Inequivalent handlebody-knots with homeomorphic complements

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We distinguish the handlebody-knots $5_1, 6_4$ and $5_2, 6_{13}$ in the table, due to Ishii et al, of irreducible handlebody-knots up to six crossings. Furthermore, we construct two infinite families of handlebody-knots, each containing one of the pairs $5_1, 6_4$ and $5_2, 6_{13}$, and show that any two handlebody-knots in each family have homeomorphic complements but they are not equivalent.

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1 Introduction

Given a knot in S^3, its regular neighborhood is a knotted solid torus. Conversely, an embedded solid torus in S^3 uniquely determines a knot. Thus we may regard an embedded solid torus as a knot in S^3. Instead of an embedded solid torus in S^3, consider an embedded handlebody. We may regard it as a kind of a knot. Following Ishii, Kishimoto, Moriuchi and Suzuki [3], we say that a handlebody embedded in S^3 is a *handlebody-knot*.

Throughout this paper, by a handlebody-knot we will mean a genus two handlebody embedded in S^3. A handcuff graph or a θ–curve Γ in a handlebody-knot H is called a *spine* if H is a regular neighborhood of Γ. The spine of H is not uniquely determined, but any two spines are related by a finite sequence of isotopies and IH-moves (see Ishii [2]), where an IH-move is a local move on a spatial trivalent graph depicted in Figure 1.

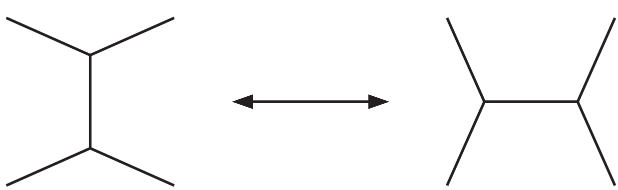


Figure 1

Published: 7 May 2012 DOI: 10.2140/agt.2012.12.1059

Two handlebody-knots H_1 and H_2 are said to be *equivalent* if there exists an isotopy of S^3 that takes H_1 to H_2, or equivalently if there exists an orientation-preserving automorphism h of S^3 such that $h(H_1) = H_2$. A handlebody-knot H is *reducible* if there exists a 2–sphere S in S^3 such that $S \cap H$ is a disk separating H into two solid tori. Otherwise, it is *irreducible*. Note that H is irreducible if $S^3 - \mathrm{int}(H)$ is ∂–irreducible.

As done for knots, we can use regular diagrams of spines of a handlebody-knot to define the crossing number of the handlebody-knot. Ishii, Kishimoto, Moriuchi and Suzuki recently give a table of handlebody-knots such that any irreducible handlebody-knot with six or fewer crossings or its mirror image is equivalent to one of the handlebody-knots in the table. See [3, Table 1]. By using some invariants, they distinguish all handlebody-knots in their table except only for the two pairs $(5_1, 6_4)$ and $(5_2, 6_{13})$. See Figure 2.

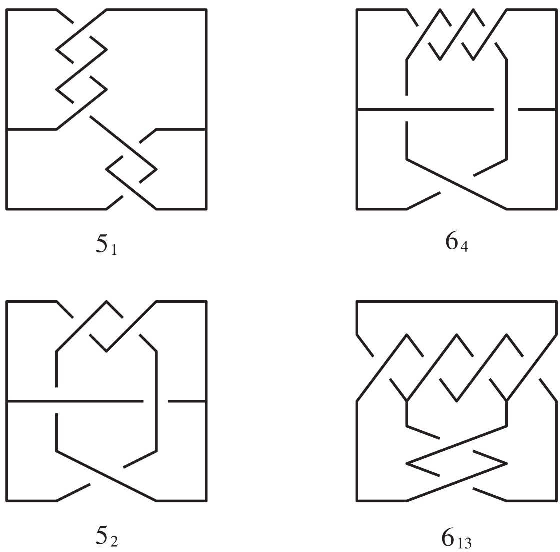


Figure 2

Consider the handcuff graphs Φ_n, Ψ_n in S^3, shown in Figure 3, where a rectangle labeled by an integer n denotes a vertical right-handed twist of two strings with $2n$ crossings. Let V_n and W_n denote regular neighborhoods of Φ_n and Ψ_n, respectively. Put $X_n = S^3 - \mathrm{int}(V_n)$ and $Y_n = S^3 - \mathrm{int}(W_n)$.

Let $\Theta_n = \Phi_n$ or Ψ_n, and let $Z_n = X_n$ or Y_n correspondingly. The handcuff graph Θ_n consists of two vertices and three edges, two forming loops and one connecting the two loops. One of the two loops bounds a disk intersecting the vertical twist in two points.

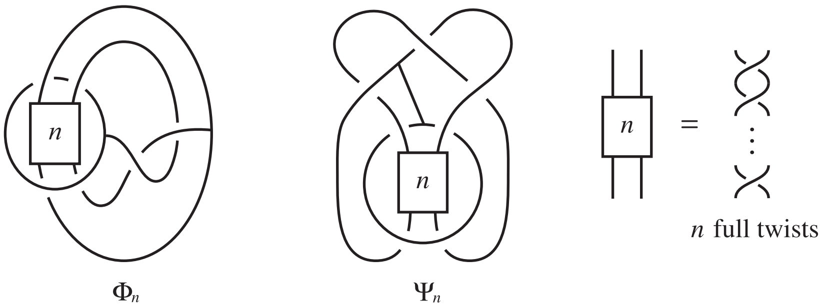


Figure 3

By twisting along the disk, one can transform Θ_n into Θ_m for any other integer m. This shows that Z_n is homeomorphic to Z_m.

For any submanifold M of S^3, denote by M^* the mirror image of M. We say that M is *amphicheiral* if an isotopy of S^3 takes M to M^*. The main result of the present paper is the following.

Theorem 1.1 *Let n and m be distinct integers.*

1. *No two of V_n, V_n^*, V_m, V_m^* are equivalent.*
2. *No two of W_n, W_n^*, W_m, W_m^* are equivalent.*

In particular, V_n and W_n are not amphicheiral for each integer n.

By calculating fundamental groups, one can show that X_0 and Y_0 are not homeomorphic. This implies that V_n and W_m are not equivalent for any integers n and m.

It is a celebrated result of Gordon and Luecke that if two knots in S^3 have homeomorphic complements then the homeomorphism between the two complements extends to an automorphism of S^3 [1]. In contrast, Motto [5] showed that handlebody-knots are not determined by their complements. We remark that our infinite families of inequivalent handlebody-knots are also of this type.

We can now distinguish the handlebody-knots $5_1, 6_4$, and $5_2, 6_{13}$ in the table due to Ishii et al.

Corollary 1.2 (1) *No two of $5_1, 5_1^*, 6_4, 6_4^*$ are equivalent.*

(2) *No two of $5_2, 5_2^*, 6_{13}, 6_{13}^*$ are equivalent.*

In particular, $5_1, 5_2, 6_4, 6_{13}$ are not amphicheiral.

