

## Random groups arising as graph products

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In this paper we study the hyperbolicity properties of a class of random groups arising as graph products associated to random graphs. Recall, that the construction of a graph product is a generalization of the constructions of right-angled Artin and Coxeter groups. We adopt the Erdős and Rényi model of a random graph and find precise threshold functions for hyperbolicity (or relative hyperbolicity). We also study automorphism groups of right-angled Artin groups associated to random graphs. We show that with probability tending to one as  $n \rightarrow \infty$ , random right-angled Artin groups have finite outer automorphism groups, assuming that the probability parameter  $p$  is constant and satisfies  $0.2929 < p < 1$ .

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### 1 Introduction

The needs of mathematical modeling of large systems of various nature raise the problem of studying random geometric and algebraic objects. For a system of great complexity it is unrealistic to assume that one is able to have a precise description of its configuration space; the latter should be viewed as being a partially known or a random space.

The most developed stochastic–topological object is a random graph. The theory of random graphs, initiated in 1959 by Erdős and Rényi [17], is nowadays a well-developed and fast growing branch of discrete mathematics. The theory of random graphs (see Alon and Spencer [1], Bollobás [4], and Janson, Łuczak and Rucinski [24]) offers a plethora of spectacular results and predictions playing an essential role in various engineering and computer science applications.

Configuration spaces of mechanical linkages with bars of random lengths were studied in Farber [19] and Farber–Kappeler [20]. These are closed smooth manifolds depending on a large number of independent random parameters. Although the number of homeomorphism types of these manifolds grows extremely fast with the dimension, their topological characteristics can be predicted with high probability when the number of links tends to infinity.

Random simplicial complexes of higher dimension were recently introduced and studied by Linial–Meshulam in [28] and Meshulam–Wallach in [30]. The fundamental groups of random 2–complexes are random groups of a fairly general type. Random groups arising in this way are the focus of recent work by Babson–Hoffman–Kahle [2].

The theory of random groups introduced by Gromov in [21; 22] depends on a density parameter  $0 \leq d \leq 1$ . In this model, a random group is given by a presentation with randomly chosen relations where each of the relations has a fixed length and the number of relations depends on the density. Gromov proved that for  $d < 1/2$ , a random group is infinite and hyperbolic, while it is trivial for  $d > 1/2$  (see [21, page 273]). For more on this model we refer to the work of Zuk [35] and Ollivier [32].

Automorphism groups of 1– and 2–relator random groups were studied by Champetier [5] and Kapovich, Schupp, and Shpilrain [26] (see also Kapovich–Schupp [25]). In [26], for example, they show that a random 1–relator group is torsion-free word hyperbolic and has trivial outer automorphism group. Other types of probabilistic questions in group theory, such as the existence of homomorphisms between groups of a specific type, have been investigated by Shalev, Liebeck, and others (see for example, Shalev [34]).

Random right-angled Artin groups, which were first studied in Costa–Farber [12], represent a different class of random groups. The right-angled Artin group associated to a graph  $\Gamma$  is the group generated by the vertex set of  $\Gamma$  with commutation relations between adjacent vertices. A random graph in the Erdős–Rényi model gives rise to a random right-angled Artin group in which each pair of generators commutes with a fixed probability  $0 \leq p \leq 1$ . A random right-angled Coxeter group is obtained by adding additional relations specifying that each generator has order two.

In this paper we study a class of groups, called graph products, that includes both right-angled Artin groups and right-angled Coxeter groups. The paper has two main goals. First, we study hyperbolicity (or relative hyperbolicity) of graph products associated to random graphs. Second, we study automorphisms of random right-angled Artin groups and more generally, random graph products of cyclic groups. The interest in these results lies not in the random graph theory itself, but in the applications to random group theory. Graph products form a large and interesting class of groups which, as this paper demonstrates, lend themselves naturally to probabilistic methods.

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## 2 The hyperbolicity of random graph products

The construction of graph products is a generalization of the constructions of right-angled Artin and Coxeter groups. Let  $\Gamma$  be a finite simplicial graph with vertex set  $V$  and let  $\{G_v\}$  be a collection of nontrivial, finitely generated groups indexed by  $V$ . Then the *graph product*  $\mathcal{G}_\Gamma = \mathcal{G}(\Gamma, \{G_v\})$  is the quotient of the free product  $\prod G_v$  by commutator relations between the generators of  $G_v$  and  $G_w$  whenever  $v, w$  are connected by an edge in  $\Gamma$ . Well known examples of these groups are the right-angled Coxeter groups (when  $G_v = \mathbb{Z}/2\mathbb{Z}$  for all  $v$ ) and the right-angled Artin groups (when  $G_v = \mathbb{Z}$  for all  $v$ ). We will denote these groups by  $W_\Gamma$  and  $A_\Gamma$  respectively.

We use the Erdős and Rényi model of random graphs in which each edge of the complete graph on  $n$  vertices is included with probability  $0 < p < 1$  independently of all other edges, where  $p$  is a function of  $n$ . In other words, we consider the probability space  $G(n, p)$  of all  $2^{\binom{n}{2}}$  subgraphs of the complete graph on  $n$  vertices  $\{1, 2, \dots, n\}$  and the probability that a specific graph  $\Gamma \in G(n, p)$  appears as a result of a random process equals

$$(1) \quad \text{Prob}(\Gamma) = p^{E_\Gamma} (1 - p)^{\binom{n}{2} - E_\Gamma},$$

where  $E_\Gamma$  denotes the number of edges of  $\Gamma$ , see Janson, Łuczak and Rucinski [24].

