\mathbb{B}^*(X) := \mathbb{B}(\mathrm{id}\colon X \to X)$ become a covariant functor and a contravariant functor, respectively. And a Grothendieck transformation $\gamma\colon \mathbb{B} \to \mathbb{B}'$ induces natural transformations $\gamma_*\colon \mathbb{B}_* \to \mathbb{B}'_*$ and $\gamma^*\colon \mathbb{B}^* \to \mathbb{B}'^*$.

4.2.1 Lemma *Let $\gamma\colon \mathbb{B} \to \mathbb{B}'$ be a Grothendieck transformation $\gamma\colon \mathbb{B} \to \mathbb{B}'$. Then any bivariant class $b \in \mathbb{B}(f\colon X \to Y)$ gives rise to the following commutative diagram*

$$\begin{array}{ccc} \mathbb{B}_*(Y) & \xrightarrow{\gamma_*} & \mathbb{B}'_*(Y) \\ {\scriptstyle b\bullet}\downarrow & & \downarrow{\scriptstyle \gamma(b)\bullet} \\ \mathbb{B}_*(X) & \xrightarrow[\gamma_*]{} & \mathbb{B}'_*(X). \end{array}$$

It is called a Verdier-type Riemann–Roch formula associated to the bivariant class b.

4.2.2 Example The Fulton–MacPherson bivariant group $\mathbb{F}(f\colon X \to Y)$ of constructible functions consists of all the constructible functions on X which satisfy the local Euler condition with respect to f. Here a constructible function $\alpha \in F(X)$ is said to satisfy the *local Euler condition with respect to* f if for any point $x \in X$ and for any local embedding $(X, x) \to (\mathbf{C}^N, 0)$ the equality $\alpha(x) = \chi(B_\epsilon \cap f^{-1}(z); \alpha)$ holds, where B_ϵ is a sufficiently small open ball of the origin 0 with radius ϵ and z is any point close to $f(x)$ (cf Brasselet [10] and Sabbah [59]). In particular, if $\mathbb{1}_f := \mathbb{1}_X$ belongs to the bivariant group $\mathbb{F}(f\colon X \to Y)$, then the morphism $f\colon X \to Y$ is called an *Euler morphism*. And any constructible function in the bivariant group $\mathbb{F}(f\colon X \to Y)$ is called a *bivariant constructible function*.

The three operations on $\mathbb{F}$ are defined as follows:

(1) The product $\bullet$: $\mathbb{F}(f\colon X \to Y) \otimes \mathbb{F}(g\colon Y \to Z) \to \mathbb{F}(gf\colon X \to Z)$ is defined by $\alpha \bullet \beta = \alpha \cdot f^*\beta$,

(2) The pushforward $f_\star\colon \mathbb{F}(gf\colon X \to Z) \to \mathbb{F}(g\colon Y \to Z)$ is defined by $f_\star(\alpha)(y) = \int c_*(\alpha|_{f^{-1}})$,

(3) For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

the pullback $g^\star\colon \mathbb{F}(f\colon X \to Y) \to \mathbb{F}(f'\colon X' \to Y')$ is the functional pullback g'^*, ie, $g^\star(\alpha)(x') := \alpha(g'(x'))$.

4.2.3 Remark The group $\mathbb{F}(\mathrm{id}_X\colon X \to X)$ consists of all locally constant functions and $\mathbb{F}(X \to \mathrm{pt}) = F(X)$.

4.2.4 Proposition *For any bivariant constructible function $\alpha \in \mathbb{F}(f\colon X \to Y)$, the Euler–Poincaré characteristic $\chi(f^{-1}(y);\alpha) = \int c_*(\alpha|_{f^{-1}(y)})$ is locally constant, ie, constant along connected components of the base variety Y. In particular, if $f\colon X \to Y$ is an Euler proper morphism, then the Euler–Poincaré characteristic of the fibers are locally constant.*

4.2.5 Corollary *Let $\{X_i, \pi_{ij}\colon X_j \to X_i\}$ be a pro-algebraic variety such that for each $i < j$ the structure morphism $\pi_{ij}\colon X_j \to X_i$ is an Euler proper morphism (hence surjective) of topologically connected algebraic varieties. Let e_{ij} be the constant Euler–Poincaré characteristic e_{ij} of the fiber of the morphism π_{ij}. Then we get the inductive Euler–Poincaré characteristic homomorphism $\chi^{\mathrm{ind}}\colon F^{\mathrm{ind}}(X_\infty) \to \varinjlim\{\times e_{ij}\colon \mathbb{Z} \to \mathbb{Z}\}$.*

4.3 Canonically oriented projective systems

4.3.1 Definition (A canonically oriented projective system (cf [34])) Let $\mathbb{B}$ be a bivariant theory having units and let $\{X_i, \pi_{ij}\colon X_j \to X_i\}$ be a projective system. If there exists an assignment $\Theta(\pi_{ij}) \in \mathbb{B}(\pi_{ij}\colon X_j \to X_i)$ such that

$$\Theta(\pi_{ii}) = 1_{X_i} \quad \text{and} \quad \Theta(\pi_{jk}) \bullet \Theta(\pi_{ij}) = \Theta(\pi_{ik}) \ (i < j < k),$$

then we say that Θ is a canonical orientation on the projective system $\{X_i\}$ and that $\{X_i, \pi_{ij}\colon X_j \to X_i, \Theta\}$ is a *canonically oriented projective system.* We set $\Theta_{ij} := \Theta(\pi_{ij})$ and the map $\Theta_{ij}\bullet\colon \mathbb{B}_*(X_i) \to \mathbb{B}_*(X_j)$ is denoted by $\pi_{ij}^!$, called the Gysin map induced by the canonical orientation Θ_{ij}.

4.3.2 Remark For the sake of convenience of presentation below, we sometimes write $\{X_i, \pi_{ij}\colon X_j \to X_i, \{\Theta_{ij}\}\}$ instead of $\{X_i, \pi_{ij}\colon X_j \to X_i, \Theta\}$.

