Free group automorphisms with parabolic boundary orbits

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For $N \geqslant 4$, we show that there exist automorphisms of the free group F_N which have a parabolic orbit in ∂F_N. In fact, we exhibit a technology for producing infinitely many such examples.

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1 Introduction

An automorphism φ of the free group F_N of rank N induces a homeomorphism $\partial\varphi$ of the (Gromov) boundary ∂F_N of F_N. The dynamics of the map $\partial\varphi$ on ∂F_N has been studied a lot; see Levitt and Lustig [13; 14; 15; 16] and the author's thesis [10]. We give a survey of the known results relevant in our context in Section 3. In this paper, we focus on the following question:

> *Does there exist an automorphism φ of F_N such that there is a parabolic orbit for the homeomorphism $\partial\varphi$?*

We say that an automorphism φ *has a parabolic orbit* if there exists two points $X, Y \in \partial F_N$, $X \neq Y$, such that

$$\lim_{k\to\pm\infty} \partial\varphi^k(Y) = X.$$

We note that this implies that X is a fixed point of $\partial\varphi$. In such a situation, the point $X \in \partial F_N$ is called a *parabolic fixed point* for φ, and the set $\{\partial\varphi^k(Y) \mid k \in \mathbb{Z}\}$ is called a *parabolic orbit* for φ. We prove:

Theorem 1.1 *For $N \geqslant 4$ there exists an infinite family $\{\varphi_k \mid k \in \mathbb{N}\}$ of automorphisms of F_N which have a parabolic orbit, such that for any $k, k', p, p' \in \mathbb{N}$, φ_k^p and $\varphi_{k'}^{p'}$ are conjugate if and only if $k = k'$ and $p = p'$.*

Discussions with some of the experts of the subject have led the author to feel that the existence of such parabolic orbits come somehow as a surprise. To put Theorem 1.1 in perspective, we would like to mention the following three facts.

First, given a compact set K and a homeomorphism f of K, one says that f has *North-South dynamics*, if (i) f has precisely two distinct fixed points x^+ and x^-,

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(ii) $\lim_{k\to+\infty} f^k(y) = x^+$ and $\lim_{k\to+\infty} f^{-k}(y) = x^-$ for all $y \in K \smallsetminus \{x^-, x^+\}$, and (iii) the limit of f^k when k tends to infinity is uniform on compact subsets of $K \smallsetminus \{x^-\}$ and the limit of f^{-k} is uniform on compact subsets of $K \smallsetminus \{x^+\}$. It is proved in Levitt and Lustig [13] that "most" automorphisms of F_N, in a precise sense we do not explain here, have North-South dynamics on ∂F_N. In particular, they cannot have a parabolic orbit.

Second, let δ be the automorphism of $F_2 = \langle a, b\rangle$ defined by $\delta(a) = a$ and $\delta(b) = ba$. The outer automorphism class D of δ is sometimes called a *Dehn twist automorphism*. The reader, who has in mind the action by isometries of $\mathrm{SL}_2(\mathbb{Z})$ on the hyperbolic plane, should be warned that Dehn twist automorphisms do not give rise to parabolic orbits in ∂F_2. We give in Section 6 a description of all possible dynamics of automorphisms of F_2 in the outer class D^n, for $n \in \mathbb{Z}$.

Third, more generally, it is known that geometric automorphisms of F_N do not have parabolic orbits in ∂F_N. We recall that an automorphism φ of F_N is *geometric* if there exist a surface S (with nonempty boundary) with fundamental group $\pi_1(S)$ isomorphic to F_N and a homeomorphism f of S which induces φ on $F_N \cong \pi_1(S)$. More details are given in Section 4.2. As a consequence, since all automorphisms of F_2 are known to be geometric, one obtains:

Proposition 1.2 *No automorphism of F_2 has a parabolic orbit.*

To our knowledge, the question of the existence of automorphisms with a parabolic orbit is still open for F_3.

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2 A first example

For the impatient reader, we give a first example of an automorphism of $F_4 = \langle a, b, c, d\rangle$ with a parabolic orbit "inside F_4" (using Proposition 3.5, this gives immediately a parabolic orbit in ∂F_4).

Let φ be the automorphism defined by

$$\begin{aligned} \varphi\colon a &\mapsto a \\ b &\mapsto ba \\ c &\mapsto ca^2 \\ d &\mapsto dc. \end{aligned}$$

The inverse of φ is given by

$$\begin{aligned} \varphi^{-1}\colon a &\mapsto a \\ b &\mapsto ba^{-1} \\ c &\mapsto ca^{-2} \\ d &\mapsto da^2c^{-1}. \end{aligned}$$

The common limit point of the forward and backward iteration of φ (called a "parabolic fixed point") will be the element $ba^{-\infty} = ba^{-1}a^{-1}a^{-1}a^{-1}\cdots \in \partial F_4$. The element of F_4 which gives rise to a parabolic orbit with this limit point is bd^{-1}. We calculate:

$$\begin{aligned} bd^{-1} \overset{\varphi}{\mapsto} bac^{-1}\cdot d^{-1} \overset{\varphi}{\mapsto} bc^{-1}\cdot c^{-1}d^{-1} \overset{\varphi}{\mapsto} ba^{-1}c^{-1}\cdot a^{-2}c^{-1}c^{-1}d^{-1} \\ \overset{\varphi}{\mapsto} ba^{-2}c^{-1}\cdot a^{-4}c^{-1}a^{-2}c^{-1}c^{-1}d^{-1} \overset{\varphi}{\mapsto} \cdots \end{aligned}$$

$$\begin{aligned} b\cdot d^{-1} \overset{\varphi^{-1}}{\mapsto} ba^{-1}\cdot ca^{-2}d^{-1} \overset{\varphi^{-1}}{\mapsto} ba^{-2}\cdot ca^{-4}ca^{-2}d^{-1} \\ \overset{\varphi^{-1}}{\mapsto} ba^{-3}\cdot ca^{-6}ca^{-4}ca^{-2}d^{-1} \overset{\varphi^{-1}}{\mapsto} \cdots \end{aligned}$$

In these calculations, we help the reader to follow through the iteration by introducing an extra $\cdot$ which is "mapped" to the $\cdot$ in the next iteration step. The crucial feature is that at any of these $\cdot$ no cancellation does occur. We see that $\lim_{k\to+\infty}\varphi^k(bd^{-1}) = \lim_{k\to+\infty}\varphi^{-k}(bd^{-1}) = ba^{-\infty}$. A more formal justification is given in Section 5.