Proof The sequences of pictures in Figure 4(a),(b) show that V_0 and V_{-1} are respectively equivalent to 5_1 and 6_4, and the sequences of pictures in Figure 4(c),(d) show that W_0 and W_1 are respectively equivalent to 5_2 and 6^*_{13}. Hence the result immediately follows from Theorem 1.1. □

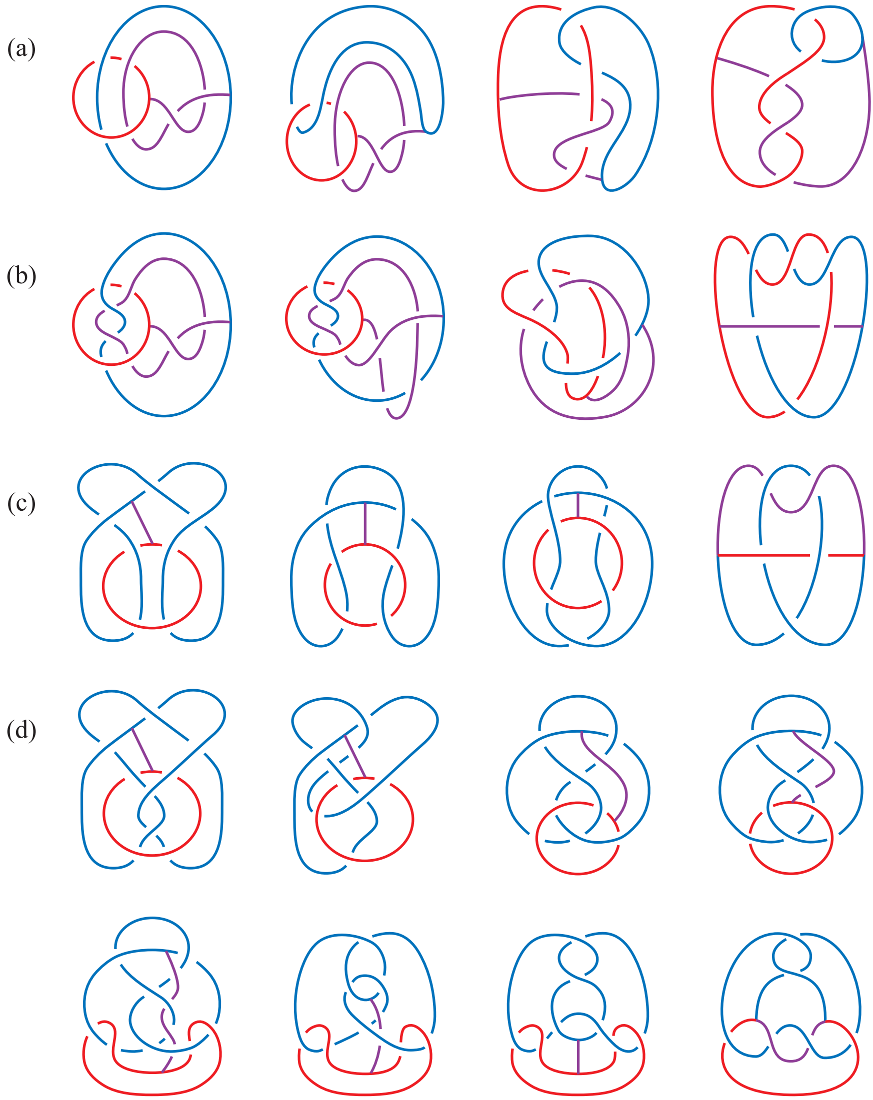


Figure 4

Some figures in this paper are best viewed in color; readers confused by figures in a black-and-white version are recommended to view the electronic version.

2 Curves in the boundary of a genus two handlebody

A properly embedded disk in a 3–manifold M is *essential* if it is not isotopic to a disk in ∂M. A properly embedded compact surface in M, which is neither a disk nor a sphere, is *essential* if it is incompressible and is not ∂–parallel. Given a set $\{c_1, \dots, c_n\}$ of disjoint simple loops in ∂M, $M[c_1 \cup \cdots \cup c_n]$ will denote the 3–manifold obtained by attaching 2–handles to M along disjoint neighborhoods of $c_1, \dots, c_n$.

Throughout this section, H will denote a genus two handlebody. A simple loop in ∂H is called a *primitive curve* if there exists a disk in H, called a *dual disk*, that intersects the loop in a single point.

Lemma 2.1 *Let c_1, c_2 be two disjoint nonisotopic primitive curves in ∂H. If there are two disjoint nonisotopic essential disks D_1, D_2 of H each of which is a common dual disk of c_1 and c_2, then the fundamental group of $H[c_1 \cup c_2]$ is either the infinite cyclic group or the cyclic group of order 2.*

Proof The two disks D_1, D_2 cut H into a 3–ball B and $c_1 \cup c_2$ into four arcs. Let D_i^+, D_i^- be the copies of D_i on ∂B for $i = 1, 2$. There are two cases; the four arcs together with the four disks $D_1^\pm, D_2^\pm$ form two cycles of length 2 or a single cycle of length 4. See Figure 5. One easily sees that the fundamental group of $H[c_1 \cup c_2]$ is the infinite cyclic group in the first case and it is the cyclic group of order 2 in the latter case. □

An element x of the free group F of rank 2 is called a *primitive element* if there exists an element $y \in F$ such that x, y generate F.

Lemma 2.2 *Let A be an essential separating annulus in H. Let c_1, c_2 be two essential simple loops in ∂H which are disjoint from ∂A. Suppose that A separates c_1 and c_2. Then one of c_1 and c_2 represents a proper power of a primitive element of the free group $\pi_1(H)$.*

Proof By Kobayashi [4, Lemma 3.2(i)], A cuts H into a solid torus H_1 and a genus two handlebody H_2. Since A separates c_1 and c_2, we may assume $c_1 \subset H_1$ and $c_2 \subset H_2$. Let A_i be the copy of A in ∂H_i for $i = 1, 2$. Then the core of A_1 is parallel to c_1 in ∂H_1, and the core of A_2 represents a primitive element of the free group $\pi_1(H_2)$.

If c_1 were a meridian curve of H_1 then A would be compressible in H. If c_1 were homotopic to the core of H_1 then A would be ∂–parallel in H. Hence c_1 is homotopic in H_1 to $n\,(\geq 2)$ times around the core of H_1.

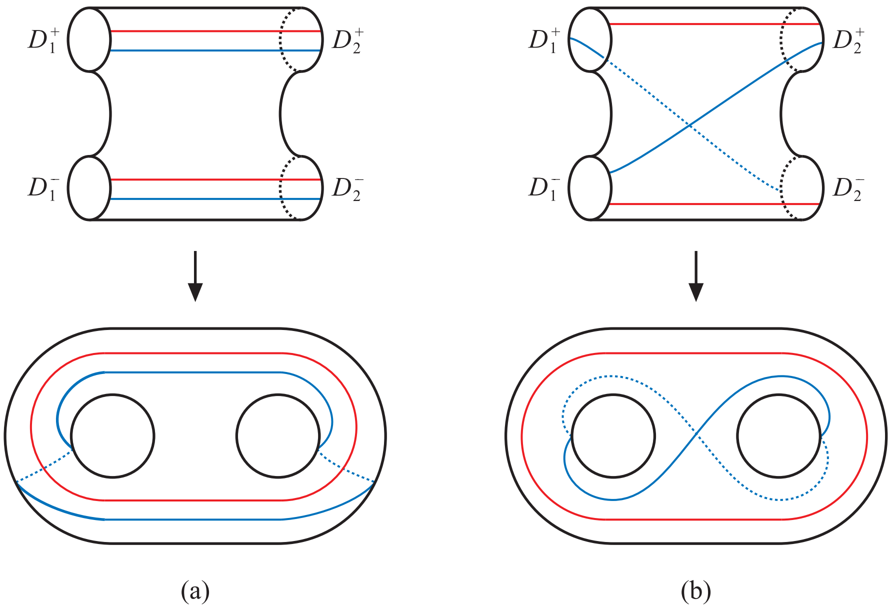


Figure 5

Let x be a generator of the infinite cyclic group $\pi_1(H_1)$, and let y, z be two elements generating the free group $\pi_1(H_2)$. Here, we may assume that x^n is represented by the core of A_1 (or c_1) and y is represented by the core of A_2. By the Van Kampen's theorem, $\pi_1(H)$ has three generators x, y, z and one relation $x^n = y$. Thus $\pi_1(H)$ is the free group on x and z, and c_1 represents x^n in the group $\pi_1(H)$. □

Lemma 2.3 *Let c_1, c_2 be two simple loops in ∂H which are not contractible in H. Suppose that there exists a properly embedded disk D in $H - c_1 \cup c_2$ which splits H into two solid tori, each containing one of c_1 and c_2. Then any such disk is isotopic to D in $H - c_1 \cup c_2$.*

Proof Let E be a properly embedded disk in $H - c_1 \cup c_2$ which splits H into two solid tori H_1 and H_2 with $c_i \subset H_i$ for each $i = 1, 2$. Suppose that E is not isotopic to D in $H - c_1 \cup c_2$.