4.3.3 Definition If $\{X_i, \pi_{ij}\colon X_j \to X_i, \Theta\}$ is a canonically oriented projective system, then the inductive limit $\varinjlim\{\mathbb{B}_*(X_i), \pi_{ij}^!\colon \mathbb{B}_*(X_i) \to \mathbb{B}_*(X_j)\}$ is denoted by $\mathbb{B}_*^{\mathrm{ind}}(X_\infty; \Theta)$.

4.3.4 Example In Corollary 4.2.5 we have that $F^{\mathrm{ind}}(X_\infty) = \mathbb{F}_*^{\mathrm{ind}}(X_\infty; \{\mathbb{1}_{\pi_{ij}}\})$.y

4.3.5 Proposition *Let $\{X_n, \pi_{n(n+1)}\colon X_{n+1} \to X_n, \Theta\}$ be a canonically oriented projective system of topologically connected algebraic varieties with $\Theta_{n(n+1)} \in \mathbb{F}(X_{n+1} \to X_n)$. Assume that the (constant) Euler–Poincaré characteristic of $\Theta_{n(n+1)}$ restricted to each fiber ${\pi_{n(n+1)}}^{-1}(y)$, ie, $\chi({\pi_{n(n+1)}}^{-1}(y); \Theta_{n(n+1)})$ is nonzero and it shall be denoted by $e_f(\Theta_{n(n+1)})$. And we set $e_f(\Theta_{01}) := 1$. Then the canonical inductive Euler–Poincaré characteristic homomorphism $\chi^{\mathrm{ind}}\colon \mathbb{F}_*^{\mathrm{ind}}(X_\infty; \Theta\}) \to \mathbb{Q}$ is described by*

$$\chi^{\mathrm{ind}}([\alpha_n]) = \frac{\chi(\alpha_n)}{e_f(\Theta_{01}) \cdot e_f(\Theta_{12}) \cdots e_f(\Theta_{(n-1)n})}.$$

Proof This can be seen as follows. Let $(f, \alpha)\colon X \to Y$ be a morphism of topologically connected algebraic varieties with $\alpha \in \mathbb{F}(f\colon X \to Y)$. It follows from Proposition 4.2.4 that the Euler–Poincaré characteristic $\chi(f^{-1}(y); \alpha)$ of α restricted to each fiber $f^{-1}(y)$ is constant (and nonzero by assumption). So $f_*\alpha = e_f(\alpha) \cdot \mathbb{1}_Y$. Then to prove the above statement, it suffices to see that we have the commutative diagram

$$\begin{array}{ccc} F(Y) & \xrightarrow{\chi} & \mathbb{Z} \\ {\scriptstyle \alpha\bullet}\downarrow & & \downarrow{\scriptstyle \times e_f(\alpha)} \\ F(X) & \xrightarrow[\chi]{} & \mathbb{Z}. \end{array}$$

To see this, we need the *projection formula* $f_*(\alpha \cdot f^*\beta) = (f_*\alpha) \cdot \beta$ for a morphism $f\colon X \to Y$ and constructible functions $\alpha \in F(X)$ and $\beta \in F(Y)$. Then, using this projection formula we have $\chi(\alpha \bullet \beta) = \chi(\alpha \cdot f^*\beta) = \chi(f_*\alpha \cdot \beta) = \chi\big((e_f(\alpha) \cdot \mathbb{1}_Y) \cdot \beta\big) = e_f(\alpha) \cdot \chi(\beta)$. Thus we get the above commutative diagram. □

Mimicking the above proof, we can show the following:

4.3.6 Corollary *Let $\{X_i, \pi_{ij}\colon X_j \to X_i, \{\Theta_{ij}\}\}$ be a canonically oriented projective system. Assume that $\mathbb{B}_*(\mathrm{pt})$ is a commutative ring with a unit, denoted by $\mathcal{R}^{\mathbb{B}}$, and let $P = \{p_{ij}\}$ be a projective system of nonzero elements $p_{ij} \in \mathcal{R}^{\mathbb{B}}$. Then, if we set*

$$\mathbb{B}_{*,P}^{\mathrm{st.ind}}(X_\infty; \{\Theta_{ij}\}) = \big\{[\alpha_\lambda] \in \mathbb{B}_*^{\mathrm{ind}}(X_\infty; \{\Theta_{ij}\}) \,|\, \chi_{\mathbb{B}_*}(\Theta_{ij} \bullet \alpha_i) = p_{ij} \cdot \chi_{\mathbb{B}_*}(\alpha_i)\ (i < j)\big\}$$

we get the inductive $\chi_{\mathbb{B}_*}$*–characteristic homomorphism*

$$\chi^{\mathrm{ind}}_{\mathbb{B}_*}\colon \mathbb{B}^{\mathrm{st.ind}}_{*,P}(X_\infty;\{\Theta_{ij}\}) \to \varinjlim\{\times p_{ij}\colon \mathcal{R}^{\mathbb{B}} \to \mathcal{R}^{\mathbb{B}}\}.$$

4.4 Characteristic classes of projective systems

Fulton and MacPherson conjectured the existence of a bivariant Chern class [34] and Brasselet [10] found it (cf Sabbah [59], Yokura [76; 77] and Zhou [83; 84]):

4.4.1 Theorem *On the category of embeddable complex analytic varieties and cellular morphisms, there exists a Grothendieck transformation* $\gamma^{\mathrm{Br}}\colon \mathbb{F} \to \mathbb{H}$ *satisfying the normalization condition that* $\gamma^{\mathrm{Br}}(\mathbb{1}_\pi) = c(TX)\cap[X]$ *for* X *smooth, where* $\pi\colon X \to \mathrm{pt}$ *and* $\mathbb{1}_\pi = \mathbb{1}_X$*.*

4.4.2 Corollary *For a bivariant constructible function* $\alpha \in \mathbb{F}(f\colon X \to Y)$ *we have the commutative diagram*

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y) \\ {\scriptstyle \alpha\bullet_{\mathbb{F}}=\alpha\cdot f^*}\big\downarrow & & \big\downarrow{\scriptstyle \gamma^{\mathrm{Br}}(\alpha)\bullet_{\mathbb{H}}} \\ F(X) & \xrightarrow[c_*]{} & H_*(X). \end{array}$$