3 Basics

This section serves sort of as glossary: We summarize in a sequence of brief subsections the basic definitions and facts which are needed to follow the arguments in the subsequent sections. The expert reader is encouraged to skip the first few subsections (and to go back later to them, if need be). However, the terminology introduced in the last subsections is nonstandard and should be read carefully.

3.1 The induced boundary homeomorphism

Let F_N denote the free group of finite rank $N \geqslant 2$. The boundary ∂F_N of F_N is a Cantor set. If $\mathcal{A} = \{a_1, \ldots, a_N\}$ is a basis of F_N, we denote by $\mathcal{A}^{\pm 1}$ the set $\{a_1, \ldots, a_N, a_1^{-1}, \ldots, a_N^{-1}\}$. A word $w = w_1 \cdots w_p$ ($w_i \in \mathcal{A}^{\pm 1}$) is *reduced* if $w_{i+1} \neq w_i^{-1}$. The free group F_N can be understood as the set of (finite) reduced words in $\mathcal{A}^{\pm 1}$. Then the boundary ∂F_N is naturally identified to the set of (right) infinite reduced words $X = x_1 \cdots x_p \ldots$ with $x_i \in \mathcal{A}^{\pm 1}$, $x_{i+1} \neq x_i^{-1}$. The cylinder defined by a reduced word $w = w_1 \cdots w_p$ is the set of right-infinite reduced words $X = x_1 \cdots x_k \cdots$ which admit w as prefix: $x_i = w_i$ for $i \in \{1, \ldots, p\}$. A basis of topology of ∂F_N is given by the set of all such cylinders.

An automorphism φ of a free group F_N induces a homeomorphism $\partial\varphi$ of the boundary ∂F_N. This can easily be checked by considering a standard set of generators of the automorphisms group $\mathrm{Aut}(F_N)$ of F_N. Alternatively, this can be seen as a consequence of the fact that a quasi-isometry of a proper Gromov-hyperbolic space induces a homeomorphism on the boundary of this space; see Ghys and de la Harpe [7]. Indeed, F_N equipped with the word metric associated to a basis $\mathcal{A}$, is a proper Gromov–0–hyperbolic space, and any automorphism of F_N is a quasi-isometry of F_N with respect to this metric.

3.2 Compactification of F_N

Let $\overline{F}_N$ denote the union of F_N and its boundary ∂F_N, ie $\overline{F}_N = F_N \cup \partial F_N$. Given a basis of F_N, if w is a reduced word, let C_w be the set of reduced finite or infinite words which have w as prefix. A basis of topology of $\overline{F}_N$ is given by the finite subsets of F_N and the sets C_w (with w describing all the reduced words of F_N). Then $\overline{F}_N$ is a compact set, and the inclusions of F_N and ∂F_N in $\overline{F}_N$ are embeddings. If φ is an automorphism of F_N, $\overline{\varphi}$ will denote the map defined by $\overline{\varphi}(g) = \varphi(g)$ if $g \in F_N$ and $\overline{\varphi}(X) = \partial\varphi(X)$ if $X \in \partial F_N$. The map $\overline{\varphi}$ is a homeomorphism of $\overline{F}_N$.

3.3 Getting rid of periodicity

Let f be a homeomorphism of a topological space $\mathcal{X}$. We denote by $\mathrm{Fix}(f) = \{x \in \mathcal{X} \mid f(x) = x\}$ the set of fixed points of f, and by $\mathrm{Per}(f) = \bigcup_{k \in \mathbb{N}} \mathrm{Fix}(f^k)$ the set of periodic points of f.

Levitt and Lustig have proved in [14] that there exists an integer p, which depends only on the rank N of F_N, such that for all $\varphi \in \mathrm{Aut}(F_N)$, the periodic points of $\overline{\varphi}^p$ are fixed points: $\mathrm{Fix}(\overline{\varphi}^p) = \mathrm{Per}(\overline{\varphi}^p)$. This result has been refined by Feighn and

Handel in [5], where the notion of "forward rotationless" outer automorphism has been introduced. This lead us to say, in this paper, that an automorphism $\varphi \in \mathrm{Aut}(F_N)$ is *rotationless* if $\mathrm{Fix}(\overline{\varphi}) = \mathrm{Per}(\overline{\varphi})$. The previously mentioned result can be rephrased as follows:

Theorem 3.1 (Levitt–Lustig) *Any automorphism $\varphi \in \mathrm{Aut}(F_N)$ has a power φ^p ($p \in \mathbb{N}$) which is rotationless.*

3.4 Nature of fixed points

Let φ be a rotationless automorphism of F_N. The set $\mathrm{Fix}(\varphi)$ is a subgroup of F_N, which is called the *fixed subgroup* of φ. This fixed subgroup has finite rank; see Cooper [4]. More precisely, Bestvina and Handel [2] proved that $\mathrm{rank}(\mathrm{Fix}(\varphi)) \leqslant N$. In particular, $\mathrm{Fix}(\varphi)$ is a quasiconvex subgroup of F_N, and thus its boundary $\partial\,\mathrm{Fix}(\varphi)$ naturally injects into ∂F_N. By continuity of $\overline{\varphi}$, every point of $\partial\,\mathrm{Fix}(\varphi)$ is contained in $\mathrm{Fix}(\partial\varphi)$. Following Nielsen, these fixed points of $\partial\varphi$ are called *singular*; the fixed points of $\partial\varphi$ which are not singular are called *regular*.

A fixed point X of $\partial\varphi$ is *attracting* if there exists a neighbourhood U of X in $\overline{F}_N$ such that the sequence $\overline{\varphi}^k(x)$ converges to X for all x in U. A fixed point X of $\partial\varphi$ is *repulsing* if it is attracting for $\partial\varphi^{-1}$. Gaboriau et al [6] proved that:

Lemma 3.2 *Let $\varphi \in \mathrm{Aut}(F_N)$. A regular fixed point of $\partial\varphi$ is either attracting or repulsing.* □

However, outside of the regular fixed point set, ie for singular fixed points, the dynamics can be quite a bit more complicated. In particular, there may exist *mixed* fixed points, ie fixed points which serve as attractor for some orbits, and simultaneously as repeller for others. This phenomenon is rather common; some concrete examples will be spelled out in the subsequent sections.