If E is disjoint from D then D and E are parallel in H, that is, they cut off a 1–handle $D \times I$ from H. Since neither c_1 nor c_2 is contractible in H, $\partial D \times I$ does not meet any of c_1 and c_2. This means that $D \times I$ is, in fact, the parallelism between D and E in $H - c_1 \cup c_2$. This contradicts our assumption on E.

We may assume that the intersection $D \cap E$ is transverse and minimal up to isotopy of E. Then a standard disk swapping argument shows that $D \cap E$ has no circle components. An arc component of $D \cap E$, outermost in D, cuts off a subdisk of D. Surgery on E along the subdisk yields two disks, both of which are disjoint from $c_1 \cup c_2$. Let E' be any of these disks. Then E' lies in a solid torus H_i for some $i = 1, 2$. By the minimality of $|D \cap E|$, E' is parallel in $H - c_1 \cup c_2$ to neither E nor a disk in ∂H. Hence E' is a meridian disk of the solid torus H_i, cutting it into a 3–ball in which c_i lies. This implies that c_i is contractible in H, a contradiction. □

3 V_n and V_m ($n \neq m$) are not equivalent

Consider Φ_0. The drawings in Figure 4(a) depict an isotopy from V_0 to 5_1, showing that there exists a properly embedded nonseparating annulus A_0 in X_0 as shown in Figure 6(a). Cutting X_0 along A_0 gives a new compact 3–manifold U as shown in Figure 6(b), where the two loops in ∂U are the cores of the two copies A_0^+ and A_0^- of A_0 in ∂U. Let $c^{\pm}$ be the loops. After an isotopy, U becomes the complement of a standardly embedded genus two handlebody in S^3 (see Figure 7), so U itself is a genus two handlebody.

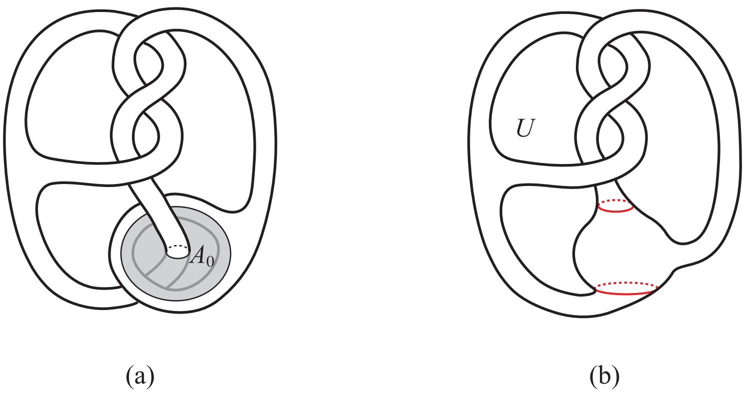


Figure 6

Let $C = c^+ \cup c^-$. Take three essential nonseparating disks X, Y, Z in U as shown in Figure 8(a). These three disks divide U into two 3–balls $B^{\pm}$ and C into arcs. See Figure 8(b). Let $X^{\pm}, Y^{\pm}, Z^{\pm}$ be copies of X, Y, Z in $\partial B^{\pm}$. Then $C^{\pm} = C \cap B^{\pm}$ consists of five arcs, two connecting $X^{\pm}$ and $Y^{\pm}$, two connecting $X^{\pm}$ and $Z^{\pm}$, and one connecting $Y^{\pm}$ and $Z^{\pm}$. Set $\Delta = X \cup Y \cup Z$ and $\Delta^{\pm} = X^{\pm} \cup Y^{\pm} \cup Z^{\pm}$. Then $\partial B^{\pm} - (\Delta^{\pm} \cup C^{\pm})$ is a union of (open) disks.

Lemma 3.1 *U does not contain an essential disk or annulus or a properly embedded Möbius band which is disjoint from C.*

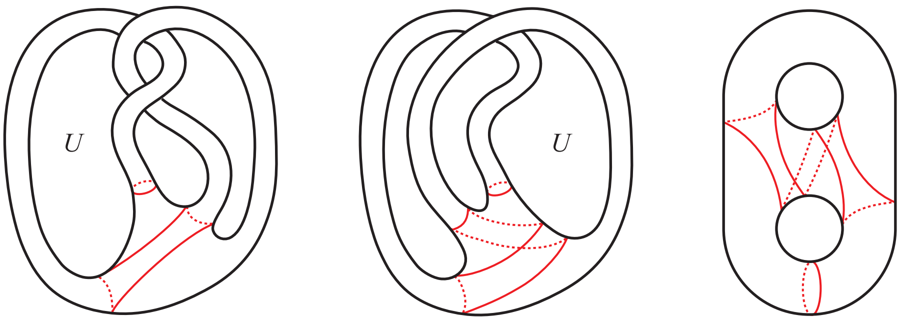
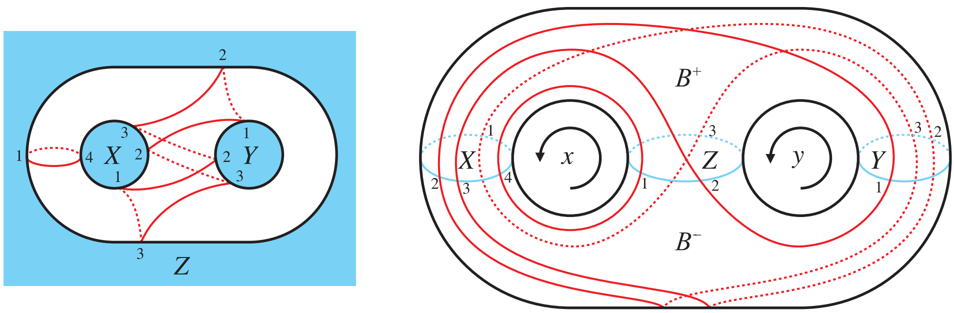


Figure 7



(a) (b)

Figure 8

Proof Assume for contradiction that U contains such a surface F.

First, suppose that F is a disk. The intersection $F \cap \Delta$ may be assumed to be transverse and minimal among all essential disks of U that are disjoint from C. Note that $F \cap \Delta \neq \varnothing$, since otherwise F would be properly embedded in either B^+ or B^- with $\partial F \cap (\Delta^{\pm} \cup C^{\pm}) = \varnothing$ and hence F would be parallel to a disk in ∂U. By the minimality of $|F \cap \Delta|$, F has no circle components of intersection with Δ. An arc component of intersection, outermost in F, cuts off a disk F' from F. Any two disks in $\Delta^{\pm}$ are joined by an arc in $C^{\pm}$, so the arc $F' \cap \partial U$ together with an arc in $\partial\Delta$ bounds a disk in ∂U that is disjoint from C. This disk could be used to reduce $|F \cap \Delta|$, contradicting the minimality assumption. Hence F is not a disk.