In particular, for an Euler morphism we have the commutative diagram

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y) \\ {\scriptstyle \mathbb{1}_f\bullet_{\mathbb{F}}=f^*}\big\downarrow & & \big\downarrow{\scriptstyle \gamma^{\mathrm{Br}}(\mathbb{1}_f)\bullet_{\mathbb{H}}} \\ F(X) & \xrightarrow[c_*]{} & H_*(X). \end{array}$$

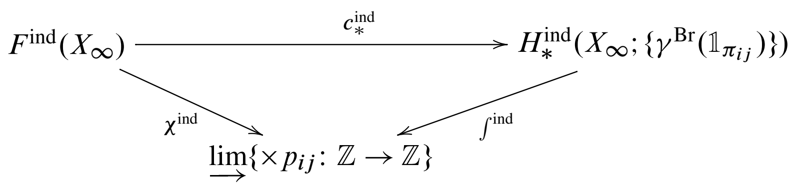
The homomorphism $\gamma^{\mathrm{Br}}(\mathbb{1}_f)\bullet_{\mathbb{H}}$ shall be denoted by f^{**}. Using Corollary 4.4.2, we get the following:

4.4.3 Theorem *Let* $\mathcal{V}$ *be the category of embeddable complex analytic varieties and cellular morphisms.*

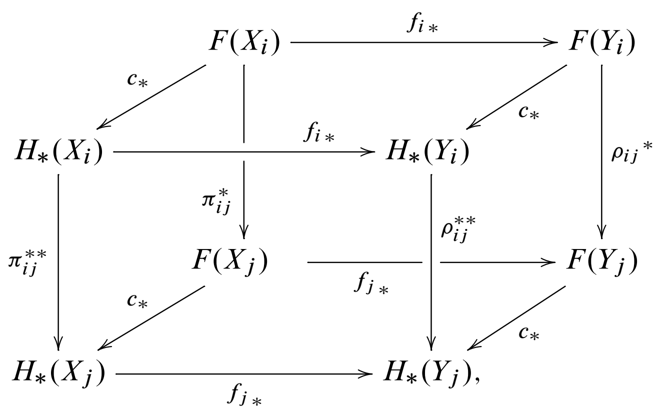
(1) *Let* $\{f_i\colon X_i \to Y_i\}$ *be a fiber-square pro-morphism between two pro-algebraic varieties* $\{X_i, \pi_{ij}\colon X_j \to X_i\}$ *and* $\{Y_i, \rho_{ij}\colon Y_j \to Y_i\}$ *with structure morphisms being Euler morphisms. Then we have the commutative diagram*

$$\begin{array}{ccc} F^{\mathrm{ind}}(X_\infty) & \xrightarrow{c_*^{\mathrm{ind}}} & H_*^{\mathrm{ind}}(X_\infty;\{\gamma^{\mathrm{Br}}(\mathbb{1}_{\pi_{ij}})\}) \\ {\scriptstyle f_{\infty *}}\big\downarrow & & \big\downarrow{\scriptstyle f_{\infty *}} \\ F^{\mathrm{ind}}(Y_\infty) & \xrightarrow{c_*^{\mathrm{ind}}} & H_*^{\mathrm{ind}}(Y_\infty;\{\gamma^{\mathrm{Br}}(\mathbb{1}_{\rho_{ij}})\}). \end{array}$$

(2) *Let* $X_\infty = \varprojlim\{X_i, \pi_{ij}\colon X_j \to X_i\}$ *be a proalgebraic variety such that for each* $i < j$ *the structure morphism* $\pi_{ij}\colon X_j \to X_i$ *is an Euler proper morphism (hence surjective) of topologically connected algebraic varieties with the constant Euler–Poincaré characteristic* p_{ij} *of the fiber of the morphism* π_{ij} *being nonzero. Then we get the commutative diagram:*



Proof It suffices to show the commutativity of the square in the front of the diagram



ie, for any $x \in H_*(X_i)$, $\rho_{ij}^{**}(f_{i*}(x)) = f_{j*}(\pi_{ij}^{**}(x))$, which is more precisely

$$\gamma^{\mathrm{Br}}(\mathbb{1}_{\rho_{ij}}) \bullet_{\mathbb{H}} f_{i\star}(x) = f_{j\star}(\gamma^{\mathrm{Br}}(\mathbb{1}_{\pi_{ij}}) \bullet_{\mathbb{H}} x) = f_{j\star}(f_i^\star \gamma^{\mathrm{Br}}(\mathbb{1}_{\rho_{ij}}) \bullet_{\mathbb{H}} x),$$

since $\mathbb{1}_{\pi_{ij}} = f_i^\star \mathbb{1}_{\rho_{ij}}$. This is nothing but the *projection formula* of the Bivariant Theory [34, Section 2.2, (A_{123})]. Thus we get the theorem. □

Following the above construction, similarly we can get an inductive limit version of Baum, Fulton and MacPherson's Riemann–Roch $\tau_*\colon G_0(-) \to H_*(-) \otimes \mathbb{Q}$, using the bivariant Riemann–Roch theorem [33; 34]. And much more general is the following theorem. Below, if each fiber square in a fiber-square pro-morphism is replaced by an independent square, then we call it an *independent-square pro-morphism*.

