Bushy pseudocharacters and group actions on quasitrees

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Given a group acting on a graph quasi-isometric to a tree, we give sufficient conditions for a pseudocharacter to be bushy. We relate this with the conditions studied by Bestvina and Fujiwara on their work on bounded cohomology and obtain some results on the space of pseudocharacters.

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1 Introduction

If $G$ is a finitely presented group, then $f: G \to \mathbb{R}$ is a quasihomomorphism or quasicharacter if $f(\alpha) + f(\beta) - f(\alpha\beta)$ is bounded independent of $\alpha, \beta$.

If $G$ is a finitely presented group, then $f: G \to \mathbb{R}$ is a pseudocharacter if it has the following properties:

- $f(\alpha^n) = n\alpha$ for all $\alpha \in G$, $n \in \mathbb{Z}$.
- $\delta f(\alpha, \beta) = f(\alpha) + f(\beta) - f(\alpha\beta)$ is bounded independent of $\alpha, \beta$.

Clearly the constant map $f(G) = 0$ is a trivial pseudocharacter.

Remark  Note that if $f$ is a quasicharacter and $\phi$ is given by

$$\phi(g) = \lim_{n \to \infty} f(g^n)/n,$$

then $\phi$ is a pseudocharacter with $\phi - f$ bounded.

Let $S$ be a finite generating set for $G$. If $\Gamma(G, S)$ is the Cayley graph associated to the generating set $S$, then $f$ can be extended affinely over the edges of $\Gamma(G, S)$.

If $\phi: \mathbb{R}_+ \to \Gamma(G, S)$ is an infinite ray, then the sign of $\phi$ is

$$\sigma(\phi) = \begin{cases} +1 & \text{if } \lim_{t \to \infty} f \circ \phi(t) = \infty, \\ -1 & \text{if } \lim_{t \to \infty} f \circ \phi(t) = -\infty, \\ 0 & \text{otherwise}. \end{cases}$$

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If \( w \) is some infinite word in the generators \( S \), there is a path \( \phi_w : \mathbb{R}_+ \rightarrow \Gamma(G, S) \) beginning at 1 and realizing the word. Define \( \sigma(w) = \sigma(\phi_w) \). If \( g \) is a group element, let \( \sigma(g) \) be the sign of \( f(g) \). Notice that if we pick a word \( w \) representing \( g \) then 

\[ \sigma(ww\cdots) = \sigma(w^\infty) = \sigma(g). \]

**Definition 1.1** Let

\[
E(f, S) = \{ w = w_1 w_2 \cdots \mid w_i \in S \cup S^{-1} \text{ and } \sigma(w) \in \{+1, -1\} \}/\sim,
\]

where \( w = w_1 w_2 \cdots \sim \varepsilon \) \( v = v_1 v_2 \cdots \) if \( \sigma(w) = \sigma(v) \) and for all \( D \) with \( \sigma(w)D > C \) there is a word \( d = d_1 \cdots d_n \) in the letters \( S \cup S^{-1} \) such that:

- \( w_p d = v_p \) in \( G \) for some prefix \( w_p \) of \( w \) and some prefix \( v_p \) of \( v \),
- \( |f(w_PD) - D| \leq C \) for all prefixes \( d_p \) of \( d \).

The word \( d \) will be referred to as a **connecting word** and \( w \sim v \) if \( w \sim \varepsilon \) \( v \) for some \( C \). This is an equivalence relation.

Since the set \( E(f, S) \) is invariant under change of generators (see Manning [12, 2.3]) it can be denoted just by \( E(f) \).

Let \( f : G \rightarrow \mathbb{R} \) be a pseudocharacter. \( E(f)^+ \subset E(f) \) denotes the set of positive elements of \( E(f) \), and \( E(f)^- \subset E(f) \) the set of negative elements. If \( |E(f)| = 2 \), \( f \) is said to be **uniform**. If \( |E(f)^+| = 1 \) or \( |E(f)^-| = 1 \) but \( f \) is not uniform, then \( f \) is said to be **unipotent**. Otherwise, \( f \) is said to be **bushy**.

This work is mainly based in two papers. The first is due to Bestvina and Fujiwara [2]. In the first part of that work they consider a group acting on a \( \delta \)-hyperbolic graph by isometries. There, they finish the work started by Fujiwara in [5; 6; 7] proving that if the action holds certain conditions (Manning called this a Bestvina–Fujiwara action), then the dimension of the second bounded cohomology of \( G \) as a vector space over \( \mathbb{R} \) is the cardinal of the continuum.

On the other hand, Manning proves in [12] two interesting results about pseudocharacters. In the first one the author proves that if for a given group \( G \) there is a nonuniform pseudocharacter, then \( G \) admits a cobounded quasiaction on a bushy tree. To do that, he also defines **Bottleneck Property** characterizing when a metric space is quasi-isometric to a tree.

The second one relates the existence of a **bushy** pseudocharacter with the conditions on the action studied in [2].

**Proposition 1.2** [12, 4.27] If \( f : G \rightarrow \mathbb{R} \) is a bushy pseudocharacter, then there is a Bestvina–Fujiwara action of \( G \) on a quasitree.
Herein we work in the opposite direction. In Section 2, we give some sufficient conditions for the existence of nonuniform pseudocharacters.

**Proposition 2.7** Let $G$ be a group acting on a quasitree $X$. Let $g_1, g_2$ be two hyperbolic elements of $G$ such that $d_H(\Gamma_1(g_1, x_0, \gamma_1), \Gamma_2(g_2, x_0, \gamma_2)) = \infty$. Then, if $h$ is a pseudocharacter such that $h(g_1) > 0$ and $h(g_2) > 0$ and $h$ is bornologous on the action then $g_1^\infty \not\sim g_2^\infty$ in $E(h)$.

**Corollary 2.11** Let $G$ be a group acting by isometries on a quasitree $X$ so that the action is metrically proper. Let $g_1, g_2$ be two hyperbolic elements of $G$ such that $d_H(\Gamma_1(g_1, x_0, \gamma_1), \Gamma_2(g_2, x_0, \gamma_2)) = \infty$. Then, if $h$ is a pseudocharacter such that $h(g_1) > 0$ and $h(g_2) > 0$ then $g_1^\infty \not\sim g_2^\infty$ in $E(h)$.

**Corollary 2.15** Consider a nonelementary action of a group $G$ on a quasitree $X$. If the action is metrically proper then every nonelementary pseudocharacter is bushy.

