A homological characterization of topological amenability

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Generalizing Block and Weinberger's characterization of amenability we introduce the notion of uniformly finite homology for a group action on a compact space and use it to give a homological characterization of topological amenability for actions. By considering the case of the natural action of G on its Stone–Čech compactification we obtain a homological characterization of exactness of the group.

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There are two well known homological characterizations of amenability for a countable discrete group G. One, given by Johnson [10], states that a group is amenable if and only if a certain cohomology class in the first bounded cohomology $H_b^1\big(G, \ell_0^1(G)^{**}\big)$ vanishes, where $\ell_0^1(G)$ is the augmentation ideal. By contrast Block and Weinberger [2] described amenability in terms of the non-vanishing of a homology class in the 0–dimensional uniformly finite homology of G, $H_0^{\mathrm{uf}}(G, \mathbb{R})$. The relationship between these characterizations is explored by the first, second and fourth authors in [4].

Amenable actions on a compact space were extensively studied by Anantharaman-Delaroche and Renault in [1] as a generalization of amenability which is sufficiently strong for applications and yet is exhibited by almost all known groups. A group is amenable if and only if the action on a point is amenable and it is exact if and only if it acts amenably on its Stone–Čech compactification, βG, (see Higson and Roe [9], Guentner and Kaminker [8] and Ozawa [12]). It is natural to consider the question of whether or not the Johnson and Block–Weinberger characterizations of amenability can be generalized to this much broader context. In particular Higson asked for such a characterization of exactness.

In [3] we showed how to generalize Johnson's result in terms of bounded cohomology with coefficients in a specific module $N_0(G,X)^{**}$ associated to the action. In this paper we turn our attention to the Block–Weinberger theorem, studying a related module $W_0(G,X)$ (the *standard module of the action*), and define the *uniformly finite homology of the action*, $H_*^{\mathrm{uf}}(G \curvearrowright X)$ as the group homology with coefficients in $W_0(G,X)^*$.

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The modules $N_0(G,X)^{**}$ and $W_0(G,X)^*$ should be thought of as analogues of the modules $(\ell^\infty(G)/\mathbb{R})^*$ and $\ell^\infty(G)$ respectively, which play a key role in the definition of the uniformly finite homology for groups. The two characterizations are intimately related, and we consider this relationship in Section 5.

In the case of Block and Weinberger's uniformly finite homology the vanishing of the 0–dimensional homology group is equivalent to vanishing of a fundamental class $\left[\sum_{g\in G} g\right] \in H_0^{\mathrm{uf}}(G,\mathbb{R})$, however the homology group $H_0^{\mathrm{uf}}(G \curvearrowright X)$ is rarely trivial even when the action is topologically non-amenable. Indeed if X is a compactification of G then the homology group is always non-zero, see Theorem 6 below. A similar phenomenon can be observed for controlled coarse homology (see Nowak and Špakula [11]), which is another generalization of uniformly finite homology: only the vanishing of the fundamental class has geometric applications. Here we show that topological amenability is detected by a fundamental class $[G \curvearrowright X] \in H_0^{\mathrm{uf}}(G \curvearrowright X)$ for the action, and we obtain a homological characterization of topological amenability generalizing the Block–Weinberger theorem, Theorem 9, which may be summarized as follows:

Theorem *Let G be a finitely generated group acting by homeomorphisms on a compact Hausdorff topological space X. The action of G on X is topologically amenable if and only if the fundamental class $[G \curvearrowright X]$ is non-zero in $H_0^{\mathrm{uf}}(G \curvearrowright X)$.*

When the space X is a point, the uniformly finite homology of the action $H_n^{\mathrm{uf}}(G \curvearrowright X)$ reduces to $H_n^{\mathrm{uf}}(G,\mathbb{R})$, the uniformly finite homology of G with real coefficients, recovering the characterization proved by Block and Weinberger [2].

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1 The uniformly finite homology of an action

Let G be a finitely generated group acting by homeomorphisms on a compact Hausdorff space X. The space $C(X,\ell^1(G))$ of continuous $\ell^1(G)$ valued functions on X is equipped with the $\sup -\ell^1$ norm

$$\|\xi\| = \sup_{x\in X} \sum_{g\in G} |\xi(x)(g)|.$$

The summation map on $\ell^1(G)$ induces a continuous map $\sigma\colon C(X,\ell^1(G)) \to C(X)$, where $C(X)$ is equipped with the ℓ^∞ norm. The space $N_0(G,X)$ is defined to be the pre-image $\sigma^{-1}(0)$ which we identify as $C\big(X, \ell^1_0(G)\big)$, while, identifying $\mathbb{R}$ with the constant functions on X we define $W_0(G,X)$ to be the subspace $N_0(G,X) + \mathbb{R} = \sigma^{-1}(\mathbb{R})$. Restricting σ to the subspace $W_0(G,X)$ we can regard it as a map $W_0(G,X) \to \mathbb{R}$, and with this convention we may regard σ as an element of the dual space $W_0(G,X)^*$.

