# Bushy pseudocharacters and group actions on quasitrees

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Given a group acting on a graph quasi-isometric to a tree, we give sufficient conditions for a pseudocharacter to be bushy. We relate this with the conditions studied by Bestvina and Fujiwara on their work on bounded cohomology and obtain some results on the space of pseudocharacters.

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## 1 Introduction

If $G$ is a finitely presented group, then $f\colon G \to \mathbb{R}$ is a *quasihomomorphism* or *quasicharacter* if $f(\alpha)+f(\beta)-f(\alpha\beta)$ is bounded independent of $\alpha, \beta$.

If $G$ is a finitely presented group, then $f\colon G \to \mathbb{R}$ is a *pseudocharacter* if it has the following properties:

- $f(\alpha^n) = n\alpha$ for all $\alpha \in G$, $n \in \mathbb{Z}$.
- $\delta f(\alpha,\beta) = f(\alpha)+f(\beta)-f(\alpha\beta)$ is bounded independent of $\alpha$, $\beta$.

Clearly the constant map $f(G)=0$ is a trivial pseudocharacter.

**Remark** Note that if $f$ is a quasicharacter and $\phi$ is given by

$$\phi(g) = \lim_{n\to\infty} f(g^n)/n,$$

then $\phi$ is a pseudocharacter with $\phi - f$ bounded.

Let $S$ be a finite generating set for $G$. If $\Gamma(G,S)$ is the Cayley graph associated to the generating set $S$, then $f$ can be extended affinely over the edges of $\Gamma(G,S)$.

If $\phi\colon \mathbb{R}_+ \to \Gamma(G,S)$ is an infinite ray, then the *sign* of $\phi$ is

$$\sigma(\phi) = \begin{cases} +1 & \text{if } \lim_{t\to\infty} f\circ\phi(t) = \infty, \\ -1 & \text{if } \lim_{t\to\infty} f\circ\phi(t) = -\infty, \\ 0 & \text{otherwise.} \end{cases}$$

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If $w$ is some infinite word in the generators $S$, there is a path $\phi_w\colon \mathbb{R}_+ \to \Gamma(G,S)$ beginning at 1 and realizing the word. Define $\sigma(w) = \sigma(\phi_w)$. If $g$ is a group element, let $\sigma(g)$ be the sign of $f(g)$. Notice that if we pick a word $w$ representing $g$ then $\sigma(www\cdots) = \sigma(w^\infty) = \sigma(g)$.

**Definition 1.1** Let

$$E(f,S) = \{w = w_1 w_2 \cdots \mid w_i \in S \cup S^{-1} \text{ and } \sigma(w) \in \{+1,-1\}\}/\sim,$$

where $w = w_1 w_2 \cdots \sim_C v = v_1 v_2 \cdots$ if $\sigma(w) = \sigma(v)$ and for all $D$ with $\sigma(w) D > C$ there is a word $d = d_1 \cdots d_n$ in the letters $S \cup S^{-1}$ such that:

- $w_p d = v_p$ in $G$ for some prefix $w_p$ of $w$ and some prefix $v_p$ of $v$,
- $|f(w_p d_p) - D| \leq C$ for all prefixes $d_p$ of $d$.

The word $d$ will be referred to as a *connecting word* and $w \sim v$ if $w \sim_C v$ for some $C$. This is an equivalence relation.

Since the set $E(f,S)$ is invariant under change of generators (see Manning [12, 2.3]) it can be denoted just by $E(f)$.

Let $f\colon G \to \mathbb{R}$ be a pseudocharacter. $E(f)^+ \subset E(f)$ denotes the set of positive elements of $E(f)$, and $E(f)^- \subset E(f)$ the set of negative elements. If $|E(f)| = 2$, $f$ is said to be *uniform*. If $|E(f)^+| = 1$ or $|E(f)^-| = 1$ but $f$ is not uniform, then $f$ is said to be *unipotent*. Otherwise, $f$ is said to be *bushy*.

This work is mainly based in two papers. The first is due to Bestvina and Fujiwara [2]. In the first part of that work they consider a group acting on a $\delta$–hyperbolic graph by isometries. There, they finish the work started by Fujiwara in [5; 6; 7] proving that if the action holds certain conditions (Manning called this a *Bestvina–Fujiwara action*), then the dimension of the second bounded cohomology of $G$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.

On the other hand, Manning proves in [12] two interesting results about pseudocharacters. In the first one the author proves that if for a given group $G$ there is a nonuniform pseudocharacter, then $G$ admits a cobounded quasiaction on a bushy tree. To do that, he also defines *Bottleneck Property* characterizing when a metric space is quasi-isometric to a tree.

The second one relates the existence of a *bushy* pseudocharacter with the conditions on the action studied in [2].

**Proposition 1.2** [12, 4.27] *If $f\colon G \to \mathbb{R}$ is a bushy pseudocharacter, then there is a Bestvina–Fujiwara action of $G$ on a quasitree.*

Herein we work in the opposite direction. In Section 2, we give some sufficient conditions for the existence of nonuniform pseudocharacters.

**Proposition 2.7** *Let $G$ be a group acting on a quasitree $X$. Let $g_1, g_2$ be two hyperbolic elements of $G$ such that $d_H(\Gamma_1(g_1, x_0, \gamma_1), \Gamma_2(g_2, x_0, \gamma_2)) = \infty$. Then, if $h$ is a pseudocharacter such that $h(g_1) > 0$ and $h(g_2) > 0$ and $h$ is bornologous on the action then $g_1^\infty \not\sim g_2^\infty$ in $E(h)$.*

**Corollary 2.11** *Let $G$ be a group acting by isometries on a quasitree $X$ so that the action is metrically proper. Let $g_1, g_2$ be two hyperbolic elements of $G$ such that $d_H(\Gamma_1(g_1, x_0, \gamma_1), \Gamma_2(g_2, x_0, \gamma_2)) = \infty$. Then, if $h$ is a pseudocharacter such that $h(g_1) > 0$ and $h(g_2) > 0$ then $g_1^\infty \not\sim g_2^\infty$ in $E(h)$.*

**Corollary 2.15** *Consider a nonelementary action of a group $G$ on a quasitree $X$. If the action is metrically proper then every nonelementary pseudocharacter is bushy.*

In Section 3 we prove that given a Bestvina–Fujiwara action, it holds the conditions of Proposition 2.7. Moreover:

**Theorem 3.10** *Let $G$ be a group acting on a quasitree. If it is a Bestvina–Fujiwara action, then there is a bushy pseudocharacter $h\colon G \to \mathbb{R}$.*

Examples of Bestvina–Fujiwara actions on quasitrees may be built using some axiomatic construction defined by Bestvina, Bromberg and Fujiwara in [1]. We include in this section a short explanation of how that works.