We say a statement holds *asymptotically almost surely* or *a.a.s.* if the probability that it holds tends to one as  $n \rightarrow \infty$ . For  $f, g$  functions of  $n$  and  $c$  a constant, we use the notation

- $f \rightarrow c$  to mean  $f$  tends to  $c$  as  $n$  goes to infinity, and
- $f \sim g$  to mean  $f/g \rightarrow 1$ .

Our main results concerning hyperbolicity are as follows:

**Theorem 2.1** *Fix a collection of nontrivial, finite groups indexed by the natural numbers  $\{G_i\}$ . For a random graph  $\Gamma \in G(n, p)$ , let  $\mathcal{G}_\Gamma = \mathcal{G}(\Gamma, \{G_i\})$  be the associated graph product. Then*

- (1) *If  $(1 - p)n^2 \rightarrow 0$  then  $\mathcal{G}_\Gamma$  is finite, a.a.s*
- (2) *If  $pn \rightarrow 0$  then  $\mathcal{G}_\Gamma$  is hyperbolic, a.a.s.*
- (3) *If  $pn \rightarrow \infty$  and  $(1 - p)n^2 \rightarrow \infty$ , then  $\Gamma$  is not hyperbolic, a.a.s.*

Applying this to the case where all vertex groups are cyclic of order 2, we obtain the following.

**Corollary 2.2** *The right-angled Coxeter group  $W_\Gamma$  corresponding to a random Erdős and Rényi graph  $\Gamma \in G(n, p)$  is hyperbolic (a.a.s) if either  $(1 - p)n^2 \rightarrow 0$  or  $pn \rightarrow 0$ . If, however,  $pn \rightarrow \infty$  and  $(1 - p)n^2 \rightarrow \infty$  then the random right-angled Coxeter group  $W_\Gamma$  is not hyperbolic, a.a.s.*

If the vertex groups  $G_v$  are not necessarily finite, one may speak about hyperbolicity relative to the family of vertex subgroups  $\{G_v\}$ . We establish the following result:

**Theorem 2.3** *Fix a collection of nontrivial, finitely generated groups indexed by the natural numbers  $\{G_i\}$ . For a random graph  $\Gamma \in G(n, p)$ , let  $\mathcal{G}_\Gamma = \mathcal{G}(\Gamma, \{G_i\})$  be the associated graph group. Then*

- (1) *If  $(1 - p)n^2 \rightarrow 0$  then  $\mathcal{G}_\Gamma$  is isomorphic to the direct product  $G_1 \times \cdots \times G_n$ , a.a.s.*
- (2) *If  $pn \rightarrow 0$  then  $\mathcal{G}_\Gamma$  is weakly hyperbolic relative to the family of subgroups  $\{G_i\}_{i \leq n}$ , a.a.s.*
- (3) *If  $pn \rightarrow \infty$  and  $(1 - p)n^2 \rightarrow \infty$  then  $\mathcal{G}_\Gamma$  is not weakly hyperbolic relative to the family of subgroups  $\{G_i\}_{i \leq n}$ , a.a.s.*

Since the vertex groups in [Theorem 2.3](#) are arbitrary finitely generated groups, one could add an additional random element by taking them to be some class of random groups. The same result still holds. Likewise in [Theorem 2.1](#), the vertex groups can be random finite groups.

The proofs of [Theorems 2.1](#) and [2.3](#) will be given in [Section 4](#).

### 3 Graph products and buildings

Let  $\mathcal{G}_\Gamma$  be a graph product of groups  $\{G_v\}$  where  $v$  runs over vertices of a graph  $\Gamma$ . We call a complete subgraph of  $\Gamma$  a *clique*.<sup>1</sup> We can associate a simplicial complex  $X_\Gamma = X(\Gamma, \{G_v\})$  with  $\mathcal{G}_\Gamma$  as follows. Define two posets, partially ordered by inclusion,

$$S_\Gamma = \{G_T \mid T \subseteq V, T = \emptyset \text{ or } T \text{ spans a clique in } \Gamma\}$$

$$\mathcal{G}S_\Gamma = \{gG_T \mid g \in \mathcal{G}_\Gamma, G_T \in S_\Gamma\}.$$

Recall that the geometric realization (or flag complex) associated to a poset  $\mathcal{P}$  is the simplicial complex whose vertices are the elements of  $\mathcal{P}$  and whose  $k$ -simplices are totally ordered subsets  $(p_0 < p_1 < \cdots < p_k)$ . Let  $X_\Gamma$  be the geometric realization of

<sup>1</sup>Some authors reserve the word clique for maximal complete subgraphs. We do not require maximality.