In Section 3 we prove that given a Bestvina–Fujiwara action, it holds the conditions of Proposition 2.7. Moreover:

**Theorem 3.10** Let $G$ be a group acting on a quasitree. If it is a Bestvina–Fujiwara action, then there is a bushy pseudocharacter $h: G \to \mathbb{R}$.

Examples of Bestvina–Fujiwara actions on quasitrees may be built using some axiomatic construction defined by Bestvina, Bromberg and Fujiwara in [1]. We include in this section a short explanation of how that works.

A quasiaction of a group $G$ on a metric space $X$ associates to each $g \in G$ a quasi-isometry $A_g: X \to X$ with uniform quasi-isometry constants so that $A_{1d} = \text{Id}_X$ and such that the distance between $A_h \circ A_g$ and $A_{hg}$ in the sup norm is uniformly bounded independent of $g, h \in G$. This is a natural and interesting extension of group actions and it has been relevant in relation to trees. In [15], Mosher, Sageev, and Whyte prove that every cobounded quasi-structure on a bounded valence bushy tree is quasiconjugate to an action on a tree. However, there are examples of quasiactions on simplicial trees which are not quasiconjugate to actions on $\mathbb{R}$-trees. See [12] for the examples and [13] for further results on quasiactions on trees.

Given a nonuniform pseudocharacter $h$, Manning builds in [12] a cobounded quasi-structure on a bushy tree $T$. In Section 4 we show that this construction can be made by adding a condition to the relation between the space $E(h)$ and the boundary of the tree $\partial T$.

In the last section we state some implications on the space of pseudocharacters and therefore, in the cobounded cohomology of the group.
Corollary 5.4  If there is a bushy pseudocharacter $h: G \to \mathbb{R}$ then the dimension of the space generated by the bushy pseudocharacters on $G$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.

All groups are assumed to be finitely presented.

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2 Actions and pseudocharacters

Definition 2.1  A map between metric spaces, $f: (X, d_X) \to (Y, d_Y)$, is said to be quasi-isometric if there are constants $\lambda \geq 1$ and $C > 0$ such that for all $x, x' \in X$, $(1/\lambda)d_X(x, x') - A \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + A$. If there is a constant $B > 0$ such that $N_B(f(X)) = Y$ where $N_B(f(X)) = \{y \in Y \mid d_Y(y, f(x)) < B\}$, then $f$ is a quasi-isometry and $X, Y$ are quasi-isometric.

Theorem 2.2  [12, Theorem 4.6]  Let $Y$ be a geodesic metric space. The following are equivalent:

1. $Y$ is quasi-isometric to some simplicial tree $\Gamma$.

2. (Bottleneck Property)  There is some $\Delta > 0$ so that for all $x, y \in Y$ there is a midpoint $m = m(x, y)$ with $d(x, m) = d(y, m) = \frac{1}{2}d(x, y)$ and the property that any path from $x$ to $y$ must pass within less than $\Delta$ of the point $m$.

Let $(X, d)$ be a metric space. Fix a base point $o \in X$ and for $x, x' \in X$ put $(x|x')_o = \frac{1}{2}(d(x, o) + d(x', o) - d(x, x'))$. The number $(x|x')_o$ is nonnegative and it is called the Gromov product of $x, x'$ with respect to $o$. See Gromov [9].

Definition 2.3  A metric space $X$ is (Gromov) hyperbolic if it satisfies the $\delta$-inequality

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta$$

for some $\delta \geq 0$, for every base point $o \in X$ and all $x, y, z \in X$. 

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Let \( X \) be a hyperbolic space and \( o \in X \) a base point. A sequence of points \( \{x_i\} \subset X \) converges to infinity if
\[
\lim_{i,j \to \infty} (x_i|x_j)_o = \infty.
\]
This property is independent of the choice of \( o \) since
\[
|x|x' - (x|x')_o' | \leq |oo'|
\]
for any \( x, x', o, o' \in X \).

Two sequences \( \{x_i\}, \{x'_i\} \) that converge to infinity are equivalent if
\[
\lim_{i \to \infty} (x_i|x'_i)_o = \infty.
\]
Using the \( \delta \)-inequality, we easily see that this defines an equivalence relation for sequences in \( X \) converging to infinity. The boundary at infinity \( \partial_\infty X \) of \( X \) is defined to be the set of equivalence classes of sequences converging to infinity.

The following lemma is a well known property of quasigeodesics (see Bowditch [3] or Fujiwara [6]). The statement with the proof can be found in Manning [13].

**Lemma 2.4** Let \( X \) be a \( \delta \)-hyperbolic space. Given \( K \geq 1 \) and \( L \geq 0 \), there exists \( B(K, L, \delta) \geq 0 \) such that if \( \gamma_1, \gamma_2 \) are two \((K, L)\)-quasigeodesics with the same endpoints in \( X \cup \partial X \), then \( \gamma_1 \subset N_B(\gamma_2) \) and \( \gamma_2 \subset N_B(\gamma_1) \).

**Definition 2.5** Fix \( x_0 \in X \), where \( X \) is a \( \delta \)-hyperbolic metric space on which \( G \) quasiacts. Let \( O_{g,x} : \mathbb{R} \to X \) be defined by \( O_{g,x}(t) = g^\lfloor t \rfloor x \) where \( \lfloor t \rfloor \) is the largest integer smaller than \( t \). Then it is said that \( g \) quasiacts elliptically if \( O_{g,x} \) has bounded image, and \( g \) quasiacts hyperbolically if \( O_{g,x} \) is a quasigeodesic. If \( G \) acts isometrically on \( X \) then it is said that \( g \) acts elliptically or hyperbolically or that \( g \) is elliptic or hyperbolic.

It is readily seen that this definition is independent of \( x \) and agrees with the standard definitions in case \( G \) acts isometrically.

If \( g \in G \) is hyperbolic \( x \in X \), and \( \gamma_0 : [0, 1] \to X \) is a geodesic segment with \( \gamma_0(0) = x \) and \( \gamma_0(1) = g(x) \), then it is not hard to check that \( \Gamma_{g,x,\gamma_0} : \mathbb{R} \to X \) given by
\[
\Gamma_{g,x,\gamma_0}(t) = g^\lfloor t \rfloor \gamma_0(t - \lfloor t \rfloor)
\]
is a continuous quasigeodesic. Moreover, \( g \) is an isometry of \( X \) which maps this quasigeodesic to itself by a nontrivial translation. See Figure 1.