Given an element $\xi \in C(X, \ell^1(G))$ we obtain a family of functions $\xi_g \in C(X)$ indexed by the elements of G by setting $\xi_g(x) = \xi(x)(g)$.

In this notation, the Banach space $C(X, \ell^1(G))$ is equipped with a natural action of G,

$$(g \cdot \xi)_h = g * \xi_{g^{-1}h},$$

for each $g, h \in G$, where $*$ denotes the translation action of G on $C(X)$: $g * f(x) = f(g^{-1}x)$ for $f \in C(X)$. We note that with these actions on $C(X, \ell^1(G))$ and $C(X)$, the map σ is equivariant which implies that N_0, W_0 are G–invariant subspaces.

Definition 1 (Brodzki, Niblo, Nowak and Wright [3]) We call $W_0(G,X)$, with the above action of G, *the standard module of the action of G on X*.

We have the following short exact sequence of G–modules:

$$0 \longrightarrow N_0(G,X) \xrightarrow{i} W_0(G,X) \xrightarrow{\sigma} \mathbb{R} \longrightarrow 0.$$

It is also worth pointing out that when X is a point we have $W_0(G,X) = \ell^1(G)$ and $N_0(G,X) = \ell^1_0(G)$. The above modules and decompositions were introduced, with a slightly different but equivalent description, in [3] for a compact X and in Douglas and Nowak [6] in the case when $X = \beta G$, the Stone–Čech compactification of G.

Recall that if V is a G–module then V^* is a also a G–module with the action of G given by $(g\psi)(\xi) = \psi(g^{-1}\xi)$ for $\psi \in V^*$ and $\xi \in V$. We now introduce the notion of uniformly finite homology for a group action.

Definition 2 Let G be a finitely generated group acting by homeomorphisms on a compact space X. We define the *uniformly finite homology of the action* to be

$$H_n^{\mathrm{uf}}(G \curvearrowright X) = H_n(G, W_0(G,X)^*),$$

for every $n \geq 0$, where H_n denotes group homology.

For the sake of computation we choose a finite generating set $S = S^{-1}$ for G, and let Γ denote the corresponding Cayley graph of G. Now we take a projective resolution starting

$$\cdots \to C_1(\Gamma, \mathbb{Z}) \to C_0(\Gamma, \mathbb{Z}) \to \mathbb{Z}$$

where C_i denotes the cellular chain complex, and tensor over $\mathbb{Z}[G]$ with the module $W_0(G,X)^*$. The first two terms of the chain complex can be explicitly identified as

$$\bigoplus_{s\in S} W_0(G,X)^* \to W_0(G,X)^* \tag{1}$$

where the (finite) direct sum is equipped with an ℓ^1–norm and the boundary map is given by

$$\psi \mapsto \sum_{s\in S} \psi_s - s^{-1}\cdot\psi_s.$$

As the coefficient module is a dual module, we observe that the boundary map in Equation (1) is the dual δ^* of the coboundary map δ for group cohomology with coefficients in $W_0(G,X)$ using the above resolution:

$$W_0(G,X) \xrightarrow{\delta} \bigoplus_{s\in S} W_0(G,X), \qquad\qquad \xi \longmapsto \bigoplus_{s\in S} \xi - s\cdot\xi$$

where the direct sum is now equipped with a supremum norm.

A certain homology class in the uniformly finite homology of the action will be of particular importance to us.

Definition 3 Let G act by homeomorphisms on a compact space X. The *fundamental class of the action*, denoted $[G \curvearrowright X]$, is the homology class in $H_0^{\mathrm{uf}}(G \curvearrowright X)$ represented by the summation map σ viewed as an element of $W_0(G,X)^*$.