A particular case of a mixed fixed point is the case (defined in the Introduction) of a parabolic fixed point. Thus we obtain as special case the following consequence of Lemma 3.2:

Remark 3.3 Any parabolic fixed point of φ is singular.

3.5 Limit points

Let φ be a rotationless automorphism of F_N. For any $x \in \overline{F}_N$, if $\lim_{k\to+\infty} \overline{\varphi}^k(x)$ exists, we denote it by $\omega_\varphi(x)$. In [16], Levitt and Lustig proved:

Theorem 3.4 (Levitt–Lustig) *Let $\varphi \in \mathrm{Aut}(F_N)$ be rotationless. Then for any $x \in \bar{F}_N$ the sequence $\bar{\varphi}^k(x)$ converges to some element $\omega_\varphi(x) \in \mathrm{Fix}(\bar{\varphi})$.* □

A point $X \in \partial\,\mathrm{Fix}(\varphi)$ is a *ω–limit point* of φ if there exists $x \in \bar{F}_N$ such that $X = \omega_\varphi(x)$. A point $X \in \partial\,\mathrm{Fix}(\varphi)$ is a *limit point* of φ if it is a ω–limit point of φ or φ^{-1}. Let L^ω_φ denote the set of ω–limit points of φ and let L_φ denote the set of limit points of φ.

For any $g \in F_N$, $g \neq 1$, the sequence g^k has a limit in ∂F_N when $k \to +\infty$: this limit is denoted by g^∞.

Proposition 3.5 *Let $\varphi \in \mathrm{Aut}(F_N)$ be a rotationless automorphism. If $g \in F_N \setminus \mathrm{Fix}(\varphi)$, then*

$$\omega_\varphi(g) = \omega_\varphi(g^\infty).$$

Proof The proof is a simple adaptation of the arguments in the proof of [13, Proposition 2.3]. We fix a basis $\mathcal{A}$ of F_N. We note that for all $g \in F_N \setminus \{1\}$, the Gromov product (g, g^∞) (ie the length of longest common prefix) of g and g^∞ is bigger than $\frac{1}{2}(|g|+1)$ (where $|g|$ denotes the length of g in the basis $\mathcal{A}$). If $g \notin \mathrm{Fix}(\varphi)$, then the length of $\varphi^k(g)$, and thus also the Gromov product $(\varphi^k(g), (\varphi^k(g))^\infty)$, tend to infinity. Theorem 3.4 implies that $\omega_\varphi(g) = \omega_\varphi(g^\infty)$. □

Proposition 3.5 shows that $L^\omega_\varphi = \{\omega_\varphi(X) \mid X \in \partial F_N\}$. We do not know whether $L^\omega_\varphi = \{\omega_\varphi(g) \mid g \in F_N\}$ holds.

3.6 Isoglossy classes

For any $\varphi \in \mathrm{Aut}(F_N)$, two points $X, Y \in \partial F_N$ are called *isogloss* (with respect to φ) if there exists some $g \in \mathrm{Fix}(\varphi)$ such that $X = gY$. It follows directly from this definition that isoglossy is an equivalence relation. The fixed subgroup $\mathrm{Fix}(\varphi)$ acts naturally on the fixed point set $\mathrm{Fix}(\partial\varphi)$, which is thus naturally partitioned into isoglossy classes. If $X, Y \in \mathrm{Fix}(\partial\varphi)$ are isogloss, then they are of same "dynamical type": they are simultaneously singular, attracting, repulsing, mixed, parabolic or limit points.

3.7 Dynamics graph

Let $\varphi \in \mathrm{Aut}(F_N)$ be a rotationless automorphism. We associate to φ a graph Γ_φ, called the *dynamics graph* of φ. The vertices of Γ_φ are the isoglossy classes of points of L_φ. There is an oriented edge from the isoglossy class x_1 to the isoglossy class x_2 if there exists some representatives X_i of x_i and $X \in \partial F_N$ such that $\omega_{\varphi^{-1}}(X) = X_1$

Figure 1. North-South dynamics graph

and $\omega_\varphi(X) = X_2$. The main theorem of [10] states that Γ_φ is a finite graph. We give in Figure 1 the dynamics graph of an automorphism which has North-South dynamics on ∂F_N.

Finally, we note that, for a rotationless automorphism φ, the existence of parabolic orbit is equivalent to the fact that there is an edge of the dynamics graph Γ_φ which is a loop.

Remark 3.6 In [11], Levitt introduces a graph in order to code the dynamics of so-called "simple-dynamics homeomorphisms" of the Cantor set C: a homeomorphism $f\colon C \to C$ has *simple dynamics* if the set $\mathrm{Fix}(f)$ of its fixed points is finite, and if the sequence f^n uniformly converges on any compact set disjoint from $\mathrm{Fix}(f)$. If $\varphi \in \mathrm{Aut}(F_N)$ is a rotationless automorphism with trivial fixed subgroup, then $\partial\varphi$ has simple dynamics, and the graph Γ_φ is the same as the one defined in [11]. In this case, the fixed points of $\partial\varphi$ are either attracting or repulsing. Thus, if one is interested in parabolic orbits, which are the main focus of the present paper, one has to purposefully leave to world of "simple dynamics" homeomorphisms.

4 Examples

4.1 Inner automorphisms

Let $i_u \in \mathrm{Aut}(F_N)$ denote the *conjugation*, or *inner automorphism*, by $u \in F_N$, ie $i_u(g) = ugu^{-1}$ for all $g \in F_N$. The set $\mathrm{Inn}(F_N)$ of inner automorphisms of F_N is a normal subgroup of $\mathrm{Aut}(F_N)$. The quotient group, denoted by $\mathrm{Out}(F_N)$, is the group of *outer automorphisms* of F_N.

The homeomorphism $\partial i_u\colon \partial F_N \to \partial F_N$ induced by i_u is the left translation by u: $\partial i_u(X) = uX$. If $u \neq 1$, the map ∂i_u has precisely 2 fixed points: u^∞ and $u^{-\infty}$ (where u^∞ is the limit of the sequence u^k, and $u^{-\infty}$ is the limit of the sequence u^{-k}, for $k \to +\infty$). Moreover, for any point $X \in \partial F_N$ different from $u^{-\infty}$, the sequence $\partial\varphi^k(X)$ converges to u^∞ when k tends to infinity. One checks easily that the map ∂i_u has North-South dynamics, from $u^{-\infty}$ to u^∞, on ∂F_N; see [13] for instance.