A quasiaction of a group $G$ on a metric space $X$ associates to each $g \in G$ a quasi-isometry $A_g\colon X \to X$ with uniform quasi-isometry constants so that $A_{\mathrm{Id}} = \mathrm{Id}_X$ and such that the distance between $A_h \circ A_g$ and $A_{hg}$ in the sup norm is uniformly bounded independent of $g, h \in G$. This is a natural and interesting extension of group actions and it has been relevant in relation to trees. In [15], Mosher, Sageev, and Whyte prove that every cobounded quasiaction on a bounded valence bushy tree is quasiconjugate to an action on a tree. However, there are examples of quasiactions on simplicial trees which are not quasiconjugate to actions on $\mathbb{R}$-trees. See [12] for the examples and [13] for further results on quasiactions on trees.

Given a nonuniform pseudocharacter $h$, Manning builds in [12] a cobounded quasiaction on a bushy tree $T$. In Section 4 we show that this construction can be made by adding a condition to the relation between the space $E(h)$ and the boundary of the tree $\partial T$.

In the last section we state some implications on the space of pseudocharacters and therefore, in the cobounded cohomology of the group.

**Corollary 5.4** *If there is a bushy pseudocharacter $h\colon G \to \mathbb{R}$ then the dimension of the space generated by the bushy pseudocharacters on $G$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

All groups are assumed to be finitely presented.

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## 2 Actions and pseudocharacters

**Definition 2.1** A map between metric spaces, $f\colon (X, d_X) \to (Y, d_Y)$, is said to be *quasi-isometric* if there are constants $\lambda \geq 1$ and $C > 0$ such that for all $x, x' \in X$, $(1/\lambda) d_X(x, x') - A \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + A$. If there is a constant $B > 0$ such that $N_B(f(X)) = Y$ where $N_B(f(X)) = \{y \in Y \mid d_Y(y, f(X)) < B\}$, then $f$ is a *quasi-isometry* and $X, Y$ are *quasi-isometric*.

**Theorem 2.2** [12, Theorem 4.6] *Let $Y$ be a geodesic metric space. The following are equivalent:*

(1) *$Y$ is quasi-isometric to some simplicial tree $\Gamma$.*

(2) (Bottleneck Property) *There is some $\Delta > 0$ so that for all $x, y \in Y$ there is a midpoint $m = m(x, y)$ with $d(x, m) = d(y, m) = \frac{1}{2} d(x, y)$ and the property that any path from $x$ to $y$ must pass within less than $\Delta$ of the point $m$.*

Let $(X, d)$ be a metric space. Fix a base point $o \in X$ and for $x, x' \in X$ put $(x|x')_o = \frac{1}{2}(d(x, o) + d(x', o) - d(x, x'))$. The number $(x|x')_o$ is nonnegative and it is called the *Gromov product* of $x, x'$ with respect to $o$. See Gromov [9].

**Definition 2.3** A metric space $X$ is *(Gromov) hyperbolic* if it satisfies the $\delta$-inequality

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta$$

for some $\delta \geq 0$, for every base point $o \in X$ and all $x, y, z \in X$.

Let $X$ be a hyperbolic space and $o \in X$ a base point. A sequence of points $\{x_i\} \subset X$ *converges to infinity* if

$$\lim_{i,j\to\infty} (x_i|x_j)_o = \infty.$$

This property is independent of the choice of $o$ since

$$|(x|x')_o - (x|x')_{o'}| \le |oo'|$$

for any $x, x', o, o' \in X$.

Two sequences $\{x_i\}, \{x'_i\}$ that converge to infinity are *equivalent* if

$$\lim_{i\to\infty} (x_i|x'_i)_o = \infty.$$

Using the $\delta$–inequality, we easily see that this defines an equivalence relation for sequences in $X$ converging to infinity. The *boundary at infinity* $\partial_\infty X$ of $X$ is defined to be the set of equivalence classes of sequences converging to infinity.

The following lemma is a well known property of quasigeodesics (see Bowditch [3] or Fujiwara [6]). The statement with the proof can be found in Manning [13].

**Lemma 2.4** *Let $X$ be a $\delta$–hyperbolic space. Given $K \ge 1$ and $L \ge 0$, there exists $B(K, L, \delta) \ge 0$ such that if $\gamma_1, \gamma_2$ are two $(K, L)$–quasigeodesics with the same endpoints in $X \cup \partial X$, then $\gamma_1 \subset N_B(\gamma_2)$ and $\gamma_2 \subset N_B(\gamma_1)$.*

**Definition 2.5** Fix $x_0 \in X$, where $X$ is a $\delta$–hyperbolic metric space on which $G$ quasiacts. Let $O_{g,x}\colon \mathbb{R} \to X$ be defined by $O_{g,x}(t) = g^{\lfloor t\rfloor}x$ where $\lfloor t\rfloor$ is the largest integer smaller than $t$. Then it is said that $g$ *quasiacts elliptically* if $O_{g,x}$ has bounded image, and $g$ *quasiacts hyperbolically* if $O_{g,x}$ is a quasigeodesic. If $G$ acts isometrically on $X$ then it is said that $g$ *acts* elliptically or hyperbolically or that $g$ is *elliptic* or *hyperbolic*.

It is readily seen that this definition is independent of $x$ and agrees with the standard definitions in case $G$ acts isometrically.

If $g \in G$ is hyperbolic $x \in X$, and $\gamma_0\colon [0, 1] \to X$ is a geodesic segment with $\gamma_0(0) = x$ and $\gamma_0(1) = g(x)$, then it is not hard to check that $\Gamma_{g,x,\gamma_0}\colon \mathbb{R} \to X$ given by

$$\Gamma_{g,x,\gamma_0}(t) = g^{\lfloor t\rfloor}\gamma_0(t - \lfloor t\rfloor) \tag{1}$$

is a continuous quasigeodesic. Moreover, $g$ is an isometry of $X$ which maps this quasigeodesic to itself by a nontrivial translation. See Figure 1.

A quasigeodesic where $g$ acts by nontrivial translation will be referred to as a *quasiaxis* (or $(K{-}L)$–*quasiaxis* if the constants are relevant). A quasiaxis of $g$ is given the $g$–orientation by the requirement that $g$ acts as a positive translation.

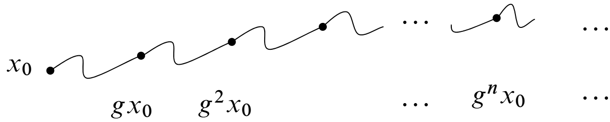


Figure 1: The isometry $g$ acts on $\Gamma_{g,x,\gamma_0}$ by a nontrivial translation.