As noted above, when X is a point we have $W_0(G,X) = \ell^1(G)$, so $W_0(G,X)^* = \ell^\infty(G)$, $[G \curvearrowright X] = \big[\sum_{g\in G} g\big]$, and

$$H_0^{\mathrm{uf}}(G,\mathbb{R}) \simeq H_0(G,\ell^\infty(G)) \simeq H_0^{\mathrm{uf}}(G, W_0(G,\mathrm{pt})^*) = H_0^{\mathrm{uf}}(G \curvearrowright \mathrm{pt}).$$

Consider the dual of the short exact sequence of coefficients above:

$$0 \longrightarrow \mathbb{R}^* \xrightarrow{\sigma^*} W_0(G,X)^* \longrightarrow N_0(G,X)^* \longrightarrow 0.$$

The map σ is always split as a vector space map, and hence its dual σ^* is also split. We now consider the question of when we can split the map σ^* equivariantly. Identifying $\mathbb{R}^*$ with $\mathbb{R}$, the map σ^* takes 1 to σ, hence the condition that $\mu\colon W_0(G,X)^* \to \mathbb{R}$ splits σ^* is the condition $\mu(\sigma) = 1$. Hence a G–equivariant splitting of σ^* can be regarded as a G–invariant functional $\mu \in W_0(G,X)^{**}$ such that $\mu(\sigma) = 1$. But this is precisely an invariant mean for the action as described in [3, Definition 13], so we obtain the following:

Lemma 4 *Let G be a group acting by homeomorphisms on a compact Hausdorff space X. Then the action is topologically amenable if and only if there is a G–equivariant splitting of the map σ^* in the short exact sequence*

$$0 \longrightarrow \mathbb{R}^* \xrightarrow{\sigma^*} W_0(G,X)^* \longrightarrow N_0(G,X)^* \longrightarrow 0.$$

Applying this lemma to the long exact sequence in group homology arising from the short exact sequence above we obtain:

Corollary 5 *If the group G acts topologically amenably on the compact Hausdorff space X, then for each n there is a short exact sequence*

$$0 \longrightarrow H_n(G,\mathbb{R}) \longrightarrow H_n(G, W_0(G,X)^*) \longrightarrow H_n(G, N_0(G,X)^*) \longrightarrow 0,$$

mapping the fundamental class $[1] \in H_0(G,\mathbb{R})$ to the fundamental class $[G \curvearrowright X]$ of the action. This gives us an isomorphism

$$H_n^{\mathrm{uf}}(G \curvearrowright X) \cong H_n(G,\mathbb{R}) \oplus H_n(G, N_0(G,X)^*).$$

In Theorem 9 we characterize topological amenability in terms of the 0–dimensional homology. In particular when the action is not topologically amenable we will show (Corollary 10) that $H_0^{\mathrm{uf}}(G \curvearrowright X)$ is isomorphic to $H_0(G, N_0(G,X)^*)$.

2 Non-vanishing elements in $H_0^{\mathrm{uf}}(G \curvearrowright X)$

Unlike the Block–Weinberger case, vanishing of the fundamental class does not in general imply the vanishing of $H_0^{\mathrm{uf}}(G \curvearrowright X)$.

Theorem 6 *Let X be a compact G space containing an open G–invariant subspace U on which G acts properly. Then $H_0^{\mathrm{uf}}(G \curvearrowright X)$ is non-zero. In particular $H_0^{\mathrm{uf}}(G \curvearrowright \overline{G})$ is non-zero for any compactification $\overline{G}$ of G.*

Proof If G is finite, and the action of G on X is trivial, then $H_0^{\mathrm{uf}}(G \curvearrowright X) = W_0(G,X)^*$ which is non-zero.

Otherwise we may assume that the action of G on U is non-trivial, replacing U with X if G is finite. Thus we may pick a point x_0 in U, and $x_1 = g_1 x_0$ in $G x_0$ with $x_0 \neq x_1$. The evaluation functional on $W_0(G,X)$ defined by $\eta \mapsto \eta_e(x_0)$ is an element of $W_0(G,X)^*$ and hence gives a homology class $[\mathrm{ev}_{e,x_0}] \in H_0^{\mathrm{uf}}(G \curvearrowright X)$. We will show that this is always non-zero. We establish this by constructing a sequence ξ^n

of elements of $W_0(G,X)$ which have pairing 1 with ev_{e,x_0}, but have asymptotically trivial pairing with all boundaries. Hence we will deduce that $[\mathrm{ev}_{e,x_0}]$ is non-zero.

Let $f \in C(X)$ be a positive function of norm 1, with $f(x_0)=1$ and with the support K of f contained in $U \setminus \{x_1\}$. By construction $x_0 \notin g_1^{-1}K$. Define $\xi \in W_0(G,X)$ by $\xi_e = f, \xi_{g_1} = -f$, and $\xi_g = 0$ for $g \neq e, g_1$. We note that ξ is in $W_0(G,X)$ as required, indeed it is in $N_0(G,X)$, since $\sum_{g\in G} \xi_g$ is identically zero.