4.4.4 Theorem *Let* $\gamma\colon \mathbb{B} \to \mathbb{B}'$ *be a Grothendieck transformation between two bivariant theories* $\mathbb{B}, \mathbb{B}'\colon \mathcal{C} \to \mathcal{A}$ *and let* $\gamma_*\colon \mathcal{R} = \mathbb{B}_*(\mathrm{pt}) \to \mathcal{R}' = \mathbb{B}'_*(\mathrm{pt})$ *be an* R*–module homomorphism with a commutative ring* R *with a unit. Let* $P = \{p_{ij}\}$ *be a projective system of nonzero elements* $p_{ij} \in R$*.*

(1) *Let* $\{X_i, \pi_{ij}\colon X_j \to X_i, \{\Theta_{ij}\}\}$ *be a canonically oriented projective system. Then we get the following inductive limit version of the natural transformation* $\gamma_*\colon \mathbb{B}_* \to \mathbb{B}'_*$*:*

$$\gamma_*^{\mathrm{ind}}\colon \mathbb{B}_*^{\mathrm{ind}}(X_\infty; \{\Theta_{ij}\}) \to \mathbb{B}'^{\,\mathrm{ind}}_*(X_\infty; \{\gamma(\Theta_{ij})\}),$$
$$\gamma_*^{\mathrm{ind}}\colon \mathbb{B}_{*,P}^{\mathrm{st.ind}}(X_\infty; \{\Theta_{ij}\}) \to \mathbb{B}'^{\,\mathrm{st.ind}}_{*,P}(X_\infty; \{\gamma(\Theta_{ij})\}).$$

(2) *Let* $\{f_i\colon X_i \to Y_i\}$ *be an independent-square pro-morphism between two oriented projective systems* $\{X_i, \pi_{ij}\colon X_j \to X_i, \{\Theta_{ij}\}\}$ *and* $\{Y_i, \rho_{ij}\colon Y_j \to Y_i, \{\Theta'_{ij}\}\}$ *such that* $\Theta_{ij} = f_i^\star \Theta'_{ij}$*. Then we have the following commutative diagrams:*

$$\begin{array}{ccc} \mathbb{B}_*^{\mathrm{ind}}(X_\infty; \{\Theta_{ij}\}) & \xrightarrow{\gamma_*^{\mathrm{ind}}} & \mathbb{B}'^{\mathrm{ind}}_*(X_\infty; \{\gamma(\Theta_{ij})\}) \\ {\scriptstyle f_{\infty *}}\downarrow & & \downarrow{\scriptstyle f_{\infty *}} \\ \mathbb{B}_*^{\mathrm{ind}}(Y_\infty; \{\Theta'_{ij}\}) & \xrightarrow[\gamma_*^{\mathrm{ind}}]{} & \mathbb{B}'^{\mathrm{ind}}_*(Y_\infty; \{\gamma(\Theta'_{ij})\}), \end{array}$$

$$\begin{array}{ccc} \mathbb{B}_{*,P}^{\mathrm{st.ind}}(X_\infty; \{\Theta_{ij}\}) & \xrightarrow{\gamma_*^{\mathrm{ind}}} & \mathbb{B}'^{\,\mathrm{st.ind}}_{*,P}(X_\infty; \{\gamma(\Theta_{ij})\}) \\ {\scriptstyle f_{\infty *}}\downarrow & & \downarrow{\scriptstyle f_{\infty *}} \\ \mathbb{B}_{*,P}^{\mathrm{st.ind}}(Y_\infty; \{\Theta'_{ij}\}) & \xrightarrow[\gamma_*^{\mathrm{ind}}]{} & \mathbb{B}'^{\,\mathrm{st.ind}}_{*,P}(Y_\infty; \{\gamma(\Theta'_{ij})\}). \end{array}$$

(3) *Let* $\mathbb{B}_*(\mathrm{pt}) = \mathbb{B}'_*(\mathrm{pt})$ *be a commutative ring* $\mathcal{R}$ *with a unit and we assume that the homomorphism* $\gamma\colon \mathbb{B}_*(\mathrm{pt}) \to \mathbb{B}'_*(\mathrm{pt})$ *is the identity. Let* $P = \{p_{\lambda\mu}\}$ *be a projective system of nonzero elements* $p_{\lambda\mu} \in \mathcal{R}$*. Then we get the commutative diagram*

$$\begin{array}{ccc} \mathbb{B}_{*,P}^{\mathrm{st.ind}}(X_\infty; \{\Theta_{ij}\}) & \xrightarrow{\gamma_*^{\mathrm{ind}}} & \mathbb{B}'^{\,\mathrm{st.ind}}_{*,P}(X_\infty; \{\gamma(\Theta_{ij})\}) \\ {\scriptstyle \chi_{\mathbb{B}_*}^{\mathrm{ind}}}\downarrow & & \downarrow{\scriptstyle \chi_{\mathbb{B}'_*}^{\mathrm{ind}}} \\ \varinjlim\{\times p_{ij}\colon \mathcal{R} \to \mathcal{R}\} & \xrightarrow[\gamma^{\mathrm{ind}}]{} & \varinjlim\{\times p_{ij}\colon \mathcal{R}' \to \mathcal{R}'\}. \end{array}$$

As shown by Brasselet, Schürmann and Yokura [12] (cf Brasselet, Schürmann and Yokura [13], Ernström and Yokura [29; 30], Schürmann [61] and Yokura [77]), a natural transformation between two covariant functors commuting with exterior products is always extended to a Grothendieck transformation between their associated bivariant theories. Therefore we get the following:

4.4.5 Corollary *If a canonical orientation is defined for pro-objects, then a natural transformation between two covariant functors commuting with exterior products can* *be extended to a natural transformation between the inductive limit versions of the covariant functors.*

5 Green functors and Grothendieck–Green functors

In this section we discuss a uniqueness of the homomorphism $e\colon K_0(\mathcal{V}/X) \to F(X)$ defined by $e([f\colon Y \to X]) := f_*\mathbb{1}_Y$. A good reference for this section is Bouc [9].

5.1 Green functors

5.1.1 Definition (Green functor) A Green functor $G = (G^*, G_*)$ is a Mackey functor endowed with a bilinear map (or an exterior product) $G(X) \times G(Y) \to G(X \times Y)$ denoted by $(x, y) \mapsto x \times y$ which are bifunctorial, associative and unitary, in the following sense:

(G-I) (bifunctoriality) For morphisms $f\colon X \to X'$ and $g\colon Y \to Y'$ the following diagrams commute:

$$\begin{array}{ccc} G(X)\times G(Y) & \xrightarrow{\times} & G(X\times Y) \\ {\scriptstyle f_*\times g_*}\downarrow & & \downarrow{\scriptstyle (f\times g)_*} \\ G(X')\times G(Y') & \xrightarrow[\times]{} & G(X'\times Y'), \end{array} \qquad \begin{array}{ccc} G(X')\times G(Y') & \xrightarrow{\times} & G(X'\times Y') \\ {\scriptstyle f^*\times g^*}\downarrow & & \downarrow{\scriptstyle (f\times g)^*} \\ G(X)\times G(Y) & \xrightarrow[\times]{} & G(X\times Y). \end{array}$$