**Definition 2.6** Let $G$ be a group acting by isometries on a metric space $X$. We say that a pseudocharacter $h\colon G \to \mathbb{R}$ is *bornologous on the action* if given any $x_0 \in X$ and any $g \in G$, for all $R > 0$ there exists $S > 0$ such that for all $g' \in G$ with $g'(x_0) \in B(g(x_0), R)$, $|h(g') - h(g)| \le S$.

A *quasitree* is a complete geodesic metric space quasi-isometric to some simplicial tree. These spaces satisfy bottleneck property; see Theorem 2.2. Herein, we will add the assumption that the quasitree is a graph. This is not a restrictive assumption since we are working in a coarse setting but it has obvious technical advantages. Therefore, from now on, a *quasitree* will be a graph satisfying bottleneck property.

**Proposition 2.7** *Let $G$ be a group acting on a quasitree $X$. Let $g_1, g_2$ be two hyperbolic elements of $G$ such that $d_H(\Gamma_1(g_1, x_0, \gamma_1), \Gamma_2(g_2, x_0, \gamma_2)) = \infty$. Then, if $h$ is a pseudocharacter such that $h(g_1) > 0$ and $h(g_2) > 0$ and $h$ is bornologous on the action then $g_1^\infty \not\sim g_2^\infty$ in $E(h)$.*

**Proof** Since $h(g_1) > 0$ and $h(g_2) > 0$, $\sigma_h(g_1) > 0$ and $\sigma_h(g_2) > 0$. Let $w_j$ be the word representing $g_j$ in the letters of the generating set $S \cup S^{-1}$. Then $w_j w_j \cdots = w_j^\infty$ is an element of $E(h)$ fixed by $g_j$ for $j = 1, 2$. Note that, with the assumptions taken, $\sigma(w_j^\infty) = +1$.

Let us see that $w_1^\infty \not\sim w_2^\infty$ in $E(h)$.

Let us denote, for simplicity, $w = w_1^\infty$ and $v = w_2^\infty$ and suppose $w \sim_C v$ for some $C > 0$. Then, given $D_1 > C$ there is a connecting word $d = d_1 \cdots d_n$ such that:

- $w_p d = v_p$ in $G$ for some prefix $w_p$ of $w$ and some prefix $v_p$ of $v$,
- $|h(w_p d_p) - D_1| \le C$ for all prefixes $d_p$ of $d$.

By abuse of notation let us identify the group element $g$ with the word representing it, $w$. Therefore, we write $w(x_0)$ for the image of the isometric action $g$ on $x_0$. Let $\gamma$ be a geodesic path from $w_p(x_0)$ to $v_p(x_0)$ and let $m$ be the midpoint in $\gamma$.

Let $\gamma_1$ be a geodesic path from $x_0$ to $w_1(x_0)$ and $\gamma_2$ be a geodesic path from $x_0$ to $w_2(x_0)$ and consider $\Gamma_1 := \Gamma_{g_1,x_0,\gamma_1}(t)$, $\Gamma_2 := \Gamma_{g_2,x_0,\gamma_2}(t)$ two continuous quasi-geodesics defined as in (1). Let $\Gamma_1(w_p, w_q)$ be the restriction of $\Gamma_{g_1,x_0,\gamma_1}(t)$ to a (quasi-isometric) path from $w_p(x_0)$ to $w_q(x_0)$ for any prefixes $w_p, w_q$ of $w$. Also, let $\Gamma_2(v_p, v_q)$ be the restriction of $\Gamma_{g_2,x_0,\gamma_2}(t)$ to a (quasi-isometric) path from $v_p(x_0)$ to $v_q(x_0)$ for any prefixes $v_p, v_q$ of $v$.

Let $\Delta$ be the bottleneck property constant.

**Claim** Since $d_H(\Gamma_1(g_1, x_0, \gamma_1), \Gamma_2(g_2, x_0, \gamma_2)) = \infty$, we may assume $D_1$ big enough to assure that for any $w_p \subset w_q$ and $v_p \subset v_q$, then $\Gamma_1(w_p, w_q) \cap B(m, \Delta) = \varnothing$ and $\Gamma_2(v_p, v_q) \cap B(m, \Delta) = \varnothing$. See Figure 2.

Since $d_H(\Gamma_1(g_1, x_0, \gamma_1), \Gamma_2(g_2, x_0, \gamma_2)) = \infty$, we assume $D_1 = D_1(h, \Delta, g_1, g_2)$ is big enough to guarantee that $A := d(w_p, v_p)$ is as big as we want. Notice that $d(w_1^i(x_0), w_1^{i+1}(x_0)) = d(x_0, w_1(x_0)) =: d_1$ so $d(w_1^i(x_0), w_1^{i+k}(x_0)) \leq k \cdot d_1$. Respectively, for $w_2$, we have $d(w_2^i(x_0), w_2^{i+1}(x_0)) = d(x_0, w_2(x_0)) =: d_2$ therefore $d(w_2^i(x_0), w_2^{i+k}(x_0)) \leq k \cdot d_1$. So, if $A$ is big enough, either $k$ is also big enough (depending on the distance $(A/2-\Delta)/\max\{d_1, d_2\}$) or we can assure the quasigeodesic $\Gamma_1(w_p, w_p \cdot w_1^k)$ from $w_p(x_0)$ to $w_p \cdot w_1^k(x_0)$ satisfies $\Gamma_1(w_p, w_p \cdot w_1^k) \cap B(m, \Delta) = \varnothing$. Also, either $k$ is big enough or the quasigeodesic $\Gamma_2(v_p, v_p \cdot w_2^k)$ from $v_p(x_0)$ to $v_p \cdot w_2^k(x_0)$ satisfies $\Gamma_2(v_p, v_p \cdot w_2^k) \cap B(m, \Delta) = \varnothing$.

Now, let us assume $k = k(h, \Delta, g_1, g_2)$ as big as we want and fix it assuming that the corresponding quasigeodesics $\Gamma_1(w_p, w_p \cdot w_1^k)$ and $\Gamma_2(v_p, v_p \cdot w_2^k)$ do not intersect the ball $B(m, \Delta)$. By hypothesis we have that $h(g_j) > 0$ and $h(g_j^n) = nh(g_j)$ for $j = 1, 2$. Then, for any $j \geq k$, we have $h(w_p \cdot w_1^j(x_0)) = h(w_p) + jh(g_1)$ and $h(v_p \cdot w_2^j) = h(v_p) + jh(g_2)$ are much bigger than $D_1$. Then, since $h$ is bornologous on the action, we can assume $k$ big enough so that $w_p \cdot w_1^j(x_0) \notin B(m, \Delta + \max\{d_1, d_2\})$ and $v_p \cdot w_2^j(x_0) \notin B(m, \Delta + \max\{d_1, d_2\})$ for any $j \geq k$. So, the quasigeodesic segments $\Gamma_1(w_p \cdot w_1^j, w_p \cdot w_1^{j+1}) \cap B(m, \Delta) = \varnothing$ and $\Gamma_2(v_p \cdot w_2^j, v_p \cdot w_2^{j+1}) \cap B(m, \Delta) = \varnothing$ for every $j \geq 0$, proving the claim. □

Now, let $D_2 >> h(m), D_1$. Then, we will reach a contradiction finding a uniform bound for $D_2 - D_1$.

