Inequivalent handlebody-knots with homeomorphic complements

JUNG HOON LEE
SANGYOP LEE

We distinguish the handlebody-knots $5_1, 6_4$ and $5_2, 6_{13}$ in the table, due to Ishii et al, of irreducible handlebody-knots up to six crossings. Furthermore, we construct two infinite families of handlebody-knots, each containing one of the pairs $5_1, 6_4$ and $5_2, 6_{13}$, and show that any two handlebody-knots in each family have homeomorphic complements but they are not equivalent.

57M50

1 Introduction

Given a knot in $S^3$, its regular neighborhood is a knotted solid torus. Conversely, an embedded solid torus in $S^3$ uniquely determines a knot. Thus we may regard an embedded solid torus as a knot in $S^3$. Instead of an embedded solid torus in $S^3$, consider an embedded handlebody. We may regard it as a kind of a knot. Following Ishii, Kishimoto, Moriuchi and Suzuki [3], we say that a handlebody embedded in $S^3$ is a handlebody-knot.

Throughout this paper, by a handlebody-knot we will mean a genus two handlebody embedded in $S^3$. A handcuff graph or a $\theta$–curve $\Gamma$ in a handlebody-knot $H$ is called a spine if $H$ is a regular neighborhood of $\Gamma$. The spine of $H$ is not uniquely determined, but any two spines are related by a finite sequence of isotopies and IH-moves (see Ishii [2]), where an IH-move is a local move on a spatial trivalent graph depicted in Figure 1.

![Figure 1](image-url)
Two handlebody-knots $H_1$ and $H_2$ are said to be equivalent if there exists an isotopy of $S^3$ that takes $H_1$ to $H_2$, or equivalently if there exists an orientation-preserving automorphism $h$ of $S^3$ such that $h(H_1) = H_2$. A handlebody-knot $H$ is reducible if there exists a 2–sphere $S$ in $S^3$ such that $S \cap H$ is a disk separating $H$ into two solid tori. Otherwise, it is irreducible. Note that $H$ is irreducible if $S^3 - \text{int}(H)$ is $\partial$–irreducible.

As done for knots, we can use regular diagrams of spines of a handlebody-knot to define the crossing number of the handlebody-knot. Ishii, Kishimoto, Moriuchi and Suzuki recently give a table of handlebody-knots such that any irreducible handlebody-knot with six or fewer crossings or its mirror image is equivalent to one of the handlebody-knots in the table. See [3, Table 1]. By using some invariants, they distinguish all handlebody-knots in their table except only for the two pairs $(5_1, 6_4)$ and $(5_2, 6_{13})$. See Figure 2.

Consider the handcuff graphs $\Phi_n$, $\Psi_n$ in $S^3$, shown in Figure 3, where a rectangle labeled by an integer $n$ denotes a vertical right-handed twist of two strings with $2n$ crossings. Let $V_n$ and $W_n$ denote regular neighborhoods of $\Phi_n$ and $\Psi_n$, respectively. Put $X_n = S^3 - \text{int}(V_n)$ and $Y_n = S^3 - \text{int}(W_n)$.

Let $\Theta_n = \Phi_n$ or $\Psi_n$, and let $Z_n = X_n$ or $Y_n$ correspondingly. The handcuff graph $\Theta_n$ consists of two vertices and three edges, two forming loops and one connecting the two loops. One of the two loops bounds a disk intersecting the vertical twist in two points.
By twisting along the disk, one can transform $\Theta_n$ into $\Theta_m$ for any other integer $m$. This shows that $Z_n$ is homeomorphic to $Z_m$.

For any submanifold $M$ of $S^3$, denote by $M^*$ the mirror image of $M$. We say that $M$ is amphicheiral if an isotopy of $S^3$ takes $M$ to $M^*$. The main result of the present paper is the following.

**Theorem 1.1** Let $n$ and $m$ be distinct integers.

1. No two of $V_n, V_n^*, V_m, V_m^*$ are equivalent.
2. No two of $W_n, W_n^*, W_m, W_m^*$ are equivalent.

In particular, $V_n$ and $W_n$ are not amphicheiral for each integer $n$.

By calculating fundamental groups, one can show that $X_0$ and $Y_0$ are not homeomorphic. This implies that $V_n$ and $W_m$ are not equivalent for any integers $n$ and $m$.

It is a celebrated result of Gordon and Luecke that if two knots in $S^3$ have homeomorphic complements then the homeomorphism between the two complements extends to an automorphism of $S^3$ [1]. In contrast, Motto [5] showed that handlebody-knots are not determined by their complements. We remark that our infinite families of inequivalent handlebody-knots are also of this type.

We can now distinguish the handlebody-knots $5_1, 6_4$, and $5_2, 6_{13}$ in the table due to Ishii et al.

**Corollary 1.2**

1. No two of $5_1, 5_1^*, 6_4, 6_4^*$ are equivalent.
2. No two of $5_2, 5_2^*, 6_{13}, 6_{13}^*$ are equivalent.

In particular, $5_1, 5_2, 6_4, 6_{13}$ are not amphicheiral.
Proof The sequences of pictures in Figure 4(a),(b) show that \( V_0 \) and \( V_{-1} \) are respectively equivalent to \( 5_1 \) and \( 6_4 \), and the sequences of pictures in Figure 4(c),(d) show that \( W_0 \) and \( W_1 \) are respectively equivalent to \( 5_2 \) and \( 6_{13}^* \). Hence the result immediately follows from Theorem 1.1. \( \square \)

Figure 4

Some figures in this paper are best viewed in color; readers confused by figures in a black-and-white version are recommended to view the electronic version.
2 Curves in the boundary of a genus two handlebody

A properly embedded disk in a 3–manifold $M$ is essential if it is not isotopic to a disk in $\partial M$. A properly embedded compact surface in $M$, which is neither a disk nor a sphere, is essential if it is incompressible and is not $\partial$–parallel. Given a set $\{c_1, \ldots, c_n\}$ of disjoint simple loops in $\partial M$, $M[c_1 \cup \cdots \cup c_n]$ will denote the 3–manifold obtained by attaching 2–handles to $M$ along disjoint neighborhoods of $c_1, \ldots, c_n$.