(G-II) (associativity) $(x \times y) \times z = x \times (y \times z)$ for $x \in G(X), y \in G(Y), z \in G(Z)$. To be more precise, the square

$$\begin{array}{ccc} G(X)\times G(Y)\times G(Z) & \xrightarrow{\mathrm{Id}_{G(X)}\times(\times)} & G(X)\times G(Y\times Z) \\ {\scriptstyle (\times)\times \mathrm{Id}_{G(Z)}}\downarrow & & \downarrow{\scriptstyle \times} \\ G(X\times Y)\times G(Z) & \xrightarrow[\times]{} & G(X\times Y\times Z) \end{array}$$

is commutative, up to identifications $(X \times Y) \times Z \cong X \times Y \times Z \cong X \times (Y \times Z)$.

(G-III) (unitarity) For a point pt there exists a unit $1_G \in G(\mathrm{pt})$ such that for any $x \in G(X)$, $p_{1*}(x \times 1_G) = x = p_{2*}(1_G \times x)$. Here $p_1\colon X \times \mathrm{pt} \to X$ and $p_2\colon \mathrm{pt} \times X \to X$ are the projections (which are in fact isomorphisms).

The corresponding ones in the representations of finite groups is called the Burnside ring or the Burnside functor [9].

5.1.2 Remark For a Green functor G, by the identification $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$, the abelian group $G(\mathrm{pt})$ becomes a ring with the exterior product operation and the other abelian group $G(X)$ is a $G(\mathrm{pt})$–module.

5.1.3 Remark The theory of *algebraic cobordism* that was introduced by Levine and Morel [48] is a much finer theory of Green functors in the sense that the pushforward homomorphisms are considered only for projective morphisms and the pullback homomorphisms are considered only for smooth morphisms. In such a restricted situation, it shall be called a *restricted Green functor*. Such a theory is sometimes called a *Borel–Moore functor with products*; see Levine and Pandharipande [49].

The constructible function functor $F(X)$ and the relative Grothendieck group $K_0(\mathcal{V}/X)$ are both Green functors by considering the usual exterior products.

If G, G' are Green functors on a category $\mathcal{C}$, a morphism or a natural transformation τ from G to G' is a natural transformation of Mackey functors G and G' which is compatible with exterior products, ie, such that for a variety X the following diagram commutes:

$$\begin{array}{ccc} G(X) \times G(Y) & \xrightarrow{\times} & G(X \times Y) \\ {\scriptstyle \tau_X \times \tau_Y}\downarrow & & \downarrow{\scriptstyle \tau_{X \times Y}} \\ G'(X) \times G'(Y) & \xrightarrow[\times]{} & G'(X \times Y). \end{array}$$

If moreover $\tau_{\mathrm{pt}}\colon G(\mathrm{pt}) \to G'(\mathrm{pt})$ sends the unit to the unit, then the natural transformation τ is called *unitary*.

5.2 Grothendieck–Green functor

5.2.1 Definition If a Green functor $G = (G^*, G_*)$ satisfies that for a closed subvariety $Z \subset Y$,

$$p_Y^*(1_G) = i_{Y-Z*} i_{Y-Z}^* p_Y^*(1_G) + i_{Z*} i_Z^* p_Y^*(1_G),$$

then it is called a *Grothendieck–Green* functor. Here we let $p_W\colon X \to \mathrm{pt}$ be the map to a point for a variety W.

The constructible function functor $F(X)$ and the relative Grothendieck group functor $K_0(\mathcal{V}/X)$ are both Grothendieck–Green functors. Another highly nontrivial example of a Grothendieck–Green functor is the Grothendieck ring $K_0(D^b(MHM(X))$ of the derived category of mixed Hodge modules with the natural t-structure (see Getzler [35, Proposition 3.9 and Definition 4.3]).

The following theorem is an algebro-geometric analogue of [9, Proposition 2.4.4]:

5.2.2 Theorem *For any unitary Grothendieck–Green functor $G\colon \mathcal{V} \to \mathcal{A}$, there exists a unique unitary natural transformation of Grothendieck–Green functors*

$$\tau\colon K_0(\mathcal{V}/-) \to G(-).$$

Proof Let $[h\colon W \to X] \in K_0(\mathcal{V}/X)$ and let $p_W\colon W \to \mathrm{pt}$ be the map to a point. Then $[h\colon W \to X]$ can be expressed as $[h\colon W \to X] = h_* p_W^*([\mathrm{pt} \to \mathrm{pt}])$.y Let G be another Grothendieck–Green functor. If there exists a unitary natural transformation $\tau\colon K_0(\mathcal{V}/-) \to G(-)$, then it follows from the naturality and unitarity that we have to have $\tau_X([h\colon W \to X]) = \tau_X\big(h_* p_W^*([\mathrm{pt} \to \mathrm{pt}])\big) = h_* p_W^*(1_G)$. So, all we have to do is to show that $\tau_X([h\colon W \to X]) := h_* p_W^*(1_G)$ gives us a natural transformation between two Grothendieck–Green functors, and then we are done. Since the proof is straightforward, it is left for the reader. □

As a corollary of this theorem, a unitary natural transformation from $e\colon K_0(\mathcal{V}/X) \to F(X)$ *has to* be defined by $e([f\colon Y \to X]) := f_* \mathbb{1}_Y$.