$\mathcal{G}\mathcal{S}_\Gamma$  and let  $K \subset X_\Gamma$  be the geometric realization of  $\mathcal{S}_\Gamma$ . Left multiplication induces an action of  $\mathcal{G}_\Gamma$  on  $X_\Gamma$  with fundamental domain  $K$ . The stabilizer of the vertex  $gG_T$  is conjugate to  $G_T$ . Thus, the action of  $\mathcal{G}_\Gamma$  on  $X_\Gamma$  is always cocompact, and it is proper if and only if all vertex groups are finite.

In the case that every vertex group is cyclic of order 2,  $X_\Gamma$  is the well-known Davis complex for the right-angled Coxeter group  $W_\Gamma$ . For a more general graph product  $\mathcal{G}_\Gamma$ ,  $X_\Gamma$  is a right-angled building whose apartments are isomorphic to this Davis complex. (See Davis [13] or Charney, Ruane, Stambaugh and Vijayan [8] for a discussion of these buildings.)

The complexes  $X_\Gamma$  have a natural metric. Though  $X_\Gamma$  was defined as a simplicial complex, it also has a natural cubical structure. To see this, it suffices to describe the cubical structure on the fundamental domain  $K$ . For a pair  $G_T \subseteq G_{T'}$  in  $\mathcal{S}_\Gamma$ , let

$$[G_T, G_{T'}] = \{G_R \in \mathcal{S}_\Gamma \mid T \subseteq R \subseteq T'\}.$$

It is easily seen that  $[G_T, G_{T'}]$  spans a cube of dimension  $|T'| - |T|$  in  $K$ . The cubical structure induces a piecewise Euclidean metric on  $X_\Gamma$  which was shown by M Davis to be CAT(0) [13].

Moussong [31] showed that in some cases, the Davis complex could be given a CAT(-1) metric and used this to find precise conditions on when an arbitrary Coxeter group is word hyperbolic (see also Davis [14]). In the case of right-angled Coxeter groups, his conditions reduce to the requirement that  $\Gamma$  does not contain an *empty square*, that is, a 4-cycle such that neither diagonal spans an edge in  $\Gamma$ . This theorem was generalized by J. Meier to graph products of finite groups.

**Theorem 3.1** (Meier [29]) *Let  $\mathcal{G}_\Gamma = \mathcal{G}(\Gamma, \{G_v\})$  be a graph product where each  $G_v$  is a nontrivial, finite group. Then  $\mathcal{G}_\Gamma$  is Gromov hyperbolic if and only if  $\Gamma$  has no empty squares.*

To do this, Meier (following Moussong) proves that if  $\Gamma$  has no empty squares, then the cubical metric on  $X_\Gamma$  can be deformed to a CAT(-1) metric. Conversely, if  $\Gamma$  has an empty square, it is easy to show that the apartments in  $X_\Gamma$  contain 2-flats, so  $X_\Gamma$  cannot be hyperbolic. Thus,  $X_\Gamma$  is hyperbolic if and only if  $\Gamma$  has no empty squares. If the vertex groups are all finite, then the action of  $\mathcal{G}_\Gamma$  on  $X_\Gamma$  is proper and cocompact, so by the Milnor-Svarc Lemma,  $\mathcal{G}_\Gamma$  is quasi-isometric to  $X_\Gamma$  and Meier’s theorem follows.

If the vertex groups are not required to be finite, one can still draw some conclusions about  $\mathcal{G}_\Gamma$ , albeit weaker ones. While the action is no longer proper, it is discontinuous,

that is, the orbit of every point is discrete. In [7], the first author and J Crisp prove a relative version of the Milnor–Svarc Lemma. It states that if a finitely generated group  $G$  acts discontinuously and cocompactly on a geodesic metric space  $X$  with stabilizers conjugate to a collection of subgroups  $\mathcal{H}$ , then the graph obtained from the Cayley graph of  $G$  by coning off all cosets of the subgroups in  $\mathcal{H}$  is quasi-isometric to  $X$ . A group  $G$  is said to be *weakly hyperbolic relative to  $\mathcal{H}$*  if this coned off Cayley graph is hyperbolic (see Farb [18]). Applying this to  $\mathcal{G}_\Gamma$  acting on  $X_\Gamma$ , we see that  $\mathcal{G}_\Gamma$  is weakly hyperbolic relative to  $\{G_v\}$  if and only if  $X_\Gamma$  is hyperbolic.

**Theorem 3.2** *Let  $\mathcal{G}_\Gamma = \mathcal{G}(\Gamma, \{G_v\})$  be a graph group where each  $G_v$  is a nontrivial, finitely generated group. Then  $\mathcal{G}_\Gamma$  is weakly hyperbolic relative to  $\{G_v\}$  if and only if  $\Gamma$  has no empty squares.*

We remark that for a group to be (strongly) relatively hyperbolic requires an additional condition on quasi-geodesics in the Cayley graph. In general, one does not expect this additional condition to hold for  $\mathcal{G}_\Gamma$ . Indeed, Behrstock, Drutu, and Mosher [3] have shown that right-angled Artin groups associated to connected graphs  $\Gamma$  are not (strongly) relatively hyperbolic with respect to *any* collection of proper subgroups.

## 4 Proofs of Theorems 2.1 and 2.3

The discussion in the previous section shows that the hyperbolicity (or relative hyperbolicity) of a graph product depends only on the existence of empty squares in the graph. In this section, we consider the probability that a random graph contains an empty square.