A quasigeodesic where \( g \) acts by nontrivial translation will be referred to as a quasiaxis (or \((K-L)\)-quasiaxis if the constants are relevant). A quasiaxis of \( g \) is given the \( g \)-orientation by the requirement that \( g \) acts as a positive translation.
we are working in a coarse setting but it has obvious technical advantages. Therefore, a quasitree
Proof
Let us see that
Proposition 2.7
Let a pseudocharacter $h$ be such that $h(g_1) > 0$ and $h(g_2) > 0$ and $h$ is bornologous on the action then $g_1^\infty g_2^\infty$ in $E(h)$.
Proof Since $h(g_1) > 0$ and $h(g_2) > 0$, $\sigma_h(g_1) > 0$ and $\sigma_h(g_2) > 0$. Let $w_j$ be the word representing $g_j$ in the letters of the generating set $S \cup S^{-1}$ Then $w_j w_j \cdots = w_j^\infty$ is an element of $E(h)$ fixed by $g_j$ for $j = 1, 2$. Note that, with the assumptions taken, $\sigma(w_j^\infty) = +1$.

Let us see that $w_1^\infty \not\sim w_2^\infty$ in $E(h)$.

Let us denote, for simplicity, $w = w_1^\infty$ and $v = w_2^\infty$ and suppose $w \sim_C v$ for some $C > 0$. Then, given $D_1 > C$ there is a connecting word $d = d_1 \cdots d_n$ such that:
- $w_p d = v_p$ in $G$ for some prefix $w_p$ of $w$ and some prefix $v_p$ of $v$,
- $|h(w_p d_p) - D_1| \leq C$ for all prefixes $d_p$ of $d$.

By abuse of notation let us identify the group element $g$ with the word representing it, $w$. Therefore, we write $w(x_0)$ for the image of the isometric action $g$ on $x_0$. Let $\gamma$ be a geodesic path from $w_p(x_0)$ to $v_p(x_0)$ and let $m$ be the midpoint in $\gamma$.

Let $\gamma_1$ be a geodesic path from $x_0$ to $w_1(x_0)$ and $\gamma_2$ be a geodesic path from $x_0$ to $w_2(x_0)$ and consider $\Gamma_1 := \Gamma_{g_1,x_0,\gamma_1}(t)$, $\Gamma_2 := \Gamma_{g_2,x_0,\gamma_2}(t)$ two continuous quasi-geodesics defined as in (1). Let $\Gamma_1(w_p, w_q)$ be the restriction of $\Gamma_{g_1,x_0,\gamma_1}(t)$ to a
(quasi-isometric) path from \(w_p(x_0)\) to \(w_q(x_0)\) for any prefixes \(w_p, w_q\) of \(w\). Also, let \(\Gamma_2(v_p, v_q)\) be the restriction of \(\Gamma_{g_2, x_0, y_2}(t)\) to a (quasi-isometric) path from \(v_p(x_0)\) to \(v_q(x_0)\) for any prefixes \(v_p, v_q\) of \(v\).

Let \(\Delta\) be the bottleneck property constant.

**Claim** Since \(d_H(\Gamma_{1}(g_1, x_0, \gamma_1), \Gamma_{2}(g_2, x_0, \gamma_2)) = \infty\), we may assume \(D_1\) big enough to assure that for any \(w_p \subset w_q\) and \(v_p \subset v_q\), then \(\Gamma_1(w_p, w_q) \cap B(m, \Delta) = \emptyset\) and \(\Gamma_2(v_p, v_q) \cap B(m, \Delta) = \emptyset\). See Figure 2.

Since \(d_H(\Gamma_{1}(g_1, x_0, \gamma_1), \Gamma_{2}(g_2, x_0, \gamma_2)) = \infty\), we assume \(D_1 = D_1(h, \Delta, g_1, g_2)\) is big enough to guarantee that \(A := d(w_p, v_p)\) is as big as we want. Notice that 
\[
d(w^1_p(x_0), w^{i+1}_p(x_0)) = d(x_0, w_1(x_0)) =: d_1 \quad \text{so} \quad d(w^i_p(x_0), w^{i+k}_p(x_0)) \leq k \cdot d_1.
\]

respectively, for \(w_2\), we have 
\[
d(w^1_p(x_0), w^{i+1}_p(x_0)) = d(x_0, w_2(x_0)) =: d_2 \quad \text{therefore}
\]
\[
d(w^i_p(x_0), w^{i+k}_p(x_0)) \leq k \cdot d_1.
\]

So, if \(A\) is big enough, either \(k\) is also big enough (depending on the distance \((A/2-\Delta)/\max\{d_1, d_2\}\)) or we can assure the quasi-isodesics \(\Gamma_1(w_p, w_p, w_k^1)\) from \(w_p(x_0)\) to \(w_p, w_k^1(x_0)\) satisfies \(\Gamma_1(w_p, w_p, w_k^1) \cap B(m, \Delta) = \emptyset\).

Also, either \(k\) is big enough or the quasi-isodesics \(\Gamma_2(v_p, v_p, w_k^2)\) from \(v_p(x_0)\) to \(v_p, w_k^2(x_0)\) satisfies \(\Gamma_2(v_p, v_p, w_k^2) \cap B(m, \Delta) = \emptyset\).

Now, let us assume \(k = k(h, \Delta, g_1, g_2)\) as big as we want and fix it assuming that the corresponding quasiodesics \(\Gamma_1(w_p, w_p, w_k^1)\) and \(\Gamma_2(v_p, v_p, w_k^2)\) do not intersect the ball \(B(m, \Delta)\). By hypothesis we have that \(h(g_j) > 0\) and \(h(g^n_j) = nh(g_j)\) for \(j = 1, 2\). Then, for any \(j \geq k\), we have \(h(w_p, w^j_p(x_0)) = h(w_p) + j h(g_1)\) and \(h(v_p, w^j_p(x_0)) = h(v_p) + j h(g_2)\) are much bigger than \(D_1\). Then, since \(h\) is bornological on the action, we can assume \(k\) big enough so that \(w_p, w^j_p(x_0) \notin B(m, \Delta + \max\{d_1, d_2\})\) and \(v_p, w^j_p(x_0) \notin B(m, \Delta + \max\{d_1, d_2\})\) for any \(j \geq k\). So, the quasiodesics segments \(\Gamma_1(w_p, w^j_p, w_p, w^j_{p+1}) \cap B(m, \Delta) = \emptyset\) and \(\Gamma_2(v_p, w^j_p, v_p, w^j_{p+1}) \cap B(m, \Delta) = \emptyset\) for every \(j \geq 0\), proving the claim.

\[\square\]

Now, let \(D_2 >> h(m), D_1\). Then, we will reach a contradiction finding a uniform bound for \(D_2 - D_1\).