Now form the sequence

$$\xi^n = \sum_{k\in G} \phi_n(k) k\cdot\xi, \text{ where } \phi_n(k) = \max\left\{\frac{n-d(e,k)}{n}, 0\right\}.$$

If $\xi^n_g(x)$ is non-zero then x is in gK or $gg_1^{-1}K$. By properness of the action there are only finitely many $h\in G$ such that hK meets K. Let N be the number of such h. If $x \in hK$, then $x \in gK \cup gg_1^{-1}K$ for at most $2N$ values of g, hence for each $x\in X$, the set of g with $\xi^n_g(x) \neq 0$ has cardinality at most $2N$. Since $|\xi^n_g(x)| \leq 2$ for each g,n,x it follows that $\|\xi^n\| \leq 4N$ for all n.

For $s \in S$ consider

$$\xi^n - s\cdot\xi^n = \sum_{g\in G} \phi_n(g)(g\cdot\xi - sg\cdot\xi) = \sum_{g\in G} (\phi_n(g) - \phi_n(s^{-1}g))g\cdot\xi.$$

Since $|(g\cdot\xi)_h(x)| \leq 2$ for all x and $|\phi_n(g)-\phi_n(s^{-1}g)| \leq \frac{1}{n}$ it follows that

$$\left|\left(\xi^n - s\cdot\xi^n\right)_h(x)\right| \leq \tfrac{2}{n}$$

for all h,x. On the other hand, for a given x, $(\xi^n - s\cdot\xi^n)_h(x)$ is non-zero for at most $4N$ values of h, hence $\|\xi^n - s\cdot\xi^n\| \leq \frac{8N}{n}$. We thus have a sequence ξ^n in $W_0(G,X)$ with $\|\delta\xi^n\| \to 0$. It follows that for any 1–chain ψ we have $(\delta^*\psi)(\xi^n) = \psi(\delta\xi^n) \to 0$ as $n\to\infty$.

Note that

$$\mathrm{ev}_{e,x_0}(\xi^n) = \xi^n_e(x_0) = \phi_n(e)(e\cdot\xi)_e(x_0) + \phi_n(g_1^{-1})(g_1^{-1}\cdot\xi)_e(x_0)$$

since the other terms in the sum vanish. The first term is $\phi_n(e)f(x_0) = 1$, while $(g_1^{-1}\cdot\xi)_e(x_0) = (g_1^{-1} * \xi_{g_1})(x_0) = 0$ since x_0 is not in $g_1^{-1}K$. Thus $\mathrm{ev}_{e,x_0}(\xi^n) = 1$ for all n, hence ev_{e,x_0} cannot be a boundary. □

We remark that the proof of nontriviality of the cycle ev_{e,x_0} in $H_0^{\mathrm{uf}}(G \curvearrowright X) = H_0(G, W_0(G,X)^*)$ can alternatively be phrased in terms of the pairing with the cohomology group $H^0(G, W_0(G,X)^{**})$: Let ζ be a weak–$*$ limit point of the sequence ξ^n in $W_0(G,X)^{**}$. Then the observation that $\delta\xi^n$ tends to zero as $n\to\infty$ implies that ζ is a cocycle. Pairing the cohomology class $[\zeta]$ with the homology class $[\mathrm{ev}_{e,x_0}]$ yields 1, hence $[\mathrm{ev}_{e,x_0}]$ must be non-zero.

We also observe that there is a surjection from $H_0^{\mathrm{uf}}(G \curvearrowright X)$ onto $H_0(G, N_0(G,X)^*)$, induced by the surjection $W_0(G,X)^* \to N_0(G,X)^*$. Since the sequence ξ^n lives in $N_0(G,X)$, the non-trivial elements constructed in the proposition remain non-trivial after applying this map.

3 Characterizing amenability

We recall the definition of a (topologically) amenable action. Let $W_{00}(G,X)$ denote the dense subspace of $W_0(G,X)$ consisting of elements ξ with ξ_g non-zero for only finitely many $g \in G$.

Definition 7 Let G be a group acting by homeomorphisms on a compact Hausdorff space. The action of G on X is said to be *topologically amenable* if there exists a sequence of elements $\xi^n \in W_{00}(G,X)$ such that

(1) $\xi^n_g \geq 0$ in $C(X)$ for every $n \in \mathbb{N}$ and $g \in G$,

(2) $\sigma(\xi^n) = 1$ for every n,

(3) $\sup_{s\in S} \|\xi^n - s\cdot\xi^n\| \to 0$.

Universality of the Stone–Čech compactification leads to the observation that a group acts amenably on some compact space if and only if it acts amenably on βG, which is equivalent to exactness. Amenable actions on compact spaces (lying between the point and βG) form a spectrum of generalized amenability properties interpolating between amenability and exactness. We will return to this point later. As we will now see the summation functional σ can be used to detect amenability of an action.