By assumption, there is a connecting word $d' = d'_1 \cdots d'_n$ such that:

- $w_{p'} d' = v_{p'}$ in $G$ for some prefix $w_{p'}$ of $w$ and some prefix $v_{p'}$ of $v$,
- $|h(w_{p'} d'_{p'}) - D_2| \leq C$ for all prefixes $d'_{p'}$ of $d$.

Let $\Gamma_1 := \Gamma_1(w_p, w'_p)$, $\Gamma_2 := \Gamma_2(v_p, v'_p)$ be quasigeodesic paths defined as above from $w_p(x_0)$ to $w_{p'}(x_0)$ and from $v_p(x_0)$ to $v_{p'}(x_0)$. Let $\gamma'_j\colon [0,1] \to X$ be a geodesic path from $w_{p'} \cdot d'_{p'} \cdot d'_{j-1}(x_0)$ to $w_{p'} \cdot d'_{p'} \cdot d'_j(x_0)$ for $1 \leq j \leq n$ and let $\gamma'\colon [0,n] \to X$ be the path from $w_{p'}(x_0)$ to $v_{p'}(x_0)$ defined by $d'$ where $\gamma'(t) = \gamma'_{\lfloor t \rfloor}(t - \lfloor t \rfloor)$.

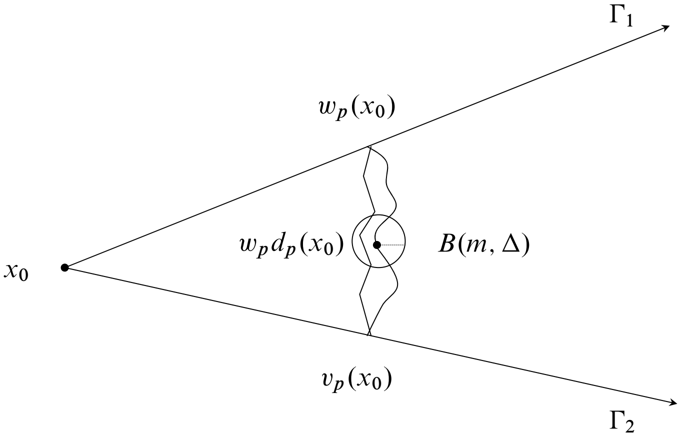


Figure 2: Assuming $D_1$ big enough, we obtain that $\Gamma_1(w_p, w_p \cdot w_1^k)$ and $\Gamma_2(v_p, v_p \cdot w_2^k)$ do not intersect the ball $B(m,\Delta)$.

Then, $\Gamma = \Gamma_1 \cup \Gamma' \cup \Gamma_2^{-1}$ is a path from $w_p(x_0)$ to $v_p(x_0)$. By bottleneck property (see Theorem 2.2) there is a point $x \in \Gamma$ such that $d(x,m) \leq \Delta$. By the previous claim, we can assume that $x \in \gamma_j'[0,1]$ for some $1 \leq j \leq n$. See Figure 3.

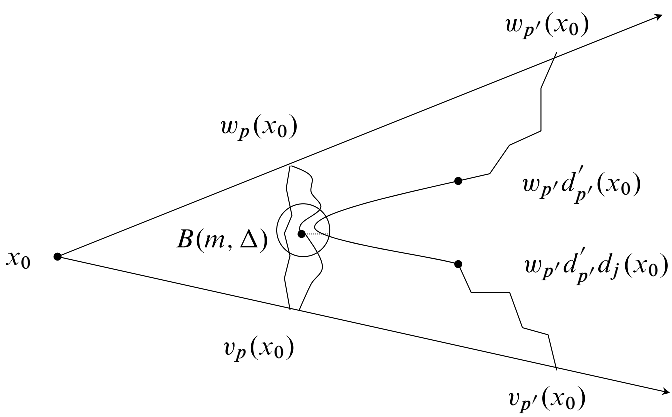


Figure 3: The path $[w_p(x_0), w_{p'}(x_0)] \cup [w_{p'}(x_0), v_{p'}(x_0)] \cup [v_{p'}(x_0), v_p(x_0)]$ intersects the ball $B(m,\Delta)$ by bottleneck property.

By hypothesis, $h(w_{p'}d'_{p'}), h(w_{p'}d'_{p'}d'_j) \in (D_2 - C, D_2 + C)$. Let $F = F(S, x_0)$ be a constant so for all $s \in S$, $d(x_0, s(x_0)) \leq F$. Then, $d(w_{p'}d'_{p'}(x_0), w_{p'}d'_{p'}d'_j(x_0)) \leq F$. Therefore, $d(w_{p'}d'_{p'}(x_0), m) \leq \Delta + F$ and $d(w_p d'_p d'_j(x_0), m) \leq \Delta + F$ which implies $d(w_{p'}d'_{p'}(x_0), w_p d_p(x_0)) \leq 2\Delta + 2F$ and that $d(w_p d'_p d'_j(x_0), w_p d_p(x_0)) \leq 2\Delta + 2F$ for some prefix $d_p$ of $d$. But since $h$ is bornologous on the action, there is some $S = S(2\Delta + 2F)$ such that $|h(w_p d_p) - h(w_p d'_p d'_j)| < S$ and $D_2 - D_1 < S + 2C$.

Thus, $D_2 - D_1$ is bounded by a constant depending only on $h, C, F, \gamma_1, \gamma_2$ and $\Delta$, leading to contradiction. □

**Definition 2.8** An action of a group by isometries on a metric space is *metrically proper* if for all $x \in X$ and for all $R > 0$ the set $\{g \in G \mid g(N(x, R)) \cap N(x, R) \neq \varnothing\}$ is finite.

**Lemma 2.9** *An action of a group $G$ by isometries on a metric space $X$ is metrically proper if and only if for all $x_0 \in X$ and for all $g \in G$ then for all $R > 0$ the set $\{g' \in G \mid g'x_0 \in N(g(x_0), R)\}$ is finite.*

**Proof** The set $\{h \in G \mid h(B(g(x_0), R)) \cap B(g(x_0), R) \neq \varnothing\}$ is finite if the action is metrically proper. If $g' = hg$, then $\{g' \in G \mid g'g^{-1}B(g(x_0), R) \cap B(g(x_0), R) \neq \varnothing\} = \{g' \in G \mid g'B(x_0, R) \cap B(g(x_0), R) \neq \varnothing\}$ is finite. In particular, the set $\{g' \in G \mid g'x_0 \in B(g(x_0), R)\}$ is finite.