Throughout this section, $H$ will denote a genus two handlebody. A simple loop in $\partial H$ is called a primitive curve if there exists a disk in $H$, called a dual disk, that intersects the loop in a single point.

**Lemma 2.1** Let $c_1, c_2$ be two disjoint nonisotopic primitive curves in $\partial H$. If there are two disjoint nonisotopic essential disks $D_1, D_2$ of $H$ each of which is a common dual disk of $c_1$ and $c_2$, then the fundamental group of $H[c_1 \cup c_2]$ is either the infinite cyclic group or the cyclic group of order 2.

**Proof** The two disks $D_1, D_2$ cut $H$ into a 3–ball $B$ and $c_1 \cup c_2$ into four arcs. Let $D_i^+, D_i^-$ be the copies of $D_i$ on $\partial B$ for $i = 1, 2$. There are two cases; the four arcs together with the four disks $D_1^\pm, D_2^\pm$ form two cycles of length 2 or a single cycle of length 4. See Figure 5. One easily sees that the fundamental group of $H[c_1 \cup c_2]$ is the infinite cyclic group in the first case and it is the cyclic group of order 2 in the latter case. \( \square \)

An element $x$ of the free group $F$ of rank 2 is called a primitive element if there exists an element $y \in F$ such that $x, y$ generate $F$.

**Lemma 2.2** Let $A$ be an essential separating annulus in $H$. Let $c_1, c_2$ be two essential simple loops in $\partial H$ which are disjoint from $\partial A$. Suppose that $A$ separates $c_1$ and $c_2$. Then one of $c_1$ and $c_2$ represents a proper power of a primitive element of the free group $\pi_1(H)$.

**Proof** By Kobayashi [4, Lemma 3.2(i)], $A$ cuts $H$ into a solid torus $H_1$ and a genus two handlebody $H_2$. Since $A$ separates $c_1$ and $c_2$, we may assume $c_1 \subset H_1$ and $c_2 \subset H_2$. Let $A_i$ be the copy of $A$ in $\partial H_i$ for $i = 1, 2$. Then the core of $A_1$ is parallel to $c_1$ in $\partial H_1$, and the core of $A_2$ represents a primitive element of the free group $\pi_1(H_2)$.

If $c_1$ were a meridian curve of $H_1$ then $A$ would be compressible in $H$. If $c_1$ were homotopic to the core of $H_1$ then $A$ would be $\partial$–parallel in $H$. Hence $c_1$ is homotopic in $H_1$ to $n \ (\geq 2)$ times around the core of $H_1$. 

*Algebraic & Geometric Topology, Volume 12 (2012)*
Let $x$ be a generator of the infinite cyclic group $\pi_1(H_1)$, and let $y, z$ be two elements generating the free group $\pi_1(H_2)$. Here, we may assume that $x^n$ is represented by the core of $A_1$ (or $c_1$) and $y$ is represented by the core of $A_2$. By the Van Kampen’s theorem, $\pi_1(H)$ has three generators $x, y, z$ and one relation $x^n = y$. Thus $\pi_1(H)$ is the free group on $x$ and $z$, and $c_1$ represents $x^n$ in the group $\pi_1(H)$.

**Lemma 2.3** Let $c_1, c_2$ be two simple loops in $\partial H$ which are not contractible in $H$. Suppose that there exists a properly embedded disk $D$ in $H - c_1 \cup c_2$ which splits $H$ into two solid tori, each containing one of $c_1$ and $c_2$. Then any such disk is isotopic to $D$ in $H - c_1 \cup c_2$.

**Proof** Let $E$ be a properly embedded disk in $H - c_1 \cup c_2$ which splits $H$ into two solid tori $H_1$ and $H_2$ with $c_i \subset H_i$ for each $i = 1, 2$. Suppose that $E$ is not isotopic to $D$ in $H - c_1 \cup c_2$.

If $E$ is disjoint from $D$ then $D$ and $E$ are parallel in $H$, that is, they cut off a 1-handle $D \times I$ from $H$. Since neither $c_1$ nor $c_2$ is contractible in $H$, $\partial D \times I$ does not meet any of $c_1$ and $c_2$. This means that $D \times I$ is, in fact, the parallelism between $D$ and $E$ in $H - c_1 \cup c_2$. This contradicts our assumption on $E$.

We may assume that the intersection $D \cap E$ is transverse and minimal up to isotopy of $E$. Then a standard disk swapping argument shows that $D \cap E$ has no circle
components. An arc component of $D \cap E$, outermost in $D$, cuts off a subdisk of $D$. Surgery on $E$ along the subdisk yields two disks, both of which are disjoint from $c_1 \cup c_2$. Let $E'$ be any of these disks. Then $E'$ lies in a solid torus $H_i$ for some $i = 1, 2$. By the minimality of $|D \cap E|$, $E'$ is parallel in $H - c_1 \cup c_2$ to neither $E$ nor a disk in $\partial H$. Hence $E'$ is a meridian disk of the solid torus $H_i$, cutting it into a 3–ball in which $c_i$ lies. This implies that $c_i$ is contractible in $H$, a contradiction. \( \Box \)

### 3. $V_n$ and $V_m$ ($n \neq m$) are not equivalent

Consider $\Phi_0$. The drawings in Figure 4(a) depict an isotopy from $V_0$ to $S_1$, showing that there exists a properly embedded nonseparating annulus $A_0$ in $X_0$ as shown in Figure 6(a). Cutting $X_0$ along $A_0$ gives a new compact 3–manifold $U$ as shown in Figure 6(b), where the two loops in $\partial U$ are the cores of the two copies $A_0^+$ and $A_0^-$ of $A_0$ in $\partial U$. Let $c^\pm$ be the loops. After an isotopy, $U$ becomes the complement of a standardly embedded genus two handlebody in $S^3$ (see Figure 7), so $U$ itself is a genus two handlebody.

![Figure 6](image)

Let $C = c^+ \cup c^-$. Take three essential nonseparating disks $X, Y, Z$ in $U$ as shown in Figure 8(a). These three disks divide $U$ into two 3–balls $B^\pm$ and $C$ into arcs. See Figure 8(b). Let $X^\pm, Y^\pm, Z^\pm$ be copies of $X, Y, Z$ in $\partial B^\pm$. Then $C^\pm = C \cap B^\pm$ consists of five arcs, two connecting $X^\pm$ and $Y^\pm$, two connecting $X^\pm$ and $Z^\pm$, and one connecting $Y^\pm$ and $Z^\pm$. Set $\Delta = X \cup Y \cup Z$ and $\Delta^\pm = X^\pm \cup Y^\pm \cup Z^\pm$. Then $\partial B^\pm - (\Delta^\pm \cup C^\pm)$ is a union of (open) disks.