Remark 4.1 We note that the fixed subgroup of i_u is cyclic, generated by the root of u (ie the element $v \in F_N$ such that $u = v^p$ with $p \in \mathbb{N}$ maximal). In particular, u^∞ and $u^{-\infty}$ are singular fixed points of i_u. This shows that when defining "X is an attracting fixed point of φ" in Section 3.4, it makes a crucial difference that we request the neighbourhood U of X to be taken in $\overline{F}_N$ and not just in ∂F_N.

4.2 Geometric automorphisms

Let Σ be a compact surface with fundamental group $\pi_1(\Sigma)$ isomorphic to F_N (in particular, Σ has nonempty boundary). The surface Σ can be equipped with a hyperbolic metric (ie a metric of constant curvature equal to -1) in such a way that every boundary component of the boundary of Σ is a geodesic. The universal cover $\widetilde{\Sigma}$ of Σ is then identified with a closed convex subset of the hyperbolic plane $\mathbb{H}^2$, and the Gromov boundary $\partial\widetilde{\Sigma}$ of $\widetilde{\Sigma}$, which is naturally identified with the boundary ∂F_N of F_N, injects in the boundary (or circle at infinity) S_∞ of $\mathbb{H}^2$. Since S_∞ is a circle, it can be equipped with a natural cyclic order. This order on S_∞ induces a cyclic order on ∂F_N.

In his fundamental work [19; 20; 21], Nielsen proposed an original and fruitful point of view to study homeomorphisms of surfaces. The basic idea is that the behaviour of a homeomorphism f of a surface Σ is well reflected by the collection of all the lifts $\widetilde{f}$ of f to $\widetilde{\Sigma}$ which have each much simpler individual behaviour. This idea is at the origin of what is now called "Nielsen–Thurston classification" of homeomorphisms of surfaces (see Handel and Thurston [9]), and it has much influenced the study of (outer) automorphisms of free groups (see Gaboriau et al [6], Feighn and Handel [5], Handel and Mosher [8]). The key fact is that any lift $\widetilde{f}$ of f induces a homeomorphism $\partial\widetilde{f}$ of $\partial\widetilde{\Sigma}$. A basic (but rather fundamental) remark is that $\partial\widetilde{f}$ preserves the cyclic order on $\partial\widetilde{\Sigma} \subseteq S_\infty$.

An homeomorphism f of Σ induces an outer automorphism of $\pi_1(\Sigma)$, and thus an outer automorphism $\Phi \in \mathrm{Out}(F_N)$ (in fact, this outer automorphism Φ only depends on the mapping class of f). Such an outer automorphism Φ of F_N (and also any automorphism $\varphi \in \Phi$) is called *geometric*. Classical Galois theory for covering spaces states that the lifts of f are in bijective correspondence with the automorphisms in the outer class Φ. More precisely, an automorphism $\varphi \in \Phi$ and a lift $\widetilde{f}$ of f are in correspondence if and only if

$$\varphi(g) \circ \widetilde{f} = \widetilde{f} \circ g \quad \forall g \in F_N,$$

where the elements of F_N are considered as deck transformations of $\widetilde{\Sigma}$. As a consequence, the dynamics of $\partial\widetilde{f}$ on $\partial\widetilde{\Sigma}$ and the dynamics of $\partial\varphi$ on ∂F_N are conjugate via the natural identification between $\partial\widetilde{\Sigma}$ and ∂F_N.

It follows from the previous discussion that, for any geometric automorphism $\varphi \in \mathrm{Aut}(F_N)$, the homeomorphism $\partial\varphi$ of ∂F_N must preserve a cyclic order on ∂F_N.

Another fact proved by Nielsen is that $\partial\tilde{f}$ has at least 2 periodic points on ∂F_N (for a proof in the context of free groups; see [14]). This means that there exists a positive power of $\partial\tilde{f}$ which has at least 2 fixed points on ∂F_N. Both these facts (existence of 2 fixed points and preservation of a cyclic order) yield directly:

Proposition 4.2 *A geometric automorphism of F_N cannot have a parabolic orbit in ∂F_N.*

This fact is particularly meaningful for the free group of rank 2. Indeed, it is well known that any outer automorphism of F_2 can be induced by a homeomorphism of a torus with one boundary component; see Nielsen [18]. This is precisely how Proposition 1.2 is proved.

4.3 Outer automorphisms

Although well known, we believe that at this point it might be wise to alert the less expert reader about a common misunderstanding. It is by no means true that any two automorphisms φ, φ' which belong to the same outer automorphism class Φ, must have conjugate dynamics. Indeed, their dynamics graphs Γ_φ and $\Gamma_{\varphi'}$ may look quite different. Concrete examples are easy to come by, and some are given in the subsequent sections.

The reader who wants to be more subtle can easily check that indeed some automorphisms in Φ have naturally conjugate dynamics. The resulting *isogredience classes* go again all the way back to Nielsen (see also [13]), and one could associate to Φ a *total dynamics graph* which is the disjoint union of the Γ_φ over a set of representatives for the single isogredience classes. However, this goes beyond the scope of this paper.