Conversely, suppose $\{g' \in G \mid g'x_0 \in N(g(x_0), 2R)\}$ is finite. Then, if $h = g'g^{-1}$, the set $\{h \in G \mid (hg)(x_0) \cap N(g(x_0), 2R)) \neq \varnothing\} = \{h \in G \mid h(g(x_0)) \cap N(g(x_0), 2R)) \neq \varnothing\}$ is finite which implies $\{h \in G \mid h(N(g(x_0), R)) \cap N(g(x_0), R)) \neq \varnothing\}$ is finite. □

**Proposition 2.10** *Let $G$ be a group acting by isometries on a metric space $X$ and let $h\colon G \to \mathbb{R}$ be a pseudocharacter. If the action is metrically proper then the pseudocharacter is bornologous on the action.*

**Proof** Let $x_0 \in X$, $g \in G$ and $R > 0$. Since the action is metrically proper the set $K = \{g' \in G \mid g'x_0 \in N(g(x_0), R)\}$ is finite. Therefore, it suffices to take $S := \max_{g' \in K}\{|h(g') - h(g)|\}$. □

**Corollary 2.11** *Let $G$ be a group acting by isometries on a quasitree $X$ so that the action is metrically proper. Let $g_1, g_2$ be two hyperbolic elements of $G$ such that $d_H(\Gamma_1(g_1, x_0, \gamma_1), \Gamma_2(g_2, x_0, \gamma_2)) = \infty$. Then, if $h$ is a pseudocharacter such that $h(g_1) > 0$ and $h(g_2) > 0$ then $g_1^\infty \not\sim g_2^\infty$ in $E(h)$.*

**Definition 2.12** Two hyperbolic isometries $g_1, g_2$ are said to be *independent* if their quasiaxis do not contain rays which are a finite Hausdorff distance apart. Equivalently the fixed point sets of $g_1, g_2$ in $\partial X$ are disjoint. An action is *nonelementary* if there are at least two independent hyperbolic elements.

**Definition 2.13** We say that a pseudocharacter $h\colon G \to \mathbb{R}$ is *nonelementary* if there is a pair of independent $g_1, g_2 \in G$ such that $h(g_1) \neq 0$ and $h(g_2) \neq 0$.

**Corollary 2.14** *Consider a nonelementary action of a group $G$ on a quasitree $X$ and a nonelementary pseudocharacter $h\colon G \to \mathbb{R}$. Then, if $h$ is bornologous on the action, it is bushy.*

**Corollary 2.15** *Consider a nonelementary action of a group $G$ on a quasitree $X$. If the action is metrically proper then every nonelementary pseudocharacter is bushy.*

## 3 Existence of bushy pseudocharacters

Note any two $(K-L)$–*quasiaxis* of $g$ are within some universal $B = B(\delta, K, L)$ of one another and any sufficiently long $(K, L)$–quasigeodesic arc $J$ in a $B$–neighborhood of a quasiaxis $l$ of $g$ inherits a natural $g$–orientation: a point of $l$ within $B(\delta, K, L)$ of the terminal endpoint of $J$ is ahead (with respect to the $g$–orientation of $l$) of a point of $l$ within $B(\delta, K, L)$ of the initial endpoint of $J$. This orientation of $J$ will be denoted by $g$–orientation of $J$.

**Definition 3.1** (Bestvina and Fujiwara [2]) If $g_1, g_2$ are hyperbolic elements of $G$, let $g_1 \sim g_2$ if for an arbitrarily long segment $J$ in a $(K, L)$–quasiaxis for $g_1$ there is a $g \in G$ such that $g(J)$ is within $B(\delta, K, L)$ of a $(K, L)$–quasiaxis of $g_2$ and $g\colon J \to g(J)$ is orientation-preserving with respect to the $g_2$–orientation on $g(J)$.

This defines an equivalence relation. As it is said in [2], the concept does not change if $B$ is replaced by a larger constant.

**Definition 3.2** A *Bestvina–Fujiwara* action is a nonelementary action of a group $G$ on a hyperbolic graph $X$ so that there exist independent $g_1, g_2$ such that $g_1 \not\sim g_2$.

**Lemma 3.3** *Let $X$ be a (geodesic) Gromov hyperbolic space and let $G$ be a group acting by isometries on $X$. Let $g_1, g_2 \in G$ such that $g_1 \not\sim g_2$. Then, for any point $x_0 \in X$, and any pair of geodesics $\gamma_1 := [x_0, g_1(x_0)]$, $\gamma_2 := [x_0, g_2(x_0)]$, $d_H(\Gamma_1(g_1, x_0, \gamma_1), \Gamma_2(g_2, x_0, \gamma_2)) = \infty$.*

**Proof** Suppose there is some $B > 0$ such that $\Gamma_1 \subset N_B(\Gamma_2)$ and $\Gamma_2 \subset N_B(\Gamma_1)$. Then, the action of the identity $i$ on $\Gamma_1$ is contained in $N_B(\Gamma_2)$ and $i\colon \Gamma_1 \to i(\Gamma_1) = \Gamma_1$ is orientation preserving with respect to the $g_1$–orientation and the $g_2$–orientation on $\Gamma_1$. This contradicts the fact that $g_1 \not\sim g_2$. □

Let us recall now the basic construction of quasihomomorphisms associated to the action as presented in [6].

Let $X$ be a hyperbolic graph and a group $G$ acting on $X$. Let $w$ be a finite (oriented) path in $X$. By $|w|$ denote the length of $w$, by $i(w)$ the starting point and by $t(w)$ the finishing point. For $g \in G$, $g \circ w$ is a path starting at $g(i(w))$ and finishing at $g(t(w))$ and it is called a *copy* of $w$. Obviously, $|g \circ w| = |w|$.

Let $\alpha$ be a finite path. Define

$$|\alpha|_w = \{\text{the maximal number of nonoverlapping copies of } w \text{ in } \alpha\}.$$

Suppose $x, y \in X$ are two vertices and that $W$ is an integer with $0 < W < |w|$. Then

$$c_{w,W}(x,y) = d(x,y) - \inf_{\alpha}(|\alpha| - W|\alpha|_w),$$

where $\alpha$ ranges over all paths from $x$ to $y$.