**Lemma 3.1** $U$ does not contain an essential disk or annulus or a properly embedded Möbius band which is disjoint from $C$.

*Algebraic & Geometric Topology, Volume 12 (2012)*
Proof Assume for contradiction that $U$ contains such a surface $F$.

First, suppose that $F$ is a disk. The intersection $F \cap \Delta$ may be assumed to be transverse and minimal among all essential disks of $U$ that are disjoint from $C$. Note that $F \cap \Delta \neq \emptyset$, since otherwise $F$ would be properly embedded in either $B^+ \cup C^-$ or $B^- \cup C^+$ and hence $F$ would be parallel to a disk in $\partial U$. By the minimality of $|F \cap \Delta|$, $F$ has no circle components of intersection with $\Delta$. An arc component of intersection, outermost in $F$, cuts off a disk $F_0$ from $F$. Any two disks in $\Delta^\pm$ are joined by an arc in $\Delta^\pm$, so the arc $F_0 \cap \partial U$ together with an arc in $\partial \Delta$ bounds a disk in $\partial U$ that is disjoint from $C$. This disk could be used to reduce $|F \cap \Delta|$, contradicting the minimality assumption. Hence $F$ is not a disk.

The fundamental group $\pi_1(U)$ is a free group generated by two elements $x$ and $y$, where $x$ and $y$ are respectively represented by the cores of the 1–handles $N(X)$ and $N(Y)$, attached to the 3–ball $N(Z)$. See Figure 8(b). The two loops $c^+$ and $c^-$ represent two group elements $x$ and $x y x y^{-1} x^{-1} y^{-1}$. Hence the 3–manifold
$Q = U[c^+ \cup c^-]$ has a trivial fundamental group, so it is a 3–ball. Since $F$ is disjoint from $C$, $F$ is properly embedded in $Q$. No Möbius bands can be properly embedded in a 3–ball, so $F$ must be an annulus. Since every properly embedded annulus in a 3–ball is separating, $F$ must be separating in $U$. Splitting $U$ along $F$, we get a solid torus $U_1$ and a genus two handlebody $U_2$, where the core of the copy of $F$ in $\partial U_1$ winds the solid torus $U_1$ at least two times in the longitudinal direction. See [4, Lemma 3.2(i)].

Neither $x$ nor $xyxy^{-1}x^{-1}y^{-1}$ is a proper power of a primitive element of the group $\pi_1(U)$. Thus it follows from Lemma 2.2 that the two loops $c^+$ and $c^-$ are not separated by $F$. Since $c^+$ and $c^-$ are not parallel in $\partial U$, they are contained in $U_2$. Hence $F$ splits $Q$ into $U_1$ and $U_2[c^+ \cup c^-]$. In particular, $F$ cuts off the solid torus $U_1$ from the 3–ball $Q$ so that the core of the copy of $F$ in $\partial U_1$ is homotopic to at least two times around the core of $U_1$. This is impossible. 

Lemma 3.2 $A_0$ is incompressible and $\partial$–incompressible in $X_0$.

Proof Since each of $c^+$ and $c^-$ represents a nontrivial element of the free group $\pi_1(U)$, $A_0$ is incompressible. Suppose that $A_0$ is $\partial$–compressible. Then there exists a properly embedded disk $D$ in $U$ intersecting $C$ in a single point. We may assume that $D$ intersects $c^+$ Then the frontier of a neighborhood of $D \cup c^+$ in $U$ is an essential separating disk in $U$ that is disjoint from $C$, contradicting Lemma 3.1. Hence $A_0$ is $\partial$–incompressible.

Lemma 3.3 $X_0$ is irreducible and $\partial$–irreducible. Hence $X_n$ is irreducible and $\partial$–irreducible for any integer $n$.

Proof It is clear that $X_0$ is irreducible. If $X_0$ is $\partial$–reducible then any compressing disk for $\partial X_0$ can be isotoped to be disjoint from $A_0$. Then it lies in $U$ as an essential disk disjoint from $c^+ \cup c^-$. This contradicts Lemma 3.1.

Since $X_n$ is $\partial$–irreducible, $V_n$ is an irreducible handlebody-knot.

Lemma 3.4 $A_0$ is a unique properly embedded nonseparating annulus in $X_0$ up to isotopy.

Proof Let $A$ be a properly embedded nonseparating annulus in $X_0$ that is not isotopic to $A_0$. The $\partial$–irreducibility of $X_0$ implies that $A$ is incompressible and $\partial$–incompressible.
We may assume that \( A \) had been chosen to intersect \( A_0 \) transversely and minimally among all properly embedded nonseparating annuli in \( X_0 \). Note that \( A \) must intersect \( A_0 \), otherwise \( A \) would survive in \( U \) and be incompressible, so by Lemma 3.1 \( A \) would be parallel to either \( A_0^+ \) or \( A_0^- \) in \( U \) and hence be parallel to \( A_0 \) in \( X_0 \), contradicting the choice of \( A \).

Suppose that there are circle components of \( A \cap A_0 \) that are inessential on both \( A \) and \( A_0 \). Let \( \alpha \) be a circle component of \( A \cap A_0 \) that is innermost on \( A_0 \) among all such circle components. Then \( \alpha \) bounds a disk \( D \) in \( A \) and a disk \( D_0 \) in \( A_0 \). Note that the interior of \( D_0 \) is disjoint from \( A \), since otherwise an innermost component of \( A \cap D_0 \) on \( D_0 \) would bound a compressing disk for \( A \). We now obtain a new nonseparating annulus \( (A - D) \cup D_0 \), which is properly embedded in \( X_0 \) and can be isotoped so as to intersect \( A_0 \) transversely with fewer components of intersection. This contradicts the choice of \( A \). Hence each circle component of \( A \cap A_0 \), if it exists, is essential on at least one of \( A \) and \( A_0 \). Suppose that there are circle components of \( A \cap A_0 \) that are essential on one of the annuli \( A \) and \( A_0 \), and inessential on the other annulus. Let \( \beta \) be a circle component of \( A \cap A_0 \) that is innermost on (say) \( A \) among all such circle components (the argument for the case \( \beta \subset A_0 \) is similar). Then \( \beta \) bounds a disk \( E \) in \( A \). Since no circle components of \( A \cap A_0 \) are inessential on both \( A \) and \( A_0 \), the interior of \( E \) misses \( A_0 \) by the choice of \( \beta \). This implies that \( E \) is a compressing disk for \( A_0 \), a contradiction. We conclude that all circle components of \( A \cap A_0 \), if they exist, are essential on both \( A \) and \( A_0 \).