By assumption, there is a connecting word \(d' = d'_1 \cdots d'_n\) such that:

- \(w_p, d' = w_p\) in \(G\) for some prefix \(w_p\) of \(w\) and some prefix \(v_p\) of \(v\),
- \(|h(w_p, d'_p) - D_2| \leq C\) for all prefixes \(d'_p\) of \(d\).

Let \(\Gamma_1 := \Gamma_1(w_p, w_p')\), \(\Gamma_2 := \Gamma_2(v_p, v_p')\) be quasiodesics paths defined as above from \(w_p(x_0)\) to \(w_p'(x_0)\) and from \(v_p(x_0)\) to \(v_p'(x_0)\). Let \(\gamma'_j : [0, 1] \to X\) be a geodesic path from \(w_p', d'_p \cdots d'_{j-1}(x_0)\) to \(w_p', d'_p \cdots d'_j(x_0)\) for \(1 \leq j \leq n\) and let \(\gamma' : [0, n] \to X\) be the path from \(w_p'(x_0)\) to \(v_p'(x_0)\) defined by \(d'\) where \(\gamma'(t) = \gamma'_j (t - [t])\).
Figure 2: Assuming $D_1$ big enough, we obtain that $\Gamma_1(w_p, w_p \cdot w_1^k)$ and $\Gamma_2(v_p, v_p \cdot w_2^k)$ do not intersect the ball $B(m, \Delta)$.

Then, $\Gamma = \Gamma_1 \cup \Gamma' \cup \Gamma_2^{-1}$ is a path from $w_p(x_0)$ to $v_p(x_0)$. By bottleneck property (see Theorem 2.2) there is a point $x \in \Gamma$ such that $d(x, m) \leq \Delta$. By the previous claim, we can assume that $x \in \gamma_j'[0, 1]$ for some $1 \leq j \leq n$. See Figure 3.

Figure 3: The path $[w_p(x_0), w_p'(x_0)] \cup [w_p'(x_0), v_p'(x_0)] \cup [v_p'(x_0), v_p(x_0)]$ intersects the ball $B(m, \Delta)$ by bottleneck property.

By hypothesis, $h(w_p' d_p', h(w_p' d_p' d_j') \in (D_2 - C, D_2 + C)$. Let $F = F(S, x_0)$ be a constant so for all $s \in S$, $d(x_0, s(x_0)) \leq F$. Then, $d(w_p' d_p'(x_0), w_p' d_p' d_j'(x_0)) \leq F$. Therefore, $d(w_p' d_p'(x_0), m) \leq \Delta + F$ and $d(w_p d_p' d_j'(x_0), m) \leq \Delta + F$ which implies $d(w_p' d_p'(x_0), w_p d_p(x_0)) \leq 2\Delta + 2F$ and that $d(w_p' d_p' d_j'(x_0), w_p d_p(x_0)) \leq 2\Delta + 2F$ for some prefix $d_p$ of $d$. But since $h$ is bornologous on the action, there is some $S = S(2 \Delta + 2F)$ such that $|h(w_p d_p) - h(w_p' d_p')| < S$ and $D_2 - D_1 < S + 2C$. 

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Thus, $D_2 - D_1$ is bounded by a constant depending only on $h, C, F, \gamma_1, \gamma_2$ and $\Delta$, leading to contradiction. \qed

**Definition 2.8** An action of a group by isometries on a metric space is **metrically proper** if for all $x \in X$ and for all $R > 0$ the set $\{g \in G \mid g(N(x), R)) \cap N(x, R) \neq \emptyset\}$ is finite.

**Lemma 2.9** An action of a group $G$ by isometries on a metric space $X$ is metrically proper if and only if for all $x_0 \in X$ and for all $g \in G$ then for all $R > 0$ the set $\{g' \in G \mid g'x_0 \in N(g(x_0), R)\}$ is finite.

**Proof** The set $\{h \in G \mid h(B(g(x_0), R)) \cap B(g(x_0), R) \neq \emptyset\}$ is finite if the action is metrically proper. If $g' = hg$, then $\{g' \in G \mid g'g^{-1}B(g(x_0), R) \cap B(g(x_0), R) \neq \emptyset\} = \{g' \in G \mid g'B(x_0, R) \cap B(g(x_0), R) \neq \emptyset\}$ is finite. In particular, the set $\{g' \in G \mid g'x_0 \in B(g(x_0), R)\}$ is finite. Conversely, suppose $\{g' \in G \mid g'x_0 \in N(g(x_0), 2R)\}$ is finite. Then, if $h = g'g^{-1}$, the set $\{h \in G \mid (hg)(x_0) \cap N(g(x_0), 2R) \neq \emptyset\} = \{h \in G \mid h(g(x_0)) \cap N(g(x_0), 2R) \neq \emptyset\}$ is finite which implies $\{h \in G \mid h(N(g(x_0), R)) \cap N(g(x_0), R) \neq \emptyset\}$ is finite. \qed

**Proposition 2.10** Let $G$ be a group acting by isometries on a metric space $X$ and let $h: G \to \mathbb{R}$ be a pseudocharacter. If the action is metrically proper then the pseudocharacter is bornologous on the action.

**Proof** Let $x_0 \in X$, $g \in G$ and $R > 0$. Since the action is metrically proper the set $K = \{g' \in G \mid g'x_0 \in N(g(x_0), R)\}$ is finite. Therefore, it suffices to take $S := \max_{g' \in K} |h(g') - h(g)|$. \qed

**Corollary 2.11** Let $G$ be a group acting by isometries on a quasitree $X$ so that the action is metrically proper. Let $g_1, g_2$ be two hyperbolic elements of $G$ such that $d_H(\Gamma_1(g_1, x_0, \gamma_1), \Gamma_2(g_2, x_0, \gamma_2)) = \infty$. Then, if $h$ is a pseudocharacter such that $h(g_1) > 0$ and $h(g_2) > 0$ then $g_1^\infty \not\sim g_2^\infty$ in $E(h)$.

