Todd genera of complex torus manifolds

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We prove that the Todd genus of a compact complex manifold X of complex dimension n with vanishing odd degree cohomology is one if the automorphism group of X contains a compact n–dimensional torus T^n as a subgroup. This implies that if a quasitoric manifold admits an invariant complex structure, then it is equivariantly homeomorphic to a compact smooth toric variety, which gives a negative answer to a problem posed by Buchstaber and Panov.

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1 Introduction

A *torus manifold* is a connected closed oriented smooth manifold of even dimension, say $2n$, endowed with an effective action of an n–dimensional torus T^n having a fixed point. A typical example of a torus manifold is a *compact smooth toric variety* which we call a *toric manifold* in this paper. Every toric manifold is a complex manifold. However, a torus manifold does not necessarily admit a complex (even an almost complex) structure. For example, the 4–dimensional sphere S^4 with a natural T^2–action is a torus manifold but admits no almost complex structure.

On the other hand, there are infinitely many nontoric torus manifolds of dimension $2n$ which admit T^n–invariant almost complex structures when $n \geq 2$. For instance, for any positive integer k, there exists a torus manifold of dimension 4 with an invariant almost complex structure whose Todd genus is equal to k (see Masuda [10, Theorem 5.1]) while the Todd genus of a toric manifold is always one. One can produce higher dimensional examples by taking products of those 4–dimensional examples with toric manifolds. The cohomology rings of the torus manifolds in these examples are generated by its degree-two part like toric manifolds.

In this paper, we consider a torus manifold with a T^n–invariant (genuine) complex structure. We will call such a torus manifold a *complex torus manifold*. The following is our main theorem.

Theorem 1.1 *If a complex torus manifold has vanishing odd degree cohomology, then its Todd genus is equal to one.*

Remark 1.2 If a closed smooth manifold M has vanishing odd degree cohomology, then any smooth T^n–action on M has a fixed point (see Bredon [2, Corollary 10.11, page 164]). In particular, a connected closed oriented smooth manifold M of dimension $2n$ with an effective T^n–action is a torus manifold if M has vanishing odd degree cohomology. This implies that Theorem 1.1 is equivalent to the statement in the abstract.

Other important examples of torus manifolds are *quasitoric manifolds*[1] introduced by Davis and Januskiewicz in [4]. A quasitoric manifold of dimension $2n$ is a closed smooth manifold with a locally standard T^n–action, whose orbit space is an n–dimensional simple polytope. It is unknown whether any toric manifold is a quasitoric manifold. However, if a toric manifold is projective, then it is a quasitoric manifold because a projective toric manifold with the restricted compact torus action admits a moment map which identifies the orbit space with a simple polytope.

Kustarev gives a criterion [9, Theorem 1] of when a quasitoric manifold admits an invariant almost complex structure. It also follows from his criterion that there are many nontoric quasitoric manifolds which have invariant almost complex structures. However, it has been unknown whether there is a quasitoric manifold which admits an invariant complex structure, and Buchstaber and Panov posed the following problem in [3, Problem 5.23], which motivated the study in this paper.

Problem 1.3 (Buchstaber–Panov) *Find an example of nontoric quasitoric manifold that admits an invariant complex structure.*

As a consequence of Theorem 1.1, we obtain the following, which gives a negative answer to Problem 1.3.

Theorem 1.4 *If a quasitoric manifold admits an invariant complex structure, then it is equivariantly homeomorphic to a toric manifold.*

This paper is organized as follows. In Section 2, we study simply connected compact complex surfaces with torus actions. In Section 3, we review the notion of multi-fan and recall a result on Todd genus. In Section 4, we define a map associated with the multi-fan of a complex torus manifold X and give a criterion of when the Todd genus of X is one in terms of the map. Theorems 1.1 and 1.4 are proved in Sections 5 and 6, respectively. Throughout this paper, all cohomology rings and homology groups are taken with $\mathbb{Z}$–coefficients.

[1] Davis and Januszkiewicz use the terminology *toric manifold* [4] but it was already used in algebraic geometry as the meaning of (compact) smooth toric variety, so Buchstaber and Panov started using the word *quasitoric manifold* [3].

While preparing this paper, the first author and Karshon proved that a complex torus manifold is equivariantly biholomorphic to a toric manifold [7]. Although Theorem 1.1 is contained in the result, the argument in this paper is completely different from that in [7] and we believe that this paper is worth publishing.