The fundamental group $\pi_1(U)$ is a free group generated by two elements x and y, where x and y are respectively represented by the cores of the 1–handles $N(X)$ and $N(Y)$, attached to the 3–ball $N(Z)$. See Figure 8(b). The two loops c^+ and c^- represent two group elements x and $xyxy^{-1}x^{-1}y^{-1}$. Hence the 3–manifold

$Q = U[c^+ \cup c^-]$ has a trivial fundamental group, so it is a 3–ball. Since F is disjoint from C, F is properly embedded in Q. No Möbius bands can be properly embedded in a 3–ball, so F must be an annulus. Since every properly embedded annulus in a 3–ball is separating, F must be separating in U. Splitting U along F, we get a solid torus U_1 and a genus two handlebody U_2, where the core of the copy of F in ∂U_1 winds the solid torus U_1 at least two times in the longitudinal direction. See [4, Lemma 3.2(i)].

Neither x nor $xyxy^{-1}x^{-1}y^{-1}$ is a proper power of a primitive element of the group $\pi_1(U)$. Thus it follows from Lemma 2.2 that the two loops c^+ and c^- are not separated by F. Since c^+ and c^- are not parallel in ∂U, they are contained in U_2. Hence F splits Q into U_1 and $U_2[c^+ \cup c^-]$. In particular, F cuts off the solid torus U_1 from the 3–ball Q so that the core of the copy of F in ∂U_1 is homotopic to at least two times around the core of U_1. This is impossible. □

Lemma 3.2 *A_0 is incompressible and ∂–incompressible in X_0.*

Proof Since each of c^+ and c^- represents a nontrivial element of the free group $\pi_1(U)$, A_0 is incompressible. Suppose that A_0 is ∂–compressible. Then there exists a properly embedded disk D in U intersecting C in a single point. We may assume that D intersects c^+. Then the frontier of a neighborhood of $D \cup c^+$ in U is an essential separating disk in U that is disjoint from C, contradicting Lemma 3.1. Hence A_0 is ∂–incompressible. □

Lemma 3.3 *X_0 is irreducible and ∂–irreducible. Hence X_n is irreducible and ∂–irreducible for any integer n.*

Proof It is clear that X_0 is irreducible. If X_0 is ∂–reducible then any compressing disk for ∂X_0 can be isotoped to be disjoint from A_0. Then it lies in U as an essential disk disjoint from $c^+ \cup c^-$. This contradicts Lemma 3.1. □

Since X_n is ∂–irreducible, V_n is an irreducible handlebody-knot.

Lemma 3.4 *A_0 is a unique properly embedded nonseparating annulus in X_0 up to isotopy.*

Proof Let A be a properly embedded nonseparating annulus in X_0 that is not isotopic to A_0. The ∂–irreducibility of X_0 implies that A is incompressible and ∂–incompressible.

We may assume that A had been chosen to intersect A_0 transversely and minimally among all properly embedded nonseparating annuli in X_0. Note that A must intersect A_0, otherwise A would survive in U and be incompressible, so by Lemma 3.1 A would be parallel to either A_0^+ or A_0^- in U and hence be parallel to A_0 in X_0, contradicting the choice of A.

Suppose that there are circle components of $A \cap A_0$ that are inessential on both A and A_0. Let α be a circle component of $A \cap A_0$ that is innermost on A_0 among all such circle components. Then α bounds a disk D in A and a disk D_0 in A_0. Note that the interior of D_0 is disjoint from A, since otherwise an innermost component of $A \cap D_0$ on D_0 would bound a compressing disk for A. We now obtain a new nonseparating annulus $(A - D) \cup D_0$, which is properly embedded in X_0 and can be isotoped so as to intersect A_0 transversely with fewer components of intersection. This contradicts the choice of A. Hence each circle component of $A \cap A_0$, if it exists, is essential on at least one of A and A_0. Suppose that there are circle components of $A \cap A_0$ that are essential on one of the annuli A and A_0, and inessential on the other annulus. Let β be a circle component of $A \cap A_0$ that is innermost on (say) A among all such circle components (the argument for the case $\beta \subset A_0$ is similar). Then β bounds a disk E in A. Since no circle components of $A \cap A_0$ are inessential on both A and A_0, the interior of E misses A_0 by the choice of β. This implies that E is a compressing disk for A_0, a contradiction. We conclude that all circle components of $A \cap A_0$, if they exist, are essential on both A and A_0.

A similar argument, using an outermost arc component of intersection instead of an innermost circle component and using the ∂–incompressibility of $A \cup A_0$ instead of the incompressibility, shows that all arc components of $A \cap A_0$, if they exist, are essential on both A and A_0. Thus all the components of $A \cap A_0$ are either circles or arcs.

First, suppose that they are all circles. Take an annulus cut off from A by an outermost component of $A \cap A_0$ in A, and surger A_0 along this annulus. The resulting surface is a union of two annuli disjoint from A_0. Let A_0' be any one of these two annuli. Since one boundary circle of A_0' is isotopic to that of A_0 (or A), A_0' must be incompressible in X_0 and hence in U. By Lemma 3.1, A_0' must be ∂–parallel in U, which implies that A_0' is either ∂–parallel in X_0 or parallel to A_0. In any case, we can reduce $|A \cap A_0|$, giving a contradiction.

Now suppose that all components of $A \cap A_0$ are arcs that are essential on both A and A_0. The arcs divide A into rectangles $R_1, \dots, R_n$, where $n = |A \cap A_0|$. Consider $R = R_1$. We may regard R as a properly embedded disk in U whose boundary intersects $C = c^+ \cup c^-$ in two points. There are two cases; ∂R intersects each of c^+ and c^- in a single point, or ∂R intersects only one of c^+ and c^-, say, c^+. In the former case, each of c^+ and c^- is a primitive curve in U, that is, it is a generator of the free group $\pi_1(U)$ of rank two, but it is easy to see from Figure 8(b) that one of c^+ and c^- is not a generator.

In the latter case, the two points in $\partial R \cap c^+$ split c^+ into two arcs a_1 and a_2. Let S_i $(i = 1, 2)$ be a properly embedded surface in U obtained from R by attaching a band along a_i and then pushing the interior of the resulting surface into the interior of U. Note that S_i is disjoint from C for each $i = 1, 2$. The two ends of a_i must lie on the same side of R (then S_i is an annulus), otherwise S_i would be a Möbius band, contradicting Lemma 3.1.

If R were ∂–parallel in U then we could reduce $|A \cap A_0|$. Thus R is an essential disk in U. First, suppose that R is a nonseparating disk in U. Consider any S_i and recall that S_i is obtained from the nonseparating disk R by attaching a band. Any such annulus has boundary circles which are not mutually parallel in ∂U and at least one of which is essential in ∂U. Since the two boundary circles of S_i are not mutually parallel in ∂U, S_i is not ∂–parallel in U. Since at least one boundary circle of S_i is essential in ∂U, S_i is incompressible in U, otherwise a compression of S_i would yield an essential disk in U disjoint from C, contradicting Lemma 3.1. Hence S_i is an essential annulus. This contradicts Lemma 3.1 again.

