On symplectic uniruling of Hamiltonian fibrations

CLÉMENT HYVRIER

Under certain conditions of technical order, we show that closed connected Hamiltonian fibrations over symplectically uniruled manifolds are also symplectically uniruled. As a consequence, we partially extend to nontrivial Hamiltonian fibrations a result of Lu [14], stating that any trivial symplectic product of two closed symplectic manifolds with one of them being symplectically uniruled verifies the Weinstein Conjecture for closed separating hypersurfaces of contact type. The proof of our result is based on the product formula for Gromov–Witten invariants of Hamiltonian fibrations derived by the author in [5].

1 Introduction

A Hamiltonian fibration is a special case of symplectically fibered space where structure group reduces to the group of Hamiltonian diffeomorphisms of the fiber. Here we will consider closed connected Hamiltonian fibrations over symplectic bases:

$$(F^{2n_F}, \omega) \overset{\iota}{\hookrightarrow} P^{2n_P} \overset{\pi}{\to} (B^{2n_B}, \omega_B).$$

An important feature of such fibrations is that their total spaces admit symplectic structures, denoted by $\omega_{P,\kappa}$, that are compatible with symplectic structures of the fibers (see Section 2.3.1 for more details). In particular, this class of fibrations provides a nice framework to define Gromov–Witten invariants (GW–invariants).

In the present note we are interested in the existence for such fibrations of nonvanishing genus zero Gromov–Witten invariants with a least one point as constraint, namely *symplectic uniruledness* (see Definition 2.2). In the case the Hamiltonian fibration is *rationally cohomologically split* (c–split), meaning that the embedding of the fiber induces an injective map in rational cohomology, it is known that symplectic uniruledness of the fiber induces symplectic uniruledness of the total space as shown by Ruan and Li. More precisely:

Theorem 1.1 [12, Proposition 2.10]   *If $(F, \omega) \overset{\iota}{\hookrightarrow} (P, \omega_{P,\kappa}) \overset{\pi}{\to} (B, \omega_B)$ is a c–splitting Hamiltonian fibration, then $(P, \omega_{P,\kappa})$ is symplectically uniruled for a class $\sigma \in H_2(P; \mathbb{Z})$ such that $\pi_*\sigma = 0$ if and only if (F, ω) is symplectically uniruled.*

In this note we make the following observation, which, in some sense, is complementary to Theorem 1.1:

(O) *For c–splitting Hamiltonian fibrations, symplectic uniruledness of the base is sufficient to ensure symplectic uniruledness of the total space.*

The c–splitting property actually holds in many Hamiltonian fibrations (see Blanchard [1], Hyvrier [5], Kędra [6] and Lalonde and McDuff [9]); it is in fact conjectured by Lalonde and McDuff [9] that every Hamiltonian fibration is c–split.

Although we think **(O)** should hold in full generality, we only provide a proof under the following technical assumptions. First, we assume that (F, ω) is *semipositive relative to the total space*, ie (F, ω) verifies condition $(\star)$ described in Section 2.3.2. This condition, verified by Hamiltonian fibrations of real dimension at most six, is a fibered analog of the "standard" semipositivity condition used by McDuff and Salamon [17].

Secondly, we assume that the base is symplectically uniruled for a class $\sigma_B \in H_2(B; \mathbb{Z})$ which *only admits simple decompositions for some ω_B–tame almost complex structure J_B on B*. This condition, made explicit in Definition 2.1, guarantees that, for suitable almost complex structure J_P on P lifting J_B, there are no simple stable J_P–holomorphic maps in P projecting to a nonsimple stable J_B–holomorphic maps in B representing σ_B. This is in particular realized by primitive classes admitting pseudo-holomorphic representatives. Here is our main result:

Theorem 1.2 Let $(F, \omega) \overset{\iota}{\hookrightarrow} (P, \omega_{P,\kappa}) \overset{\pi}{\to} (B, \omega_B)$ be a c–splitting Hamiltonian fibration. Assume (F, ω) is semipositive relatively to P and that (B, ω_B) is symplectically uniruled for some class $\sigma_B \in H_2(B; \mathbb{Z})$ admitting only simple decompositions. Then $(P, \omega_{P,\kappa})$ is also symplectically uniruled.

This generalizes our result in [5] stating that symplectic uniruling holds for total spaces of Hamiltonian fibrations over symplectically rationally connected base (for a class σ_B as above), ie when the base has a nonvanishing genus zero Gromov–Witten invariant with a least two points as constraints. However, in that latter case, no c–splitting assumption is required.

One of the main ingredients of the proof of Theorem 1.2 is to use *fibered almost complex structure on P* in order to relate pseudo-holomorphic maps in P with pseudo-holomorphic maps in B. Roughly speaking, a fibered almost complex structure J_P on P is an almost complex structure lifting some ω_B–tame almost complex structure J_B on B, and preserving both horizontal and vertical directions in some splitting of TP induced by a Hamiltonian connection on P (cf Definition 2.3). This geometrical setup

enables us to relate, generically, some GW–invariant of P with some GW–invariant of B via a product-type formula involving GW–invariants of some Hamiltonian fibration over S^2. The technical assumptions of the theorem are there to ensure that transversality can be realized within the realm of fibered almost complex structures.

It is believed that the ad hoc hypothesis of Theorem 1.2 can be ruled out using virtual perturbations (see Chen and Li [2], Li and Tian [11] and Ruan [18], amongst others). The problem is to show that one can make perturbations compatibly with the fiber bundle projection π. Removing these assumptions is part of a joint work in progress with Shengda Hu. In some circumstances however, we can remove the assumption on σ_B. For instance, it can be removed if the base is a smooth projective manifold (B, J_B, ω_B). In such case, Ruan and Kollàr showed that *strong symplectic uniruledness* is equivalent to *projective uniruledness*, ie that there is a rational curve passing through every point of B. The following proposition then follows from Theorem 1.2.