2 Simply connected complex surfaces with torus actions

We first recall two results on simply connected 4–manifolds.

Theorem 2.1 (Orlik and Raymond [12]) *If a simply connected closed smooth manifold of dimension* 4 *admits an effective smooth action of* T^2*, then it is diffeomorphic to a connected sum of copies of* $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ *(*$\mathbb{C}P^2$ *with reversed orientation) and* $S^2 \times S^2$.

Theorem 2.2 (Donaldson [5]) *If a simply connected projective complex surface is decomposed into* $Y_1 \# Y_2$ *as oriented smooth manifolds, then either* Y_1 *or* Y_2 *has a negative definite cup-product form.*

Let X be a simply connected compact complex surface whose automorphism group contains T^2 as a subgroup. By Theorem 2.1, we have

$$X \cong k\mathbb{C}P^2 \# \ell\overline{\mathbb{C}P^2} \# m(S^2 \times S^2), \quad k, \ell, m \geq 0 \tag{2-1}$$

as oriented smooth manifolds. Therefore, the Euler characteristic $\chi(X)$ and the signature $\sigma(X)$ of X are respectively given by

$$\chi(X) = k + \ell + 2m + 2, \quad \sigma(X) = k - \ell$$

and hence the Todd genus $\mathrm{Todd}(X)$ of X is given by

$$\mathrm{Todd}(X) = \frac{1}{4}(\chi(X) + \sigma(X)) = \frac{1}{2}(k + m + 1). \tag{2-2}$$

The following proposition is a key step toward Theorem 1.1.

Proposition 2.3 *Let* X *be as above. Then* $\mathrm{Todd}(X) = 1$.

Proof Since X is simply connected, the first Betti number of X is 0, and in particular, even. Thus, X is a deformation of an algebraic surface; see Kodaira [8, Theorem 25]. Since any algebraic surface is projective (see Barth, Peters and Van de Ven [1, Chapter IV, Corollary 5.6]), we can apply Theorem 2.2 to our X.

Unless $(k,m)=(1,0)$ or $(0,1)$, it follows from Equation (2-1) that X can be decomposed into $Y_1 \# Y_2$ as oriented smooth manifolds, where

$$\begin{array}{lll} Y_1 = \mathbb{C}P^2, & Y_2 = (k-1)\mathbb{C}P^2 \# \ell\overline{\mathbb{C}P^2} \# m(S^2\times S^2) & \text{if } k\geq 2, \\ Y_1 = S^2\times S^2, & Y_2 = k\mathbb{C}P^2 \# \ell\overline{\mathbb{C}P^2} \# (m-1)(S^2\times S^2) & \text{if } m\geq 2, \\ Y_1 = \mathbb{C}P^2 \# \ell\overline{\mathbb{C}P^2}, & Y_2 = S^2\times S^2 & \text{if } (k,m)=(1,1). \end{array}$$

In any case, neither of Y_1 and Y_2 has a negative cup product form and this contradicts Theorem 2.2. Therefore, $(k,m)=(1,0)$ or $(0,1)$ and hence $\mathrm{Todd}(X)=1$ by Equation (2-2). □

3 Torus manifolds and multi-fans

In this section, we review the notion of multi-fans introduced by Hattori and the second author in [6] and the second author in [10] and recall a result on Todd genus.

A *torus manifold* X of dimension $2n$ is a connected closed oriented manifold endowed with an effective action of T^n having a fixed point. In this paper, we are concerned with the case when X has a complex structure invariant under the action. We will call such a torus manifold a *complex torus manifold*.

Throughout this section, X will denote a complex torus manifold of complex dimension n unless otherwise stated. We define a combinatorial object $\Delta_X := (\Sigma_X, C_X, w_X)$ called the *multi-fan* of X. A *characteristic submanifold* of X is a connected complex codimension 1 holomorphic submanifold of X fixed pointwise by a circle subgroup of T^n. Characteristic submanifolds are T^n–invariant and intersect transversally. Since X is compact, there are only finitely many characteristic submanifolds, denoted $X_1,\dots,X_m$. We set

$$\Sigma_X := \left\{ I \in \{1,2,\dots,m\} \,\middle|\, X_I := \bigcap_{i\in I} X_i \neq \varnothing \right\},$$

which is an abstract simplicial complex of dimension $n-1$.