Theorem 8 *Let G be a finitely generated group acting on a compact space X by homeomorphisms. The following conditions are equivalent:*

(1) *the action of G on X is topologically amenable,*

(2) $\sigma \notin \overline{\mathrm{Image}(\delta^*)}^{\|\cdot\|}$,

(3) $\sigma \notin \mathrm{Image}(\delta^*)$.

Proof (1) $\Longrightarrow$ (2) Assume first that the action is amenable. Take a sequence $\xi^n \in W_{00}(G,X)$ as in Definition 7 and let μ be the weak-$*$ limit of a convergent subnet ξ_β. Then

$$\mu(\sigma) = \lim_\beta \sigma(\xi_\beta) = 1,$$

and in particular σ is not in the kernel of μ. On the other hand

$$|\mu(\delta^*\psi)| = \lim_\beta |\delta^*\psi(\xi_\beta)| = \lim_\beta |\psi(\delta\xi_\beta)| \le \lim_\beta \Big(\|\psi\| \sup_{s\in S} \|\xi_\beta - s\cdot\xi_\beta\| \Big) = 0,$$

for every $\psi \in \bigoplus_{s\in S} W_0(G,X)^*$. Thus $\mathrm{Image}(\delta^*) \subseteq \ker\mu$. Since $\ker\mu$ is norm-closed, we conclude that

$$\overline{\mathrm{Image}(\delta^*)}^{\|\cdot\|} \subseteq \ker\mu.$$

Thus $\sigma \notin \overline{\mathrm{Image}(\delta*)}^{\|\cdot\|}$ and (2) follows.

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1) We begin by showing that for every $D > 0$ there exists $\xi \in W_0(G,X)$ such that $\|\delta\xi\| < D|\sigma(\xi)|$. Let us suppose not, whence there exists a constant $D > 0$ such that

$$\|\delta\xi\| \ge D|\sigma(\xi)| \tag{2}$$

for all ξ. Consider the functional $\psi\colon \delta(W_0(G,X)) \to \mathbb{R}$, defined by

$$\psi(\delta\xi) = \sigma(\xi).$$

This is well defined, since $\delta\colon W_0(G,X) \to \bigoplus_{s\in S} W_0(G,X)$ is injective. By inequality (2), ψ is continuous on $\delta(W_0(G,X))$ and, by the Hahn–Banach theorem, we can extend it to a continuous functional Ψ on $\bigoplus_{s\in S} W_0(G,X)$. By definition, for $\xi \in W_0(G,X)$ we have

$$[\delta^*(\Psi)](\xi) = \Psi(\delta\xi) = \psi(\delta\xi) = \sigma(\xi),$$

hence σ is in the image of δ^*, contradicting (3).

We have thus shown that (3) implies for every $D > 0$ there is a ξ with $\|\delta\xi\| < D|\sigma(\xi)|$. Hence there exists a sequence $\xi^n \in W_0(G,X)$ such that $\sigma(\xi^n) = 1$ for all n, and $\|\delta\xi^n\| \to 0$. Since $W_{00}(G,X)$ is dense in $W_0(G,X)$, we may assume without loss of generality that $\xi^n \in W_{00}(G,X)$.

To obtain a positive element, as in Definition 7, we replace $\xi^n_g(x)$ by

$$\zeta^n_g(x) = \frac{|\xi^n_g(x)|}{\sum_{h\in G} |\xi^n_h(x)|}.$$

Note that as $\sigma(\xi^n) = 1$ we have $\sum_{h\in G} |\xi^n_h(x)| \ge 1$ for all x, n.

It is clear that ζ^n satisfies conditions 1 and 2 of the definition, while condition 3 follows from the inequality $\|\zeta^n - s\cdot\zeta^n\| \le 2\|\xi^n - s\cdot\xi^n\|$. Hence we deduce that the action is amenable. □

We are now in the position to prove the main theorem, which is stated here in a more general form. The *reduced homology* $\bar{H}_n^{\mathrm{uf}}(G \curvearrowright X) = \bar{H}_n(G, W_0(G,X)^*)$ in the statement is defined, as in the context of L^2–(co)homology, by taking the closure of the images in the chain complex.