A similar argument, using an outermost arc component of intersection instead of an innermost circle component and using the \( \partial \)-incompressibility of \( A \cup A_0 \) instead of the incompressibility, shows that all arc components of \( A \cap A_0 \), if they exist, are essential on both \( A \) and \( A_0 \). Thus all the components of \( A \cap A_0 \) are either circles or arcs.

First, suppose that they are all circles. Take an annulus cut off from \( A \) by an outermost component of \( A \cap A_0 \) in \( A \), and surger \( A_0 \) along this annulus. The resulting surface is a union of two annuli disjoint from \( A_0 \). Let \( A_0' \) be any one of these two annuli. Since one boundary circle of \( A_0' \) is isotopic to that of \( A_0 \) (or \( A \)), \( A_0' \) must be incompressible in \( X_0 \) and hence in \( U \). By Lemma 3.1, \( A_0' \) must be \( \partial \)-parallel in \( U \), which implies that \( A_0' \) is either \( \partial \)-parallel in \( X_0 \) or parallel to \( A_0 \). In any case, we can reduce \( |A \cap A_0| \), giving a contradiction.

Now suppose that all components of \( A \cap A_0 \) are arcs that are essential on both \( A \) and \( A_0 \). The arcs divide \( A \) into rectangles \( R_1, \ldots, R_n \), where \( n = |A \cap A_0| \). Consider \( R = R_1 \). We may regard \( R \) as a properly embedded disk in \( U \) whose boundary intersects \( C = c^+ \cup c^- \) in two points. There are two cases; \( \partial R \) intersects each of \( c^+ \) and \( c^- \) in a single point, or \( \partial R \) intersects only one of \( c^+ \) and \( c^- \), say, \( c^+ \). In the

*Algebraic & Geometric Topology, Volume 12 (2012)*
former case, each of $c^+$ and $c^-$ is a primitive curve in $U$, that is, it is a generator of the free group $\pi_1(U)$ of rank two, but it is easy to see from Figure 8(b) that one of $c^+$ and $c^-$ is not a generator.

In the latter case, the two points in $\partial R \cap c^+$ split $c^+$ into two arcs $a_1$ and $a_2$. Let $S_i$ ($i = 1, 2$) be a properly embedded surface in $U$ obtained from $R$ by attaching a band along $a_i$ and then pushing the interior of the resulting surface into the interior of $U$. Note that $S_i$ is disjoint from $C$ for each $i = 1, 2$. The two ends of $a_i$ must lie on the same side of $R$ (then $S_i$ is an annulus), otherwise $S_i$ would be a Möbius band, contradicting Lemma 3.1.

If $R$ were $\partial$–parallel in $U$ then we could reduce $|A \cap A_0|$. Thus $R$ is an essential disk in $U$. First, suppose that $R$ is a nonseparating disk in $U$. Consider any $S_i$ and recall that $S_i$ is obtained from the nonseparating disk $R$ by attaching a band. Any such annulus has boundary circles which are not mutually parallel in $\partial U$ and at least one of which is essential in $\partial U$. Since the two boundary circles of $S_i$ are not mutually parallel in $\partial U$, $S_i$ is not $\partial$–parallel in $U$. Since at least one boundary circle of $S_i$ is essential in $\partial U$, $S_i$ is incompressible in $U$, otherwise a compression of $S_i$ would yield an essential disk in $U$ disjoint from $C$, contradicting Lemma 3.1. Hence $S_i$ is an essential annulus. This contradicts Lemma 3.1 again.

Suppose that $R$ is an essential separating disk in $U$. Then $R$ splits $U$ into two solid tori $U_1$ and $U_2$, where $S_i$ can be pushed into $U_i$. If the core of some $S_i$ winds $U_i$ at least two times in the longitudinal direction, then $S_i$ is an essential annulus in $U$, contradicting Lemma 3.1. Thus the core of each $S_i$ is homotopic to the core of $U_i$. This implies that $c^+ = a_1 \cup a_2$ is a primitive curve in $U$. Since $c^-$ does not intersect $R \cup c^+$, $c^-$ is also a primitive curve in $U$. See Figure 9. This contradicts our observation that one of $c^+$ and $c^-$ is not a primitive curve in $U$. \[\square\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9.png}
\caption{Figure 9}
\end{figure}

**Lemma 3.5** $V_0$ is not amphicheiral.
Proof Assume that there exists an orientation-preserving automorphism \( h \) of \( S^3 \) that takes \( V_0 \) to \( V_0^* \) (and then \( X_0 \) to \( X_0^* \)). Take a regular neighborhood \( N(A_0) \) of the nonseparating annulus \( A_0 \) in \( X_0 \). Put \( A_h = h(A_0) \) and \( N(A_h) = h(N(A_0)) \). Then \( \widetilde{V}_h = V_0^* \cup N(A_h) \) is the image of \( \widetilde{V}_0 = V_0 \cup N(A_0) \) under the automorphism \( h \). The frontier of \( N(A_0) \) in \( X_0 \) consists of two annuli whose cores \( c^+ \) and \( c^- \) run along \( \partial \widetilde{V}_0 \) as shown in Figure 6(b), where \( U \) in the figure may be considered as the closed complement of \( \widetilde{V}_0 \). Each core \( c^\pm \) bounds a disk \( D^\pm \) in \( \widetilde{V}_0 \). Let \( c_h^\pm = h(c^\pm) \) and \( D_h^\pm = h(D^\pm) \). Then \( c_h^\pm \) are the cores of the frontier annuli of \( N(A_h) \) in \( X_0^* \) and they bound disks \( D_h^\pm \).