Let S^1 be the unit circle group of complex numbers and T_i the circle subgroup of T^n which fixes X_i pointwise. We take the isomorphism $\lambda_i\colon S^1 \to T_i \subset T^n$ such that

(3-1) $$\lambda_i(g)_*(\xi) = g\xi \quad \forall g\in S^1 \text{ and } \forall \xi \in TX|_{X_i}/TX_i$$

where $\lambda_i(g)_*$ denotes the differential of $\lambda_i(g)$ and the right hand side of Equation (3-1) above is the scalar multiplication with the complex number g on the normal bundle

$TX|_{X_i}/TX_i$ of X_i. We regard λ_i as an element of the Lie algebra $\operatorname{Lie} T^n$ of T^n through the differential and assign a cone

$$C_X(I) := \operatorname{pos}(\lambda_i \mid i \in I) \subset \operatorname{Lie} T^n \tag{3-2}$$

to each simplex $I \in \Sigma_X$, where $\operatorname{pos}(A)$ denotes the positive hull spanned by elements in the set A. This defines a map C_X from Σ_X to the set of cones in $\operatorname{Lie} T^n$.

We denote the set of $(n-1)$–dimensional simplices in Σ_X by $\Sigma_X^{(n)}$. For $I \in \Sigma^{(n)}$, X_I is a subset of the T^n–fixed point set of X. The *weight function* $w_X\colon \Sigma_X^{(n)} \to \mathbb{Z}_{>0}$ is given by

$$w_X(I) := \# X_I,$$

where $\# A$ denotes the cardinality of the finite set A.

The triple $\Delta_X := (\Sigma_X, C_X, w_X)$ is called the multi-fan of X. The Todd genus $\operatorname{Todd}(X)$ of X can be read from the multi-fan Δ_X as follows.

Theorem 3.1 [10] *Let v be an arbitrary vector in $\operatorname{Lie} T^n$ which is not contained in $C_X(J)$ for any $J \in \Sigma_X \backslash \Sigma_X^{(n)}$. Then*

$$\operatorname{Todd}(X) = \sum w_X(I),$$

where the summation runs over all $I \in \Sigma_X^{(n)}$ such that $C_X(I)$ contains v.

The following corollary follows immediately from Theorem 3.1.

Corollary 3.2 $\operatorname{Todd}(X) = 1$ *if and only if the pair (Σ_X, C_X) forms an ordinary complete nonsingular fan and $w_X(I) = 1$ for every $I \in \Sigma_X^{(n)}$.*

Suppose X_J is connected for every $J \in \Sigma_X$. Then X_J is a complex codimension $\# J$ holomorphic submanifold of X with a T^n–fixed point. Moreover, the induced action of the quotient torus T^n/T_J on X_J is effective and preserves the complex structure of X_J, where T_J is the $\# J$–dimensional subtorus of T^n generated by T_j for $j \in J$. Therefore, X_J is a complex torus manifold of complex dimension $n - \# J$ with the effective action of the quotient torus T^n/T_J.

In this case, the multi-fan $\Delta_{X_J} = (\Sigma_{X_J}, C_{X_J}, w_{X_J})$ of X_J for $J \in \Sigma$ can be obtained from the multi-fan Δ_X of X as discussed in [6], which we shall review. We note $X_J \cap X_i$ is nonempty if and only if $J \cup \{i\}$ is a simplex in Σ_X and each characteristic submanifold of X_J can be written as the nonempty intersection $X_J \cap X_i$. Hence, the simplicial complex Σ_{X_J} coincides with the link $\operatorname{link}(J; \Sigma_X)$ of J in Σ_X and

$$C_{X_J}(I) = \operatorname{pos}(\overline{\lambda_i} \mid i \in I)$$

for $I \in \operatorname{link}(J;\Sigma_X)$, where $\overline{\lambda_i}$ denotes the image of λ_i under the quotient map $\operatorname{Lie} T^n \to \operatorname{Lie} T^n/T_J$. The weight function w_{X_J} is the constant function 1.