Corollary 1.3 *Let $(F, \omega) \overset{\iota}{\hookrightarrow} (P, \omega_{P,\kappa}) \overset{\pi}{\to} (B, \omega_B)$ be a c–splitting Hamiltonian fibration over a smooth projective manifold (B, J_B, ω_B). Assume (F, ω) is semipositive relatively to $(P, \omega_{P,\kappa})$ and that (B, J_B, ω_B) is uniruled. Then $(P, \omega_{P,\kappa})$ is strongly symplectically uniruled.*

In view of Theorem 1.2 and Theorem 1.1, it is natural to ask the following question: "If neither F or B is symplectically uniruled, can we conclude that P is not symplectically uniruled?" Here is a partial answer:

Theorem 1.4 *Assume (B, J_B, ω_B) is a smooth projective manifold and that P is c–split. Suppose that neither (B, J_B, ω_B), nor (F, ω), is symplectically uniruled. Then $(P, \omega_{P,\kappa})$ is not symplectically uniruled for any (generic) fibered almost complex structure J_P on P lifting J_B.*

Proof Assume by contradiction that P is symplectically uniruled for some class $\sigma \in H_2(P; \mathbb{Z})$. Two possibilities may occur: $\pi_* \sigma$ is zero or not. In case $\pi_*(\sigma) = 0$, we can use the c–splitting hypothesis to apply Theorem 1.1 and obtain a contradiction. Now, let's consider the case where $\pi_* \sigma \neq 0$. Since P is symplectically uniruled, through every point $p \in P$ there exists a σ–rational J_P pseudo-holomorphic map u passing through p. Since J_P is fibered, the map $u_B := \pi \circ u$ is J_B pseudo-holomorphic. Moreover, u_B passes through $\pi(p)$. Consequently, B is covered by rational curves. Since B is projective this implies that B is symplectically uniruled; thus we have a contradiction. □

Before proceeding to the applications, we make a remark concerning possible generalizations of Theorem 1.2, when we replace symplectic uniruledness by k–*symplectic rational connectedness* with $k > 1$ (cf Definition 2.2). Roughly speaking, a symplectic manifold is said to be k–symplectically rationally connected if there exists a nonvanishing Gromov–Witten invariant with k points as constraints.

It is easy to find examples where Theorem 1.1 and observation (O) do not generalize to $k > 1$, for instance when P is a trivial symplectic fibration. Nevertheless, it is possible to combine symplectic rational connectedness of the fiber and of the base in order to obtain symplectic rational connectedness of the total space. Before making a precise statement, recall that the relative semipositivity condition on (F, ω) implies that the fiber is actually semipositive. It is well-known that under such hypothesis, the *quantum homology* of (F, ω) is generically well-defined. This quantum homology is endowed with a ring structure called *quantum product*, which can be seen as a deformation of the intersection product in homology. We introduce the following notation: let [pt] be the homology class of a point in F, we write $[pt]_Q$ to denote the class of the point seen as an element of the Quantum homology of F. Also, let $[pt]_Q^k$ denote the k–th power of $[pt]_Q$ with respect to the quantum product.

Theorem 1.5 *Suppose (F, ω) is semipositive relatively to $(P, \omega_{P,\kappa})$, and assume that $[pt]_Q^l \neq 0$, $l > 1$. Furthermore, suppose that (B, ω_B) is $(l+1)$–symplectically rationally connected for some class $\sigma_B \in H_2(B, \mathbb{Z})$ admitting only simple decompositions. Then P is l–symplectically rationally connected.*

Theorem 1.2 can be applied to the Weinstein Conjecture for separating hypersurfaces, or in shorter terms the *shW–Conjecture*. In 1979, A Weinstein [22] conjectured that: "*Every closed hypersurface S of contact type in a given symplectic manifold (X, ω) carries a closed characteristic.*" Since Viterbo's proof of the conjecture for $(\mathbb{R}^{2n}, \omega_0)$ in 1986 [21], many results followed. In particular, Hofer and Viterbo highlighted in [3] the strong interplay between genus zero Gromov–Witten invariants and the conjecture, in cases where the hypersurface S *separates* X, ie when there exist submanifolds X^+ and X^- of X having common boundary and such that $X = X^+ \cup X^-$ and $S = X^+ \cap X^-$. Note that this latter condition is realized whenever $H^1(X, \mathbb{Z}_2) = 0$. Shortly after, these results were extended by Liu and Tian in [13], to any symplectic manifold and any genus. In 2000, using the results of Liu and Tian, Guangcun Lu proved:

Theorem 1.6 (Lu [14, Corollary 3]) *Any separating hypersurface of contact type in a symplectically uniruled closed symplectic manifold (B, ω_B) admits a closed characteristic. In particular, the shW–Conjecture holds in products $(B \times F, \omega_B \oplus \omega)$ of a symplectically uniruled closed symplectic manifold (B, ω_B) with any symplectic manifold (F, ω).*

As a direct consequence of the first assertion of Theorem 1.6 and of Theorem 1.2 we obtain the following generalization to nontrivial Hamiltonian fibrations of the second assertion of Theorem 1.6:

Corollary 1.7 *The shW–Conjecture holds in c–splitting Hamiltonian fibrations $(P, \omega_{P,\kappa})$ with relatively semipositive fiber (F, ω) and with symplectically uniruled base (B, ω_B) for some class σ_B admitting only simple decompositions.*

In fact, as it was shown in [5, Corollary], c–splitting holds for Hamiltonian fibrations over 2–symplectically rationally connected base for some class σ_B as in the corollary above. It then follows from Corollary 1.7 that the shW–Conjecture holds in Hamiltonian fibrations with relatively semipositive fiber and with 2–symplectically rationally connected base for some class admitting only simple decompositions. In particular, the shW–Conjecture holds in Hamiltonian fibrations (with relatively semipositive fiber) over $(\mathbb{C}P^n, \omega_{FS})$ where ω_{FS} denotes the Fubini–Study Kähler form, and over $(S^2 \times S^2, \omega \oplus \omega)$ where ω is an area form on S^2. It also holds for c–splitting Hamiltonian fibrations over ruled surfaces. Finally it should also hold for c–splitting Hamiltonian fibrations over blow-ups of such bases, since symplectic uniruledness is a symplectic cobordism invariant by Hu, Li and Ruan [4].

This note is organized as follows: in Section 2 we introduce the notions and notation needed for the proofs of the results, and in Section 3 we proceed with the proofs.

2 Framework and tools

2.1 Gromov–Witten invariants

Let J be an ω–tame almost complex structure on (X^{2n_X}, ω), ie J is a smooth endomorphism of TX such that

$$J^2 = -\mathrm{Id}_{TX} \quad \text{and} \quad \omega(v, Jv) > 0 \text{ for all } v \in TX.$$

The set of all such almost complex structures is nonempty and contractible. We denote by c_1^{TX} the associated first Chern class of TX, and say that (X, ω) is *semipositive* if

and only if there are no spherical class $A \in H_2(X; \mathbb{Z})$ with positive area $\omega(A)$ such that

$$3 - n_X \le c_1^{TX}(A) < 0.$$

We begin by recalling the definition of genus zero Gromov–Witten invariant for (X, ω) assuming it is semipositive. We follow the expositions of McDuff and Salamon [17, Chapter 7] or Ruan and Tian [19]. At the end we introduce the definition of simple decomposability.