**Definition 2.12** Two hyperbolic isometries $g_1, g_2$ are said to be **independent** if their quasiaxis do not contain rays which are a finite Hausdorff distance apart. Equivalently the fixed point sets of $g_1, g_2$ in $\partial X$ are disjoint. An action is **nonelementary** if there are at least two independent hyperbolic elements.

**Definition 2.13** We say that a pseudocharacter $h: G \to \mathbb{R}$ is nonelementary if there is a pair of independent $g_1, g_2 \in G$ such that $h(g_1) \neq 0$ and $h(g_2) \neq 0$. 

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Corollary 2.14 Consider a nonelementary action of a group $G$ on a quasitree $X$ and a nonelementary pseudocharacter $h: G \to \mathbb{R}$. Then, if $h$ is bornologous on the action, it is bushy.

Corollary 2.15 Consider a nonelementary action of a group $G$ on a quasitree $X$. If the action is metrically proper then every nonelementary pseudocharacter is bushy.

3 Existence of bushy pseudocharacters

Note any two $(K - L)$–quasiaxis of $g$ are within some universal $B = B(\delta, K, L)$ of one another and any sufficiently long $(K, L)$–quasigeodesic arc $J$ in a $B$–neighborhood of a quasiaxis $l$ of $g$ inherits a natural $g$–orientation: a point of $l$ within $B(\delta, K, L)$ of the terminal endpoint of $J$ is ahead (with respect to the $g$–orientation of $l$) of a point of $l$ within $B(\delta, K, L)$ of the initial endpoint of $J$. This orientation of $J$ will be denoted by $g$–orientation of $J$.

Definition 3.1 (Bestvina and Fujiwara [2]) If $g_1, g_2$ are hyperbolic elements of $G$, let $g_1 \sim g_2$ if for an arbitrarily long segment $J$ in a $(K, L)$–quasiaxis for $g_1$ there is a $g \in G$ such that $g(J)$ is within $B(\delta, K, L)$ of a $(K, L)$–quasiaxis of $g_2$ and $g: J \to g(J)$ is orientation-preserving with respect to the $g_2$–orientation on $g(J)$.

This defines an equivalence relation. As it is said in [2], the concept does not change if $B$ is replaced by a larger constant.

Definition 3.2 A Bestvina–Fujiwara action is a nonelementary action of a group $G$ on a hyperbolic graph $X$ so that there exist independent $g_1, g_2$ such that $g_1 \not\sim g_2$.

Lemma 3.3 Let $X$ be a (geodesic) Gromov hyperbolic space and let $G$ be a group acting by isometries on $X$. Let $g_1, g_2 \in G$ such that $g_1 \not\sim g_2$. Then, for any point $x_0 \in X$, and any pair of geodesics $\gamma_1 := [x_0, g_1(x_0)], \gamma_2 := [x_0, g_2(x_0)]$, $d_H(\Gamma_1(g_1, x_0, \gamma_1), \Gamma_2(g_2, x_0, \gamma_2)) = \infty$.

Proof Suppose there is some $B > 0$ such that $\Gamma_1 \subset N_B(\Gamma_2)$ and $\Gamma_2 \subset N_B(\Gamma_1)$. Then, the action of the identity $i$ on $\Gamma_1$ is contained in $N_B(\Gamma_2)$ and $i: \Gamma_1 \to i(\Gamma_1) = \Gamma_1$ is orientation-preserving with respect to the $g_1$–orientation and the $g_2$–orientation on $\Gamma_1$. This contradicts the fact that $g_1 \not\sim g_2$. \hfill \Box
Let us recall now the basic construction of quasihomomorphisms associated to the action as presented in [6].

Let $X$ be a hyperbolic graph and a group $G$ acting on $X$. Let $w$ be a finite (oriented) path in $X$. By $|w|$ denote the length of $w$, by $i(w)$ the starting point and by $t(w)$ the finishing point. For $g \in G$, $g \circ w$ is a path starting at $g(i(w))$ and finishing at $g(t(w))$ and it is called a copy of $w$. Obviously, $|g \circ w| = |w|$.  

Let $\alpha$ be a finite path. Define

$$|\alpha|_w = \{\text{the maximal number of nonoverlapping copies of } w \text{ in } \alpha\}.$$ 

Suppose $x, y \in X$ are two vertices and that $W$ is an integer with $0 < W < |w|$. Then

$$c_{w,W}(x, y) = d(x, y) - \inf_{\alpha}(|\alpha| - W|\alpha|_w),$$

where $\alpha$ ranges over all paths from $x$ to $y$.

**Lemma 3.4** [6, Lemma 3.4] Let $x, y, z$ be three points in $X$. Then

$$|c_{w,W}(x, y) - c_{w,W}(x, z)| \leq 2d(y, z).$$

Let us omit $W$ from the notation and write $c_w$. Define $h_w: G \to \mathbb{R}$ by

$$h_w(g) = c_w(x_0, g(x_0)) - c_{w-1}(x_0, g(x_0)).$$

**Proposition 3.5** [6, Proposition 3.10] If $X$ is a $\delta$–hyperbolic space $h_w$ is a quasihomomorphism (ie quasicharacter).

From Lemma 3.4, it is immediate to obtain the following.

**Lemma 3.6** Let $X$ be a (geodesic) Gromov hyperbolic space. For any word $w$, any points $x_0, x \in X$ and any constants $0 < W < |w|$, $R > 0$, the subset of the real line $\{h_w(g) \mid g(x_0) \in B(x, R)\}$ is bounded. In particular, it has diameter at most $8R$.

Therefore, the following is immediate.

**Proposition 3.7** Let $X$ be a (geodesic) Gromov hyperbolic space. For any word $w$, any points $x_0, x \in X$ and any constants $0 < W < |w|$, $h_w$ is bornologous on the action.

Let us recall two propositions from [2].
Proposition 3.8 [2, Proposition 2] Suppose a group $G$ acts on a $\delta$-hyperbolic graph $X$ by isometries. Suppose also that the action is nonelementary and that there exist independent hyperbolic elements $g_1, g_2 \in G$ such that $g_1 \not\sim g_2$. Then, there is a sequence $f_1, f_2, \ldots \in G$ of hyperbolic elements such that:

- $f_i \not\sim f_i^{-1}$ for $i = 1, 2, \ldots$,
- $f_i \not\sim f_j^\pm 1$ for $j < i$.