**Lemma 3.4** [6, Lemma 3.4] *Let $x$, $y$, $z$ be three points in $X$. Then*

$$|c_{w,W}(x,y) - c_{w,W}(x,z)| \le 2d(y,z). \qquad \square$$

Let us omit $W$ from the notation and write $c_w$. Define $h_w\colon G \to \mathbb{R}$ by

$$h_w(g) = c_w(x_0, g(x_0)) - c_{w^{-1}}(x_0, g(x_0)).$$

**Proposition 3.5** [6, Proposition 3.10] *If $X$ is a $\delta$–hyperbolic space $h_w$ is a quasihomomorphism (ie quasicharacter).*

From Lemma 3.4, it is immediate to obtain the following.

**Lemma 3.6** *Let $X$ be a (geodesic) Gromov hyperbolic space. For any word $w$, any points $x_0, x \in X$ and any constants $0 < W < |w|$, $R > 0$, the subset of the real line $\{h_w(g) \mid g(x_0) \in B(x,R)\}$ is bounded. In particular, it has diameter at most $8R$.*

Therefore, the following is immediate.

**Proposition 3.7** *Let $X$ be a (geodesic) Gromov hyperbolic space. For any word $w$, any points $x_0, x \in X$ and any constants $0 < W < |w|$, $h_w$ is bornologous on the action.*

Let us recall two propositions from [2].

**Proposition 3.8** [2, Proposition 2] *Suppose a group $G$ acts on a $\delta$–hyperbolic graph $X$ by isometries. Suppose also that the action is nonelementary and that there exist independent hyperbolic elements $g_1, g_2 \in G$ such that $g_1 \not\sim g_2$. Then, there is a sequence $f_1, f_2, \ldots \in G$ of hyperbolic elements such that:*

- $f_i \not\sim f_i^{-1}$ for $i = 1, 2, \ldots,$
- $f_i \not\sim f_j^{\pm 1}$ for $j < i$.

Replacing if necessary $g_1, g_2$ by high positive powers of conjugates, let $F$ be a free subgroup of $G$ with basis $\{g_1, g_2\}$ such that each nontrivial element of $F$ is hyperbolic and $F$ is quasiconvex with respect to the action on $X$. See the proof of [2, Propostion 2] and [9, Section 5.3] for details. Such free groups are called *Schottky groups*.

**Proposition 3.9** [2, Proposition 5] *Suppose $1 \neq f \in F$ is cyclically reduced and $f \not\sim f^{-1}$. Then there is $a > 0$ such that $h_{f^a}$ is unbounded on $\langle f \rangle$. Moreover, if $f^{\pm 1} \not\sim f' \in F$ then $h_{f^a}$ is 0 on $\langle f' \rangle$ for sufficiently large $a > 0$.*

**Theorem 3.10** *Let $G$ be a group acting on a quasitree. If it is a Bestvina–Fujiwara action, then there is a bushy pseudocharacter $h\colon G \to \mathbb{R}$.*

**Proof** Consider the sequence $f_1, f_1, \ldots$ from Proposition 3.8 and assume in addition (without loss of generality) that each $f_i$ is cyclically reduced. Define $h_i\colon G \to \mathbb{R}$ as $h_i = h_{f^{a_i}}$ where $a_i$ is chosen as in Proposition 3.9 so that $h_i$ is unbounded on $\langle f_i \rangle$ and so that it is 0 on $\langle f_j \rangle$ for $j < i$ and also 0 on $\langle f_{i+1} \rangle$. With the same argument, we may also assume that $\lim_{k \to \infty} h_i(f_i^k) = +\infty$.

Let $h$ be a pseudocharacter at a bounded distance (see Remark) from the quasicharacter $h_1 + h_2$. (Notice that everything works if we consider $h_i$ and $h_{i+1}$ instead). Clearly, $h(f_1) > 0$ and $h(f_2) > 0$.

Thus $\sigma_h(f_j) > 0$ and $f_j^{\pm\infty}$ defines an element in $E(h)$ for $j = 1, 2$. Let $w_j$ be the word representing $f_j$ in the letters $S \cup S^{-1}$. Then $w_j w_j \cdots = w_j^{\infty}$ is an element of $E(f)$ fixed by $f_j$ for $j = 1, 2$. Note that, with the assumptions taken, $\sigma(w_j^{\infty}) = +1$.

Let us see that $w_1^{\infty} \not\sim w_2^{\infty}$ in $E(h)$. It suffices to check that by Proposition 3.7 and Lemma 3.3 we are in the conditions of Proposition 2.7.

The same argument proves that $w_1^{-\infty} \not\sim w_2^{-\infty}$ in $E(h)$. □

**Corollary 3.11** *If a Cayley graph $X = \Gamma(G, S)$ satisfies the bottleneck property and the canonical action of the group is a Bestvina–Fujiwara action, then there is a bushy pseudocharacter $h\colon G \to \mathbb{R}$.*

Bestvina, Bromberg and Fujiwara give in [1] a general construction of actions of groups on quasitrees. They proceed axiomatically defining a distance function and then modifying it to define what they call *projection complex*.

Let $\mathbf{Y}$ be a set and assume that for each $Y \in \mathbf{Y}$ there is a function

$$d_Y^{\pi}\colon (\mathbf{Y}\backslash\{Y\}) \times (\mathbf{Y}\backslash\{Y\}) \to [0,\infty)$$

and a constant $\xi > 0$ that satisfies the following axioms:

(A1) $d_Y^{\pi}(X,Z) = d_Y^{\pi}(Z,X)$

(A2) $d_Y^{\pi}(X,Z) + d_Y^{\pi}(Z,W) \geq d_Y^{\pi}(X,W)$

(A3) $\min\{d_Y^{\pi}(X,Z), d_Z^{\pi}(X,Y)\} < \xi$

(A4) $\#\{Y \mid d_Y^{\pi}(X,Z) \geq \xi\}$ is finite for all $X, Z \in \mathbf{Y}$

Given this distance function they define $\mathcal{H}(X,Z)$ to be the set of pairs $(X',Z') \in \mathbf{Y}\times\mathbf{Y}$ such that one of the following holds:

- $d_X^{\pi}(X',Z'), d_Z^{\pi}(X',Z') > 2\xi$
- $X = X'$ and $d_Z^{\pi}(X,Z') > 2\xi$
- $Z = Z'$ and $d_X^{\pi}(X',Z) > 2\xi$
- $(X',Z') = (X,Z)$

Then, they define the function

$$d_Y(X,Z) = \min_{(X',Z')\in\mathcal{H}(X,Z)} d_Y^{\pi}(X',Z')$$

and the set $\mathbf{Y}_K(X,Z)$ to be the set of $Y \in \mathbf{Y}$ such that $d_Y(X,Z) > K$.

For some (big enough) constant $K > 0$, they define the *projection complex* $\mathcal{P}_K(\mathbf{Y})$ as a 1–complex whose vertex set is $\mathbf{Y}$ and such that there is an edge connecting two vertices $X$ and $Z$ if $\mathbf{Y}_K(X,Z)$ is empty.