5.2.3 Remark In the above theorem, one cannot replace the Grothendieck–Green functor $K_0(\mathcal{V}/-)$ by the constructible function Grothendieck–Green functor F. For the characteristic function $\mathbb{1}_W \in F(X)$ for a subvariety $W \subset X$ we have that, as in the above proof, $\mathbb{1}_W$ can be expressed as $\mathbb{1}_W = i_{W*} p_W^*(\mathbb{1}_{\mathrm{pt}})$, where $i_W\colon W \to X$ be the inclusion. Hence, as in the above proof, we could define $\tau_X(\mathbb{1}_W) := (i_W)_* p_W^*(1_G)$. Then, all the arguments of the above proof perfectly work even for the constructible function Grothendieck–Green functor F, except for the naturality of the pushforward:

$$\begin{array}{ccc} F(X) & \xrightarrow{\tau_X} & G(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle G(f)_*} \\ F(Y) & \xrightarrow[\tau_Y]{} & G(Y). \end{array}$$

In fact, one can see that this does not already hold for $G = K_0(\mathcal{V}/-)$. Indeed, if it were the case, the uniqueness of such a unitary natural transformation would imply that for any variety X we should have the isomorphism $K_0(\mathcal{V}/X) \cong F(X)$ and hence, in particular, we would have the isomorphism $K_0(\mathcal{V}/\mathrm{pt}) \cong F(\mathrm{pt}) \cong \mathbb{Z}$, which is not the case.

5.2.4 Remark Surely $i_F\colon F(X) \to K_0(\mathcal{V}/X)$ defined by $i_F(\mathbb{1}_W) := [i_W\colon W \to X]$ is injective. However, the above remark implies that this injective transformation cannot be a unitary natural transformation between the two Grothendieck–Green functors.

Acknowledgements I would like to thank Jörg Schürmann and Willem Veys for their valuable comments and suggestions on an earlier version [72] and the referee for his/her valuable comments and suggestions. I would also like to give my sincere gratitude to Clint McCrory for having introduced me to characteristic classes of singular spaces. This work is partially supported by Grant-in-Aid for Scientific Research number 21540088, the Ministry of Education, Culture, Sports, Science and Technology (MEXT), Japan.

References

[1] **P Aluffi**, *Chern classes of birational varieties*, Int. Math. Res. Not. 2004 (2004) 3367–3377 MR2098642

[2] **P Aluffi**, *Modification systems and integration in their Chow groups*, Selecta Math. 11 (2005) 155–202 MR2183846

[3] **M Artin**, **B Mazur**, *Etale homotopy*, Lecture Notes in Math. 100, Springer, Berlin (1969) MR0245577

[4] **M F Atiyah**, *The signature of fibre-bundles*, from: "Global Analysis (Papers in Honor of K Kodaira)", (D C Spencer, S Iyanaga, editors), Univ. Tokyo Press (1969) 73–84 MR0254864

[5] **J Ax**, *Injective endomorphisms of varieties and schemes*, Pacific J. Math. 31 (1969) 1–7 MR0251036

[6] **P Baum**, **W Fulton**, **R MacPherson**, *Riemann–Roch for singular varieties*, Inst. Hautes Études Sci. Publ. Math. (1975) 101–145 MR0412190

[7] **F Bittner**, *The universal Euler characteristic for varieties of characteristic zero*, Compos. Math. 140 (2004) 1011–1032 MR2059227

[8] **K Borsuk**, *Theory of shape*, Lecture Notes Series 28, Matematisk Institut, Aarhus Universitet (1971) MR0293602

[9] **S Bouc**, *Green functors and G–sets*, Lecture Notes in Math. 1671, Springer, Berlin (1997) MR1483069

[10] **J-P Brasselet**, *Existence des classes de Chern en théorie bivariante*, from: "Analysis and topology on singular spaces, II, III (Luminy, 1981)", Astérisque 101, Soc. Math. France (1983) 7–22 MR737926

[11] **J-P Brasselet**, **J Schürmann**, **S Yokura**, *Classes de Hirzebruch et classes de Chern motiviques*, C. R. Math. Acad. Sci. Paris 342 (2006) 325–328 MR2201957

[12] **J-P Brasselet**, **J Schürmann**, **S Yokura**, *On Grothendieck transformations in Fulton–MacPherson's bivariant theory*, J. Pure Appl. Algebra 211 (2007) 665–684 MR2344222

[13] **J-P Brasselet**, **J Schürmann**, **S Yokura**, *On the uniqueness of bivariant Chern class and bivariant Riemann–Roch transformations*, Adv. Math. 210 (2007) 797–812 MR2303240

[14] **J-P Brasselet**, **J Schürmann**, **S Yokura**, *Hirzebruch classes and motivic Chern classes for singular spaces*, J. Topol. Anal. 2 (2010) 1–55 MR2646988

[15] **J-P Brasselet**, **M-H Schwartz**, *Sur les classes de Chern d'un ensemble analytique complexe*, from: "The Euler–Poincaré characteristic", Astérisque 82, Soc. Math. France (1981) 93–147 MR629125

[16] **S E Cappell**, **A Libgober**, **L G Maxim**, **J L Shaneson**, *Hodge genera of algebraic varieties, II*, Math. Ann. 345 (2009) 925–972 MR2545873

[17] **S E Cappell**, **L Maxim**, **J Schürmann**, **J L Shaneson**, *Equivariant characteristic classes of complex algebraic varieties*, to appear in Comm. Pure Appl. Math. `arXiv:1004.1844`

[18] **S E Cappell**, **L Maxim**, **J Schürmann**, **J L Shaneson**, *Characteristic classes of complex hypersurfaces*, Adv. Math. 225 (2010) 2616–2647 MR2680178

[19] **S E Cappell**, **L Maxim**, **J Schürmann**, **J L Shaneson**, **S Yokura**, *Characteristic classes of symmetric products of complex quasi-projective varieties* `arXiv:1008.4299`

[20] **S E Cappell**, **L Maxim**, **J L Shaneson**, *Hodge genera of algebraic varieties I*, Comm. Pure Appl. Math. 61 (2008) 422–449 MR2376848

[21] **S E Cappell**, **J L Shaneson**, *Stratifiable maps and topological invariants*, J. Amer. Math. Soc. 4 (1991) 521–551 MR1102578

[22] **S S Chern**, **F Hirzebruch**, **J-P Serre**, *On the index of a fibered manifold*, Proc. Amer. Math. Soc. 8 (1957) 587–596 MR0087943

[23] **A Craw**, *An introduction to motivic integration*, from: "Strings and geometry", (M Douglas, J Gauntlett, M Gross, editors), Clay Math. Proc. 3, Amer. Math. Soc. (2004) 203–225 MR2103724