Theorem 9 *Let G be a finitely generated group acting by homeomorphisms on a compact space X. The following conditions are equivalent:*

(1) *the action of G on X is topologically amenable,*

(2) $[G \curvearrowright X] \neq 0$ *in* $\bar{H}_0^{\mathrm{uf}}(G \curvearrowright X)$,

(3) $[G \curvearrowright X] \neq 0$ *in* $H_0^{\mathrm{uf}}(G \curvearrowright X)$,

(4) *the map* $(i^*)_*\colon H_0^{\mathrm{uf}}(G \curvearrowright X) \to H_0(G, N_0(G,X)^*)$ *is not injective,*

(5) *the map* $(i^*)_*\colon H_1^{\mathrm{uf}}(G \curvearrowright X) \to H_1(G, N_0(G,X)^*)$ *is surjective.*

Proof The equivalence (1)$\iff$(2)$\iff$(3) follows from Theorem 8. As noted in Section 1, the boundary map in homology is precisely the map δ^*, hence $H_0^{\mathrm{uf}}(G \curvearrowright X)$ is exactly the quotient of $W_0(G,X)^*$ by the image of δ^*, while the reduced homology is the quotient of $W_0(G,X)^*$ by the closure of the image of δ^*. Hence conditions (1), (2) and (3) correspond exactly to conditions (1), (2) and (3) of Theorem 8.

As in the proof of Corollary 5 the short exact sequence of coefficients yields a long exact sequence which terminates as

$$\longrightarrow H_0(G,\mathbb{R}^*) \xrightarrow{(\sigma^*)_*} H_0(G, W_0(G,X)^*) \xrightarrow{(i^*)_*} H_0(G, N_0(G,X)^*) \longrightarrow 0,$$

and in which the fundamental class $[1] \in H_0(G,\mathbb{R}^*)$ maps to the class $[G \curvearrowright X]$. Thus $[G \curvearrowright X] \neq 0$ if and only if the map $(\sigma^*)_*$ is non-zero, or equivalently the kernel of $(i^*)_*$ is non-zero. Thus it follows that (3) is equivalent to (4).

Also by exactness of the sequence $[G \curvearrowright X] \neq 0$ if and only if [1] is not in the image of the connecting map, or equivalently the connecting map is zero, and we obtain the equivalence of (3) and (5). □

Combining this with Corollary 5 we obtain:

Corollary 10 *Let G be a group acting by homeomorphisms on a compact Hausdorff topological space X.*

$$H_0^{\mathrm{uf}}(G \curvearrowright X) \cong \begin{cases} H_0(G,\mathbb{R}) \oplus H_0(G, N_0(G,X)^*) & \text{when the action is amenable,} \\ H_0(G, N_0(G,X)^*) & \text{when the action is not amenable.} \end{cases}$$

4 Functoriality

We return to the remark that we made earlier that the actions of G on compact spaces form a spectrum, with the single point at one end of the spectrum and the Stone–Čech compatification of G at the other end. We can make sense of this statement homologically as follows.

Lemma 11 *Let G be a finitely generated group, let X, Y be compact G–spaces and let $f\colon X \to Y$ be a continuous G–map. Then f induces a pull-back $f^*\colon W_0(G,Y) \to W_0(G,X)$ on standard modules. Dualising, f induces a push-forward*

$$f_*\colon H_n^{\mathrm{uf}}(G \curvearrowright X) \to H_n^{\mathrm{uf}}(G \curvearrowright Y).$$

Moreover f_* takes the fundamental class $[G \curvearrowright X]$ to the fundamental class $[G \curvearrowright Y]$.

Proof The map f induces pull-back maps $f^*\colon C(Y, \ell^1(G)) \to C(X, \ell^1(G))$ and $f^*\colon C(Y) \to C(X)$ defined by $f^*(\xi) = \xi \circ f$. It is easy to see that these are continuous, indeed contractive. These are compatible with the summation maps in the sense that $\sigma_X \circ f^* = f^* \circ \sigma_Y$, where σ_X and σ_Y denote the summation maps for X, Y respectively. It follows that f restricts to a continuous map $f^*\colon W_0(G,Y) \to W_0(G,X)$.

Dualising we have a map $W_0(G,X)^* \to W_0(G,Y)^*$. As f is equivariant, so are f^* and its adjoint, hence the latter induces a map f_* on homology.

The fact that $f_*[G \curvearrowright X] = [G \curvearrowright Y]$ follows from the identity $\sigma_X \circ f^* = f^* \circ \sigma_Y$. □

It follows from Lemma 11 that if $[G \curvearrowright Y]$ is non-trivial then so is $[G \curvearrowright X]$, recovering the statement that if the action on Y is topologically amenable then so is the action on X.

Now suppose that X is an arbitrary compact space on which G acts by homeomorphisms so by universality there are equivariant continuous maps

$$\beta G \to X \to \{*\}.$$

If G is amenable, which is to say that the action on a point is amenable, then the action on X is topologically amenable. On the other hand if the action on X is topologically amenable then the action on βG is also topologically amenable, hence G is exact. Hence we recover two well known facts.