The goal of this section is to establish threshold functions for the existence of an empty square in a random graph  $\Gamma$ . We will prove

**Theorem 4.1** *Let  $\Gamma \in G(n, p)$  be a random graph.*

- (1) *If  $(1-p)n^2 \rightarrow 0$  then  $\Gamma$  is a complete graph, a.a.s.*
- (2) *If  $pn \rightarrow 0$  then  $\Gamma$  has no empty squares, a.a.s.*
- (3) *If  $pn \rightarrow \infty$  and  $(1-p)n^2 \rightarrow \infty$  then  $\Gamma$  has an empty square, a.a.s.*

**Proof** The first statement is easy to prove. Namely, for a random graph  $\Gamma \in G(n, p)$ , the probability that a given pair of vertices is not connected by an edge is  $1-p$ , hence the expected number of missing edges in  $\Gamma$  is

$$\binom{n}{2}(1-p) \sim \frac{1}{2}n^2(1-p).$$

By the first moment method (see Janson, Łuczak and Rucinski [24, page 54]), if this expectation goes to 0, then the probability that there exists a missing edge also goes to 0. Thus  $n^2(1 - p) \rightarrow 0$  implies that  $\Gamma$  is a complete graph with probability tending to one as  $n \rightarrow \infty$ .

For the second statement, fix a set of  $n$  vertices and consider an ordered 4-tuple of distinct vertices  $(v_1, v_2, w_1, w_2)$ . Let  $I_{(v_1, v_2, w_1, w_2)}: G(n, p) \rightarrow \{0, 1\}$  be the random variable which takes the value 1 on  $\Gamma \in G(n, p)$  if and only if  $(v_1, v_2, w_1, w_2)$  span an empty square in  $\Gamma$  with  $\{v_1, v_2\}$  and  $\{w_1, w_2\}$  appearing as diagonal pairs (see Figure 1). In more detail, one has  $I_{(v_1, v_2, w_1, w_2)}(\Gamma) = 1$  if and only if the edges

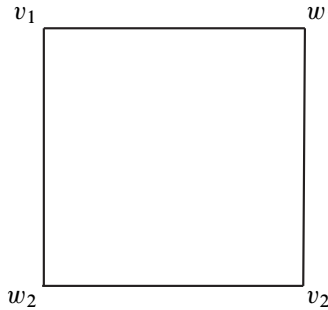


Figure 1

$v_1 w_1, v_1 w_2, v_2 w_1$  and  $v_2 w_2$  are included in  $\Gamma$  and the edges  $v_1 v_2$  and  $w_1 w_2$  are not included in  $\Gamma$ . The sum

$$X = \sum I_{(v_1, v_2, w_1, w_2)}: G(n, p) \rightarrow \mathbb{Z},$$

over all 4-tuples of vertices counts each empty square in  $\Gamma$  eight times, corresponding to the 8 reflections of the square. The expectation  $\mathbb{E}(I_{(v_1, v_2, w_1, w_2)})$  equals  $p^4(1 - p)^2$ , hence

$$\begin{aligned} \mathbb{E}(X) &= \sum \mathbb{E}(I_{(v_1, v_2, w_1, w_2)}) \\ &= n(n - 1)(n - 2)(n - 3)p^4(1 - p)^2 \\ &\sim n^4 p^4(1 - p)^2 \end{aligned}$$

If  $pn \rightarrow 0$  as  $n \rightarrow \infty$ , then  $(1 - p) \rightarrow 1$  and  $\mathbb{E}(X) \rightarrow 0$ , hence the probability that  $\Gamma$  has an empty square goes to 0. This proves statement (2).

For the third statement, we use the second moment method. Namely, the probability that  $X \neq 0$  satisfies

$$\text{Prob}(X \neq 0) \geq \frac{(\mathbb{E} X)^2}{\mathbb{E}(X^2)}$$

(see [24, page 54]). Thus to prove (3), it suffices to show that  $(\mathbb{E}X)^2 \sim \mathbb{E}(X^2)$ . By the previous paragraph, we have

$$(\mathbb{E}X)^2 \sim n^8 p^8 (1-p)^4.$$

To compute  $\mathbb{E}(X^2)$ , we divide the sum

$$X^2 = \sum I_{(v_1, v_2, w_1, w_2)} I_{(x_1, x_2, y_1, y_2)}$$

into several cases. Write  $I_{\mathbf{v}, \mathbf{w}} = I_{(v_1, v_2, w_1, w_2)}$  and  $I_{\mathbf{x}, \mathbf{y}} = I_{(x_1, x_2, y_1, y_2)}$ .