**Theorem 3.12** [1, Theorem 2.9] *For $K$ sufficiently large $\mathcal{P}_K(\mathbf{Y})$ is a quasitree.*

Also, suppose $G$ is a group acting on the set $\mathbf{Y}$, that there exists a function $d_Y^{\pi}$ satisfying (A1)–(A4) and *projection distances are preserved*, ie $d_{g(A)^{\pi}}(g(B), g(C)) = d_A^{\pi}(B,C)$ for all $A, B, C \in \mathbf{Y}$ and $g \in G$. Then, $G$ acts naturally on $\mathcal{P}_K(\mathbf{Y})$. See Theorem 3.15.

Using this construction, several examples of groups which act on quasitrees are given in [1]. To obtain examples for Theorem 3.10 it suffices to check that there exist two independent hyperbolic elements $g_1, g_2$ such that $g_1 \not\sim g_2$ (on $P_K(\mathbf{Y})$).

In [1, Lemma 2.11], a technical lemma, there is certain constant $K'$ involved which roughly depends on $K$. Fixing that constant and assuming $g \in G$ and $Y \in \mathbf{Y}$ such that

$$d_Y(g^{-N}(Y), g^N(Y)) > K'$$

for some $N > 0$ (see [1, Lemma 2.13]), they define the *combinatorial axis* as

$$\mathbf{Y}_{K'}(g) = \{Y \in \mathbf{Y} \mid d_Y(g^{-N}(Y), g^N(Y)) > K' \text{ for some } N > 0\}.$$

Also, fixing $K'$, they consider the following axioms on $g \in G$:

(B1) The element $g$ is contained in a unique maximal virtually cyclic subgroup, $\mathrm{EC}(g)$, the *elementary closure* of $g$.

(B2) $\mathrm{EC}(g)$ is malnormal, ie $\psi\,\mathrm{EC}(g)\psi^{-1} = \mathrm{EC}(g)$ for $\psi \in G$ implies $\psi \in \mathrm{EC}(g)$.

(B3) There is $Y \in \mathbf{Y}_{K'}(g)$ and $m > 0$ such that if $\psi \in G$ fixes $Y, g(Y), \dots, g^m(Y)$, then $\psi \in \mathrm{EC}(g)$.

The following definition also comes from [1]. Assume $G$ is acting on a geodesic metric space $X$. Then, $g \in G$ is a WPD element if $\langle g\rangle$ has a bounded orbit in $X$ and for every $x \in X$ and $D > 0$ there is $M > 0$ such that the set

$$\{\phi \in G \mid d(g^i(x), \phi(g^i(x))) \le D,\ i = \pm M\}$$

is finite.

**Proposition 3.13** [1, Proposition 2.16] *Suppose $g \in G$ satisfies (B1)–B3. Then $g$ is a WPD element with respect to the action of $G$ on $\mathcal{P}_K(\mathbf{Y})$.*

In [2] the authors define what they call *weak proper discontinuity* or WPD. A group action satisfies WPD if and only if

(i) there exists a hyperbolic element,

(ii) $G$ is not virtually cyclic,

(iii) every hyperbolic element in $G$ is a WPD element.

Then, they prove that if the action of a group $G$ on a $\delta$–hyperbolic graph $X$ satisfies WPD, then the action is nonelementary and there exist independent hyperbolic $g_1, g_2$ such that $g_1 \not\sim g_2$. See [2, Proposition 6].

However, in the proof of [2, Proposition 6(5)] the key is to find two independent WPD elements, more than having the condition on every hyperbolic element of the group. Therefore, this can be restated as follows.

**Proposition 3.14** *Let $G$ be a group acting on a $\delta$–hyperbolic graph $X$ such that $G$ is not virtually cyclic and there exist two independent hyperbolic elements $f_1, f_2 \in G$ which are WPD elements. Then there exist independent hyperbolic $g_1, g_2$ such that $g_1 \not\sim g_2$.*

**Theorem 3.15** [1, Theorem 2.17] *Suppose a group $G$ acts on a set $\mathbf{Y}$ satisfying axioms (A1)–(A4) such that projection distances are preserved. Therefore, $G$ acts on the quasitree $\mathcal{P}_K(\mathbf{Y})$. Further, assume that there exist independent elements $g_1, g_2 \in G$ that satisfy (B1)–(B3). Then, there is a nonabelian free subgroup $F \subset G$ all of whose nontrivial elements act on $\mathcal{P}_K(\mathbf{Y})$ hyperbolically as WPD elements.*

Suppose then that there is a group $G$ and two independent elements $g_1, g_2 \in G$ satisfying the conditions of the theorem. By Proposition 3.14 it follows that there are two independent hyperbolic elements $f_1, f_2$ such that $f_1 \not\sim f_2$, this is, the action of $G$ on $\mathbf{Y}$ is a Bestvina–Fujiwara action on a quasitree.

Hence, by Theorem 3.10:

**Corollary 3.16** *Suppose a group $G$ acts on a set $\mathbf{Y}$ satisfying axioms (A1)–(A4) such that projection distances are preserved. Further, assume that there exist independent elements $g_1, g_2 \in G$ that satisfy (B1)–(B3). Then, there exists a bushy pseudocharacter $h\colon G \to \mathbb{R}$.*

**Example 3.17** Consider any nonelementary word hyperbolic group $G$. For every hyperbolic element $g \in G$, (B1)–(B2) hold. See Lück and Weiermann [11, Example 3.5] (or Theorem 3.2 in III.$\Gamma$.3 on page 459 and Corollary 3.10 in III.$\Gamma$.3 on page 462 from Bridson and Haefliger [4]). Also, the action may be chosen so (B3) holds.

For any nonelementary word hyperbolic group $G$ there is an action on a set $\mathbf{Y}$ satisfying axioms (A1)–(A4) so that projection distances are preserved (see [2, Examples 2.1]). Therefore, there exists a bushy pseudocharacter $h\colon G \to \mathbb{R}$.

## 4 Quasiactions on trees

Given a pseudocharacter $f\colon G \to \mathbb{R}$, Manning introduces the following constructions. The first one gives a tree obtained from the Cayley graph of the group.

Consider an (unambiguous) triangular generating set $S$. Then scale $f$ so that $f(G)$ misses $\mathbb{Z} + \frac{1}{2}$ and so that $f$ changes by at most $\frac{1}{4}$ over each edge. Let $\widetilde{K}$ be the simply connected 2–complex obtained by attaching 2–cells according to the relations of the presentation. Then, a tree is built with vertex set in one-to-one correspondence with the components of $\widetilde{K}\backslash f^{-1}(\mathbb{Z}+\frac{1}{2})$. The edges correspond to components of $f^{-1}(\mathbb{Z}+\frac{1}{2})$, each of which is some possibly infinite track which separates $\widetilde{K}$ into two components. This construction is also the starting point in the author's [14] where given a real valued function on a geodesic space we give a sufficient condition for the space to be quasi-isometric to a tree.