Suppose that R is an essential separating disk in U. Then R splits U into two solid tori U_1 and U_2, where S_i can be pushed into U_i. If the core of some S_i winds U_i at least two times in the longitudinal direction, then S_i is an essential annulus in U, contradicting Lemma 3.1. Thus the core of each S_i is homotopic to the core of U_i. This implies that $c^+ = a_1 \cup a_2$ is a primitive curve in U. Since c^- does not intersect $R \cup c^+$, c^- is also a primitive curve in U. See Figure 9. This contradicts our observation that one of c^+ and c^- is not a primitive curve in U. □

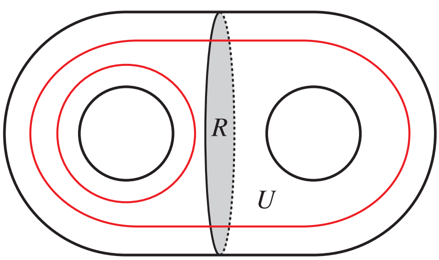


Figure 9

Lemma 3.5 *V_0 is not amphicheiral.*

Proof Assume that there exists an orientation-preserving automorphism h of S^3 that takes V_0 to V_0^* (and then X_0 to X_0^*). Take a regular neighborhood $N(A_0)$ of the nonseparating annulus A_0 in X_0. Put $A_h = h(A_0)$ and $N(A_h) = h(N(A_0))$. Then $\widetilde{V}_h = V_0^* \cup N(A_h)$ is the image of $\widetilde{V}_0 = V_0 \cup N(A_0)$ under the automorphism h. The frontier of $N(A_0)$ in X_0 consists of two annuli whose cores c^+ and c^- run along $\partial \widetilde{V}_0$ as shown in Figure 6(b), where U in the figure may be considered as the closed complement of $\widetilde{V}_0$. Each core $c^\pm$ bounds a disk $D^\pm$ in $\widetilde{V}_0$. Let $c_h^\pm = h(c^\pm)$ and $D_h^\pm = h(D^\pm)$. Then $c_h^\pm$ are the cores of the frontier annuli of $N(A_h)$ in X_0^* and they bound disks $D_h^\pm$.

Note that A_h is a properly embedded nonseparating annulus in X_0^*. By Lemma 3.4 A_0^* is a unique properly embedded nonseparating annulus in X_0^* up to isotopy. Hence A_h and A_0^* are isotopic in X_0^*.

Note that $\mathrm{cl}(\widetilde{V}_0 - N(D^\pm))$ is an embedded solid torus in S^3. The core of the solid torus is either the unknot or the right-handed trefoil according to the choice of the disks D^+ and D^-. We may assume that the core is the unknot for D^- and the right-handed trefoil for D^+. See Figure 10. Similarly, $\mathrm{cl}(\widetilde{V}_h - N(D_h^\pm))$ is a solid torus embedded in S^3 whose core is either the unknot or the left-handed trefoil. The orientation-preserving automorphism h takes $\mathrm{cl}(\widetilde{V}_0 - N(D^+))$ to $\mathrm{cl}(\widetilde{V}_h - N(D_h^+))$ or $\mathrm{cl}(\widetilde{V}_h - N(D_h^-))$. This implies that the right-handed trefoil is equivalent to the unknot or the left-handed trefoil, both of which are impossible. □

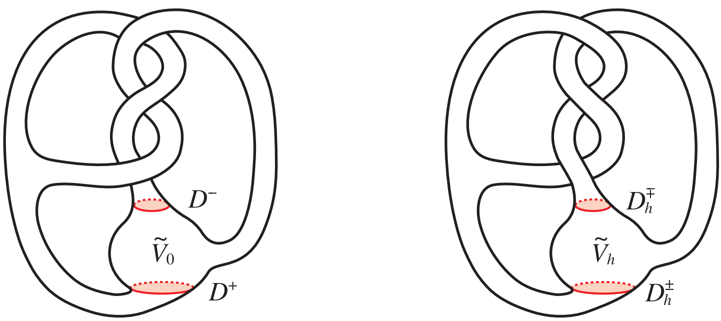


Figure 10

Recall that twisting V_0 n times along the shaded disk in Figure 11(a) defines a homeomorphism $\sigma_k \colon X_0 \to X_k$. By Lemma 3.4, $A_k = \sigma_k(A_0)$ is up to isotopy a unique nonseparating annulus in X_k. Note that $A_k \subset S^3$ is an unknotted annulus with k full twists and its boundary is the $(2, 2k)$–torus link (if $k = \pm 1$, the boundary is the Hopf link). See Figure 11(b).

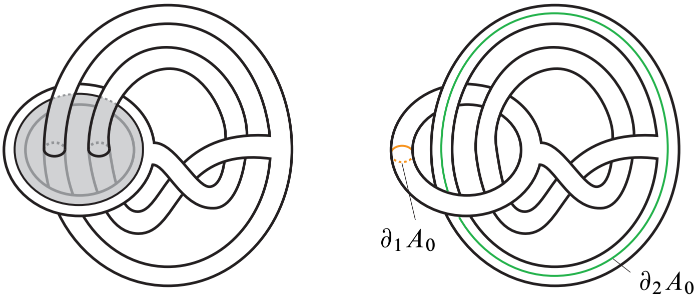


Figure 11

Let c_k, d_k be the two loop edges of Φ_k and e_k the nonloop edge. Then V_k is a union of two solid tori $V_{k,1} = N(c_k), V_{k,2} = N(d_k)$, and a 1–handle $H_k = \mathrm{cl}(N(e_k) - V_{k,1} \cup V_{k,2})$. It may be assumed that $V_{k,1}$ contains the boundary of the shaded disk in Figure 11(a). Each boundary component of A_k is not contractible in V_k if $k \neq 0$, and a cocore disk D_k for the 1–handle H_k splits V_k into two solid tori, isotopic to $V_{k,1}$ and $V_{k,2}$, each of which contains one boundary component of A_k. Let $\partial_i A_k (i = 1, 2)$ denote the boundary component of A_k lying in $V_{k,i}$. See Figure 11(b).

Lemma 3.6 *There exists an orientation-preserving automorphism of the pair (S^3, V_{-1}) which interchanges $V_{-1,1}$ and $V_{-1,2}$.*

Proof Figure 4(b) allows us to regard V_{-1} as 6_4. It is easy to see that an involution on $(S^3, 6_4)$ is defined by rotating 6_4 through π about a vertical axis. The involution is the desired automorphism. □

Proof of Theorem 1.1(1) First, assume that V_n is amphicheiral for some nonzero integer n (V_0 is not amphicheiral by Lemma 3.5), that is, there is an orientation-preserving homeomorphism of pairs $(S^3, V_n) \to (S^3, V_n^*)$. Note that A_n and A_n^* are up to isotopy unique nonseparating annuli in X_n and X_n^*, respectively. Hence composing with an orientation-preserving automorphism of the pair (S^3, V_n^*), if necessary, we may assume that the homeomorphism takes A_n to A_n^*. In other words, A_n and A_n^* are isotopic in S^3. However, one of the annuli A_n and A_n^* has right-handed $|n|$ full twists and the other left-handed $|n|$ full twists, so they cannot be isotopic. This gives a contradiction. Therefore V_n is not equivalent to its mirror image for any integer n.

Let n, m be distinct integers, and assume that there is a homeomorphism of pairs $h\colon (S^3, V_n) \to (S^3, V_m)$, where h may or may not preserve the orientation of S^3.

Similarly as above, we may assume that $h(A_n) = A_m$. Then $h(\partial A_n) = \partial A_m$, which means that h takes a $(2, 2n)$–torus link to a $(2, 2m)$–torus link. Hence $m = n$ or $m = -n$. The former contradicts the assumption that n and m are distinct. If $n = 0$ then h must preserve the orientation of S^3 by Lemma 3.5, so h is isotopic to the identity of S^3 and we have nothing to prove. Hence we may assume that $m = -n$ and $n \neq 0$. Since the twists of A_n and A_{-n} are reversed, h must be orientation-reversing.