We now remark that when f is surjective then the induced map on uniformly finite homology is also surjective in dimension 0. To see this note that for $\xi \in C(Y, \ell^1(G))$ we have

$$\|f^*\xi\| = \sup_{x \in X} \sum_{g \in G} |\xi_g(f(x))| = \sup_{y \in Y} \sum_{g \in G} |\xi_g(y)| = \|\xi\|$$

hence f^* is an isometry onto its image. It follows by an application of the Hahn–Banach Theorem that the dual map $(f^*)^*\colon W_0(G,X)^* \to W_0(G,Y)^*$ is surjective: given an element $\phi \in W_0(G,Y)^*$ we define $\psi\colon f^*(W_0(G,Y)) \to \mathbb{R}$ by $\psi(f^*(\xi)) = \phi(\xi)$. As f^* is isometric it follows that ψ is continuous and hence extends to $W_0(G,X)^*$. By construction $(f^*)^*(\psi) = \phi$ hence $(f^*)^*$ is surjective and we obtain a short exact sequence of G–modules

$$0 \longrightarrow W_0(G,f)^* \longrightarrow W_0(G,X)^* \xrightarrow{(f^*)^*} W_0(G,Y)^* \longrightarrow 0$$

where $W_0(G,f)$ denotes the quotient space $W_0(G,X)/f^*W_0(G,Y)$.

This induces a long exact sequence in group homology from which we extract the following fragment.

$$\cdots \longrightarrow H_0^{\mathrm{uf}}(G \curvearrowright X) \xrightarrow{f_*} H_0^{\mathrm{uf}}(G \curvearrowright Y) \longrightarrow 0.$$

Thus surjectivity of f implies surjectivity of the map f_* on homology in dimension 0.

Consider again the general situation of a continuous G–map $f\colon X \to Y$. We have seen that topological amenability automatically transfers from Y to X, but in general it does not transfer in the opposite direction. In order to transfer it from X to Y we need to place additional constraints on the map f.

Anantharaman-Delaroche and Renault [1] introduce the notion of an amenable equivariant map between measure spaces, see also Buneci [5]. Let G be a groupoid, let T,S be Borel G–spaces, and $\pi\colon T \to S$ a Borel G map. Equipping T,S with suitable measures one can form $L^\infty(T)$ and $L^\infty(S)$, and the map π gives $L^\infty(T)$ the structure of an $L^\infty(S)$–module. The map π is said to be amenable if there exists a positive unital $L^\infty(S)$–linear map $\mu\colon L^\infty(T) \to L^\infty(S)$, commuting with convolution by functions on G. For our purposes we will restrict to the case that G is a discrete group, whence the latter condition is simply equivariance of the map. Since $L^\infty(T), L^\infty(S)$ are commutative unital C^*–algebras they can be identified with function spaces $C(X), C(Y)$, for some compact Hausdorff spaces X,Y (as a simple example, if T,S are discrete with counting measure then X,Y are the Stone–Čech compactifications $\beta T, \beta S$). In this context μ is a $C(Y)$–linear equivariant map. We take this as the definition of an amenable map between compact Hausdorff spaces.

Definition 12 Let G be a discrete group and X,Y be compact Hausdorff topological spaces on which G acts by homeomorphisms. A continuous G–equivariant map $f\colon X \to Y$ induces a $G - C(Y)$–module structure on $C(X)$ by pullback. The map f is said to be *amenable* if there is a bounded $C(Y)$–linear G–equivariant map $\mu\colon C(X) \to C(Y)$ with $\mu(\mathbf{1}_X) = \mathbf{1}_Y$.

It is easy to see that amenability of the map f implies that f is surjective, hence f^* is topologically injective; moreover μ is a splitting of f^*.

If Y is a point then the map $X \to Y$ is amenable if and only if the action of G on X is amenable in the sense of von Neumann [14], that is, there is a G–invariant mean on $C(X)$. In the special case when X is the Stone–Čech compactification $\beta(G/H)$ of a coset space G/H the map $\beta(G/H) \to \{*\}$ is amenable precisely when H is co-amenable in G (see Eymard [7]).