The next appears as [12, Definition 4.9].

Let $V$ be the set of components of $f^{-1}(\mathbb{Z}+\frac{1}{2})$. Then $V$ is in one-to-one correspondence with the set of edges of $T$. Let $X$ be the simplicial graph with vertex set equal to $G\times V$ and the following edge condition: Two distinct vertices $(g,\tau)$ and $(g',\tau')$ are to be connected by an edge if there is some $h$ so that $hg(\tau)$ and $hg'(\tau')$ are contained in the same connected component of $f^{-1}[n-\frac{3}{2},n+\frac{1}{2}]$ for some $n$. The zero-skeleton $X^0$ is endowed with a $G$–action by setting $g(g_0,\tau_0)=(gg_0,\tau_0)$. Since this action respects the edge condition on pairs of vertices, it extends to an action on $X$. We will refer to this particular one as *Manning's action*.

**Proposition 4.1** [12, Proposition 4.27] *If $f\colon G\to\mathbb{R}$ is a bushy pseudocharacter, then Manning's action is a Bestvina–Fujiwara action.*

**Theorem 4.2** [12, Theorem 4.15] *The space $X$ satisfies the Bottleneck Property.*

Therefore, from Theorem 3.10, we can give the following corollary which would be some kind of converse to Proposition 4.1.

**Corollary 4.3** *If Manning's action is a Bestvina–Fujiwara action, then there is a bushy pseudocharacter $h\colon G\to\mathbb{R}$.*

**Lemma 4.4** [12, Lemma 4.17] *There is an injective map from $E(f)$ to $\partial X$.*

**Theorem 4.5** [12, Theorem 4.20] *If $f\colon G\to\mathbb{R}$ is a pseudocharacter which is not uniform, then $G$ admits a cobounded quasiaction on a bushy tree.*

Then, it is readily seen, from the construction of the action and the bushy tree, that Corollary 2.11 yields the following.

**Corollary 4.6** *Consider a nonelementary action of a group $G$ on a quasitree $X$. If the action is metrically proper, then for any pseudocharacter $h\colon G\to\mathbb{R}$ and any pair of independent $g_1,g_2\in G$ so that $h(g_1)>0$ and $h(g_2)>0$, there is a cobounded quasiaction of $G$ on a bushy tree $T$ so that there is an injective map from $E(h)$ to $\partial T$.*

# 5 Space of pseudocharacters

Given a group $G$, quasicharacters and pseudocharacters are major tools in the study of the bounded cohomology group $H_b^2(G;\mathbb{R})$ as we can see in [2].

The bounded cohomology group $H_b^*(G;\mathbb{R})$ of a discrete group $G$ is defined by the cochain complex $C_b^k(G;\mathbb{R})$, where

$$C_b^k(G;\mathbb{R}) = \{f\colon G^k \to \mathbb{R} \mid \sup_{G^k}|f(g_1,\dots,g_k)| < \infty\}$$

and the boundary $\delta\colon C_b^k(G;A) \to C_b^{k+1}(G;A)$ is given by

$$\delta f(g_0,\dots,g_k) = f(g_1,\dots,g_k) + \sum_{i=1}^{k}(-1)^i f(g_0,\dots,g_{i-1}g_i,\dots,g_k) + (-1)^{k+1} f(g_0,\dots,g_{k-1}).$$

See Gromov [8] and Ivanov [10] as general references for bounded cohomology.

**Remark** Note that 1–cocycles are just group homomorphisms $f\colon G \to \mathbb{R}$. In fact, $\mathrm{Hom}(G) = H^1(G;\mathbb{R})$. A *quasicharacter* is an element $f \in C^1(G;\mathbb{R})$ whose coboundary $\delta f$ lies in $C_b^2(G;\mathbb{R})$. A *pseudocharacter* is a quasicharacter such that $f(g^k) = kf(g)$ for all $k \in \mathbb{Z}$ and $g \in G$.

Let $\mathcal{V}(G)$ be the vector space of all quasihomomorphisms $G \to \mathbb{R}$ and let $\mathrm{BDD}(G)$ be the subspace of all bounded functions. Then, let $\mathrm{QH}(G) = \mathcal{V}(G)/\mathrm{BDD}(G)$.

There is an exact sequence

$$0 \to H^1(G;\mathbb{R}) \to \mathrm{QH}(G) \to H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$$

Using the sequence $f_1, f_2, \dots$ obtained in Proposition 3.8, Bestvina and Fujiwara prove that $[h_i] \in \mathrm{QH}(G)$ is not a linear combination of $[h_1],\dots,[h_{i-1}]$, ie, the sequence $[h_i]$ consists of linearly independent elements (see the proof of [2, Theorem 1]). This implies that the dimension of $H_b^2(G;\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum. See [6, Corollary 1.3] and [2, Theorem 1].

Therefore, since the argument in Theorem 3.10 works also for any pair $f_i, f_{i+1}$, the following is immediate.

**Corollary 5.1** *Let $G$ be a group acting on a (geodesic) Gromov hyperbolic graph $X$ satisfying the bottleneck property. If it is a Bestvina–Fujiwara action, then the dimension of the subspace generated by the bushy pseudocharacters as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

**Corollary 5.2** *If a Cayley graph $X = \Gamma(G, S)$ satisfies the bottleneck property and the canonical action of the group is a Bestvina–Fujiwara action, then the dimension of the subspace generated by the bushy pseudocharacters as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

**Corollary 5.3** *If Manning's action is a Bestvina–Fujiwara action, then the dimension of the subspace generated by the bushy pseudocharacters on $G$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

In particular:

**Corollary 5.4** *If there is a bushy pseudocharacter $h\colon G \to \mathbb{R}$ then the dimension of the subspace generated by the bushy pseudocharacters on $G$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

This, together with Corollary 2.14 and Corollary 2.15 yields:

**Corollary 5.5** *Consider a nonelementary action of a group $G$ on a quasitree $X$ and a nonelementary pseudocharacter $h\colon G \to \mathbb{R}$. Then, if $h$ is bornologous on the action, the dimension of the subspace generated by the bushy pseudocharacters on $G$ (in particular, the dimension of $H_b^2(G;\mathbb{R})$) as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

**Corollary 5.6** *Consider a nonelementary action of a group $G$ on a quasitree $X$. If the action is metrically proper and there exist a nonelementary pseudocharacter then the dimension of the subspace generated by the bushy pseudocharacters on $G$ (in particular, the dimension of $H_b^2(G;\mathbb{R})$) as a vector space over $\mathbb{R}$ is the cardinal of the continuum.*

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