By Lemma 2.3 $D_{\pm n}$, a cocore disk of the 1–handle $H_{\pm n}$ in $V_{\pm n}$, is up to isotopy a unique essential separating disk in $V_{\pm n}$ which separates the two boundary components of $A_{\pm n}$, so it may be assumed up to isotopy of V_{-n} that $h(D_n) = D_{-n}$ and moreover $h(H_n) = H_{-n}$. This implies that h takes each solid torus $V_{n,i} (i = 1, 2)$ to one of the two solid tori $V_{-n,1}$ and $V_{-n,2}$. Note that $\partial_1 A_{\pm n}$ is homotopic to $\pm n$ times the core of $V_{\pm n,1}$, while $\partial_2 A_{\pm n}$ is homotopic to the core of $V_{\pm n,2}$. Hence when $|n| \geq 2$, $h(\partial_i A_n) = \partial_i A_{-n}$ for each $i = 1, 2$, which implies $h(V_{n,i}) = V_{-n,i}$. When $|n| = 1$, by composing h with an orientation-preserving automorphism of the pair (S^3, V_{-1}) given in Lemma 3.6 we may assume that $h(V_{n,i}) = V_{-n,i}$ for each $i = 1, 2$. In particular, we may always assume that c_n, the core of $V_{n,1}$, is mapped by h onto c_{-n}, the core of $V_{-n,1}$. Consider the composition

$$(S^3, V_n) \xrightarrow{h} (S^3, V_{-n}) \xrightarrow{r} (S^3, V^*_{-n}),$$

where r is a reflection. See Figure 12. Let f be the restriction of the composition $r \circ h$ onto the pair $(S^3 - V_{n,1}, V_n - V_{n,1})$. Then $f\colon (S^3 - V_{n,1}, V_n - V_{n,1}) \to (S^3 - V^*_{-n,1}, V^*_{-n} - V^*_{-n,1})$ is an orientation-preserving homeomorphism of pairs.

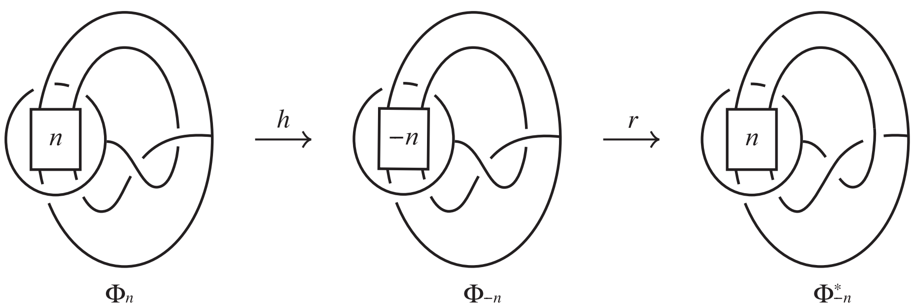


Figure 12

Note that (S^3, V_n) is obtained from (S^3, V_0) by $1/n$–surgery on c_0. Also, (S^3, V^*_{-n}) is obtained from (S^3, V^*_0) by $1/n$–surgery on c^*_0. These two surgeries define two orientation-preserving homeomorphisms of pairs as follows:

$$(S^3 - V_{0,1}, V_0 - V_{0,1}) \xrightarrow{g} (S^3 - V_{n,1}, V_n - V_{n,1}),$$
$$(S^3 - V^*_{0,1}, V^*_0 - V^*_{0,1}) \xrightarrow{g^*} (S^3 - V^*_{-n,1}, V^*_{-n} - V^*_{-n,1}).$$

For example, twisting n times along the shaded disk in Figure 11(a) defines g. The composition $(g^*)^{-1} \circ f \circ g$ is an orientation-preserving homeomorphism from $(S^3 - V_{0,1}, V_0 - V_{0,1})$ to $(S^3 - V^*_{0,1}, V^*_0 - V^*_{0,1})$. Note that the composition takes a meridian of c_0 to a meridian of c^*_0. Hence $(g^*)^{-1} \circ f \circ g$ extends to an orientation-preserving homeomorphism of pairs from (S^3, V_0) to (S^3, V^*_0). This contradicts Lemma 3.5. □

4 W_n and W_m $(n \neq m)$ are not equivalent

Consider Ψ_0. An isotopy of S^3 gives the pictures in Figure 13, showing that there exists a nonseparating annulus A_0 in Y_0. Cutting Y_0 along A_0 gives a genus two handlebody U. Let $A^{\pm}_0$ be the two copies of A_0 in ∂U and $c^{\pm}$ the cores of $A^{\pm}_0$. See Figure 14(a) for $c^{\pm}$, where U is the outside of the standardly embedded genus two surface and Y_0 can be recovered by gluing the annulus neighborhoods $A^{\pm}_0$ of $c^{\pm}$ in the manner indicated in the figure. An external view of $(U, c^{\pm})$ is illustrated in Figure 14(b), that is, U is the inside of the standardly embedded genus two surface in the figure.

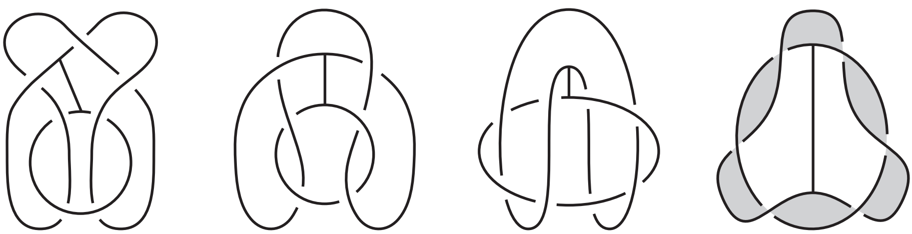


Figure 13

Lemma 4.1 *U does not contain an essential disk or a properly embedded nonseparating annulus disjoint from $c^+ \cup c^-$.*

Proof First, note that both $c^{\pm}$ are primitive curves in U, so $U[c^{\pm}]$ are solid tori. Also, it is easy to see that the fundamental group of $U[c^+ \cup c^-]$ is cyclic with order 3.

Assume that there exists an essential disk D in U disjoint from $c^+ \cup c^-$. If D is a nonseparating disk in U then it is also nonseparating in $U[c^+ \cup c^-]$ and hence

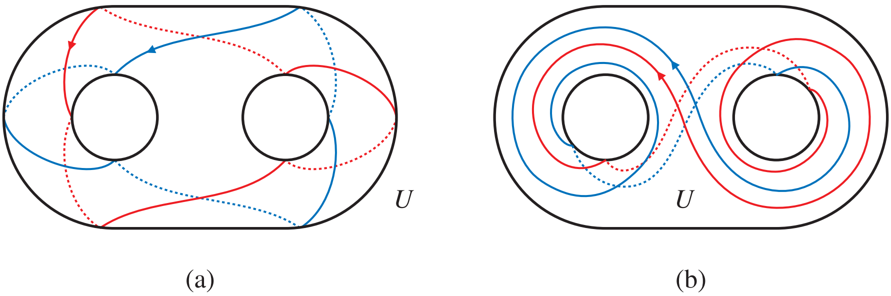


Figure 14

the fundamental group of $U[c^+ \cup c^-]$ contains an element of infinite order, contradicting the observation above. Hence D separates U into two solid tori U^+ and U^-. Since U does not contain a nonseparating disk disjoint from $c^+ \cup c^-$, both U^+ and U^- intersect $c^+ \cup c^-$ and hence we may assume that $c^\pm \subset U^\pm$. Then $\mathbb{Z}_3 \cong \pi_1(U[c^+ \cup c^-]) \cong \pi_1(U^+[c^+]) * \pi_1(U^-[c^-])$, so either $\pi_1(U^+[c^+]) \cong \mathbb{Z}_3$, $\pi_1(U^-[c^-]) = 1$ or $\pi_1(U^+[c^+]) = 1$, $\pi_1(U^-[c^-]) \cong \mathbb{Z}_3$. In the first case, since $U[c^+]$ is the union of $U^+[c^+]$ and U^- along the disk D, its fundamental group is $\pi_1(U[c^+]) \cong \pi_1(U^+[c^+]) * \pi_1(U^-) \cong \mathbb{Z}_3 * \mathbb{Z}$. This contradicts our observation that $U[c^+]$ is a solid torus. In the latter case, we get a contradiction in a similar way. Therefore we conclude that U does not contain an essential disk disjoint from $c^+ \cup c^-$.