4 Maps associated with multi-fans

Let X be a complex torus manifold of complex dimension n and $\Delta_X = (\Sigma_X, C_X, w_X)$ the multi-fan of X. Throughout this section, we assume that X_J is connected for every $J \in \Sigma_X$. We will define a continuous map f_X from the geometric realization $|\Sigma_X|$ of Σ_X to the unit sphere S^{n-1} of the vector space $\operatorname{Lie} T^n$ in which the cones $C_X(I)$ for $I \in \Sigma_X$ sit, and give a criterion of when the Todd genus of X is equal to 1 in terms of the map f_X.

We set

$$\sigma_I := \left\{ \sum_{i\in I} a_i\,\mathbf{e}_i \;\middle|\; \sum_{i\in I} a_i = 1, a_i \geq 0 \right\} \subset \mathbb{R}^m \quad \text{for } I \in \Sigma_X,$$

where $\mathbf{e}_i$ is the i–th vector in the standard basis of $\mathbb{R}^m$. The geometric realization $|\Sigma_X|$ of Σ_X is given by

$$|\Sigma_X| = \bigcup_{I\in\Sigma_X} \sigma_I.$$

Recall that the homomorphisms $\lambda_i\colon S^1 \to T^n$ for $i = 1,\dots,m$ defined in Section 3 are regarded as elements in $\operatorname{Lie} T^n$ through the differential. We take an inner product on $\operatorname{Lie} T^n$ and denote the length of an element $v \in \operatorname{Lie} T^n$ by $|v|$. We define a map $f_X\colon |\Sigma_X| \to S^{n-1}$, where S^{n-1} is the unit sphere of $\operatorname{Lie} T^n$, by

$$f_X|_{\sigma_I}\left(\sum_{i\in I} a_i\,\mathbf{e}_i\right) = \frac{\sum_{i\in I} a_i\lambda_i}{|\sum_{i\in I} a_i\lambda_i|}. \tag{4-1}$$

Clearly, f_X is a closed continuous map.

Lemma 4.1 *The map $f_X\colon |\Sigma_X| \to S^{n-1}$ is a homeomorphism if and only if* $\operatorname{Todd}(X) = 1$.

Proof We note that X_I is one point for any $I \in \Sigma_X^{(n)}$ because X_I is connected by assumption and of codimension n in X. Therefore, $w_X(I) = 1$ for any $I \in \Sigma_X^{(n)}$ and this together with Theorem 3.1 tells us that the Todd genus $\operatorname{Todd}(X)$ coincides with the number of cones $C_X(I)$ containing the vector $v \in \operatorname{Lie} T^n$ in Theorem 3.1.

The above observation implies that the cones $C_X(I)$ for $I \in \Sigma_X$ do not overlap and form an ordinary complete fan in $\operatorname{Lie} T^n$ if and only if $\operatorname{Todd}(X) = 1$ and this is equivalent to the map f_X being a homeomorphism, proving the lemma. □

For each characteristic submanifold X_i, we can also define a map $f_{X_i}\colon |\Sigma_{X_i}| \to S^{n-2}$, where S^{n-2} is the unit sphere in $\operatorname{Lie} T^n/T_i \cong (\operatorname{Lie} T_i)^{\perp}$, where $(\operatorname{Lie} T_i)^{\perp}$ denotes the orthogonal complement of a vector subspace $\operatorname{Lie} T_i$ in $\operatorname{Lie} T^n$.

Lemma 4.2 *If the map $f_{X_i}\colon |\Sigma_{X_i}| \to S^{n-2}$ is a homeomorphism, then the map $f_X|_{\mathrm{star}(\{i\};\Sigma_X)}\colon \mathrm{star}(\{i\};\Sigma_X) \to S^{n-1}$ is a homeomorphism onto its image, where $\mathrm{star}(\{i\};\Sigma_X)$ denotes the open star of $\{i\}$ in Σ_X.*

Proof It suffices to show the injectivity of $f_X|_{\mathrm{star}(\{i\};\Sigma_X)}$ because f_X is closed and continuous. Let $p_i\colon \operatorname{Lie} T^n \to (\operatorname{Lie} T_i)^{\perp}$ be the orthogonal projection. Through p_i, we identify $\operatorname{Lie} T^n/T_i$ with $(\operatorname{Lie} T_i)^{\perp}$. Recall that $\Sigma_{X_i} = \mathrm{link}(\{i\};\Sigma_X)$. For each vertex j of $\mathrm{link}(\{i\};\Sigma_X)$, we express λ_j as

$$\lambda_j = p_i(\lambda_j) + c_{i,j}\lambda_i, \quad c_{i,j} \in \mathbb{R}.$$

By the definitions of $\mathrm{link}(\{i\};\Sigma_X)$ and $\mathrm{star}(\{i\};\Sigma_X)$, we can express an element $x \in \mathrm{star}(\{i\};\Sigma_X)$ as

$$x = (1-t)\,\mathbf{e}_i + ty, \quad \text{with } y \in |\,\mathrm{link}(\{i\};\Sigma_X)|,\ 0 \le t < 1. \tag{4-2}$$