[24] **J Denef**, **F Loeser**, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. 135 (1999) 201–232 MR1664700

[25] **J Denef**, **F Loeser**, *Geometry on arc spaces of algebraic varieties*, from: "European Congress of Mathematics, Vol. I (Barcelona, 2000)", (C Casacuberta, R M Miró-Roig, J Verdera, S Xambó-Descamps, editors), Progr. Math. 201, Birkhäuser, Basel (2001) 327–348 MR1905328

[26] **A W M Dress**, *Notes on the theory of representations of finite groups. Part I: The Burnside ring of a finite group and some AGN-applications*, Universität Bielefeld, Fakultät für Mathematik (1971) MR0360771 With the aid of lecture notes, taken by M Küchler

[27] **A W M Dress**, *Contributions to the theory of induced representations*, from: "Algebraic K–theory, II: "Classical" algebraic K–theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, WA, 1972)", (H Bass, editor), Lecture Notes in Math. 342, Springer, Berlin (1973) 183–240 MR0384917

[28] **D A Edwards**, *Étale homotopy theory and shape*, from: "Algebraic and geometrical methods in topology (Conf. Topological Methods in Algebraic Topology, State Univ. New York, Binghamton, NY, 1973)", (L F McAuley, editor), Lecture Notes in Math. 428, Springer, Berlin (1974) 58–107 MR0375295

[29] **L Ernström**, **S Yokura**, *Bivariant Chern–Schwartz–MacPherson classes with values in Chow groups*, Selecta Math. 8 (2002) 1–25 MR1890193

[30] **L Ernström**, **S Yokura**, *Addendum to: "Bivariant Chern–Schwartz–MacPherson classes with values in Chow groups"* [29], Selecta Math. 10 (2004) 29–35 MR2061221

[31] **R H Fox**, *On shape*, Fund. Math. 74 (1972) 47–71 MR0296914

[32] **J H G Fu**, **C McCrory**, *Stiefel–Whitney classes and the conormal cycle of a singular variety*, Trans. Amer. Math. Soc. 349 (1997) 809–835 MR1401519

[33] **W Fulton**, *Intersection theory*, Ergeb. Math. Grenzgeb. 2, Springer, Berlin (1984) MR732620

[34] **W Fulton**, **R MacPherson**, *Categorical framework for the study of singular spaces*, Mem. Amer. Math. Soc. 31, no. 243, Amer. Math. Soc. (1981) MR609831

[35] **E Getzler**, *Resolving mixed Hodge modules on configuration spaces*, Duke Math. J. 96 (1999) 175–203 MR1663927

[36] **M Goresky**, **R MacPherson**, *Intersection homology theory*, Topology 19 (1980) 135–162 MR572580

[37] **W Gottschalk**, *Some general dynamical notions*, from: "Recent advances in topological dynamics (Proc. Conf. Topological Dynamics, Yale Univ., New Haven, CT, 1972; in honor of Gustav Arnold Hedlund)", (A Beck, editor), Lecture Notes in Math. 318, Springer, Berlin (1973) 120–125 MR0407821

[38] **M Gromov**, *Endomorphisms of symbolic algebraic varieties*, J. Eur. Math. Soc. 1 (1999) 109–197 MR1694588

[39] **M Gromov**, *Topological invariants of dynamical systems and spaces of holomorphic maps: I*, Math. Phys. Anal. Geom. 2 (1999) 323–415 MR1742309

[40] **A Grothendieck**, *Récoltes et semailles, réflexions et témoignages sur un passé de mathématicien* (1985) Available at `http://www.math.jussieu.fr/~leila/grothendieckcircle/recoltesetc.php`

[41] **A Grothendieck**, *Technique de descente et théorèmes d'existence en géométrie algébrique, II: Le théorème d'existence en théorie formelle des modules*, from: "Séminaire Bourbaki, Vol. 5", Soc. Math. France (1995) Exp. No. 195, 369–390 MR1603480

[42] **F Hirzebruch**, *Topological methods in algebraic geometry*, third edition, Grund. Math. Wiss. 131, Springer, New York (1966) MR0202713 New appendix and translation by R L E Schwarzenberger, with an additional section by A Borel

[43] **F Hirzebruch**, **T Berger**, **R Jung**, *Manifolds and modular forms*, Aspects of Math. E20, Friedr. Vieweg & Sohn, Braunschweig (1992) MR1189136 With appendices by N-P Skoruppa and by P Baum

[44] **K Kodaira**, *A certain type of irregular algebraic surfaces*, J. Analyse Math. 19 (1967) 207–215 MR0216521

[45] **M Kontsevich**, *Motivic integration*, Lecture at Orsay (1995)

[46] **M Kwieciński**, *Formule du produit pour les classes caractéristiques de Chern–Schwartz–MacPherson et homologie d'intersection*, C. R. Acad. Sci. Paris Sér. I Math. 314 (1992) 625–628 MR1158750

[47] **M Kwieciński**, **S Yokura**, *Product formula for twisted MacPherson classes*, Proc. Japan Acad. Ser. A Math. Sci. 68 (1992) 167–171 MR1193174

[48] **M Levine**, **F Morel**, *Algebraic cobordism*, Springer Monogr. in Math., Springer, Berlin (2007) MR2286826

[49] **M Levine**, **R Pandharipande**, *Algebraic cobordism revisited*, Invent. Math. 176 (2009) 63–130 MR2485880

[50] **E Looijenga**, *Motivic measures*, from: "Séminaire Bourbaki, Vol. 1999/2000", Astérisque 276 (2002) 267–297 MR1886763

[51] **R D MacPherson**, *Chern classes for singular algebraic varieties*, Ann. of Math. 100 (1974) 423–432 MR0361141

[52] **R D MacPherson**, *Characteristic classes for singular varieties*, from: "Proceedings of the Ninth Brazilian Mathematical Colloquium (Poços de Caldas, 1973), Vol. II (Portuguese)", Inst. Mat. Pura Apl., São Paulo (1977) 321–327 MR534464