Assume that there exists a properly embedded nonseparating annulus A in U which is disjoint from $c^+ \cup c^-$. Since A is disjoint from $c^+ \cup c^-$, A survives in $U[c^+ \cup c^-]$ as a properly embedded nonseparating annulus. Capping off the boundary sphere of $U[c^+ \cup c^-]$ with a 3–ball, we get a 3–manifold without boundary, in which A extends to a nonseparating sphere. But the fundamental group of the 3–manifold is the cyclic group of order 3 and hence the 3–manifold cannot contain a nonseparating sphere, a contradiction. □

Lemma 4.2 *Let $D_0 \subset U$ be the disk illustrated in Figure 15. Then up to isotopy D_0 is a unique properly embedded disk in U which is commonly dual to c^+ and c^-.*

Proof Let D be a common dual disk of c^+ and c^- that is not isotopic to D_0. We may assume that D intersects D_0 transversely and the intersection $D \cap D_0$ is minimal among all such disks. If D were disjoint from D_0, then by Lemma 2.1 $\pi_1(U[c^+ \cup c^-]) \cong \mathbb{Z}$ or $\mathbb{Z}_2$, contradicting the fact that $\pi_1(U[c^+ \cup c^-]) \cong \mathbb{Z}_3$.

By the minimality of $|D \cap D_0|$, the intersection $D \cap D_0$ has no circle components. An outermost arc of intersection in D_0 cuts off a subdisk from D_0 which intersects $c^+ \cup c^-$ in at most one point. Surgery on D along the subdisk produces two disks D_1, D_2.

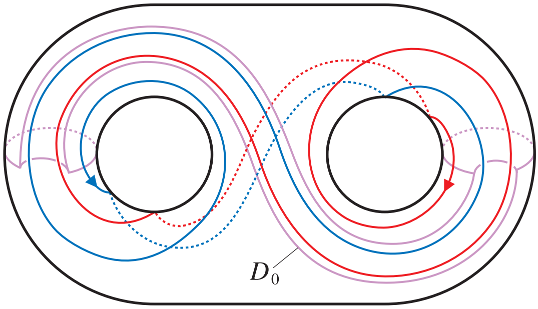


Figure 15

One of these disks, say, D_1 intersects $c^+ \cup c^-$ in at most two points. Note that D_1 is essential in U, otherwise $|D \cap D_0|$ could be reduced. By Lemma 4.1 D_1 cannot be disjoint from $c^+ \cup c^-$. If D_1 had exactly one point of intersection with $c^+ \cup c^-$ then there would exist an essential (separating) disk in U disjoint from $c^+ \cup c^-$, contradicting Lemma 4.1. Hence D_1 intersects $c^+ \cup c^-$ in two points, and so does the other disk D_2. One of the two disks D_1 and D_2 is a common dual disk of c^+ and c^-, and the other intersects one of c^+ and c^- in two points. The former disk contradicts the minimality of $|D \cap D_0|$. □

Lemma 4.3 *A_0 is incompressible and ∂–incompressible in Y_0.*

Proof One sees from Figure 14(b) that both $c^{\pm}$ are primitive curves in U, so A_0 is incompressible. Suppose that A_0 is ∂–compressible. Let D be a ∂–compressing disk for A_0. Then D is an essential disk in U which intersects $c^+ \cup c^-$ in a single point. We may assume that D intersects c^+ but not c^-. Then c^+ becomes a longitudinal curve of the solid torus $U[c^-]$, since D, a meridian disk of $U[c^-]$, intersects c^+ in a single point. This implies that $U[c^+ \cup c^-]$ is a 3–ball. But in the proof of Lemma 4.1 we already observed that the fundamental group of $U[c^+ \cup c^-]$ is the cyclic group of order 3. □

Lemma 4.4 *Y_0 is irreducible and ∂–irreducible. Hence Y_n is irreducible and ∂–irreducible for any integer n.*

Proof The same argument as in the proof of Lemma 3.3 applies here by using Lemma 4.1 instead of Lemma 3.1. □

Since Y_n is ∂–irreducible, W_n is an irreducible handlebody-knot.

Lemma 4.5 *A_0 is a unique properly embedded nonseparating annulus in Y_0 up to isotopy.*

Proof Let A be a properly embedded nonseparating annulus in Y_0 which is not isotopic to A_0. The ∂–irreducibility of Y_0 implies that A is incompressible and ∂–incompressible.

The intersection $A \cap A_0$ may be assumed to be transverse and minimal up to isotopy. Suppose that the intersection is empty. Then A lies in U and is disjoint from $c^+ \cup c^-$. Also, A is incompressible and not ∂–parallel in U, since otherwise A would be compressible in Y_0 or parallel to A_0 or an annulus in ∂Y_0. By Lemma 4.1 A is separating in U. Since A is nonseparating in Y_0, A must separate c^+ and c^-. It follows from Lemma 2.2 that one of c^+ and c^- represents a proper power of a primitive element of $\pi_1(U)$, contradicting the fact that both $c^{\pm}$ are primitive curves in U. Hence $A \cap A_0$ is not empty.

The same argument as in the third and fourth paragraphs in the proof of Lemma 3.4 applies to show that all the components of $A \cap A_0$ are essential on both A and A_0 and that they are all either circles or arcs. First, suppose that they are all circles. Then surgery on A_0 along an annulus cut off from A by an outermost component of $A \cap A_0$ in A yields two properly embedded annuli A_1, A_2 in Y_0 which are disjoint from A_0. Each annulus $A_i (i = 1, 2)$ is not isotopic to A_0 by the minimality assumption on $|A \cap A_0|$. Since we already observed that any nonseparating annulus in Y_0 which is not isotopic to A_0 cannot be disjoint from A_0, each A_i is separating in Y_0. This implies that A_0 is separating in Y_0, a contradiction.

Now suppose all the components of $A \cap A_0$ are arcs that are essential on both A and A_0. Then the arcs cut A into rectangles $R_1, \dots, R_n$. Each rectangle R_i can be considered as a properly embedded disk in U, which is essential by the minimality of $A \cap A_0$. Also, each ∂R_i intersects $c^+ \cup c^-$ in two points. There are two possibilities for the intersection of each ∂R_i with $c^+ \cup c^-$; for each i, either ∂R_i intersects each of c^+ and c^- in a single point or ∂R_i intersects one of c^+ and c^- in two points and misses the other.

Suppose that some R_i intersects one of the cores c^+ and c^- in two points. Note that each arc of $A \cap A_0$ has two copies in ∂U, one in A_0^+ and the other in A_0^-. This implies that some $R_j (j \neq i)$ intersects the other core in two points. See Figure 16(a). We may assume that R_i has two points of intersection with c^+ (and then R_j has two points of intersection with c^-). Then R_i is disjoint from c^-, implying that R_i is a properly embedded disk in the solid torus $U[c^-]$. Also, c^+ is a simple loop in $\partial U[c^-]$ intersecting R_i in two points. Since a 2–handle addition on $U[c^-]$ along c^+ results in the 3–manifold $U[c^+ \cup c^-]$ with $\pi_1(U[c^+ \cup c^-]) \cong \mathbb{Z}_3$, R_i must be ∂–parallel in $U[c^-]$. This implies that R_i is separating in U. Similarly, R_j is separating in U. Since any two disjoint separating essential disks in a genus two handlebody are parallel, R_i and R_j are parallel in U. Since R_j is disjoint from c^+, R_i can be isotoped to be disjoint from c^+ (and still from c^-). This contradicts Lemma 4.1.