Suppose $y \in \sigma_J \subset |\,\mathrm{link}(\{i\};\Sigma_X)|$ and write

$$y = \sum_{j\in J} a_j\,\mathbf{e}_j, \quad \sum_{j\in J} a_j = 1,\ a_j \ge 0.$$

Then, it follows from Equation (4-1) that

$$\begin{aligned} f_X(x) &= \frac{(1-t)\lambda_i + t\sum_{j\in J} a_j\lambda_j}{|(1-t)\lambda_i + t\sum_{j\in J} a_j\lambda_j|} \\ &= \frac{(1-t)\lambda_i + t\sum_{j\in J} a_j(p_i(\lambda_j)+c_{i,j}\lambda_i)}{|(1-t)\lambda_i + t\sum_{j\in J} a_j(p_i(\lambda_j)+c_{i,j}\lambda_i)|} \\ &= g(t,y)\,f_{X_i}(y) + h(t,y)\frac{\lambda_i}{|\lambda_i|}, \end{aligned} \tag{4-3}$$

where

$$\begin{aligned} g(t,y) &:= \frac{t|\sum_{j\in J} a_j\,p_i(\lambda_j)|}{|(1-t)\lambda_i + t\sum_{j\in J} a_j(p_i(\lambda_j)+c_{i,j}\lambda_i)|}, \\ h(t,y) &:= \frac{(1-t+t\sum_{j\in J} a_jc_{i,j})|\lambda_i|}{|(1-t)\lambda_i + t\sum_{j\in J} a_j(p_i(\lambda_j)+c_{i,j}\lambda_i)|}. \end{aligned} \tag{4-4}$$

Since $|f_X(x)| = |f_{X_i}(y)| = 1$ and $f_{X_i}(y)$ is perpendicular to $\lambda_i/|\lambda_i|$, it follows from Equation (4-3) that

$$g^2(t,y) + h^2(t,y) = 1. \tag{4-5}$$

Take another point $x' \in \operatorname{star}(\{i\}, \Sigma_X)$ and write

$$x' = (1-t)\,\mathbf{e}_i + t' y' \quad \text{with } y' \in |\operatorname{link}(\{i\}; \Sigma_X)|,\ 0 \le t' < 1,$$

similarly to Equation (4-2). Since

$$f_X(x') = g(t',y') f_{X_i}(y') + h(t',y') \frac{\lambda_i}{|\lambda_i|},$$

we have $f_X(x) = f_X(x')$ if and only if

$$g(t,y) f_{X_i}(y) = g(t',y') f_{X_i}(y'), \quad h(t,y) = h(t',y'). \tag{4-6}$$

Both $g(t,y)$ and $g(t',y')$ are nonnegative by Equation (4-4), so it follows from Equations (4-5) and (4-6) that if $f_X(x) = f_X(x')$, then

$$g(t,y) = g(t',y'), \quad f_{X_i}(y) = f_{X_i}(y'). \tag{4-7}$$

The latter identity in Equation (4-7) above implies $y = y'$ since f_{X_i} is a homeomorphism by assumption. Therefore it follows from (4-6) and (4-7) that

$$g(t,y) = g(t',y), \quad h(t,y) = h(t',y).$$

This together with (4-3) shows that

$$(1-t)\lambda_i + t \sum_{j \in J} a_j \lambda_j = (1-t')\lambda_i + t' \sum_{j \in J} a_j \lambda_j.$$

Here λ_i and $\sum_{j\in J} a_j\lambda_j$ are linearly independent, so we conclude $t = t'$. It follows $f_X|_{\operatorname{star}(\{i\};\Sigma_X)}$ is injective, which implies the lemma. □

Corollary 4.3 *If the map $f_{X_i}\colon |\Sigma_{X_i}| \to S^{n-2}$ is a homeomorphism for all i, then $f_X\colon |\Sigma_X| \to S^{n-1}$ is a covering map, and hence if $|\Sigma_X|$ is connected and $n-1 \ge 2$ in addition, then f_X is a homeomorphism.*

5 Torus manifolds with vanishing odd degree cohomology

In this section, we prove Theorem 1.1 in the Introduction.