Replacing if necessary $g_1, g_2$ by high positive powers of conjugates, let $F$ be a free subgroup of $G$ with basis $\{g_1, g_2\}$ such that each nontrivial element of $F$ is hyperbolic and $F$ is quasiconvex with respect to the action on $X$. See the proof of [2, Proposition 2] and [9, Section 5.3] for details. Such free groups are called Schottky groups.

Proposition 3.9 [2, Proposition 5] Suppose $1 \neq f \in F$ is cyclically reduced and $f \not\sim f^{-1}$. Then there is $a > 0$ such that $h_{fa}^a$ is unbounded on $\langle f \rangle$. Moreover, if $f^\pm 1 \not\sim f' \in F$ then $h_{fa}^a$ is 0 on $\langle f' \rangle$ for sufficiently large $a > 0$.

Theorem 3.10 Let $G$ be a group acting on a quasitree. If it is a Bestvina–Fujiwara action, then there is a bushy pseudocharacter $h: G \to \mathbb{R}$.

Proof Consider the sequence $f_1, f_1, \ldots$ from Proposition 3.8 and assume in addition (without loss of generality) that each $f_i$ is cyclically reduced. Define $h_i: G \to \mathbb{R}$ as $h_i = h_{fa_1}$ where $a_1$ is chosen as in Proposition 3.9 so that $h_i$ is unbounded on $\langle f_i \rangle$ and so that it is 0 on $\langle f_j \rangle$ for $j < i$ and also 0 on $\langle f_{i+1} \rangle$. With the same argument, we may also assume that $\lim_{k \to \infty} h_i(f_i^k) = +\infty$.

Let $h$ be a pseudocharacter at a bounded distance (see Remark) from the quasicharacter $h_1 \neq h_2$. (Notice that everything works if we consider $h_i$ and $h_{i+1}$ instead). Clearly, $h(f_1) > 0$ and $h(f_2) > 0$.

Thus $\sigma_h(f_1) > 0$ and $f_j^\pm \infty$ defines an element in $E(h)$ for $j = 1, 2$. Let $w_j$ be the word representing $f_j$ in the letters $S \cup S^{-1}$. Then $w_j w_j \cdots w_j^\infty$ is an element of $E(f)$ fixed by $f_j$ for $j = 1, 2$. Note that, with the assumptions taken, $\sigma(w_j^\infty) = +1$.

Let us see that $w_1^\infty \not\sim w_2^\infty$ in $E(h)$. It suffices to check that by Proposition 3.7 and Lemma 3.3 we are in the conditions of Proposition 2.7.

The same argument proves that $w_1^\infty \not\sim w_2^\infty$ in $E(h)$. □

Corollary 3.11 If a Cayley graph $X = \Gamma(G, S)$ satisfies the bottleneck property and the canonical action of the group is a Bestvina–Fujiwara action, then there is a bushy pseudocharacter $h: G \to \mathbb{R}$. 
Bestvina, Bromberg and Fujiwara give in [1] a general construction of actions of groups on quasitrees. They proceed axiomatically defining a distance function and then modifying it to define what they call projection complex.

Let \( Y \) be a set and assume that for each \( Y \in \mathbb{Y} \) there is a function
\[
d_\mathbb{Y}^Z : (\mathbb{Y}\setminus\{Y\}) \times (\mathbb{Y}\setminus\{Y\}) \to [0, \infty)
\]
and a constant \( \xi > 0 \) that satisfies the following axioms:

\begin{align*}
(A1) & \quad d_\mathbb{Y}^Z(X, Z) = d_\mathbb{Y}^Z(Z, X) \\
(A2) & \quad d_\mathbb{Y}^Z(X, Z) + d_\mathbb{Y}^Z(Z, W) \geq d_\mathbb{Y}^Z(X, W) \\
(A3) & \quad \min\{d_\mathbb{Y}^Z(X, Z), d_\mathbb{Y}^Z(X, Y)\} < \xi \\
(A4) & \quad \#\{Y \mid d_\mathbb{Y}^Z(X, Z) \geq \xi\} \text{ is finite for all } X, Z \in \mathbb{Y}
\end{align*}

Given this distance function they define \( \mathcal{H}(X, Z) \) to be the set of pairs \( (X', Z') \in \mathbb{Y} \times \mathbb{Y} \) such that one of the following holds:

\begin{itemize}
  \item \( d_\mathbb{Y}^X(X', Z'), d_\mathbb{Y}^Z(X', Z') > 2\xi \)
  \item \( X = X' \) and \( d_\mathbb{Y}^Z(X, Z') > 2\xi \)
  \item \( Z = Z' \) and \( d_\mathbb{Y}^X(X', Z) > 2\xi \)
  \item \( (X', Z') = (X, Z) \)
\end{itemize}

Then, they define the function
\[
d_\mathbb{Y}(X, Z) = \min_{(X', Z') \in \mathcal{H}(X, Z)} d_\mathbb{Y}^Z(X', Z')
\]
and the set \( \mathbb{Y}_K(X, Z) \) to be the set of \( Y \in \mathbb{Y} \) such that \( d_\mathbb{Y}(X, Z) > K \).

For some (big enough) constant \( K > 0 \), they define the projection complex \( \mathcal{P}_K(\mathbb{Y}) \) as a 1–complex whose vertex set is \( \mathbb{Y} \) and such that there is an edge connecting two vertices \( X \) and \( Z \) if \( \mathbb{Y}_K(X, Z) \) is empty.

**Theorem 3.12** [1, Theorem 2.9] For \( K \) sufficiently large \( \mathcal{P}_K(\mathbb{Y}) \) is a quasitree.

Also, suppose \( G \) is a group acting on the set \( \mathbb{Y} \), that there exists a function \( d_\mathbb{Y}^Z \) satisfying \((A1)–(A4)\) and projection distances are preserved, ie \( d_{g(A)}^Z(g(B), g(C)) = d_A^Z(B, C) \) for all \( A, B, C \in \mathbb{Y} \) and \( g \in G \). Then, \( G \) acts naturally on \( \mathcal{P}_K(\mathbb{Y}) \). See Theorem 3.15.