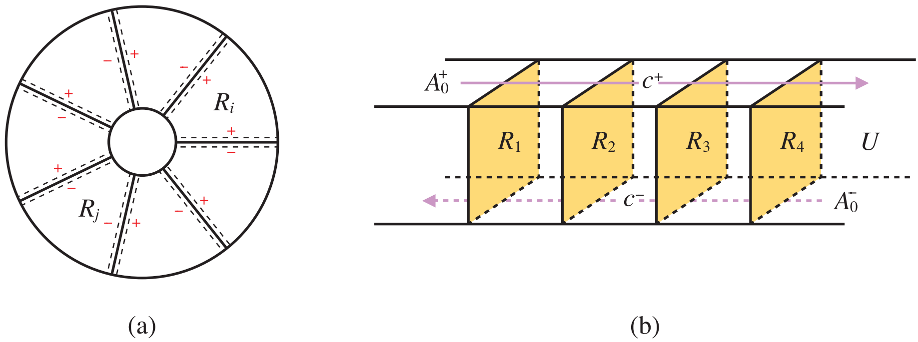


Figure 16

Hence each ∂R_i intersects each c^+ and c^- in a single point, that is, each R_i is commonly dual to c^+ and c^-. By Lemma 4.2 all the rectangles $R_1, \ldots, R_n$ are isotopic to the disk D_0 in Figure 15 and hence they are mutually parallel in U. Let $a_i^\pm = R_i \cap A_0^\pm$ for $i = 1, \ldots, n$. We may assume that $R_1, \ldots, R_n$ had been labeled so that $a_1^+, \ldots, a_n^+$ appear in A_0^+ successively along the orientation of c^+. Then $a_1^-, \ldots, a_n^-$ appear in A_0^- successively along the reversed orientation of c^-, since the algebraic intersection number of ∂D_0 with the two oriented loops $c^+ \cup c^-$ is zero. See Figure 16(b). In Y_0, the arcs $a_1^+, \ldots, a_n^+$ and the arcs $a_1^-, \ldots, a_n^-$ are identified in pair to form A. The identification defines a permutation σ of $\{1, \ldots, n\}$ such that a_i^+ is identified with $a_{\sigma(i)}^-$. In fact, $\sigma(i) \equiv -i + k \mod n$ for some integer k.

Suppose that n is odd. By replacing k with $k+n$, if necessary, we may assume that k is even. Then $\sigma(k/2) \equiv -k/2 + k \equiv k/2 \bmod n$. This implies $n = 1$, otherwise we would obtain a disconnected surface from the rectangles $R_1, \ldots, R_n$ by identifying a_i^+ and $a_{\sigma(i)}^-$ $(i = 1, \ldots, n)$. Even if $n = 1$, the identification produces a Möbius band because the two oriented loops c^+ and c^- intersect oppositely with ∂R_1. This gives a contradiction.

Suppose that n is even. The complementary regions of $R_1 \cup \cdots \cup R_n$ in U can be alternately colored black and white. If $\sigma(i) \equiv -i + k \mod n$ for some odd integer k then black regions match with black regions and white regions match with white regions, implying that A is separating in Y_0. Hence k is even. Then $\sigma(k/2) \equiv k/2 \bmod n$, and two opposite sides a_k^+ and a_k^- of R_k are identified to form a Möbius band. This is also impossible. □

Proof of Theorem 1.1(2) Let $\partial_1 A_0$ and $\partial_2 A_0$ denote the two boundary components of A_0 as shown in Figure 17. After an isotopy, the two loops appear in ∂Y_0 as shown in the last drawing in the figure.

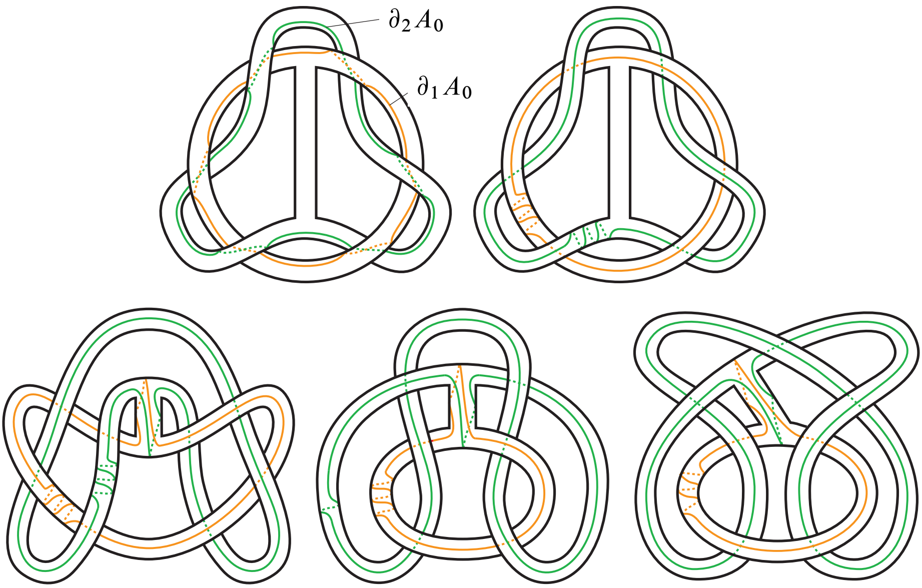


Figure 17

Recall that twisting W_0 n times along the shaded disk in Figure 18 defines a homeomorphism $\sigma_n\colon Y_0 \to Y_n$. By Lemma 4.5, $A_n = \sigma_n(A_0)$ is a unique properly embedded nonseparating annulus in Y_n up to isotopy. Let $\partial_i A_n = \sigma_n(\partial_i A_0)$ for $i = 1, 2$. The core of A_n is an embedded circle in S^3, isotopic to any boundary component of A_n in S^3 along a half of A_n. One easily sees that $\partial_1 A_n$ is a $(3, 3n{-}1)$–torus knot, and so is the core.

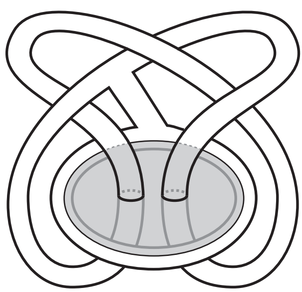


Figure 18

Assume that W_n is amphicheiral. Then there is an orientation-preserving homeomorphism of pairs $(S^3, W_n) \to (S^3, W_n^*)$. Since A_n and A_n^* are respectively up to isotopy unique nonseparating annuli in Y_n and Y_n^* by Lemma 4.5, composing with an orientation-preserving automorphism of the pair (S^3, W_n^*), if necessary, we may assume that the homeomorphism takes A_n to A_n^*. This implies that A_n and A_n^* are isotopic in S^3. In particular, their cores are isotopic. The core of A_n is a $(3, 3n-1)$–torus knot, while that of A_n^* is the mirror image of a $(3, 3n-1)$–torus knot. It is well known that every nontrivial torus knot is not amphicheiral. If $n \neq 0$ then a $(3, 3n-1)$–torus knot is not the trivial knot, so it is not amphicheiral. Hence $n = 0$. However, ∂A_0 is a $(2, -6)$–torus link (see the first drawing in Figure 17), while ∂A_0^* is the mirror image of a $(2, -6)$–torus link. The two torus links are not isotopic, a contradiction. Hence W_n is not amphicheiral for any integer n.

Let n and m be distinct integers. Then neither of the $(3, 3n-1)$–torus knot and its mirror image is isotopic to the $(3, 3m-1)$–torus knot. Hence a similar argument as above shows that neither of W_n and W_n^* is equivalent to W_m. □

Acknowledgements We would like to thank Atsushi Ishii, Kengo Kishimoto and Makoto Ozawa for their helpful conversations. The first author was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2011-0027989). The second author was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2010-0024630).

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Received: 31 October 2011 Revised: 20 February 2012