**Case 1** Let  $X_1$  denote the sum of all products  $I_{\mathbf{v}, \mathbf{w}} I_{\mathbf{x}, \mathbf{y}}$  such that no vertex appears in both 4-tuples  $(v_1, v_2, w_1, w_2)$  and  $(x_1, x_2, y_1, y_2)$ . Then

$$\mathbb{E}(X_1) = \frac{n!}{(n-8)!} p^8 (1-p)^4 \sim n^8 p^8 (1-p)^4.$$

**Case 2** Let  $X_2$  denote the sum over all  $I_{\mathbf{v}, \mathbf{w}} I_{\mathbf{x}, \mathbf{y}}$  such that exactly one vertex appears in both 4-tuples. Then

$$\mathbb{E}(X_2) = 16 \frac{n!}{(n-7)!} p^8 (1-p)^4 \sim 16n^7 p^8 (1-p)^4$$

**Case 3** Let  $X_3$  denote the sum over all  $I_{\mathbf{v}, \mathbf{w}} I_{\mathbf{x}, \mathbf{y}}$  such that 2 vertices appear in both 4-tuples making one of the pairs  $\mathbf{v}$  or  $\mathbf{w}$  equal to one of the pairs  $\mathbf{x}$  or  $\mathbf{y}$  up to permutation, i.e., the potential squares share a pair of diagonal vertices. In this case,  $(I_{\mathbf{v}, \mathbf{w}} I_{\mathbf{x}, \mathbf{y}})$  depends on the existence of 8 sides and the nonexistence of 3 diagonals, so

$$\mathbb{E}(X_3) = 8 \frac{n!}{(n-6)!} p^8 (1-p)^3 \sim 8n^6 p^8 (1-p)^3.$$

**Case 4** Let  $X_4$  denote the sum over all  $I_{\mathbf{v}, \mathbf{w}} I_{\mathbf{x}, \mathbf{y}}$  such that 2 vertices appear in both 4-tuples but in different diagonal pairs, i.e., the potential squares share an edge. Then  $(I_{\mathbf{v}, \mathbf{w}} I_{\mathbf{x}, \mathbf{y}})$  depends on the existence of 7 sides and the nonexistence of 4 diagonals, so

$$\mathbb{E}(X_4) = 32 \frac{n!}{(n-6)!} p^7 (1-p)^4 \sim 32n^6 p^7 (1-p)^4.$$

Note that there is no need to consider the case in which some pair appears as a diagonal pair in one 4-tuple and an edge pair in the other since in that case either  $I_{\mathbf{v}, \mathbf{w}}$  or  $I_{\mathbf{x}, \mathbf{y}}$  must be 0.

**Case 5** Let  $X_5$  denote the sum over all  $I_{\mathbf{v}, \mathbf{w}} I_{\mathbf{x}, \mathbf{y}}$  such that the two 4-tuples share 3 vertices. Then

$$\mathbb{E}(X_5) = 16 \frac{n!}{(n-5)!} p^6 (1-p)^3 \sim 16n^5 p^6 (1-p)^3.$$



**Case 6** Let  $X_5$  denote the sum over all  $I_{v,w}I_{x,y}$  such that up to permutation, the two 4-tuples are the same. Then

$$\mathbb{E}(X_6) = 8 \frac{n!}{(n-4)!} p^4 (1-p)^2 \sim 8n^4 p^4 (1-p)^2.$$

Now compute

$$\begin{aligned} \frac{\mathbb{E}(X^2)}{(\mathbb{E}X)^2} &= \sum_{i=1}^6 \frac{\mathbb{E}(X_i)}{(\mathbb{E}X)^2} \\ (2) \quad &\sim 1 + \frac{16}{n} + \frac{8}{n^2(1-p)} + \frac{32}{n^2 p} + \frac{16}{n^3 p^2(1-p)} + \frac{8}{n^4 p^4(1-p)^2}. \end{aligned}$$

Note that since either  $(1-p) \geq \frac{1}{2}$  or  $p \geq \frac{1}{2}$ ,

$$n^3 p^2(1-p) \geq \min\{\frac{1}{2}n^3 p^2, \frac{1}{4}n^3(1-p)\},$$

and

$$n^4 p^2(1-p)^2 \geq \min\{\frac{1}{4}n^4 p^4, \frac{1}{16}n^4(1-p)^2\}.$$

Thus, if  $pn \rightarrow \infty$  and  $(1-p)n^2 \rightarrow \infty$ , then all denominators in (2) tend to infinity implying

$$\text{Prob}(X \neq 0) \geq \frac{(\mathbb{E}X)^2}{\mathbb{E}(X^2)} \rightarrow 1.$$

This completes the proof of [Theorem 4.1](#). □

[Theorems 2.1](#) and [2.3](#) now follow by combining [Theorems 3.1, 3.2](#) and [4.1](#).

## 5 The automorphism groups of random right-angled Artin groups

A right-angled Artin group is a graph product in which vertex groups are infinite cyclic. In this section we review some basic properties of right-angled Artin groups and refer the reader to [Hermiller and Meier \[23\]](#), [Servatius \[33\]](#) and [Laurence \[27\]](#) for details. For a general survey on these groups, see [Charney \[6\]](#).

To specify a right-angled Artin group, we need only specify the graph  $\Gamma$ . If  $V$  is the vertex set of  $\Gamma$ , then the right-angled Artin group associated to  $\Gamma$  is the group with presentation

$$A_\Gamma = \langle V \mid vw = wv \text{ if } v \text{ and } w \text{ are connected by an edge in } \Gamma \rangle.$$

At one extreme, we have the case that  $\Gamma$  is a discrete graph (i.e., no edges) and  $A_\Gamma$  is the free group on  $V$ . At the other, is the case that  $\Gamma$  is a complete graph (i.e., every pair of vertices spans an edge) and  $A_\Gamma$  is the free abelian group on  $V$ .