2.1.1 Definition of Gromov–Witten invariants Recall a genus zero J–holomorphic map of X is a smooth map $u \colon S^2 \to X$ solution to the Cauchy–Riemann equation

$$\bar{\partial}_J u := 1/2(du + J \circ du \circ j_0) = 0,$$

where j_0 stands for the standard complex structure on $S^2 \cong \mathbb{C} \cup \{\infty\}$. Roughly speaking, a genus zero Gromov–Witten invariant of X is a count, up to reparametrizations of the domain, of J–holomorphic rational maps with marked points with values in X, satisfying some prescribed constraints at the marked points. More precisely, consider the moduli space of unparametrized genus zero J–holomorphic maps in X with l marked points and representing the class A

$$\mathcal{M}_{0,l}(X, A, J) := \big\{(u, \mathbf{x}) := (u, x_1, \ldots, x_l) \in C^\infty(S^2, X) \times (S^2)^l$$
$$\big| \; x_i \ne x_j \text{ if } i \ne j, \; \bar{\partial}_J u = 0, \; [u(S^2)] = A\big\} \big/ \mathrm{PSL}_2(\mathbb{C}).$$

Let $\overline{\mathcal{M}}_{0,l}(X, A, J)$ denote its compactification, in the sense of Gromov, which coincides with the moduli space of isomorphism classes $[\Sigma, u, \mathbf{x}]$ of stable J–holomorphic maps with l marked points that represent A. This is a stratified space with finitely many strata. Its top stratum is the subset $\mathcal{M}_{0,l}^*(X, A, J) \subset \mathcal{M}_{0,l}(X, A, J)$ consisting of *simple J–holomorphic maps*, ie maps that are somewhere injective. This stratum is, for generic J, a naturally oriented manifold with dimension $2n_X + 2c_1^{TX}(A) - 6 + 2l$.

There are two natural maps defined on $\overline{\mathcal{M}}_{0,l}(X, A, J)$, namely the *evaluation at the marked points map*

$$\mathrm{ev}_{l,J}^X \colon \overline{\mathcal{M}}_{0,l}(X, A, J) \to X^l, \quad [\Sigma, u, x_1, \ldots, x_l] \mapsto (u(x_1), \ldots, u(x_l)),$$

and, for $l \ge 3$, the *forgetful map*

$$\mathfrak{f} \colon \overline{\mathcal{M}}_{0,l}(X, A, J) \to \overline{\mathcal{M}}_{0,l}$$

assigning to every stable map the underlying reduced stable curve. Formally, when $l \ge 3$, Gromov–Witten invariants are the values of a multilinear homomorphism

$$\langle \cdot \rangle_{A,l}^X \colon H_*(\overline{\mathcal{M}}_{0,l}, \mathbb{Q}) \otimes (H_*(X, \mathbb{Q}))^{\otimes l} \to \mathbb{Q},$$

where

$$\langle D; \beta_1, \ldots, \beta_l \rangle^X_{A,l} := \int_{\overline{\mathcal{M}}_{0,l}(X,A,J)} (\mathrm{ev}^X_{l,J})^* \, (\mathrm{PD}_{X^l}(\beta_1 \otimes \cdots \otimes \beta_l)) \cup \mathfrak{f}^* \, \mathrm{PD}_{\overline{\mathcal{M}}_{0,l}}(D),$$

which is set to be zero unless

$$2n_X + 2c_1^{TX}(A) = \sum_{i=1}^{l} (2n_X - \deg(\beta_i)) - \deg(D).$$

When $l < 3$, there is no forgetful map and we simply "integrate" the pullback under ev^X_J of the product of the $\mathrm{PD}(\beta_i)$; in that case we use the notation $\langle \beta_1, \ldots, \beta_l \rangle^X_{A,l}$, which could be viewed as $\langle [\overline{\mathcal{M}}_{0,l}]; \beta_1, \ldots, \beta_l \rangle^X_{A,l}$ if we consider $\overline{\mathcal{M}}_{0,l}$ as a manifold of negative dimension $2l - 6$.

The integration above has to be understood as evaluation of the cohomology class with the fundamental cycle of $\overline{\mathcal{M}}_{0,l}(X, A, J)$. However, this space may not carry a fundamental class. In fact, lower strata in $\overline{\mathcal{M}}_{0,l}(X, A, J)$ may have dimensions greater than $\mathcal{M}^*_{0,l}(X, A, J)$, due to the possible presence of stable maps with multiply covered components. This problem does not show up when the manifold is semipositive. Concretely, this condition imposes that the "boundary component" $\overline{\mathcal{M}}_{0,l}(X, A, J) \setminus \mathcal{M}^*_{0,l}(X, A, J)$ is generically of codimension at least two with respect to the top stratum. Then, consider cycles $V_1, \ldots, V_l$ in X representing the β_i's, and a cycle D in $\overline{\mathcal{M}}_{0,l}$ representing D. Assume that these cycles are in general position, and such that $\mathfrak{f} \times \mathrm{ev}^X_{l,J}$ is strongly transverse to the product $D \times V_1 \times \cdots \times V_l$. The corresponding *Gromov–Witten invariant* is realized as the intersection number $(\mathfrak{f} \times \mathrm{ev}^X_{l,J}) \cdot (D \times V_1 \times \cdots \times V_l)$ (which can be seen to be $\mathbb{Z}$–valued in that case). In particular, GW–invariants generically count simple maps.

2.1.2 On the semipositivity assumption We should mention that the semipositivity assumption can be removed using virtual perturbations of the Cauchy–Riemann equation as in Li and Tian [11], Ruan [18] and Ruan and Tian [19], amongst others. However, we will not work in such generality. According to McDuff and Salamon [17] we say that a stable J–holomorphic map is *simple* if and only if it has no nonconstant multiply covered component and no two nonconstant components having the same image in X. This leads to the following definition:

Definition 2.1 Let $A \in H_2(X; \mathbb{Z})$ be a spherical class representable by J–holomorphic maps. We say that A *only admits simple decompositions* if every stable J–holomorphic map in $\overline{\mathcal{M}}_{0,l}(X, A, J)$ has no nonconstant multiply covered component and no two nonconstant components having the same image in X.

This is what we request from σ_B in Theorem 1.2. Note that the set of ω–tame almost complex structures J with respect to which a given class A only admits simple decompositions is open in the set of ω–tame almost complex structures. However, we cannot make sure that this set is connected or nonempty.

2.2 Symplectic rational connectedness

Using the notation introduced in the preceding paragraph, we define the notion of k–*symplectic rational connectedness*, following Hu, Li and Ruan [4] and Lu [14].