Using this construction, several examples of groups which act on quasitrees are given in [1]. To obtain examples for Theorem 3.10 it suffices to check that there exist two independent hyperbolic elements \( g_1, g_2 \) such that \( g_1 \not\sim g_2 \) (on \( \mathcal{P}_K(\mathbb{Y}) \)).

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In [1, Lemma 2.11], a technical lemma, there is certain constant $K'$ involved which roughly depends on $K$. Fixing that constant and assuming $g \in G$ and $Y \in \mathcal{Y}$ such that
\[ d_Y(g^{-N}(Y), g^N(Y)) > K' \]
for some $N > 0$ (see [1, Lemma 2.13]), they define the combinatorial axis as
\[ \mathcal{Y}_{K'}(g) = \{ Y \in \mathcal{Y} \mid d_Y(g^{-N}(Y), g^N(Y)) > K' \text{ for some } N > 0 \}. \]

Also, fixing $K'$, they consider the following axioms on $g \in G$:

(B1) The element $g$ is contained in a unique maximal virtually cyclic subgroup, $\text{EC}(g)$, the elementary closure of $g$.

(B2) $\text{EC}(g)$ is malnormal, i.e., $\psi \text{EC}(g)\psi^{-1} = \text{EC}(g)$ for $\psi \in G$ implies $\psi \in \text{EC}(g)$.

(B3) There is $Y \in \mathcal{Y}_{K'}(g)$ and $m > 0$ such that if $\psi \in G$ fixes $Y, g(Y), \ldots, g^m(Y)$, then $\psi \in \text{EC}(g)$.

The following definition also comes from [1]. Assume $G$ is acting on a geodesic metric space $X$. Then, $g \in G$ is a WPD element if $\langle g \rangle$ has a bounded orbit in $X$ and for every $x \in X$ and $D > 0$ there is $M > 0$ such that the set
\[ \{ \phi \in G \mid d(g^i(x), \phi(g^i(x))) \leq D, \ i = \pm M \} \]
is finite.

**Proposition 3.13** [1, Proposition 2.16] Suppose $g \in G$ satisfies (B1)–B3. Then $g$ is a WPD element with respect to the action of $G$ on $\mathcal{P}_K(Y)$.

In [2] the authors define what they call weak proper discontinuity or WPD. A group action satisfies WPD if and only if

(i) there exists a hyperbolic element,

(ii) $G$ is not virtually cyclic,

(iii) every hyperbolic element in $G$ is a WPD element.

Then, they prove that if the action of a group $G$ on a $\delta$–hyperbolic graph $X$ satisfies WPD, then the action is nonelementary and there exist independent hyperbolic $g_1, g_2$ such that $g_1 \not\sim g_2$. See [2, Proposition 6].

However, in the proof of [2, Proposition 6(5)] the key is to find two independent WPD elements, more than having the condition on every hyperbolic element of the group. Therefore, this can be restated as follows.
**Proposition 3.14** Let $G$ be a group acting on a $\delta$–hyperbolic graph $X$ such that $G$ is not virtually cyclic and there exist two independent hyperbolic elements $f_1, f_2 \in G$ which are WPD elements. Then there exist independent hyperbolic $g_1, g_2$ such that $g_1 \not\sim g_2$.

**Theorem 3.15** [1, Theorem 2.17] Suppose a group $G$ acts on a set $Y$ satisfying axioms (A1)–(A4) such that projection distances are preserved. Therefore, $G$ acts on the quasitree $\mathcal{P}_K(Y)$. Further, assume that there exist independent elements $g_1, g_2 \in G$ that satisfy (B1)–(B3). Then, there is a nonabelian free subgroup $F \subset G$ all of whose nontrivial elements act on $\mathcal{P}_K(Y)$ hyperbolically as WPD elements.

Suppose then that there is a group $G$ and two independent elements $g_1, g_2 \in G$ satisfying the conditions of the theorem. By Proposition 3.14 it follows that there are two independent hyperbolic elements $f_1, f_2$ such that $f_1 \not\sim f_2$, this is, the action of $G$ on $Y$ is a Bestvina–Fujiwara action on a quasitree.

Hence, by Theorem 3.10:

**Corollary 3.16** Suppose a group $G$ acts on a set $Y$ satisfying axioms (A1)–(A4) such that projection distances are preserved. Further, assume that there exist independent elements $g_1, g_2 \in G$ that satisfy (B1)–(B3). Then, there exists a bushy pseudocharacter $h: G \to \mathbb{R}$.

**Example 3.17** Consider any nonelementary word hyperbolic group $G$. For every hyperbolic element $g \in G$, (B1)–(B2) hold. See Lück and Weiermann [11, Example 3.5] (or Theorem 3.2 in III.$\Gamma$.3 on page 459 and Corollary 3.10 in III.$\Gamma$.3 on page 462 from Bridson and Haefliger [4]). Also, the action may be chosen so (B3) holds.

For any nonelementary word hyperbolic group $G$ there is an action on a set $Y$ satisfying axioms (A1)–(A4) so that projection distances are preserved (see [2, Examples 2.1]). Therefore, there exists a bushy pseudocharacter $h: G \to \mathbb{R}$.

## 4 Quasiactions on trees

Given a pseudocharacter $f: G \to \mathbb{R}$, Manning introduces the following constructions. The first one gives a tree obtained from the Cayley graph of the group.

Consider an (unambiguous) triangular generating set $S$. Then scale $f$ so that $f(G)$ misses $\mathbb{Z} + \frac{1}{2}$ and so that $f$ changes by at most $\frac{1}{4}$ over each edge. Let $\tilde{K}$ be the simply connected 2–complex obtained by attaching 2–cells according to the relations
of the presentation. Then, a tree is built with vertex set in one-to-one correspondence with the components of \( \tilde{K} \setminus f^{-1}(\mathbb{Z} + \frac{1}{2}) \). The edges correspond to components of \( f^{-1}(\mathbb{Z} + \frac{1}{2}) \), each of which is some possibly infinite track which separates \( \tilde{K} \) into two components. This construction is also the starting point in the author’s [14] where given a real valued function on a geodesic space we give a sufficient condition for the space to be quasi-isometric to a tree.

The next appears as [12, Definition 4.9].