The T^n–action on a torus manifold X of dimension $2n$ is said to be *locally standard* if the T^n–action on X locally looks like a faithful representation of T^n, to be more precise, any point of X has an invariant open neighborhood equivariantly diffeomorphic to an invariant open set of a faithful representation space of T^n. The orbit space X/T^n is a manifold with corners if the T^n–action on X is locally standard. A manifold with corners Q is called *face-acyclic* if every face of Q (even Q itself) is acyclic. A face–acyclic manifold with corners is called a *homology polytope* if any intersection of facets of Q is connected unless empty. The combinatorial structure of X/T^n and the topology of X are deeply related as is shown in the following theorem.

Theorem 5.1 (Masuda and Panov [11]) *Let X be a torus manifold of dimension $2n$.*

(1) *$H^{\mathrm{odd}}(X)=0$ if and only if the T^n–action on X is locally standard and X/T^n is face-acyclic.*

(2) *$H^*(X)$ is generated by its degree two part as a ring if and only if the T^n–action on X is locally standard and X/T^n is a homology polytope.*

Suppose that X is a torus manifold of dimension $2n$ with vanishing odd degree cohomology. Then $X/T^n = Q$ is a manifold with corners and face-acyclic. Let $\pi\colon X \to X/T^n = Q$ be the quotient map and let $Q_1,\dots,Q_m$ be the facets of Q. Then $\pi^{-1}(Q_1),\dots,\pi^{-1}(Q_m)$ are the characteristic submanifolds of X, denoted $X_1,\dots,X_m$ before. If Q is a homology polytope, ie any intersection of facets of Q is connected unless empty (this is equivalent to any intersection of characteristic submanifolds of X being connected unless empty), then the geometric realization $|\Sigma_X|$ of the simplicial complex Σ_X is a homology sphere of dimension $n-1$ (see [11, Lemma 8.2]), in particular, connected when $n \geq 2$. Unless Q is a homology polytope, intersections of facets are not necessarily connected. However, we can change Q into a homology polytope by cutting Q along faces of Q. This operation corresponds to blowing up along connected components of intersections of characteristic submanifolds of X equivariantly. We refer the reader to [11] for the details.

The results in Section 2 required the simply connectedness of a complex surface. Here is a criterion of the simply connectedness of a torus manifold in terms of its orbit space.

Lemma 5.2 *Suppose that the T^n–action on a torus manifold X is locally standard. Then X is simply connected if and only if the orbit space X/T^n is simply connected.*

Proof Since the group T^n is connected, the "only if" part in the lemma follows from [2, Corollary 6.3, page 91].

We shall prove the "if" part. Suppose that X/T^n is simply connected. Since each characteristic submanifold X_i of X is of real codimension two, the homomorphism

$$\pi_1\Big(X\setminus\bigcup_i X_i\Big) \to \pi_1(X) \tag{5-1}$$

induced by the inclusion map from $X \setminus \bigcup_i X_i$ to X is surjective. Here, the T^n–action on $X \setminus \bigcup_i X_i$ is free since the T^n–action on X is locally standard, so that the quotient map from $X \setminus \bigcup_i X_i$ to $(X \setminus \bigcup_i X_i)/T^n$ gives a fiber bundle with fiber T^n. The orbit space $(X \setminus \bigcup_i X_i)/T^n$ is simply connected because X/T^n is a manifold with corners, $(X \setminus \bigcup_i X_i)/T^n$ is the interior of X/T^n and X/T^n is simply connected by assumption. Therefore the inclusion map from a free T^n–orbit to $X \setminus \bigcup_i X_i$ induces an isomorphism on their fundamental groups. But any free T^n–orbit shrinks to a fixed point in X, so the epimorphism in Equation (5-1) must be trivial and hence X is simply connected. □

Now, we are in a position to prove the following main theorem stated in the Introduction.