Definition 2.2 Let $k > 0$ be an integer, and let $A \in H_2(X; \mathbb{Z})$ be a spherical homology class. A symplectic manifold (X, ω) is k–*symplectically rationally connected for σ*, or simply k–*SRC for σ*, if and only if there exist classes $\beta_{k+1}, \ldots, \beta_l \in H_*(X, \mathbb{Q})$, and $D \in H_*(\overline{\mathcal{M}}_{0,l}, \mathbb{Z})$ such that

$$\langle D; [\mathrm{pt}], \ldots, [\mathrm{pt}], \beta_{k+1}, \ldots, \beta_l \rangle^X_{A,l} \neq 0.$$

We will say that (X, ω) is k–*symplectically rationally connected* if there exists σ such that it is k–*SRC for σ*. If $k = 1$, we will say that (X, ω) is *symplectically uniruled* or *SU*. Furthermore, if (X, ω) is symplectically uniruled and $l = 3$, we say that (X, ω) is *strongly symplectically uniruled* or *SSU*.

In particular, (X, ω) is k–SRC for A only if, through every k generic points of X, there is a genus zero pseudo-holomorphic map representing A. The converse may not be true in general. Nevertheless, the equivalence holds in smooth projective varieties.

Theorem 2.1 (Kollàr [8]; Ruan [18]) *A smooth projective variety is symplectically uniruled if and only if it is* projectively uniruled, *ie through every point of the manifold there is a holomorphic map.*

Actually, it follows from the *splitting axiom for Gromov–Witten invariants* that a projective manifold is (projectively) uniruled if and only if it is strongly symplectically uniruled, as pointed out by Ruan [18]. The main ingredient of the proof is the following property of rational curves in projectively uniruled manifolds (X, J, ω) due to Kollàr, Miyaoka and Mori (see Kollàr [7, Theorem 3.11]): for a very general point $b \in X$, if $u: \mathbb{C}P^1 \to X$ is a morphism such that $[u(\mathbb{C}P^1)] \neq 0$ and $u(0) = b$, then $H^1(\mathbb{C}P^1, u^*TX) = 0$. Such morphism is said to be *free (over 0)*. Now, for $b \in X$ and a spherical class σ, let $\mathcal{M}(X, \sigma, J; b)$ denote the moduli space of σ rational J–holomorphic maps $u: \mathbb{C}P^1 \to X$ such that $b \in \mathrm{Im}(u)$. We will say that σ is *free* if, for every general point $b \in X$, the moduli space $\mathcal{M}(X, \sigma, J; b)$ only consists of free morphisms. For our purpose, we mildly refine the statement of Theorem 2.1 by the following straightforward observation.

Lemma 2.2 *If a smooth projective variety (X, J, ω) is uniruled, it is strongly symplectically uniruled for a free class σ admitting only simple decompositions.*

Proof Since X is uniruled, for every $b \in X$ there exists a morphism $u \colon \mathbb{C}P^1 \to X$ such that $b \in \operatorname{Im}(u)$. Now, fix a sufficiently general point b. In the proof of Theorem 2.1 given in [18], Ruan shows the equality between

$$N_1 := \min \{\omega(\sigma) > 0 \mid \mathcal{M}(X, J, \sigma; b) \neq \varnothing\},$$

$$N_2 := \min\{\omega(\sigma) > 0 \mid \exists \alpha_1, \ldots, \alpha_l \in H_*(X), \langle [\mathrm{pt}]; [\mathrm{pt}], \alpha_1, \ldots, \alpha_l \rangle^X_{\sigma, l+1} \neq 0\}.$$

He also points out that N_2 is realized by classes with nontrivial three point GW–invariant. Suppose $\sigma \in H_2(X; \mathbb{Z})$ realizes this minimum. Then every rational curve passing through b and representing σ is irreducible, ie any stable holomorphic map through b is simple. It follows that $\mathcal{M}(X, J, \sigma; b)$ is compact. Furthermore, by choosing b general enough, for any $u \in \mathcal{M}(X, J, \sigma; b)$ the obstruction $H^1(\mathbb{C}P^1, u^*TX)$ vanishes; thus, the moduli space $\mathcal{M}(X, J, \sigma; b)$ is a smooth oriented manifold, as desired. □

2.3 Hamiltonian fibrations and the product formula

In this section we introduce the notion of Hamiltonian fibrations which provides natural framework to study Gromov–Witten invariants.

2.3.1 Hamiltonian fibrations By definition, a *symplectic fibration* is a locally trivial smooth fibration with symplectic reference fiber (F, ω),

$$(F, \omega) \overset{\iota}{\hookrightarrow} P \overset{\pi}{\to} B,$$

and which structure group lies in the group of symplectic diffeomorphisms of the fiber, denoted $\operatorname{Symp}(F, \omega)$. It follows that each fiber $F_b := \pi^{-1}(b)$ is naturally equipped with a symplectic form ω_b. A symplectic fibration is *Hamiltonian* if the structure group can be reduced to the group $\operatorname{Ham}(F, \omega)$ of Hamiltonian diffeomorphisms. Hamiltonian fibrations are characterized as follows:

Theorem 2.3 [16, Theorem 6.36] *A symplectic fibration P as above is Hamiltonian if and only if the following two conditions are verified:*

(H_1) *P is symplectically trivial over the 1–skeleton of B.*

(H_2) *There exists a unique closed connection 2–form $\tau \in \Omega^2(P)$ extending the family $\{\omega_b\}_{b \in B}$ such that the integration of τ^{n_F+1} over the fibers of P vanishes.*

The closed 2–form in the theorem above is usually referred to as the *coupling form*. The coupling form defines a connection on P, ie a splitting at each $p \in P$,

$$T_p P = \mathrm{Hor}_{\tau, p} \oplus \ker d\pi(p).$$

The theorem states that the corresponding holonomy around any loop in B is in $\mathrm{Ham}(F, \omega)$. Any other closed extension τ' of ω generating the same horizontal distribution Hor_τ is actually uniquely obtained from τ via the equation

$$\tau' = \tau + \pi^* \varrho, \quad \varrho \in \Omega^2(B).$$

One of the main features of Hamiltonian fibrations over closed symplectic bases is that they can be given symplectic structures compatibly with the family $\{\omega_b\}_{b \in B}$. Such symplectic structures are obtained as follows:

$$\omega_{P,\kappa} := \tau + \kappa \pi^* \omega_B,$$

where $\kappa > 0$ is a large enough real number such that $\omega_{P,\kappa}$ is nondegenerate. We will also use τ to denote the deRham cohomology class corresponding to τ. There is another canonical cohomology class of P that will play an important part in the next paragraphs, namely the vertical Chern class $c_v \in H^2(P; \mathbb{Z})$. This class is defined as the first Chern class of the vertical subbundle $\ker d\pi$.