Note that \( A_h \) is a properly embedded nonseparating annulus in \( X_0^* \). By Lemma 3.4, \( A_0^* \) is a unique properly embedded nonseparating annulus in \( X_0^* \) up to isotopy. Hence \( A_h \) and \( A_0^* \) are isotopic in \( X_0^* \).

Note that \( \text{cl}(\widetilde{V}_0 - N(D^\pm)) \) is an embedded solid torus in \( S^3 \). The core of the solid torus is either the unknot or the right-handed trefoil according to the choice of the disks \( D^+ \) and \( D^- \). We may assume that the core is the unknot for \( D^- \) and the right-handed trefoil for \( D^+ \). See Figure 10. Similarly, \( \text{cl}(\widetilde{V}_h - N(D_h^\pm)) \) is a solid torus embedded in \( S^3 \) whose core is either the unknot or the left-handed trefoil. The orientation-preserving automorphism \( h \) takes \( \text{cl}(\widetilde{V}_0 - N(D^+)) \) to \( \text{cl}(\widetilde{V}_h - N(D_h^+)) \) or \( \text{cl}(\widetilde{V}_h - N(D_h^-)) \). This implies that the right-handed trefoil is equivalent to the unknot or the left-handed trefoil, both of which are impossible. \( \square \)

Recall that twisting \( V_0 \) \( n \) times along the shaded disk in Figure 11(a) defines a homeomorphism \( \sigma_k: X_0 \to X_k \). By Lemma 3.4, \( A_k = \sigma_k(A_0) \) is up to isotopy a unique nonseparating annulus in \( X_k \). Note that \( A_k \subseteq S^3 \) is an unknotted annulus with \( k \) full twists and its boundary is the \((2, 2k)\)–torus link (if \( k = \pm 1 \), the boundary is the Hopf link). See Figure 11(b).

\[\text{Figure 10}\]
Let $c_k, d_k$ be the two loop edges of $\Phi_k$ and $e_k$ the nonloop edge. Then $V_k$ is a union of two solid tori $V_{k,1} = N(c_k), V_{k,2} = N(d_k)$, and a 1–handle $H_k = \text{cl}(N(e_k) - V_{k,1} \cup V_{k,2})$. It may be assumed that $V_{k,1}$ contains the boundary of the shaded disk in Figure 11(a). Each boundary component of $A_k$ is not contractible in $V_k$ if $k \neq 0$, and a cocore disk $D_k$ for the 1–handle $H_k$ splits $V_k$ into two solid tori, isotopic to $V_{k,1}$ and $V_{k,2}$, each of which contains one boundary component of $A_k$. Let $\partial_i A_k (i = 1, 2)$ denote the boundary component of $A_k$ lying in $V_{k,i}$. See Figure 11(b).

**Lemma 3.6** There exists an orientation-preserving automorphism of the pair $(S^3, V_{-1})$ which interchanges $V_{-1,1}$ and $V_{-1,2}$.

**Proof** Figure 4(b) allows us to regard $V_{-1}$ as $6_4$. It is easy to see that an involution on $(S^3, 6_4)$ is defined by rotating 64 through $\pi$ about a vertical axis. The involution is the desired automorphism. $\Box$

**Proof of Theorem 1.1(1)** First, assume that $V_n$ is amphicheiral for some nonzero integer $n$ ($V_0$ is not amphicheiral by Lemma 3.5), that is, there is an orientation-preserving homeomorphism of pairs $(S^3, V_n) \to (S^3, V_n^*)$. Note that $A_n$ and $A_n^*$ are up to isotopy unique nonseparating annuli in $X_n$ and $X_n^*$, respectively. Hence composing with an orientation-preserving automorphism of the pair $(S^3, V_n^*)$, if necessary, we may assume that the homeomorphism takes $A_n$ to $A_n^*$. In other words, $A_n$ and $A_n^*$ are isotopic in $S^3$. However, one of the annuli $A_n$ and $A_n^*$ has right-handed $|n|$ full twists and the other left-handed $|n|$ full twists, so they cannot be isotopic. This gives a contradiction. Therefore $V_n$ is not equivalent to its mirror image for any integer $n$.

Let $n, m$ be distinct integers, and assume that there is a homeomorphism of pairs $h: (S^3, V_n) \to (S^3, V_m)$, where $h$ may or may not preserve the orientation of $S^3$. 

*Algebraic & Geometric Topology, Volume 12 (2012)*
Similarly as above, we may assume that \( h(A_n) = A_m \). Then \( h(\partial A_n) = \partial A_m \), which means that \( h \) takes a \((2, 2n)\)-torus link to a \((2, 2m)\)-torus link. Hence \( m = n \) or \( m = -n \). The former contradicts the assumption that \( n \) and \( m \) are distinct. If \( n = 0 \) then \( h \) must preserve the orientation of \( S^3 \) by Lemma 3.5, so \( h \) is isotopic to the identity of \( S^3 \) and we have nothing to prove. Hence we may assume that \( m = -n \) and \( n \neq 0 \). Since the twists of \( A_n \) and \( A_{-n} \) are reversed, \( h \) must be orientation-reversing.

By Lemma 2.3 \( D_{\pm n} \), a cocore disk of the 1–handle \( H_{\pm n} \) in \( V_{\pm n} \), is up to isotopy a unique essential separating disk in \( V_{\pm n} \) which separates the two boundary components of \( A_{\pm n} \), so it may be assumed up to isotopy of \( V_{-n} \) that \( h(D_{n}) = D_{-n} \) and moreover \( h(H_n) = H_{-n} \). This implies that \( h \) takes each solid torus \( V_{n,i} (i = 1, 2) \) to one of the two solid tori \( V_{-n,1} \) and \( V_{-n,2} \). Note that \( \partial_1 A_{\pm n} \) is homotopic to \( \pm n \) times the core of \( V_{\pm n,1} \), while \( \partial_2 A_{\pm n} \) is homotopic to the core of \( V_{\pm n,2} \). Hence when \( |n| \geq 2 \), \( h(\partial_i A_n) = \partial_i A_{-n} \) for each \( i = 1, 2 \), which implies \( h(V_{n,i}) = V_{-n,i} \). When \( |n| = 1 \), by composing \( h \) with an orientation-preserving automorphism of the pair \((S^3, V_{-1})\) given in Lemma 3.6 we may assume that \( h(V_{n,i}) = V_{-n,i} \) for each \( i = 1, 2 \). In particular, we may always assume that \( c_n \), the core of \( V_{n,1} \), is mapped by \( h \) onto \( c_{-n} \), the core of \( V_{-n,1} \). Consider the composition