Proposition 13 *Let G be a group and X, Y be compact Hausdorff topological spaces on which G acts by homeomorphisms. Let $f\colon X \to Y$ be an amenable continuous G–equivariant map. If the action of G on X is topologically amenable then so is the action of G on Y.*

Proof We use the isomorphism between the space $C(X, \ell^1(G))$ and the completed injective tensor product $C(X) \widehat{\otimes}_\epsilon \ell^1(G)$ (see, for example, Trèves [13, Theorem 44.1]) to identify $W_0(G, X)$ as a subspace of $C(X) \widehat{\otimes}_\epsilon \ell^1(G)$ and $W_0(G, Y)$ as a subspace of $C(Y) \widehat{\otimes}_\epsilon \ell^1(G)$. Since f is amenable we have a G–equivariant splitting $\mu\colon C(X) \to C(Y)$ of the map f^*, giving a map $\mu \otimes_\epsilon 1\colon C(X) \widehat{\otimes}_\epsilon \ell^1(G) \to C(Y) \widehat{\otimes}_\epsilon \ell^1(G)$. This restricts to a map $W_0(G, X) \to W_0(G, Y)$ since μ takes constant functions on X to constant functions on Y.

The corresponding dual map $W_0(G, Y)^* \to W_0(G, X)^*$ induces a map on homology that, abusing notation, we will denote $\mu^*\colon H_0^{\mathrm{uf}}(G \curvearrowright Y) \to H_0^{\mathrm{uf}}(G \curvearrowright X)$. By construction this splits the map $f_*\colon H_0^{\mathrm{uf}}(G \curvearrowright X) \to H_0^{\mathrm{uf}}(G \curvearrowright Y)$, and since $\mu(\mathbf{1}_X) = \mathbf{1}_Y$, $\mu^*([G \curvearrowright Y]) = [G \curvearrowright X]$. It follows that if the fundamental class $[G \curvearrowright X]$ is not trivial then neither is $[G \curvearrowright Y]$, and so topological amenability of the action on X implies topological amenability for the action on Y as required. □

5 The interaction between the uniformly finite homology and the bounded cohomology

We conclude with some remarks concerning the interaction of the uniformly finite homology of an action and the bounded cohomology with coefficients introduced by the authors in [3]. These illuminate the special role played by the Johnson class in $H_{\mathrm{b}}^1(G, N_0(G, X)^{**})$ and the fundamental class in $H_0^{\mathrm{uf}}(G \curvearrowright X)$ and extend the results of the first, second and fourth author in [4], which considered the special case of the action of G on a point.

In [3] we showed that topological amenability of the action is encoded by triviality of an element $[J]$ in $H_{\mathrm{b}}^1(G, N_0(G, X)^{**})$, which we call the Johnson class for the action.

This class is the image of the class $[1] \in H^0_{\mathrm{b}}(G,\mathbb{R})$ under the connecting map arising from the short exact sequence of coefficients

$$0 \longrightarrow N_0(G,X)^{**} \longrightarrow W_0(G,X)^{**} \longrightarrow \mathbb{R} \longrightarrow 0$$

which is dual to the short exact sequence appearing in Lemma 4.

By applying the forgetful functor from bounded to ordinary cohomology, we obtain a pairing of $H^1_{\mathrm{b}}(G, N_0(G,X)^{**})$ with $H_1(G, N_0(G,X)^{*})$, and clearly if the Johnson class $[J]$ is trivial then its pairing with any $[c] \in H_1(G, N_0^*)$ is zero.

Now suppose that every $[c] \in H_1(G, N_0(G,X)^*)$ pairs trivially with the Johnson class. Since the Johnson class $[J]$ is obtained by applying the connecting map to the generator $[1]$ of $H^0_{\mathrm{b}}(G,\mathbb{R}) = \mathbb{R}$, pairing $[J]$ with $[c] \in H_1(G, N_0(G,X)^*)$ is the same as pairing $[1]$ with the image of $[c]$ under the connecting map in homology. As this pairing (between $H^0(G,\mathbb{R}) = H^0_{\mathrm{b}}(G,\mathbb{R})$ and $H_0(G,\mathbb{R})$) is faithful, it follows that the image of $[c]$ under the connecting map is trivial for all $[c]$, so the connecting map is zero, which we have already noted is equivalent to amenability of the action. Thus in the case when the group is non-amenable, the non-triviality of the Johnson element must be detected by the pairing.

On the other hand, we can run a similar argument in the opposite direction: if pairing $[G \curvearrowright X]$ with every element $[\phi] \in H^0_{\mathrm{b}}(G, W_0(G,X)^{**})$ we get zero, then since $[G \curvearrowright X] = (\sigma^*)_*[1]$, we have that the pairing of $(\sigma^{**})_*[\phi] \in H^0_{\mathrm{b}}(G,\mathbb{R})$ with $[1] \in H_0(G,\mathbb{R})$ is trivial, whence $(\sigma^{**})_*[\phi] = 0$ (again by faithfulness of the pairing). Thus, by exactness, the connecting map on cohomology is injective and the Johnson class is non-trivial. So when the action is amenable, (and hence the Johnson class is trivial), non-triviality of $[G \curvearrowright X]$ must be detected by the pairing.

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