As a specific example, let us mention the case of Hamiltonian fibrations over S^2 with fiber (F, ω), which will play an important part in this note. This class of examples is particularly important in symplectic topology due to the correspondence between these fibrations and the fundamental group of $\mathrm{Ham}(F, \omega)$. More precisely, for $\gamma \in \pi_1(\mathrm{Ham}(F, \omega))$ one defines a Hamiltonian bundle $\pi \colon P_\gamma \to S^2$ with fiber F via the clutching construction: choose any representative $\tilde{\gamma} \colon [0, 1] \to \mathrm{Ham}(F, \omega)$ for γ, then

$$P_\gamma := \frac{(D^+ \times F) \sqcup (D^- \times F)}{(e^{2\pi i \theta}, x) \sim (e^{-2\pi i \theta}, \tilde{\gamma}(\theta) \cdot x) \text{ on } S^1 \times F},$$

where $D^\pm \subset \mathbb{C}$ denotes the closed unit disc. This construction is independent of the representative $\tilde{\gamma}$; moreover, any Hamiltonian fibration over S^2 with (F, ω) as fiber can be constructed in this way (see Lalonde, McDuff and Polterovich [10] or Seidel [20]). In this context, the coupling class will be denoted by τ_γ and c_γ will denote the vertical Chern class associated to P_γ.

2.3.2 The product formula Following [5] we give a product formula for GW–invariants of a Hamiltonian fibration, assuming the fiber (F, ω) is *semipositive relative to the total space*, ie

$$(\star) \qquad \forall A \in H_2^S(F), \quad \omega(A) > 0, \quad c^v(\iota(A)) \geq 3 - n_P \implies c^v(\iota(A)) \geq 0,$$

where $H_2^S(F)$ is the spherical homology subgroup of $H_2(F, \mathbb{Z})$ (ie the image of $\pi_2(F)$ under the Hurewicz map) and ι denotes the map in homology induced from the natural embedding of the fiber. For instance, this implies that the fiber is semipositive. For this purpose we equip P with an *(almost) complex structure* J_P which is compatible with the fibration structure and a Hamiltonian connection in the sense given below:

Definition 2.3 An almost complex structure J_P on P is said to be *compatible with π and τ*, or just *fibered*, if and only if there exists an ω_B–tame complex structure J_B on B and a family of ω_b–tame almost complex structures J_b in F_b such that

- $d\pi \circ J_P = J_B \circ d\pi$,

- $J_b := J_P|_{F_b}$ for all $b \in B$,

- J_P preserves the horizontal distribution induced by τ.

For fibered J_P, the projection π induces a map between moduli spaces:

$$\bar{\pi} \colon \mathcal{M}_{0,l}(P, \sigma, J_P) \to \mathcal{M}_{0,l}(B, \sigma_B, J_B), \quad [\Sigma, u, \mathbf{x}] \mapsto [\Sigma, \pi(u), \mathbf{x}],$$

where $\sigma_B := \pi_* \sigma$. In what follows we assume that σ_B is nonzero. The fiber of $\bar{\pi}$ over $[\Sigma, u_B, \mathbf{x}]$ can be described as follows. Let C denote the image of u_B in B, and let P_C denote the restriction of P along C; P_C defines a Hamiltonian fibration over S^2 with coupling form given by the pullback of τ under the natural inclusion $\iota_{P_C} \colon P_C \hookrightarrow P$. If J_C denotes the fibered almost complex structure on P_C given by the restriction of J_P to P_C we have the following identification:

$$\bar{\pi}^{-1}([\Sigma, u_B, \mathbf{x}]) \equiv \bigsqcup_{B_\sigma := \{\sigma' \in H_2(P_C; \mathbb{Z}) | \iota_{P_C} \sigma' = \sigma\}} \left(\mathcal{M}_{0,l}(P_C, J_C, \sigma') \cap \mathfrak{f}^{-1}([\Sigma, \mathbf{x}]) \right).$$

Regarding evaluation at the marked points maps we have the commutative diagram:

$$
\begin{array}{ccccc}
\bar{\pi}^{-1}([\Sigma, u_B, \mathbf{x}]) & \longrightarrow & \mathcal{M}_{0,l}(P, \sigma, J_P) & \xrightarrow{\ \bar{\pi}\ } & \mathcal{M}_{0,l}(B, \sigma_B, J_B) \\
\downarrow{\scriptstyle \mathrm{ev}_{(u_B, \mathbf{x})}} & & \downarrow{\scriptstyle \mathrm{ev}^P_{l, J_P}} & & \downarrow{\scriptstyle \mathrm{ev}^B_{l, J_B}} \\
F^l & \xrightarrow{\ (\iota)^l\ } & P^l & \xrightarrow{\ \pi^l\ } & B^l
\end{array}
$$

(2-1)

where

$$\mathrm{ev}_{(u_B, \mathbf{x})} \colon \bar{\pi}^{-1}(u_B, \mathbf{x}) \to F^l, \quad u \mapsto (u(x_1), \ldots, u(x_l)) \in \prod_{i=1}^{l} F_{u_B(x_i)}.$$

The product formula is obtained by considering the (respective) intersections of $\mathrm{ev}_{(u_B, \mathbf{x})}$, ev^B_{l, J_B}, and ev^P_{l, J_P}, with the product cycles:

$$\mathcal{C}^F := \prod_{i=1}^{l} V_i^F, \quad \mathcal{C}^B := \prod_{i=1}^{l} V_i^B, \quad \mathcal{C}^P := \prod_{i=1}^{l} V_i^P,$$

where V_i^F, V_i^B and V_i^P, respectively represent homology classes c_i^F, c_i^B and c_i^P, verifying that for some integer $0 \leq m \leq l$:

$(\star\star)$
$$\begin{cases} c_i^B = \mathrm{pt}, \quad c_i^P = \iota(c_i^F) & \text{for } i = 1, \ldots, m, \\ c_i^F = [F], \quad c_i^P = \pi^!(c_i^B) & \text{for } i = m+1, \ldots, l, \end{cases}$$

where $\pi^!$ stands for the shriek map

$$\pi^! \colon H_*(B) \to H_{2n_F + *}(P), \quad \alpha \mapsto \mathrm{PD}_P^{-1} \pi^* \mathrm{PD}_B(\alpha).$$

In other words, we consider cycles in P that are either images of cycles in F under ι, or preimages under π of cycles in B. Let ι_C denote the map in homology induced from the natural inclusion of F in P_C. We have:

Theorem 2.4 [5, Theorem A] *Let $\pi \colon P \to B$ be a Hamiltonian fibration with relative semipositive fiber (F, ω). Let $\sigma \in H_2(P, \mathbb{Z})$ and suppose $\sigma_B := \pi_*(\sigma) \neq 0$ only admits simple decompositions for some ω_B–tame almost complex structure J_B in B. Let c_i^P, c_i^B, c_i^F be as in $(\star\star)$. For generic fibered almost complex structure lifting J_B, we have*

$$(2\text{-}2) \quad \langle D; c_1^P, \ldots, c_l^P \rangle_{\sigma, l}^P = \langle D; c_1^B, \ldots, c_l^B \rangle_{\sigma_B, l}^B \cdot \sum_{\sigma' \in B_\sigma} \langle [\mathrm{pt}]; \iota_C(c_1^F), \ldots, \iota_C(c_l^F) \rangle_{\sigma', l}^{P_C},$$

where C is a curve counted in $\langle D; c_1^B, \ldots, c_l^B \rangle_{\sigma_B, l}^B$ and $D \in H_*(\overline{\mathcal{M}}_{0,l})$.