\[
(S^3, V_n) \xrightarrow{h} (S^3, V_{-n}) \xrightarrow{r} (S^3, V^*_{-n}),
\]

where \( r \) is a reflection. See Figure 12. Let \( f \) be the restriction of the composition \( r \circ h \) onto the pair \((S^3 - V_{n,1}, V_n - V_{n,1})\). Then \( f: (S^3 - V_{n,1}, V_n - V_{n,1}) \to (S^3 - V^*_{-n,1}, V^*_{n} - V^*_{-n,1}) \) is an orientation-preserving homeomorphism of pairs.

![Figure 12](image)

Note that \((S^3, V_n)\) is obtained from \((S^3, V_0)\) by \(1/n\)-surgery on \( c_0 \). Also, \((S^3, V^*_{-n})\) is obtained from \((S^3, V^*_0)\) by \(1/n\)-surgery on \( c^*_0 \). These two surgeries define two
orientation-preserving homeomorphisms of pairs as follows:

\[(S^3 - V_{0,1}, V_0 - V_{0,1}) \xrightarrow{g} (S^3 - V_{n,1}, V_n - V_{n,1}).\]

\[(S^3 - V_{0,1}^*, V_0^* - V_{0,1}^*) \xrightarrow{g^*} (S^3 - V_{-n,1}^*, V_n^* - V_{-n,1}^*).\]

For example, twisting \(n\) times along the shaded disk in Figure 11(a) defines \(g\). The composition \((g^*)^{-1} \circ f \circ g\) is an orientation-preserving homeomorphism from \((S^3 - V_{0,1}, V_0 - V_{0,1})\) to \((S^3 - V_{0,1}^*, V_0^* - V_{0,1}^*)\). Note that the composition takes a meridian of \(c_0\) to a meridian of \(c_0^*\). Hence \((g^*)^{-1} \circ f \circ g\) extends to an orientation-preserving homeomorphism of pairs from \((S^3, V_0)\) to \((S^3, V_0^*)\). This contradicts Lemma 3.5.

\[\square\]

4 \(W_n\) and \(W_m\) (\(n \neq m\)) are not equivalent

Consider \(\Psi_0\). An isotopy of \(S^3\) gives the pictures in Figure 13, showing that there exists a nonseparating annulus \(A_0\) in \(Y_0\). Cutting \(Y_0\) along \(A_0\) gives a genus two handlebody \(U\). Let \(A_0^\pm\) be the two copies of \(A_0\) in \(\partial U\) and \(c^\pm\) the cores of \(A_0^\pm\). See Figure 14(a) for \(c^\pm\), where \(U\) is the outside of the standardly embedded genus two surface and \(Y_0\) can be recovered by gluing the annulus neighborhoods \(A_0^\pm\) of \(c^\pm\) in the manner indicated in the figure. An external view of \((U, c^\pm)\) is illustrated in Figure 14(b), that is, \(U\) is the inside of the standardly embedded genus two surface in the figure.

![Figure 13](image)

**Lemma 4.1** \(U\) does not contain an essential disk or a properly embedded nonseparating annulus disjoint from \(c^+ \cup c^-\).

**Proof** First, note that both \(c^\pm\) are primitive curves in \(U\), so \(U[c^\pm]\) are solid tori. Also, it is easy to see that the fundamental group of \(U[c^+ \cup c^-]\) is cyclic with order 3. Assume that there exists an essential disk \(D\) in \(U\) disjoint from \(c^+ \cup c^-\). If \(D\) is a nonseparating disk in \(U\) then it is also nonseparating in \(U[c^+ \cup c^-]\) and hence
the fundamental group of $U[c^+ \cup c^-]$ contains an element of infinite order, contradicting the observation above. Hence $D$ separates $U$ into two solid tori $U^+$ and $U^-$. Since $U$ does not contain a nonseparating disk disjoint from $c^+ \cup c^-$, both $U^+$ and $U^-$ intersect $c^+ \cup c^-$ and hence we may assume that $c^\pm \subset U^\pm$. Then $\mathbb{Z}_3 \cong \pi_1(U[c^+ \cup c^-]) \cong \pi_1(U^+[c^+]) * \pi_1(U^-[c^-])$, so either $\pi_1(U^+[c^+]) \cong \mathbb{Z}_3$, $\pi_1(U^-[c^-]) = 1$ or $\pi_1(U^-[c^-]) = 1$, $\pi_1(U^-[c^-]) \cong \mathbb{Z}_3$. In the first case, since $U[c^+]$ is the union of $U^+[c^+]$ and $U^-$ along the disk $D$, its fundamental group is $\pi_1(U[c^+]) \cong \pi_1(U^+[c^+]) * \pi_1(U^-) \cong \mathbb{Z}_3 * \mathbb{Z}$. This contradicts our observation that $U[c^+]$ is a solid torus. In the latter case, we get a contradiction in a similar way. Therefore we conclude that $U$ does not contain an essential disk disjoint from $c^+ \cup c^-$. Assume that there exists a properly embedded nonseparating annulus $A$ in $U$ which is disjoint from $c^+ \cup c^-$. Since $A$ is disjoint from $c^+ \cup c^-$, $A$ survives in $U[c^+ \cup c^-]$ as a properly embedded nonseparating annulus. Capping off the boundary sphere of $U[c^+ \cup c^-]$ with a 3–ball, we get a 3–manifold without boundary, in which $A$ extends to a nonseparating sphere. But the fundamental group of the 3–manifold is the cyclic group of order 3 and hence the 3–manifold cannot contain a nonseparating sphere, a contradiction.

**Lemma 4.2** Let $D_0 \subset U$ be the disk illustrated in Figure 15. Then up to isotopy $D_0$ is a unique properly embedded disk in $U$ which is commonly dual to $c^+$ and $c^-$. 