5 Parabolic orbits

5.1 Structure of a parabolic fixed point

Let $\varphi \in \mathrm{Aut}(F_N)$ be an automorphism, and $X \in \mathrm{Fix}(\partial\varphi)$ be a parabolic fixed point for φ. We have seen (cf Remark 3.3) that X must be singular. A point $X \in \partial F_N$ is *rational* if it a fixed point of an inner automorphism, ie $X = u^\infty$ for some $u \in F_N \smallsetminus \{1\}$. It is proved in [10] that singular limit points of φ are rational. We deduce the following:

Lemma 5.1 *A parabolic fixed point X of $\varphi \in \mathrm{Aut}(F_N)$ is a singular rational point: $X = u^\infty$ with $u \in \mathrm{Fix}(\varphi)$.* □

Moreover, we have:

Proposition 5.2 *Let φ be an automorphism of F_N, and $X \in \mathrm{Fix}(\partial\varphi)$ be a parabolic fixed point for φ. Then any neighborhood of X in ∂F_N contains a full orbit $\{\partial\varphi^k(Y) \mid k \in \mathbb{Z}\} \subset \partial F_N$*

Proof We have seen that $X = u^\infty$, with $u \in \mathrm{Fix}(\varphi)$. We consider a given neighborhood $\mathcal{V}$ of X. Let $\vartheta = \{\partial\varphi^k(Y)) \mid k \in \mathbb{Z}\}$ be a parabolic orbit for X. We note that $\vartheta \cup \{X\}$ is a compact subset of ∂F_N. Moreover, $u^{-\infty} \notin \vartheta \cup \{X\}$ because $Y \notin \mathrm{Fix}(\partial\varphi)$. Since the sequence $(\partial i_u^p)_{p\in\mathbb{N}}$ uniformly converges on compact subsets of $\partial F_N \smallsetminus \{u^{-\infty}\}$ towards u^∞ when p tends to infinity (see Section 4.1), the set $\partial i_u^p(\vartheta)$ is contained in $\mathcal{V}$, up to taking p sufficiently large. We remark that, since $u \in \mathrm{Fix}(\varphi)$, $\partial i_u^p(\partial\varphi^k(Y)) = \partial\varphi^k(u^p Y)$, and thus $\partial i_u^p(\vartheta) = u^p\vartheta$ is a parabolic orbit for X. □

5.2 Automorphisms of F_4 which have parabolic orbits

For any $k \in \mathbb{N}$, consider the automorphism φ_k of $F_4 = \langle a, b, c, d\rangle$ given by

$$\begin{aligned} \varphi_k\colon a &\mapsto a \\ b &\mapsto ba \\ c &\mapsto ca^{k+1} \\ d &\mapsto dc \end{aligned}$$

and its inverse

$$\begin{aligned} \varphi_k^{-1}\colon a &\mapsto a \\ b &\mapsto ba^{-1} \\ c &\mapsto ca^{-k-1} \\ d &\mapsto da^{k+1}c^{-1}. \end{aligned}$$

The rose R_4 is the geometric realization of graph with one vertex and 4 edges. We put an orientation on each edge, and we label them by a, b, c and d. We can turn R_4 into a length space by declaring that each edge has length 1. As usual, the automorphisms $\varphi_k^{\pm 1}$ can be realized as homotopy equivalences $f_k^{\pm}$ of the rose R_4 where each edge is mapped linearly to the edge path with label preassigned by $\varphi_k^{\pm 1}$.

In fact, the automorphisms $\varphi_k^{\pm 1}$ define outer automorphisms which are unipotent polynomially growing in the sense of Bestvina, Feighn and Handel [1], and the maps f_k^+

satisfy the conclusions of [1, Theorem 5.1.8]. We do not quote here the statement of this theorem, which would lead us to introduce a lot of technical background, but we freely use in the sequel some consequences of it.

Let $\mathcal{A}$ be a basis of F_N. We denote by $[g]$ the reduced word, in the basis $\mathcal{A}$, representing the element $g \in F_N$. Let φ be an automorphism of F_N. A *splitting* of $g \in F_N$ for φ is a way to write $g = g_1 \cdots g_n$ such that:

(i) $n \geqslant 2$,

(ii) for all $i \in \{1, \dots, n\}$, $g_i \in F_N \smallsetminus \{1\}$,

(iii) for all $p \in \mathbb{N}$, for all $i \in \{1, \dots, n-1\}$, $[\varphi^p(g_i)][\varphi^p(g_{i+1})] = [\varphi^p(g_i g_{i+1})]$ (this means that no cancellation occurs between $[\varphi^p(g_i)]$ and $[\varphi^p(g_{i+1})]$).

In that case, we note $g = g_1 \cdots g_n$, and each g_i is called a *brick* of the splitting.

We now apply that [1, Theorem 5.1.8] to the given family φ_k and obtain:

Lemma 5.3 *For all $g \in F_4$, there exists some $p_0 \in \mathbb{N}$ such that for all $p \geqslant p_0$, $[\varphi_k^p(g)]$ and $[\varphi_k^{-p}(g)]$ have a splitting, the bricks of which are either edges or paths of the following labels: $ba^q b^{-1}$, $ca^q c^{-1}$, $ba^q c^{-1}$ or $ca^q b^{-1}$, for some $q \in \mathbb{Z}$.* □

Remark 5.4 For the reader who is familiar with the terminology of [1], the edge paths labelled by $ba^q b^{-1}$, $ca^q c^{-1}$, $ba^q c^{-1}$ or $ca^q b^{-1}$ are precisely the exceptional paths of the improved train-track map f_k.

Remark 5.5 As a consequence of Lemma 5.3, one can easily check that the sequence $(\|[\varphi_k^p(g)]\|)_{p\in\mathbb{N}}$ of lengths of $[\varphi_k^p(g)]$ is bounded above by a polynomial of degree 2 in p.

It is claimed in Maslakova [17] that there exists a general algorithm to compute the fixed subgroup of a given automorphism of F_N. There exist some easier algorithms for special cases: for instance, one could use Cohen and Lustig [3] to compute the fixed subgroup of φ_k. In fact, it is sufficient to determines the so called *indivisible Nielsen paths*; see Bestvina and Handel [2].

Let $\mathcal{NP}$ denote the set $\{a^q, ba^q b^{-1}, ca^q c^{-1} \mid q \in \mathbb{Z}\}$. We notice that $\mathcal{NP} \subseteq \mathrm{Fix}(\varphi_k)$. For an element $g \in F_4$, we consider the splitting of $[\varphi_k^p(g)] = g_1 \cdot g_2 \cdot \ldots \cdot g_r$ given by Lemma 5.3. If $g_i \in \mathcal{NP}$ for all $i \in \{1, \dots, r\}$, then $g \in \mathrm{Fix}(\varphi_k)$. Otherwise, there is some i_0 such that $g_i \in \mathcal{NP}$ for all $i \in \{1, \dots, i_0\}$ and $g_{i_0+1} \notin \mathcal{NP}$. For simplicity, we write $\omega_{\varphi_k} = \omega_k$. Then $\omega_k(g) = g_1 \ldots g_{i_0} \omega_k(g_{i_0+1}) \in \partial F_4$. This shows that $\mathrm{Fix}(\varphi_k) = \langle a, bab^{-1}, cac^{-1} \rangle$.