This formula in particular states that the sum in the righthand side of the formula is independent of the chosen C. Note that this sum is well-defined due to Gromov's compactness.

The core of the proof consists in establishing that the GW–invariants involved are generically and simultaneously well-defined, in other words the problem is to realize transversality while preserving the map $\bar{\pi}$ defined above. The proof of this is based on the relation

$$\pi_* \circ D^P = D^B \circ \pi_*,$$

where D^P and D^B respectively stand for the Fredholm operators obtained by linearizing the Cauchy Riemann operators $\bar{\partial}_{J_P}$ and $\bar{\partial}_{J_B}$. As a consequence, we derive an

exact sequence

$$0 \to \ker D^v \to \ker D^P \to \ker D^B \to \operatorname{coker} D^v \to \operatorname{coker} D^P \to \operatorname{coker} D^B \to 0,$$

where D^v denotes the restriction of D^P to vector fields along the curves that are vertically valued, ie with values in $\ker d\pi$. The vanishing, at least at the level of the universal moduli spaces, of the obstructions in the sequence above is provided by: (1) the irreducibility hypothesis on σ_B for the vanishing of the last term of the sequence; (2) perturbing the Hamiltonian connection for the vanishing of $\operatorname{coker} D^v$. It follows from standard arguments that for generic fibered almost complex structure the following holds:

- The subset $\mathcal{M}_{0,l}^{**}(P, \sigma, J_P)$ of $\mathcal{M}_{0,l}(P, \sigma, J_P)$ consisting of simple maps that project to simple maps under $\bar{\pi}$ and the moduli space $\mathcal{M}_{0,l}^*(B, \sigma_B, J_B)$ are oriented manifolds.

- For countably many $(u, \mathbf{x}) \in \mathcal{M}_{0,l}^*(B, \sigma_B, J_B)$, the preimage $\bar{\pi}^{-1}(u, \mathbf{x})$ is an oriented manifold.

More generally, $\bar{\pi}$ extends to a map, using stabilization, between the compactifications $\overline{\mathcal{M}}_{0,l}(P, \sigma, J_P)$ and $\overline{\mathcal{M}}_{0,l}(B, \sigma_B, J_B)$. We can repeat the arguments above for each stratum in $\overline{\mathcal{M}}_{0,l}(P, \sigma, J_P)$ projecting to some stratum in $\overline{\mathcal{M}}_{0,l}(B, \sigma_B, J_B)$. The irreducibility hypothesis on the decompositions of σ_B ensures that the coker D^B term always vanishes and that the image under the evaluation map $\operatorname{ev}_{l,J_B}^B$ of the lower strata in $\overline{\mathcal{M}}_{0,l}(B, \sigma_B, J_B)$ have codimension at least 2 with respect to the top stratum $\mathcal{M}^*(B, \sigma_B, J_B)$. Furthermore, condition $(\star)$ ensures that the image under the evaluation map $\operatorname{ev}_{l,J_P}^P$ of the lower strata in $\overline{\mathcal{M}}_{0,l}(P, \sigma, J_P)$ have codimension at least two with respect to $\mathcal{M}_{0,l}^{**}(P, \sigma, J_P)$. Once transversality is achieved, one recovers the formula using diagram (2-1) and the following observations:

- Since all cokernels vanish, it follows from the exact sequence above that

$$\det(\ker D^P) \cong \det(\ker D^B) \otimes \det(\ker D^v).$$

  Thus, the orientation of $\mathcal{M}_{0,l}^{**}(P, \sigma, J_P)$ is given by the product of the orientations of $\mathcal{M}_{0,l}^*(B, \sigma_B, J_B)$ and $\bar{\pi}^{-1}(u_B, \mathbf{x})$, where $(u_B, \mathbf{x}) \in \mathcal{M}_{0,l}^*(B, \sigma_B, J_B)$. Moreover, the orientations of the product pseudo-cycles are also given by a product

$$\det T\mathcal{C}_P \cong \det T\mathcal{C}_B \otimes \det T\mathcal{C}_F.$$

- It follows from symplectic triviality of P over the 1–skeleton of B, that for any two J_B–holomorphic maps representing the same class σ_B, the restrictions of P to the images of the two maps are isomorphic as Hamiltonian fibrations. Hence, $\langle \ \rangle_{\sigma',l}^{P_C}$ does not depend on C.

The proof is concluded by showing, using standard arguments, the independence with respect to the generic fibered almost complex structure of the Gromov–Witten invariants involved in the formula. See [5] for the details.

2.4 Quantum homology and Seidel elements

We start with the definition of (small) quantum homology with universal Novikov ring. As a module the quantum homology of (X, ω) is given by

$$\mathrm{QH}_*(X) \equiv \mathrm{QH}_*(X, \Lambda) := H_*(X, \mathbb{Q}) \otimes \Lambda,$$

where Λ denotes some coefficient ring that we specify. We will take $\Lambda := \Lambda^{\mathrm{univ}}[q^{-1}, q]$ where Λ^{univ} is the ring of generalized Laurent series in variable t^{-1}, ie an element $\lambda \in \Lambda^{\mathrm{univ}}$ can be written as a formal sum

$$\lambda = \sum_{i \geq 0} \lambda_i t^{r_i}, \quad \lambda_i \in \mathbb{Q}, r_i \in \mathbb{R}, \quad r_i > r_{i+1}, \quad \lim_{i \to \infty} r_i = -\infty.$$

The grading on Λ is given by imposing that $\deg(q) = 2$. Let Λ_j denote the set of elements of degree $2j$ in Λ; then we give quantum homology the following grading:

$$\mathrm{QH}_k(X) := \bigoplus_{i+2j=k} H_i(X, \mathbb{Q}) \otimes \Lambda_j.$$

Next, we introduce the (small) quantum product

$$\star \colon \mathrm{QH}_i(X) \otimes \mathrm{QH}_j(X) \to \mathrm{QH}_{i+j-2n}(X), \quad (a, b) \mapsto a \star b.$$