In this section, we consider right-angled Artin groups associated to random graphs  $\Gamma \in G(n, p)$ . It is clear that basic properties of  $\Gamma$  are reflected in properties of the group. For example,  $A_\Gamma$  decomposes as a free product if and only if  $\Gamma$  is disconnected. For  $\Gamma \in G(n, p)$ , Erdős and Rényi [17] showed that this holds asymptotically almost surely if  $p(n) = \frac{\log n - \omega(n)}{n}$ , where  $\omega: \mathbb{N} \rightarrow \mathbb{R}$  is a function such that  $\lim_{n \rightarrow \infty} \omega(n) = \infty$ . Likewise,  $A_\Gamma$  decomposes as a direct product if and only if the complementary graph is disconnected, which holds a.a.s. if  $1 - p(n) = \frac{\log n - \omega(n)}{n}$ . In [12], Costa and the second author analyze the cohomological dimension and the topological complexity of right-angled Artin groups associated to random graphs.

Automorphism groups of right-angled Artin groups have been extensively studied in recent years. (See, for example, Charney and Vogtmann [9; 10] and Day [16].) Here, we consider automorphism groups of right-angled Artin groups associated to random graphs. Our goal is to prove the following theorem:

**Theorem 5.1** *Let  $\Gamma \in G(n, p)$  be a random graph where the probability parameter  $p$  is independent on  $n$  and satisfies*

$$(3) \quad 1 - \frac{1}{\sqrt{2}} < p < 1.$$

*Then the right-angled Artin group  $A_\Gamma$  determined by  $\Gamma$  has a finite outer automorphism group  $\text{Out}(A_\Gamma)$ , asymptotically almost surely.*

Note that  $1 - \frac{1}{\sqrt{2}} \sim 0.2929$ .

Our proof shows that a random right-angled Artin group  $A_\Gamma$  admits no transvections for any fixed  $p$ ; however to exclude partial conjugations one needs the assumption (3). The definitions of transvections and partial conjugations are given below. [Theorem 5.1](#) was further strengthened by M Day [15].

## 6 Proof of [Theorem 5.1](#)

The automorphism groups of right-angled Artin groups were first studied by Servatius [33] who described a finite generating set for  $\text{Aut}(A_\Gamma)$  under certain restrictions on  $\Gamma$ . This was later extended to arbitrary  $\Gamma$  by Laurence in [27]. The generators depend on certain structures in the defining graph  $\Gamma$ . For a vertex  $v$ , the *link*  $\text{lk}(v)$  is defined as the set of all vertices which are connected to  $v$  by an edge. The *star*  $\text{st}(V)$

is the union of all edges containing  $v$ .<sup>2</sup> The Servatius-Laurence generating set consists of the following automorphisms.

- *Symmetries*: These are given by permutations of the generators arising from symmetries of the graph  $\Gamma$ .
- *Inversions*: These send a generator  $v \in V$  to its inverse.
- *Transvections*: These map  $v \mapsto vw$  where  $v, w \in V$  satisfy  $lk(v) \subseteq st(w)$ .
- *Partial conjugations*: These conjugate all of the generators in one component of  $\Gamma \setminus st(v)$  by  $v$  and occur only if  $\Gamma \setminus st(v)$  is disconnected.
- *Inner automorphisms*: These conjugate the entire group by some generator  $v$ .

Let us take a closer look at transvections. The condition that  $lk(v) \subseteq st(w)$  is necessary to guarantee that the map  $v \mapsto vw$  preserves commutator relations. With this in mind, we introduce a partial order on  $V$ . For two vertices  $v, w$ , write

$$v \leq w \text{ if } lk(v) \subset st(w).$$

It is shown in Charney and Vogtmann [9] that this relation is transitive.

Taking the quotient of  $Aut(A_\Gamma)$  by the inner automorphisms gives the outer automorphism group  $Out(A_\Gamma)$ . The first two types of automorphisms, symmetries and inversions, induce a finite subgroup of  $Aut(A_\Gamma)$  (and hence of  $Out(A_\Gamma)$ ) while transvections and partial conjugations have infinite order. Hence  $Out(A_\Gamma)$  is finite if and only if  $A_\Gamma$  does not permit any transvections or partial conjugations. To exclude transvections, we require that no two vertices are related by the partial order  $\leq$ . Consider, for example, the case where  $\Gamma$  consists of a single cycle of length  $n \geq 5$ . It is easily seen that in this case, no two vertices satisfy  $lk(v) \subset st(w)$ . To exclude partial conjugations, we require that  $\Gamma - st(v)$  is connected for every vertex  $v$ . We call such a graph *star 2-connected*.

To prove [Theorem 5.1](#), we will show that for  $\Gamma \in G(n, p)$  with  $p$  satisfying [\(3\)](#), the probability that  $A_\Gamma$  admits a transvection or a partial conjugation tends to zero as  $n$  goes to infinity.

First we show that the probability that there exists a pair of vertices  $v, w$  in  $\Gamma$  with  $lk(v) \subset st(w)$  tends to zero as  $n \rightarrow \infty$ . This would imply that  $A_\Gamma$  admits no transvections.