**Proof** Let $D$ be a common dual disk of $c^+$ and $c^-$ that is not isotopic to $D_0$. We may assume that $D$ intersects $D_0$ transversely and the intersection $D \cap D_0$ is minimal among all such disks. If $D$ were disjoint from $D_0$, then by Lemma 2.1 $\pi_1(U[c^+ \cup c^-]) \cong \mathbb{Z}$ or $\mathbb{Z}_2$, contradicting the fact that $\pi_1(U[c^+ \cup c^-]) \cong \mathbb{Z}_3$.

By the minimality of $|D \cap D_0|$, the intersection $D \cap D_0$ has no circle components. An outermost arc of intersection in $D_0$ cuts off a subdisk from $D_0$ which intersects $c^+ \cup c^-$ in at most one point. Surgery on $D$ along the subdisk produces two disks $D_1, D_2$. 

**Jung Hoon Lee and Sangyop Lee**
One of these disks, say, $D_1$ intersects $c^+ \cup c^−$ in at most two points. Note that $D_1$ is essential in $U$, otherwise $|D \cap D_0|$ could be reduced. By Lemma 4.1 $D_1$ cannot be disjoint from $c^+ \cup c^-$. If $D_1$ had exactly one point of intersection with $c^+ \cup c^-$ then there would exist an essential (separating) disk in $U$ disjoint from $c^+ \cup c^-$, contradicting Lemma 4.1. Hence $D_1$ intersects $c^+ \cup c^-$ in two points, and so does the other disk $D_2$. One of the two disks $D_1$ and $D_2$ is a common dual disk of $c^+$ and $c^-$, and the other intersects one of $c^+$ and $c^−$ in two points. The former disk contradicts the minimality of $|D \cap D_0|$.

**Lemma 4.3** $A_0$ is incompressible and $∂$–incompressible in $Y_0$.

**Proof** One sees from Figure 14(b) that both $c^\pm$ are primitive curves in $U$, so $A_0$ is incompressible. Suppose that $A_0$ is $∂$–compressible. Let $D$ be a $∂$–compressing disk for $A_0$. Then $D$ is an essential disk in $U$ which intersects $c^+ \cup c^- $ in a single point. We may assume that $D$ intersects $c^+$ but not $c^-$. Then $c^+$ becomes a longitudinal curve of the solid torus $U[c^-]$, since $D$, a meridian disk of $U[c^-]$, intersects $c^+$ in a single point. This implies that $U[c^+ \cup c^-]$ is a 3–ball. But in the proof of Lemma 4.1 we already observed that the fundamental group of $U[c^+ \cup c^-]$ is the cyclic group of order 3.

**Lemma 4.4** $Y_0$ is irreducible and $∂$–irreducible. Hence $Y_n$ is irreducible and $∂$–irreducible for any integer $n$.

**Proof** The same argument as in the proof of Lemma 3.3 applies here by using Lemma 4.1 instead of Lemma 3.1.

Since $Y_n$ is $∂$–irreducible, $W_n$ is an irreducible handlebody-knot.

**Lemma 4.5** $A_0$ is a unique properly embedded nonseparating annulus in $Y_0$ up to isotopy.
Proof Let $A$ be a properly embedded nonseparating annulus in $Y_0$ which is not isotopic to $A_0$. The $\partial$–irreducibility of $Y_0$ implies that $A$ is incompressible and $\partial$–incompressible.

The intersection $A \cap A_0$ may be assumed to be transverse and minimal up to isotopy. Suppose that the intersection is empty. Then $A$ lies in $U$ and is disjoint from $c^+ \cup c^-$. Also, $A$ is incompressible and not $\partial$–parallel in $U$, since otherwise $A$ would be compressible in $Y_0$ or parallel to $A_0$ or an annulus in $\partial Y_0$. By Lemma 4.1 $A$ is separating in $U$. Since $A$ is nonseparating in $Y_0$, $A$ must separate $c^+$ and $c^-$. It follows from Lemma 2.2 that one of $c^+$ and $c^-$ represents a proper power of a primitive element of $\pi_1(U)$, contradicting the fact that both $c^\pm$ are primitive curves in $U$. Hence $A \cap A_0$ is not empty.

The same argument as in the third and fourth paragraphs in the proof of Lemma 3.4 applies to show that all the components of $A \cap A_0$ are essential on both $A$ and $A_0$ and that they are all either circles or arcs. First, suppose that they are all circles. Then surgery on $A_0$ along an annulus cut off from $A$ by an outermost component of $A \cap A_0$ in $A$ yields two properly embedded annuli $A_1, A_2$ in $Y_0$ which are disjoint from $A_0$. Each annulus $A_i (i = 1, 2)$ is not isotopic to $A_0$ by the minimality assumption on $|A \cap A_0|$. Since we already observed that any nonseparating annulus in $Y_0$ which is not isotopic to $A_0$ cannot be disjoint from $A_0$, each $A_i$ is separating in $Y_0$. This implies that $A_0$ is separating in $Y_0$, a contradiction.

Now suppose all the components of $A \cap A_0$ are arcs that are essential on both $A$ and $A_0$. Then the arcs cut $A$ into rectangles $R_1, \ldots, R_n$. Each rectangle $R_i$ can be considered as a properly embedded disk in $U$, which is essential by the minimality of $A \cap A_0$. Also, each $\partial R_i$ intersects $c^+ \cup c^-$ in two points. There are two possibilities for the intersection of each $\partial R_i$ with $c^+ \cup c^-$; for each $i$, either $\partial R_i$ intersects each of $c^+$ and $c^-$ in a single point or $\partial R_i$ intersects one of $c^+$ and $c^-$ in two points and misses the other.

Suppose that some $R_i$ intersects one of the cores $c^+$ and $c^-$ in two points. Note that each arc of $A \cap A_0$ has two copies in $\partial U$, one in $A_0^+$ and the other in $A_0^-$. This implies that some $R_j (j \neq i)$ intersects the other core in two points. See Figure 16(a).

We may assume that $R_i$ has two points of intersection with $c^+$ (and then $R_j$ has two points of intersection with $c^-$). Then $R_i$ is disjoint from $c^-$, implying that $R_i$ is a properly embedded disk in the solid torus $U[c^-]$. Also, $c^+$ is a simple loop in $\partial U[c^-]$ intersecting $R_i$ in two points. Since a 2–handle addition on $U[c^-]$ along $c^+$ results in the 3–manifold $U[c^+ \cup c^-]$ with $\pi_1(U[c^+ \cup c^-]) \cong \mathbb{Z}_3$, $R_i$ must be $\partial$–parallel in $U[c^-]$. This implies that $R_i$ is separating in $U$. Similarly, $R_j$ is separating in $U$.