First, let $\{e_\nu\}$ be a basis of homology of $H_*(X)$, and let $\{e_\nu^*\}$ denote the corresponding dual basis with respect to the intersection pairing. Then, the quantum product is defined as follows: for $a \in H_i(X, \mathbb{Q})$ and $b \in H_j(X, \mathbb{Q})$, set

$$a \star b = \sum_{B \in H_2^S(X), \nu} \langle a, b, e_\nu \rangle_{B,3}^X e_\nu^* \otimes q^{-c_1(B)} t^{-\omega(B)}.$$

We extend this by linearity with respect to Λ. Note that $[X] = 1$ is the identity for this ring structure. Now, consider the Λ–submodule

$$\mathcal{Q}_- := \bigoplus_{i < 2n} H_i(X) \otimes \Lambda.$$

The following lemma of McDuff will be essential when proving Theorem 1.2. We give its proof for the reader's convenience:

Lemma 2.5 [15, Lemma 2.1] *If (X, ω) is not strongly uniruled then $\mathcal{Q}_-$ is an ideal in $\mathrm{QH}_*(X)$. Furthermore, if $\mathcal{Q}_-$ is an ideal $a \in \mathrm{QH}_*(X)$ is a unit, then there exists $x \in \mathcal{Q}_-$ and $\lambda \neq 0$ in Λ such that*

$$a = 1 \otimes \lambda + x.$$

Proof Let $c \in \mathrm{QH}_*(X)$ and assume by contradiction that there exists $b \in \mathcal{Q}_-$ such that $c \star b \notin \mathcal{Q}_-$. By definition of the quantum product this means that there exists $B \in H_2^S(X)$ such that

$$\langle c, b, \mathrm{pt} \rangle_{B,3}^X \neq 0.$$

Hence (X, ω) is strongly uniruled which is impossible. Next, if a is a unit, then $a \notin \mathcal{Q}_-$, for if it was we would have that 1 belongs to $\mathcal{Q}_-$. Therefore, we can write $a = 1 \otimes \lambda + x$ with $x \in \mathcal{Q}_-$ and $\lambda \neq 0$. $\square$

Seidel's representation This is a representation of $\pi_1(\mathrm{Ham}(X, \omega))$ in the subring $\mathrm{QH}_*^\times(X, \Lambda)$ of units of the quantum homology of (X, ω). Recall that $\pi \colon P_\gamma \to S^2$ denotes the Hamiltonian fibration obtained from $\gamma \in \pi_1(\mathrm{Ham}(X, \omega))$ via the clutching construction. Also, let $\mathcal{H}_\gamma \subset H_2^S(P_\gamma)$ denote the subset of section classes, ie of spherical classes that project to $[S^2]$ under the fibration projection.

Definition 2.4 The Seidel representation map

$$S \colon \pi_1(\mathrm{Ham}(X, \omega)) \to \mathrm{QH}_*^\times(X, \Lambda), \quad \gamma \mapsto S(\gamma)$$

is defined as

(2-3) $$S(\gamma) := \sum_{\sigma \in \mathcal{H}_\gamma, v} \langle \iota(e_v) \rangle_{\sigma,1}^{P_\gamma} e_v^* \otimes q^{-c_\gamma(\sigma)} t^{-\tau_\gamma(\sigma)}.$$

Geometrically, $S(\gamma)$ "counts" holomorphic sections of P_γ intersecting the cycles e_μ in the fiber above the north pole of S^2. The *splitting axiom* for fibrations over S^2 [17, Theorem 11.4.1] gives that for all section classes $\sigma \in H_2^S(P_\gamma)$ and for classes $\alpha_1, \ldots, \alpha_l \in H_*(F)$ the following holds for every integer $0 \leq k \leq l$:

$$\langle [\mathrm{pt}]; \iota(\alpha_1), \ldots, \iota(\alpha_l) \rangle_{\sigma,l}^{P_\gamma} = \sum_{A \in H_2^S(F;\mathbb{Z}), v} \langle [\mathrm{pt}]; \iota(\alpha_1), \ldots, \iota(\alpha_k), \iota(e_v) \rangle_{\sigma-\iota(A),k+1}^{P_\gamma}$$

$$\cdot \langle [\mathrm{pt}]; e_v^*, \alpha_{k+1}, \ldots, \alpha_l \rangle_{A,l-k+1}^F$$

It follows easily that the Λ linear action of $S(\gamma)$ on $a \in H_*(F)$ is given by

(2-4) $$S(\gamma)(a) := S(\gamma) \star a = \sum_{\sigma \in \mathcal{H}_\gamma, v} \langle \iota(a), \iota(e_v) \rangle_{\sigma,2}^{P_\gamma} e_v^* \otimes q^{-c_\gamma(\sigma)} t^{-\tau_\gamma(\sigma)}.$$

Note that in the above notation $S(\gamma) = S(\gamma)(1)$. Let us define the following equivalence class on section classes: we say that σ_1 is equivalent to σ_2 if and only if

$$\tau_\gamma(\sigma_1 - \sigma_2) = 0 = c_\gamma(\sigma_1 - \sigma_2).$$

We will use the notation $[\sigma]$ to denote the equivalence class of σ. From [10, Theorem 3.A] we have:

Lemma 2.6 *For every $\gamma \in \pi_1(\mathrm{Ham}(F, \omega))$ and every nonzero $a \in H_*(F)$, there exists an equivalence class of section classes $[\sigma]$ and $b \in H_*(F)$ such that*

$$\sum_{\sigma' \in [\sigma]} \langle \iota(a), \iota(b) \rangle^{P_\gamma}_{2,\sigma'} \neq 0.$$

In particular there is a section class $\sigma \in \mathcal{H}_\gamma$ such that $\langle \iota(a), \iota(b) \rangle^{P_\gamma}_{2,\sigma} \neq 0$.