[53] **S Mardešić**, **J Segal**, *Shape theory: The inverse system approach*, North-Holland Math. Library 26, North-Holland, Amsterdam (1982) MR676973

[54] **L Maxim**, **M Saito**, **J Schürmann**, *Symmetric products of mixed Hodge modules*, J. Math. Pures Appl. 96 (2011) 462–483 MR2843222

[55] **L Maxim**, **J Schürmann**, *Twisted genera of symmetric products*, Selecta Math. 18 (2012) 283–317

[56] **T Ohmoto**, *Equivariant Chern classes of singular algebraic varieties with group actions*, Math. Proc. Cambridge Philos. Soc. 140 (2006) 115–134 MR2197579

[57] **T Ohmoto**, *Generating functions of orbifold Chern classes I: Symmetric products*, Math. Proc. Cambridge Philos. Soc. 144 (2008) 423–438 MR2405899

[58] **B Poonen**, *The Grothendieck ring of varieties is not a domain*, Math. Res. Lett. 9 (2002) 493–497 MR1928868

[59] **C Sabbah**, *Espaces conormaux bivariants*, PhD thesis, Université Paris (1986)

[60] **M Saito**, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. 26 (1990) 221–333 MR1047415

[61] **J Schürmann**, *A general construction of partial Grothendieck transformations* `arXiv:math.AG/0209299`

[62] **J Schürmann**, *A generalized Verdier-type Riemann–Roch theorem for Chern–Schwartz–MacPherson classes* `arXiv:math.AG/0202175`

[63] **J Schürmann**, *Nearby cycles and characteristic classes of singular spaces* `arXiv:1003.2343`

[64] **J Schürmann**, *Characteristic classes of mixed Hodge modules*, from: "Topology of stratified spaces", (G Friedman, E Hunsicker, A Libgober, L Maxim, editors), Math. Sci. Res. Inst. Publ. 58, Cambridge Univ. Press (2011) 419–470 MR2796417

[65] **J Schürmann**, **S Yokura**, *Motivic bivariant characteristic classes* `arXiv:1110.2166v1`

[66] **J Schürmann**, **S Yokura**, *A survey of characteristic classes of singular spaces*, from: "Singularity theory", (D Chéniot, N Dutertre, C Murolo, D Trotman, A Pichon, editors), World Sci. Publ., Hackensack, NJ (2007) 865–952 MR2342943

[67] **M-H Schwartz**, *Classes caractéristiques définies par une stratification d'une variété analytique complexe. I*, C. R. Acad. Sci. Paris 260 (1965) 3262–3264 MR0212842

[68] **M-H Schwartz**, *Classes et caractères de Chern–Mather des espaces linéaires*, C. R. Acad. Sci. Paris Sér. I Math. 295 (1982) 399–402 MR684735

[69] **D Sullivan**, *Combinatorial invariants of analytic spaces*, from: "Proceedings of Liverpool Singularities—Symposium, I (1969/70)", (C T C Wall, editor), Springer, Berlin (1971) 165–168 MR0278333

[70] **W Veys**, *Arc spaces, motivic integration and stringy invariants*, from: "Singularity theory and its applications", (S Izumiya, G Ishikawa, H Tokunaga, I Shimada, T Sano, editors), Adv. Stud. Pure Math. 43, Math. Soc. Japan (2006) 529–572 MR2325153

[71] **J Woolf**, *Witt groups of sheaves on topological spaces*, Comment. Math. Helv. 83 (2008) 289–326 MR2390046

[72] **S Yokura**, *Chern classes of proalgebraic varieties and motivic measures* `arXiv:math.AG/0407237`

[73] **S Yokura**, *On Cappell–Shaneson's homology L–classes of singular algebraic varieties*, Trans. Amer. Math. Soc. 347 (1995) 1005–1012 MR1283567

[74] **S Yokura**, *A singular Riemann–Roch for Hirzebruch characteristics*, from: "Singularities Symposium—Łojasiewicz 70 (Kraków, 1996; Warsaw, 1996)", (B Jakubczyk, W Pawłucki, J Stasica, editors), Banach Center Publ. 44, Polish Acad. Sci., Warsaw (1998) 257–268 MR1677403

[75] **S Yokura**, *On a Verdier-type Riemann–Roch for Chern–Schwartz–MacPherson class*, Topology Appl. 94 (1999) 315–327 MR1695362 Special issue in memory of B J Ball

[76] **S Yokura**, *On the uniqueness problem of bivariant Chern classes*, Doc. Math. 7 (2002) 133–142 MR1911213

[77] **S Yokura**, *On Ginzburg's bivariant Chern classes*, Trans. Amer. Math. Soc. 355 (2003) 2501–2521 MR1974000

[78] **S Yokura**, *Quasi-bivariant Chern classes obtained by resolutions of singularities*, Rep. Fac. Sci. Kagoshima Univ. (2003) 17–28 MR2020372

[79] **S Yokura**, *Mackey functors on provarieties*, from: "Real and complex singularities", (J-P Brasselet, M A Soares Ruas, editors), Trends Math., Birkhäuser, Basel (2007) 343–359 MR2280150

[80] **S Yokura**, *Motivic Milnor classes*, J. Singul. 1 (2010) 39–59 MR2671765

[81] **S Yokura**, *Motivic characteristic classes*, from: "Topology of stratified spaces", (G Friedman, E Hunsicker, A Libgober, L Maxim, editors), Math. Sci. Res. Inst. Publ. 58, Cambridge Univ. Press (2011) 375–418 MR2796416

[82] **B Youssin**, *Witt groups of derived categories*, K–Theory 11 (1997) 373–395 MR1451761

[83] **J Zhou**, *Classes de Chern en théorie bivariante*, PhD thesis, Université Aix-Marseille (1995)

[84] **J Zhou**, *Morphisme cellulaire et classes de Chern bivariantes*, Ann. Fac. Sci. Toulouse Math. 9 (2000) 161–192 MR1815946

Department of Mathematics and Computer Science, Faculty of Science, Kagoshima University
1-21-35 Korimoto 1-chome, Kagoshima 890-0065, Japan

yokura@sci.kagoshima-u.ac.jp

Received: 21 April 2010 Revised: 21 November 2011