Since any two disjoint separating essential disks in a genus two handlebody are parallel, $R_i$ and $R_j$ are parallel in $U$. Since $R_j$ is disjoint from $c^+$, $R_i$ can be isotoped to be disjoint from $c^+$ (and still from $c^-$). This contradicts Lemma 4.1.
Inequivalent handlebody-knots with homeomorphic complements

Figure 16

Hence each $\partial R_i$ intersects each $c^+$ and $c^-$ in a single point, that is, each $R_i$ is commonly dual to $c^+$ and $c^-$. By Lemma 4.2 all the rectangles $R_1, \ldots, R_n$ are isotopic to the disk $D_0$ in Figure 15 and hence they are mutually parallel in $U$. Let $a_i^\pm = R_i \cap A_0^\pm$ for $i = 1, \ldots, n$. We may assume that $R_1, \ldots, R_n$ had been labeled so that $a_1^+, \ldots, a_n^+$ appear in $A_0^+$ successively along the orientation of $c^+$. Then $a_1^-, \ldots, a_n^-$ appear in $A_0^-$ successively along the reversed orientation of $c^-$, since the algebraic intersection number of $\partial D_0$ with the two oriented loops $c^+ \cup c^-$ is zero. See Figure 16(b). In $Y_0$, the arcs $a_1^+, \ldots, a_n^+$ and the arcs $a_1^-, \ldots, a_n^-$ are identified in pair to form $A$. The identification defines a permutation $\sigma$ of $\{1, \ldots, n\}$ such that $a_i^+$ is identified with $a_{\sigma(i)}^-$. In fact, $\sigma(i) \equiv -i + k \mod n$ for some integer $k$.

Suppose that $n$ is odd. By replacing $k$ with $k+n$, if necessary, we may assume that $k$ is even. Then $\sigma(k/2) \equiv -k/2 + k \equiv k/2 \mod n$. This implies $n = 1$, otherwise we would obtain a disconnected surface from the rectangles $R_1, \ldots, R_n$ by identifying $a_i^+$ and $a_{\sigma(i)}^-$ ($i = 1, \ldots, n$). Even if $n = 1$, the identification produces a Möbius band because the two oriented loops $c^+$ and $c^-$ intersect oppositely with $\partial R_1$. This gives a contradiction.

Suppose that $n$ is even. The complementary regions of $R_1 \cup \cdots \cup R_n$ in $U$ can be alternately colored black and white. If $\sigma(i) \equiv -i + k \mod n$ for some odd integer $k$ then black regions match with black regions and white regions match with white regions, implying that $A$ is separating in $Y_0$. Hence $k$ is even. Then $\sigma(k/2) \equiv k/2 \mod n$, and two opposite sides $a_k^+$ and $a_k^-$ of $R_k$ are identified to form a Möbius band. This is also impossible.

Proof of Theorem 1.1(2) Let $\partial_1 A_0$ and $\partial_2 A_0$ denote the two boundary components of $A_0$ as shown in Figure 17. After an isotopy, the two loops appear in $\partial Y_0$ as shown in the last drawing in the figure.
Recall that twisting $W_0$ $n$ times along the shaded disk in Figure 18 defines a homeomorphism $\sigma_n$: $Y_0 \to Y_n$. By Lemma 4.5, $A_n = \sigma_n(A_0)$ is a unique properly embedded nonseparating annulus in $Y_n$ up to isotopy. Let $\partial_i A_n = \sigma_n(\partial_i A_0)$ for $i = 1, 2$. The core of $A_n$ is an embedded circle in $S^3$, isotopic to any boundary component of $A_n$ in $S^3$ along a half of $A_n$. One easily sees that $\partial_1 A_n$ is a $(3, 3n-1)$–torus knot, and so is the core.

Assume that $W_n$ is amphicheiral. Then there is an orientation-preserving homeomorphism of pairs $(S^3, W_n) \to (S^3, W_n^*)$. Since $A_n$ and $A_n^*$ are respectively up to isotopy unique nonseparating annuli in $Y_n$ and $Y_n^*$ by Lemma 4.5, composing with
an orientation-preserving automorphism of the pair \((S^3, W_n^*)\), if necessary, we may assume that the homeomorphism takes \(A_n\) to \(A_n^*\). This implies that \(A_n\) and \(A_n^*\) are isotopic in \(S^3\). In particular, their cores are isotopic. The core of \(A_n\) is a \((3, 3n-1)\)-torus knot, while that of \(A_n^*\) is the mirror image of a \((3, 3n-1)\)-torus knot. It is well known that every nontrivial torus knot is not amphicheiral. If \(n \neq 0\) then a \((3, 3n-1)\)-torus knot is not the trivial knot, so it is not amphicheiral. Hence \(n = 0\). However, \(\partial A_0\) is a \((2, -6)\)-torus link (see the first drawing in Figure 17), while \(\partial A_0^*\) is the mirror image of a \((2, -6)\)-torus link. The two torus links are not isotopic, a contradiction. Hence \(W_n\) is not amphicheiral for any integer \(n\).

Let \(n\) and \(m\) be distinct integers. Then neither of the \((3, 3n-1)\)-torus knot and its mirror image is isotopic to the \((3, 3m-1)\)-torus knot. Hence a similar argument as above shows that neither of \(W_n\) and \(W_n^*\) is equivalent to \(W_m\).

\[\square\]

Acknowledgements We would like to thank Atsushi Ishii, Kengo Kishimoto and Makoto Ozawa for their helpful conversations. The first author was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2011-0027989). The second author was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2010-0024630).

References


Department of Mathematics and Inst. of Pure and Applied Math., Chonbuk National University Jeonju 561-756, Korea

Department of Mathematics, Chung-Ang University
221 Heukseok-dong, Dongjak-gu, Seoul 156-756, South Korea

junghoon@jbnu.ac.kr, sylee@cau.ac.kr

Received: 31 October 2011 Revised: 20 February 2012