Proof Assume it is not true. Equation (2-4) and linearity of Gromov–Witten invariants then imply that $S(\gamma)(a) = 0$. But since $S(\gamma)$ is invertible, this is only possible for $a = 0$; hence the contradiction. □

3 Proofs of the results

We are now ready to prove the results. We begin by proving Theorem 1.2 and Theorem 1.5. More precisely, we prove the following:

Theorem 3.1 *Assume (F, ω) verifies ($\star$) and is not strongly uniruled. If (B, ω_B) is symplectically uniruled for some class σ_B admitting only simple decompositions, then P is also symplectically uniruled. Moreover, if $[\mathrm{pt}]^l_Q \neq 0$, and if (B, ω_B) is $(l+1)$–SRC for σ_B admitting only simple decompositions, then P is at least l–SRC.*

Proof By assumption there exists a nonzero $\sigma_B \in H_2^S(B)$ and classes $c_1^B, \ldots, c_l^B \in H_*(B)$ such that

$$(3\text{-}1) \qquad\qquad \langle D; [\mathrm{pt}], c_1^B, \ldots, c_l^B \rangle^B_{\sigma_B, l+1} \neq 0.$$

Let C be the image of a map counted in (3-1). Since $\sigma_B \neq 0$, the restriction P_C of P to C is a Hamiltonian fibration over S^2. Let $\phi \in \pi_1(\mathrm{Ham}(F, \omega))$ be a Hamiltonian loop corresponding to this fibration, and let $S(\phi) \in \mathrm{QH}^\times(F; \Lambda)$ denote the corresponding Seidel element. Since (F, ω) is not strongly uniruled, it follows from Lemma 2.5 that there exists $0 \neq \lambda \in \Lambda$ and $x \in \mathcal{Q}_-$ such that

$$S(\phi) = 1 \otimes \lambda + x.$$

In particular, this directly implies that there is an equivalence class $[\sigma]$ of section classes in P_C such that

$$0 \neq \sum_{\sigma' \in [\sigma]} \langle [\mathrm{pt}] \rangle^{P_C}_{\sigma',1} = \sum_{\sigma' \in [\sigma]} \langle [\mathrm{pt}]; [\mathrm{pt}], \iota_C([F]), \ldots, \iota_C([F]) \rangle^{P_C}_{\sigma',l+1}.$$

Let ι_{P_C} denote the inclusion in homology induced by the inclusion of P_C into P. It is easy to see that the image under ι_{P_C} of $[\sigma]$ defines an equivalence class of spherical classes projecting on σ_B, in the sense that for any $\sigma_1, \sigma_2 \in \iota_{P_C}([\sigma])$ we have

$$\tau(\sigma_1 - \sigma_2) = 0 = c_v(\sigma_1 - \sigma_2).$$

By the product formula for Gromov–Witten invariants (2-2),

$$\sum_{\sigma' \in \iota_{P_C}([\sigma])} \langle D; [\mathrm{pt}], \pi^!(c_1^B), \ldots, \pi^!(c_l^B) \rangle^P_{\sigma',l+1}$$

$$= \sum_{\sigma' \in \iota_{P_C}([\sigma])} \langle D; [\mathrm{pt}], c_1^B, \ldots, c_l^B \rangle^B_{\sigma_B,l+1} \sum_{\sigma'' \in B_{\sigma'}} \langle [\mathrm{pt}]; [\mathrm{pt}], \iota_C([F]), \ldots, \iota_C([F]) \rangle^{P_C}_{\sigma'',l+1}$$

$$= \langle D; [\mathrm{pt}], c_1^B, \ldots, c_l^B \rangle^B_{\sigma_B,l+1} \sum_{\sigma' \in [\sigma]} \langle [\mathrm{pt}]; [\mathrm{pt}], \iota_C([F]), \ldots, \iota_C([F]) \rangle^{P_C}_{\sigma',l+1} \neq 0.$$

In particular, there is at least one $\sigma' \in \iota_{P_C}([\sigma])$ such that

$$\langle D; [\mathrm{pt}], \pi^!(c_1^B), \ldots, \pi^!(c_l^B) \rangle^P_{\sigma',l+1} \neq 0.$$

Now we prove the second assertion of the theorem. Since for every loop ϕ of Hamiltonian diffeomorphisms of F the Seidel element $S(\phi)$ is an unit in $\mathrm{QH}_*(F; \Lambda)$, we have that

$$0 \neq S(\phi) \star ([\mathrm{pt}]_Q^l).$$

Using the splitting axiom for Gromov–Witten invariants, and from the definition of the Seidel element, one obtains that

$$S(\phi) \star ([\mathrm{pt}]_Q^l) = \sum_{A \in H_2^S(F), v} \langle [\mathrm{pt}]; [\mathrm{pt}], \ldots, [\mathrm{pt}], e_v \rangle^F_{A,l+1} S(\phi)(e_v^*) \otimes q^{-c_1^{TF}(A)} t^{-\omega(A)}$$

$$= \sum_{\sigma \in \mathcal{H}_\phi, A \in H_2^S(F), v, \mu} \langle [\mathrm{pt}]; [\mathrm{pt}], \ldots, [\mathrm{pt}], e_v \rangle^F_{A,l+1} \langle \iota(e_v^*), \iota(e_\mu) \rangle^{P_\phi}_{\sigma,2}$$

$$\cdot e_\mu^* \otimes q^{-c_\phi(\sigma + \iota(A))} t^{-\tau_\phi(\sigma + \iota(A))}$$

$$= \sum_{\widetilde{\sigma} \in \mathcal{H}_\phi, \mu} \langle [\mathrm{pt}]; [\mathrm{pt}], \ldots, [\mathrm{pt}], \iota(e_\mu) \rangle^{P_\phi}_{\widetilde{\sigma},l+1} e_\mu^* \otimes q^{-c_\phi(\widetilde{\sigma})} t^{-\tau_\phi(\widetilde{\sigma})}.$$

Hence, there is a class $a \in H_*(F)$ and a class of section classes $[\sigma]$ such that

$$\sum_{\sigma' \in [\sigma]} \langle [\mathrm{pt}]; [\mathrm{pt}], \ldots, [\mathrm{pt}], \iota(a) \rangle^{P_\phi}_{\sigma', l+1} \neq 0.$$

We conclude by the use of the product formula as before. □

Now Theorem 1.2 follows easily:

Proof of Theorem 1.2 In case where (F, ω) is strongly uniruled, the conclusion follows from Theorem 1.1. Otherwise, we simply use Theorem 3.1 to conclude. □

As for Theorem 1.2, Corollary 1.3 follows directly from Theorem 1.1 and the proposition below. This latter proposition is a simple consequence of Theorem 2.1 asserting that projectively uniruled manifolds are strongly symplectically uniruled.

Proposition 3.2 *Assume (F, ω) verifies ($\star$) and is not strongly uniruled. Also assume that (B, J_B, ω_B) is a uniruled projective manifold. Then P is SSU.*

Proof From Lemma 2.2, (B, J_B, ω_B) is uniruled for a class σ_B verifying the assumptions of Theorem 3.1, and which is obstruction free. Hence, the product formula can be applied and the proof follows. □

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Mathematics Department, Uppsala Universitet
Box 480, SE-75106 Uppsala, Sweden

`clement.hyvrier@math.uu.se`