Moreover, we thus obtain all the isoglossy classes of limit points of φ_k by considering all the $\omega_k(h)$ with $h \in \{b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, ba^q c^{-1}, ca^q b^{-1} \mid q \in \mathbb{Z}\}$. A direct computation gives $\varphi_k^p(b) = ba^p$, $\varphi_k^p(c) = ca^{p(k+1)}$, $\varphi_k^p(ba^q c^{-1}) = ba^{q-kp}c^{-1}$, $\varphi_k^p(d) = dcca^{k+1}ca^{2(k+1)} \cdots ca^{(p-1)(k+1)}$, $\varphi_k(d^{-1}) = c^{-1} \cdot d^{-1}$. We derive that:

- $\omega_k(b) = ba^{+\infty}$,
- $\omega_k(c) = \omega_k(ca^q b^{-1}) = ca^{+\infty}$,
- $\omega_k(b^{-1}) = \omega_k(c^{-1}) = \omega_k(d^{-1}) = a^{-\infty}$,
- $\omega_k(ba^q c^{-1}) = ba^{-\infty}$,
- $\omega_k(d) = X_k^+$,

where $X_k^+ = \omega_k(d) = dcca^{k+1}ca^{2k+2}ca^{3k+3} \ldots$. We have thus shown that there are only 5 isoglossy classes in $L^{\omega}_{\varphi_k}$, given by X_k^+, $a^{-\infty}$, $ca^{+\infty}$, $ba^{+\infty}$, $ba^{-\infty}$.

Theorem 5.6 *The set $\{\varphi_k \mid k \in \mathbb{N}\}$ is a family of automorphisms of F_4, such that each φ_k has a parabolic orbit. The dynamics graph of φ_k is given in Figure 2. For any $k, k', p, p' \in \mathbb{N}$, φ_k^p and $\varphi_{k'}^{p'}$ are conjugate if and only if $k = k'$ and $p = p'$.*

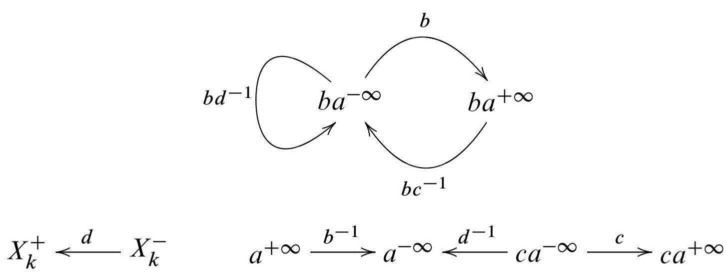

Figure 2. The dynamics graph of φ_k has 3 connected components. A label g has been added to each edge: it means that $\omega_k(g)$ is the endpoint of the edge and $\omega_k^-(g)$ is the origin of the edge.

Proof We write $\omega_{\varphi_k^{-1}} = \omega_k^-$. Arguing for φ_k^{-1} as we have done for φ_k, we obtain:

- $\omega_k^-(b^{-1}) = \omega_k^-(c^{-1}) = a^{+\infty}$,
- $\omega_k^-(c) = \omega_k^-(d^{-1}) = ca^{-\infty}$,
- $\omega_k^-(b) = ba^{-\infty}$,
- $\omega_k^-(bc^{-1}) = ba^{+\infty}$,
- $\omega_k^-(d) = X_k^-$,

where $X_k^- = \omega_k^-(d) = da^{k+1}c^{-1}a^{2k+2}c^{-1}a^{3k+3}c^{-1} \ldots$. Again, there are only 5 isoglossy classes in $L^{\omega}_{\varphi_k^{-1}}$, given by X_k^-, $a^{+\infty}$, $ca^{-\infty}$, $ba^{-\infty}$, $ba^{+\infty}$.

Note that $\varphi_k(bd^{-1}) = bac^{-1} \cdot d^{-1}$ is a splitting for φ_k. Hence $\omega_k(bd^{-1}) = \omega_k(bac^{-1}) = ba^{-\infty}$. On the other hand, $b \cdot d^{-1}$ is a splitting for φ_k^{-1}. Hence $\omega_k^-(bd^{-1}) = \omega_k^-(b) = ba^{-\infty}$. Thus $ba^{-\infty}$ is parabolic fixed point for φ_k.

Suppose that φ_k^p and $\varphi_{k'}^{p'}$ are conjugate ($k, k', p, p' \in \mathbb{N}$): there exists $\psi \in \mathrm{Aut}(F_4)$ such that $\varphi_k^p = \psi \varphi_{k'}^{p'} \psi^{-1}$. Let $M_k, M_{k'}, P \in \mathrm{GL}(4, \mathbb{Z})$ be the matrices obtained by abelianization of respectively φ_k, $\varphi_{k'}$ and ψ. Then

$$M_k^p = \begin{pmatrix} 1 & p & (k+1)p & \frac{1}{2}(k+1)p(p-1) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & p \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Computing $M_k^p P = P M_{k'}^{p'}$, one sees that P must have the following shape:

$$P = \begin{pmatrix} \lambda_1 & \mu_1 & \mu_2 & \mu_3 \\ 0 & \lambda_2 & 0 & \mu_4 \\ 0 & \mu_5 & \lambda_3 & \mu_6 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

with

$$p'(k'+1)\lambda_3 = p(k+1)\lambda_1 \quad \text{and} \quad p'\lambda_3 = p\lambda_4. \tag{1}$$

We deduce that $\det P = \lambda_1\lambda_2\lambda_3\lambda_4$, and thus $\lambda_i \in \{\pm 1\}$, since $\det P = \pm 1$. From (1) we derive $k = k'$ and $p = p'$. $\square$

5.3 Parabolic orbits for $N \geqslant 5$

For any $k \in \mathbb{N}$, consider the automorphism α_k of $F_5 = \langle a, b, c, d, e \rangle$ given by

$$\begin{aligned} \alpha_k \colon a &\mapsto a \\ b &\mapsto ba \\ c &\mapsto ca^{k+1} \\ d &\mapsto dc \\ e &\mapsto e. \end{aligned}$$

Since the restriction of α_k to $\langle a, b, c, d \rangle$ is φ_k, clearly, $\omega_{\alpha_k}(bd^{-1}) = \omega_{\alpha_k^{-1}}(bac^{-1}) = ba^{-\infty}$ is a parabolic fixed point for α_k. Considering the abelianization and arguing as previously, we check that if $k \neq k'$ and $p \neq p'$, then α_k^p and $\alpha_{k'}^{p'}$ cannot be conjugate.