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<sup>2</sup>In some contexts, the link and star are taken to be full subgraphs of  $\Gamma$ . In our context, only the vertices in these subgraphs will matter.

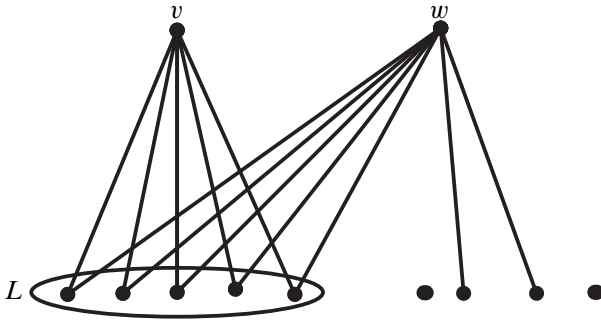


Figure 2

For a pair of distinct vertices  $v, w \in \{1, \dots, n\}$  and for a subset  $L \subset \{1, \dots, n\}$  not containing  $v, w$ , consider the following random variable

$$I_{v,w,L}: G(n, p) \rightarrow \{0, 1\},$$

where for a graph  $\Gamma \in G(n, p)$  one has  $I_{v,w,L}(\Gamma) = 1$  if and only if in  $\Gamma$  the following conditions hold: (a) the link of  $v$  equals either  $L$  or  $L \cup \{w\}$  and (b)  $L \subset \text{lk}(w)$ ; otherwise we set  $I_{v,w,L}(\Gamma) = 0$ . In other words, one has  $I_{v,w,L}(\Gamma) = 1$  if and only if (1) every vertex of  $L$  is connected by an edge to the vertices  $v$  and  $w$ ; (2) no vertex of  $\{1, \dots, n\} - L - \{v, w\}$  is connected by an edge to  $v$ . (See Figure 2.)

The sum

$$X = \sum I_{v,w,L}: G(n, p) \rightarrow \mathbb{Z},$$

where  $v, w, L$  run over all possible choices, counts the number of ordered pairs of vertices  $(v, w)$  with the property that  $\text{lk}(v) \subset \text{st}(w)$ .

The expectation  $\mathbb{E}(I_{v,w,L})$  equals

$$p^k (1 - p)^{n-2-k} p^k = p^{2k} (1 - p)^{n-2-k}$$

where  $k = |L|$ . Thus, we obtain

$$\begin{aligned} \mathbb{E}(X) &= n(n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} p^{2k} (1-p)^{n-2-k} \\ &= n(n-1)(1-p+p^2)^{n-2}. \end{aligned}$$

Now suppose that

$$p(1-p)n - 2 \log n \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

This can also be expressed by saying that

$$p(1 - p) = \frac{2 \log n + \omega(n)}{n}$$

where  $\omega(n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Note that any constant  $0 < p < 1$  satisfies this condition. Then denoting  $x = p - p^2$  one has

$$\begin{aligned} \log \mathbb{E}(X) &\leq 2 \log n + (n - 2) \log(1 - x) \\ &= 2 \log n - (n - 2) \left[ x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \right] \\ &= 2 \log n - nx + x \left[ 2 - (n - 2) \left[ \frac{x}{2} + \frac{x^2}{3} + \dots \right] \right] \\ &\leq 2 \log n - nx = -\omega(n). \end{aligned}$$

Thus, assuming (5), we have  $\mathbb{E}(X) \rightarrow 0$  as  $n \rightarrow \infty$ . By the first moment method,  $\text{Prob}(X > 0) \leq \mathbb{E}(X)$ , so we conclude that  $\text{Prob}(X > 0) \rightarrow 0$  as  $n \rightarrow \infty$ .

Now we examine conditions for the absence of partial conjugations, that is, the conditions on  $\Gamma$  such that the result of removing the star of any vertex is path-connected.

Consider a partition of the set of vertices  $\{1, \dots, n\}$  into  $v \cup L \cup S \cup T$  where  $v$  is a vertex and  $L, S, T$  are disjoint subsets not containing  $v$ . We denote  $\ell = |L|, s = |S|, t = |T|$ . Let  $J_{v,L,S,T}: G(n, p) \rightarrow \{0, 1\}$  be the random variable which associates to a random graph  $\Gamma \in G(n, p)$  one if and only if  $L$  is the link of  $v$  in  $\Gamma$  and there are no edges in  $\Gamma$  connecting a point of  $S$  to a point of  $T$ . One has

$$\mathbb{E}(J_{v,L,S,T}) = p^\ell (1 - p)^{n-1-\ell} (1 - p)^{st}.$$

Consider also the sum

$$Y = \sum J_{v,L,S,T}$$

where the sum is taken over all possible choices of  $v, L, S, T$  with  $s \geq 1$  and  $t \geq 1$ . For a random graph  $\Gamma \in G(n, p)$  the number  $Y(\Gamma)$  is positive if and only if removing the star of some vertex disconnects the graph. Thus  $\text{Prob}(Y > 0)$  is the probability that a random graph fails to be star 2-connected.