Let \( V \) be the set of components of \( f^{-1}(\mathbb{Z} + \frac{1}{2}) \). Then \( V \) is in one-to-one correspondence with the set of edges of \( T \). Let \( X \) be the simplicial graph with vertex set equal to \( G \times V \) and the following edge condition: Two distinct vertices \((g, \tau)\) and \((g', \tau')\) are to be connected by an edge if there is some \( h \) so that \( hg(\tau) \) and \( hg'(\tau') \) are contained in the same connected component of \( f^{-1}[n - \frac{3}{2}, n + \frac{1}{2}] \) for some \( n \). The zero-skeleton \( X^0 \) is endowed with a \( G \)-action by setting \( g(g_0, \tau_0) = (gg_0, \tau_0) \). Since this action respects the edge condition on pairs of vertices, it extends to an action on \( X \). We will refer to this particular one as Manning’s action.

**Proposition 4.1** [12, Proposition 4.27] If \( f: G \to \mathbb{R} \) is a bushy pseudocharacter, then Manning’s action is a Bestvina–Fujiwara action.

**Theorem 4.2** [12, Theorem 4.15] The space \( X \) satisfies the Bottleneck Property.

Therefore, from Theorem 3.10, we can give the following corollary which would be some kind of converse to Proposition 4.1.

**Corollary 4.3** If Manning’s action is a Bestvina–Fujiwara action, then there is a bushy pseudocharacter \( h: G \to \mathbb{R} \).

**Lemma 4.4** [12, Lemma 4.17] There is an injective map from \( E(f) \) to \( \partial X \).

**Theorem 4.5** [12, Theorem 4.20] If \( f: G \to \mathbb{R} \) is a pseudocharacter which is not uniform, then \( G \) admits a cobounded quasiaction on a bushy tree.

Then, it is readily seen, from the construction of the action and the bushy tree, that Corollary 2.11 yields the following.

**Corollary 4.6** Consider a nonelementary action of a group \( G \) on a quasitree \( X \). If the action is metrically proper, then for any pseudocharacter \( h: G \to \mathbb{R} \) and any pair of independent \( g_1, g_2 \in G \) so that \( h(g_1) > 0 \) and \( h(g_2) > 0 \), there is a cobounded quasiaction of \( G \) on a bushy tree \( T \) so that there is an injective map from \( E(h) \) to \( \partial T \).
5 Space of pseudocharacters

Given a group $G$, quasicharacters and pseudocharacters are major tools in the study of the bounded cohomology group $H^*_b(G; \mathbb{R})$ as we can see in [2].

The bounded cohomology group $H^*_b(G; \mathbb{R})$ of a discrete group $G$ is defined by the cochain complex $C^k_b(G; \mathbb{R})$, where

$$C^k_b(G; \mathbb{R}) = \{ f: G^k \to \mathbb{R} \mid \sup_{G^k}|f(g_1, \ldots, g_k)| < \infty \}$$

and the boundary $\delta: C^k_b(G; A) \to C^{k+1}_b(G; A)$ is given by

$$\delta f(g_0, \ldots, g_k) = f(g_1, \ldots, g_k) + \sum_{i=1}^{k} (-1)^i f(g_0, \ldots, g_{i-1}g_i, \ldots, g_k) + (-1)^{k+1} f(g_0, \ldots, g_{k-1}).$$


**Remark** Note that 1–cocycles are just group homomorphisms $f: G \to \mathbb{R}$. In fact, Hom$(G) = H^1(G; \mathbb{R})$. A quasicharacter is an element $f \in C^1(G; \mathbb{R})$ whose coboundary $\delta f$ lies in $C^2_b(G; \mathbb{R})$. A pseudocharacter is a quasicharacter such that $f(g^k) = kf(g)$ for all $k \in \mathbb{Z}$ and $g \in G$.

Let $\mathcal{V}(G)$ be the vector space of all quasihomomorphisms $G \to \mathbb{R}$ and let BDD$(G)$ be the subspace of all bounded functions. Then, let $\text{QH}(G) = \mathcal{V}(G)/\text{BDD}(G)$.

There is an exact sequence

$$0 \to H^1(G; \mathbb{R}) \to \text{QH}(G) \to H^2_b(G; \mathbb{R}) \to H^2(G; \mathbb{R}).$$

Using the sequence $f_1, f_2, \ldots$ obtained in Proposition 3.8, Bestvina and Fujiwara prove that $[h_i] \in \text{QH}(G)$ is not a linear combination of $[h_1], \ldots, [h_{i-1}]$, ie, the sequence $[h_i]$ consists of linearly independent elements (see the proof of [2, Theorem 1]). This implies that the dimension of $H^2_b(G; \mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum. See [6, Corollary 1.3] and [2, Theorem 1].

Therefore, since the argument in Theorem 3.10 works also for any pair $f_i, f_{i+1}$, the following is immediate.

**Corollary 5.1** Let $G$ be a group acting on a (geodesic) Gromov hyperbolic graph $X$ satisfying the bottleneck property. If it is a Bestvina–Fujiwara action, then the dimension of the subspace generated by the bushy pseudocharacters as a vector space over $\mathbb{R}$ is the cardinal of the continuum.
Corollary 5.2  If a Cayley graph $X = \Gamma(G, S)$ satisfies the bottleneck property and the canonical action of the group is a Bestvina–Fujiwara action, then the dimension of the subspace generated by the bushy pseudocharacters as a vector space over $\mathbb{R}$ is the cardinal of the continuum.

Corollary 5.3  If Manning’s action is a Bestvina–Fujiwara action, then the dimension of the subspace generated by the bushy pseudocharacters on $G$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.

In particular:

Corollary 5.4  If there is a bushy pseudocharacter $h: G \to \mathbb{R}$ then the dimension of the subspace generated by the bushy pseudocharacters on $G$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum.

This, together with Corollary 2.14 and Corollary 2.15 yields:

Corollary 5.5  Consider a nonelementary action of a group $G$ on a quasitree $X$ and a nonelementary pseudocharacter $h: G \to \mathbb{R}$. Then, if $h$ is bornologous on the action, the dimension of the subspace generated by the bushy pseudocharacters on $G$ (in particular, the dimension of $H^2_b(G; \mathbb{R})$) as a vector space over $\mathbb{R}$ is the cardinal of the continuum.

Corollary 5.6  Consider a nonelementary action of a group $G$ on a quasitree $X$. If the action is metrically proper and there exist a nonelementary pseudocharacter then the dimension of the subspace generated by the bushy pseudocharacters on $G$ (in particular, the dimension of $H^2_b(G; \mathbb{R})$) as a vector space over $\mathbb{R}$ is the cardinal of the continuum.

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