If $N \geqslant 6$, we split $F_N = F_4 * F_2 * F_{N-6}$. We first recall some facts about $\mathrm{Out}(F_2)$. It is well known, since Nielsen [18], that the abelianisation morphism from $\mathrm{Out}(F_2)$

to $\mathrm{GL}_2(\mathbb{Z})$ is an isomorphism. If $M \in \mathrm{SL}_2(\mathbb{Z})$ has a trace bigger than 2, then M has an eigenvalue $\lambda > 1$ which is an algebraic unity of a quadratic extension of $\mathbb{Q}$: we call λ the dilatation of M. For all $k \in \mathbb{N}$ prime, there exists $M_k \in \mathrm{SL}_2(\mathbb{Z})$ such that the dilatation λ_k of M_k belongs to $\mathbb{Q}(\sqrt{k}) \smallsetminus \mathbb{Q}$. This implies in particular that for all $p \in \mathbb{N}$, $\lambda_k^p \in \mathbb{Q}(\sqrt{k}) \smallsetminus \mathbb{Q}$. We choose $\theta_k \in \mathrm{Aut}(F_2)$ in the outer class represented by M_k. Then the automorphism θ_k^p has growth rate equal to λ_k^p.

We define $\beta_k \in \mathrm{Aut}(F_N)$ by $\beta_k = \varphi_1 * \theta_k * \mathrm{id}$, where id is the identity on F_{N-6}. Again, $\omega_{\beta_k}(bd^{-1}) = \omega_{\beta_k^{-1}}(bac^{-1}) = ba^{-\infty}$ is a parabolic fixed point for β_k. Since φ_1 is polynomially growing, it follows that the growth rate of β_k^p is λ_k^p (see for instance [12]). This proves that β_k^p is not conjugate to $\beta_{k'}^{p'}$ if $k \neq k'$ or $p \neq p'$, because the growth rate is a conjugacy invariant and because $\mathbb{Q}(\sqrt{k}) \cap \mathbb{Q}(\sqrt{k'}) = \mathbb{Q}$ (if k and k' are prime integers).

This finishes the proof Theorem 1.1. In view of Proposition 1.2, it remains to ask the following question, the answer of which we do not know:

Question 5.7 *Does there exist an automorphism of F_3 which has a parabolic orbit?*

6 Dehn twist automorphisms of F_2

In this last section, we calculate the dynamics graphs of all the automorphisms in the outer class of δ^n ($n \in \mathbb{Z}$, $n \neq 0$), where δ is the automorphism of $F_2 = \langle a, b\rangle$ defined by $\delta(a) = a$ and $\delta(b) = ba$.

Let $D \in \mathrm{Out}(F_2)$ be the outer class of δ. As explained in Section 4.2, the automorphisms in the outer class D^n ($n \in \mathbb{Z}$) cannot have parabolic orbits. We are going to describe more precisely the dynamics induced on ∂F_N by the automorphisms in the outer class D^n ($n \in \mathbb{Z}$, $n \neq 0$). For that, we pursue the strategy of [6; 13], where the interested reader will be able to find details of the following constructions.

The rose R_2 is the geometric realization of the graph with one vertex and 2 edges. We put an orientation on each edge, and we label them by a and b. We can turn R_2 in a length space by declaring that each edge has length 1. We represent D^n by an homotopy equivalence f of R_2 defined in the following way: f is the identity on the edge a and linearly sends the edge b to the edge path labelled ba^n.

The universal cover $\widetilde{R}_2$ of R_2 is a tree, equipped by the action of F_2 by deck transformations. We lift the labels of the edges of R_2 to the edges of $\widetilde{R}_2$. Equivalently, $\widetilde{R}_2$ can be considered as the Cayley graph of F_2 relative to the generating set $\{a, b\}$. Let T be the tree obtained by contracting in $\widetilde{R}_2$ all the edges labelled by a: the action of F_N on $\widetilde{R}_2$ induces an action of F_2 on T by isometries. We note that the stabilizer of a vertex of T is conjugate to the subgroup $\langle a \rangle \subset F_N$ generated by a.

As in the geometric case (see Section 4.2) the automorphisms in the outer class D^n are in 1:1 correspondence with the lifts of f to $\widetilde{R}_2$. Moreover, these lifts of f induce isometries of T. More precisely, the isometry H of T associated to the automorphism $\delta^n \in D^n$ satisfies

$$\delta^n(g) \circ H = H \circ g \quad \forall g \in F_N,$$

where the elements of F_N are considered as isometries of T. Then, for $u \in F_N$, the map $H_u = u \circ H$ is the isometry of T associated to the automorphism $i_u \circ \delta^n \in D^n$, since $(i_u \circ \delta^n)(g) \circ H_u = H_u \circ g$ holds for all $g \in F_N$.

If H_u is a hyperbolic isometry of T, then $i_u \circ \delta^n$ has North-South dynamics and the fixed points of $i_u \circ \delta^n$ are determined by the ends of the axis of H_u in T; see [13].

If H_u is an elliptic isometry, let $P \in T$ be a fixed point of H_u. There exists some $w \in F_N$ such that the stabilizer of P in F_N is $w\langle a \rangle w^{-1}$. The fact that P is a fixed point of H_u then results in the existence of an integer $k \in \mathbb{Z}$ such that $u\delta^n(w) = wa^k$. Or equivalently, such that $i_u \circ \delta^n = i_w \circ (i_{a^k} \circ \delta^n) \circ i_w^{-1}$. Indeed,

$$\begin{aligned} i_u \circ \delta^n &= i_{wa^k(\delta^n(w))^{-1}} \circ \delta^n \\ &= i_{wa^k\delta^n(w^{-1})} \circ \delta^n \\ &= i_w \circ i_{a^k} \circ i_{\delta^n(w^{-1})} \circ \delta^{n^S} \\ &= i_w \circ i_{a^k} \circ \delta^n \circ i_{w^{-1}}. \end{aligned}$$

The dynamics of $\partial(i_u \circ \delta^n)$ is thus conjugate to the dynamics of $\partial(i_{a^k} \circ \delta^n)$ for some $k \in \mathbb{Z}$. We are now going to study in more detail the automorphisms $i_{a^k} \circ \delta^n$ for $k \in \mathbb{Z}$, and in particular, to give their dynamics graphs.