For the expectation of  $Y$  we have

$$\begin{aligned} \mathbb{E}(Y) &= n \cdot \sum_{\substack{s \geq 1, t \geq 1 \\ 1 + \ell + s + t = n}} \binom{n-1}{\ell} \cdot \binom{n-1-\ell}{s} \cdot p^\ell \cdot (1 - p)^{n-\ell-1+st} \\ &= n(1 - p)^{n-1} \cdot \sum_{\ell=0}^{n-3} \binom{n-1}{\ell} \cdot \left( \frac{p}{1-p} \right)^\ell \cdot F_\ell(p), \end{aligned}$$

where

$$(6) \quad F_\ell(p) = \sum_{\substack{s+t=n-1-\ell \\ s \geq 1, t \geq 1}} \binom{s+t}{s} \cdot (1-p)^{st}.$$

Observe that for  $s \geq 2, t \geq 2$  one has  $st = s(n-1-\ell-s) \geq 2(n-3-\ell)$ . Thus separating the terms  $s = 1, t = 1$  gives

$$\begin{aligned} F_\ell(p) &= 2(n-1-\ell) \cdot (1-p)^{n-2-\ell} + \sum_{\substack{s+t=n-1-\ell \\ s \geq 2, t \geq 2}} \binom{s+t}{s} \cdot (1-p)^{st} \\ &\leq 2(n-1-\ell) \cdot (1-p)^{n-2-\ell} + (1-p)^{2(n-3-\ell)} \cdot 2^{n-1-\ell} \\ &\leq 2n \cdot (1-p)^{n-2-\ell} + (1-p)^{-4} [2(1-p)^2]^{n-1-\ell}. \end{aligned}$$

Substituting this into (6) we find

$$\begin{aligned} \mathbb{E}(Y) &\leq 2n^2(1-p)^{n-2} \cdot \sum_{\ell=0}^{n-3} \binom{n-1}{\ell} \left(\frac{p}{1-p}\right)^\ell (1-p)^{n-1-\ell} \\ &\quad + n(1-p)^{n-5} \cdot \sum_{\ell=0}^{n-3} \binom{n-1}{\ell} \left(\frac{p}{1-p}\right)^\ell (2(1-p)^2)^{n-1-\ell} \\ &\leq 2n^2 \cdot (1-p)^{n-2} \cdot \left[\frac{p}{1-p} + 1-p\right]^{n-1} \\ &\quad + n \cdot (1-p)^{n-5} \left[\frac{p}{1-p} + 2(1-p)^2\right]^{n-1} \\ &= 2n^2(1-p)^{-1} [p + (1-p)^2]^{n-1} + n(1-p)^{-4} \cdot [p + 2(1-p)^3]^{n-1} \\ &= 2n^2(1-p)^{-1} (1-x)^{n-1} + n(1-p)^{-4} (1-y)^{n-1}, \end{aligned}$$

where  $x = p - p^2$  and  $y = 2(p-1)(p-1-\frac{\sqrt{2}}{2})(p-1+\frac{\sqrt{2}}{2})$ . Clearly for all  $0 < p < 1$ , one has  $0 < x < 1$ ; and it is easy to see that for  $1 - \frac{1}{\sqrt{2}} < p < 1$ , one also has  $0 < y < 1$ .

Thus we obtain that as  $n \rightarrow \infty$ , the expectation  $\mathbb{E}(Y)$  tends to zero and by the first moment method, this implies that the probability  $\text{Prob}(Y > 0)$  tends to zero. Therefore, the right-angled Artin group corresponding to a random graph  $\Gamma$  with parameter  $p$  satisfying (3) admits no partial conjugations, a.a.s.

Since we have already proven that  $A_\Gamma$  admits no transvections, it follows that, for a random graph  $\Gamma \in G(n, p)$  with  $p$  satisfying (3) the group  $A_\Gamma$  has a finite outer automorphism group.

This completes the proof of [Theorem 5.1](#).

These results also apply to more general graph products  $\mathcal{G}_\Gamma = \mathcal{G}(\Gamma, \{G_v\})$  in which every vertex group is cyclic, but not necessarily infinite cyclic. Corredor and Gutierrez [11] have recently shown that the automorphism groups of such graph products have a generating set analogous to the Servatius-Laurence generators for a right-angled Artin group described above. The only difference is that inversions are replaced by isomorphisms of individual vertex groups (these are always of finite order) and there are additional restrictions on transvections involving the orders of the vertex groups. Once again, if there are no vertices with  $\text{lk}(v) \subset \text{st}(w)$  and no separating stars, then the automorphism group is necessarily finite. Thus we obtain,

**Theorem 6.1** *Fix a collection of nontrivial, cyclic groups indexed by the natural numbers  $\{G_i\}$ . For a random graph  $\Gamma \in G(n, p)$ , let  $\mathcal{G}_\Gamma = \mathcal{G}(\Gamma, \{G_i\})$  be the associated graph group. Suppose the probability parameter  $p$  is independent of  $n$  and satisfies*

$$(7) \quad 1 - \frac{1}{\sqrt{2}} < p < 1.$$

*Then  $\mathcal{G}_\Gamma$  has a finite outer automorphism group  $\text{Out}(A_\Gamma)$ , asymptotically almost surely.*

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