The inverse of $i_{a^k} \circ \delta^n$ is $i_{a^{-k}} \circ \delta^{-n}$. We note that

$$\begin{aligned} i_{a^k} \circ \delta^n \colon \quad a &\mapsto a & \qquad i_{a^{-k}} \circ \delta^{-n} \colon \quad a &\mapsto a \\ b &\mapsto a^k b a^{n-k} & b &\mapsto a^{-k} b a^{k-n} \\ b^{-1} &\mapsto a^{k-n} b^{-1} a^{-k} & b^{-1} &\mapsto a^{n-k} b^{-1} a^k. \end{aligned}$$

Thus the dynamics of $\partial(i_{a_2^k} \circ \delta^n)$ depends on the sign of k and of $n-k$.

Remark 6.1 Let $\sigma \in \mathrm{Aut}(F_N)$ defined by $\sigma(a) = a^{-1}$ and $\sigma(b) = b^{-1}$. We note that $i_{a^k} \circ \delta^n$ and $i_{a^{n-k}} \circ \delta^n$ are conjugate by the involution σ.

First case Assume $k(n-k) = 0$. Since δ^n and $i_{a^n} \circ \delta^n$ are conjugate by σ (see Remark 6.1), we focus on δ^n. One can check that $\mathrm{Fix}(\delta^n) = \langle a, bab^{-1} \rangle$. Let X be a point in $\partial F_2 \smallsetminus \partial\langle a, bab^{-1}\rangle$, and let x be the longest prefix of X in $\langle a, bab^{-1}\rangle$. Then $X = xY$, with no cancellation between x and Y, and the first letter of Y is equal to b or to b^{-1}. If Y begins by b, then $\omega_{\delta^n}(Y) = ba^{\infty}$ and $\omega_{\delta^{-n}}(Y) = ba^{-\infty}$. If Y begins by b^{-1}, then $\omega_{\delta^n}(Y) = a^{-\infty}$ and $\omega_{\delta^{-n}}(Y) = a^{\infty}$. Hence δ^n has 2 isoglossy classes of ω–limit points (with representatives ba^{∞} and $a^{-\infty}$), and δ^{-n} has 2 isoglossy classes of ω–limit points (with representatives $ba^{-\infty}$ and a^{∞}). The dynamics graph of δ^n is given in Figure 3.

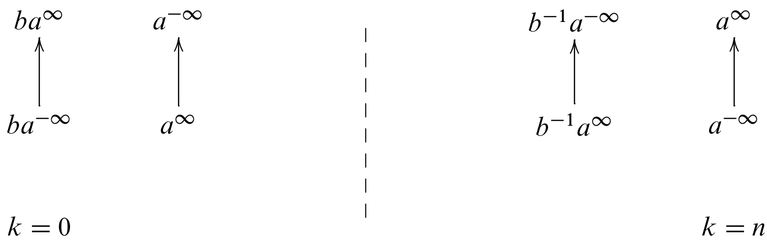

Figure 3. Dynamics graph of $i_{a^k} \circ \delta^n$ for $k(n-k) = 0$

Second case $k(n-k) < 0$. We suppose that $k > n$ (from which one deduces the case $k < 0$ by using Remark 6.1). The fixed subgroup is $\mathrm{Fix}(i_{a^k} \circ \delta^n) = \langle a\rangle$. We note that $\omega_{i_{a^k}\circ\delta^n}(b) = \omega_{i_{a^k}\circ\delta^n}(b^{-1}) = a^{\infty}$ and $\omega_{(i_{a^k}\circ\delta^n)^{-1}}(b) = \omega_{(i_{a^k}\circ\delta^n)^{-1}}(b^{-1}) = a^{-\infty}$. It follows that $\partial(i_{a^k} \circ \delta^n)$ has North-South dynamics on ∂F_2; see Figure 4.

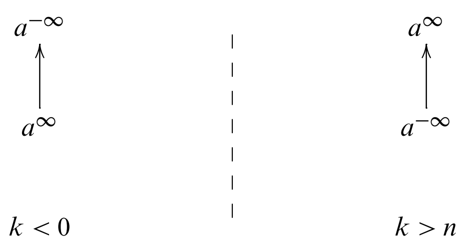

Figure 4. Dynamics graph of $i_{a^k} \circ \delta^n$ for $k(n-k) < 0$

Third case $k(n-k) > 0$, ie $0 < k < n$. We check that the fixed subgroup is equal to $\mathrm{Fix}(i_{a^k} \circ \delta^n) = \langle a\rangle$. We note that $\omega_{i_{a^k}\circ\delta^n}(b) = a^{\infty}$, $\omega_{i_{a^k}\circ\delta^n}(b^{-1}) = a^{-\infty}$, $\omega_{(i_{a^k}\circ\delta^n)^{-1}}(b) = a^{-\infty}$ and $\omega_{(i_{a^k}\circ\delta^n)^{-1}}(b^{-1}) = a^{\infty}$. For $x \in \partial F_2$, it follows that $\omega_{i_{a^k}\circ\delta^n}(X)$ and $\omega_{(i_{a^k}\circ\delta^n)^{-1}}(X)$ depend only on the first occurrence of the letter b or b^{-1} in X: if it is b, then $\omega_{i_{a^k}\circ\delta^n}(X) = a^{\infty}$ and $\omega_{(i_{a^k}\circ\delta^n)^{-1}}(X) = a^{-\infty}$; if it is b^{-1}, then $\omega_{i_{a^k}\circ\delta^n}(X) = a^{-\infty}$ and $\omega_{(i_{a^k}\circ\delta^n)^{-1}}(X) = a^{\infty}$. We say that $\partial(i_{a^k} \circ \delta^n)$ has *semi-North-South dynamics* on ∂F_2; see Figure 5.

$$\begin{tikzcd} a^{-\infty} \arrow[d, bend left] \\ a^{\infty} \arrow[u, bend left] \end{tikzcd}$$

Figure 5. Dynamics graph of $i_{a^k} \circ \delta^n$ for $k(n-k) > 0$

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