Theorem 5.3 *If a complex torus manifold X has vanishing odd degree cohomology, then the Todd genus of X is 1.*

Proof Let n be the complex dimension of X as usual. Since $H^{\mathrm{odd}}(X) = 0$, the orbit space X/T^n is face-acyclic by Theorem 5.1. As remarked after Theorem 5.1, one can change X into a complex torus manifold whose orbit space is a homology polytope by blowing up X equivariantly. Since Todd genus is a birational invariant, it remains unchanged under blowing up. Therefore we may assume that the orbit space of our X is a homology polytope, so that any intersection of characteristic submanifolds of X is connected unless empty and $|\Sigma_X|$ is a homology sphere of dimension $n-1$. Since the orbit space of X_i is a facet of X/T^n, it is also a homology polytope so that any intersection of characteristic submanifolds of X_i (viewed as a complex torus manifold) is also connected unless empty and $|\Sigma_{X_i}|$ is a homology sphere of dimension $n-2$. Therefore, the results in Section 4 are applicable to X and X_i's.

We shall prove the theorem by induction on the complex dimension n of X. If $n = 1$, then X is $\mathbb{C}P^1$ and hence $\mathrm{Todd}(X) = 1$. When $n = 2$, the orbit space X/T^2 is contractible because X/T^2 is acyclic by Theorem 5.1 and the dimension of X/T^2 is 2. Therefore, X is simply connected by Lemma 5.2 and $\mathrm{Todd}(X) = 1$ by Proposition 2.3.

Assume that $n \geq 3$ and the theorem holds when the complex dimension is equal to $n-1$. Then, $\mathrm{Todd}(X_i) = 1$ for any X_i by induction assumption and hence $f_{X_i}\colon |\Sigma_{X_i}| \to S^{n-2}$ is a homeomorphism by Lemma 4.1. Since $|\Sigma_X|$ is a homology sphere of dimension $n-1 (\geq 2)$, $|\Sigma_X|$ is connected and hence $f_X\colon |\Sigma_X| \to S^{n-1}$ is a homeomorphism by Corollary 4.3. It follows from Lemma 4.1 that $\mathrm{Todd}(X) = 1$. This completes the induction step and the theorem is proved. □

6 Proof of Theorem 1.4

A quasitoric manifold X of dimension $2n$ is a smooth closed manifold endowed with a locally standard T^n–action, whose orbit space is a simple polytope Q of dimension n. Clearly, X is a torus manifold. The characteristic submanifolds $X_1, \dots, X_m$ of X bijectively correspond to the facets $Q_1, \dots, Q_m$ of Q through the quotient map $\pi\colon X \to Q$. Therefore, for $I \subset \{1, \dots, m\}$, $X_I = \cap_{i\in I} X_i$ is nonempty if and only if $Q_I := \cap_{i\in I} Q_i$ is nonempty; so the simplicial complex

$$\Sigma_X = \{I \subset \{1, \dots, m\} \mid X_I \neq \varnothing\}$$

introduced in Section 3 is isomorphic to the boundary complex of the simplicial polytope dual to Q. As before, let T_i be the circle subgroup of T^n which fixes X_i pointwise and let $\lambda_i\colon S^1 \to T_i \subset T^n$ be an isomorphism. There are two choices of λ_i for each i.

One can recover X from the data $(Q, \{\lambda_i\}_{i=1}^m)$ up to equivariant homeomorphism as follows. Any codimension k face F of Q is written as Q_I for a unique $I \in \Sigma_X$ with cardinality k and we denote the subgroup T_I by T_F. For a point $p \in Q$, we denote by $F(p)$ the face containing p in its relative interior. Set

$$X(Q, \{\lambda_i\}_{i=1}^m) := T^n \times Q/\sim,$$

where $(t, p) \sim (s, q)$ if and only if $p = q$ and $ts^{-1} \in T_{F(p)}$. Then X and $X(Q, \{\lambda_i\}_{i=1}^m)$ are known to be equivariantly homeomorphic; see [4].

Suppose our quasitoric manifold X admits an invariant complex structure. Then, the isomorphism λ_i is unambiguously determined by requiring the identity Equation (3-1), ie

$$\lambda_i(g)_*(\xi) = g\xi, \quad \forall g \in S^1 \text{ and } \forall \xi \in TX|_{X_i}/TX_i.$$

The simplicial complex Σ_X and the elements λ_i are used to define the multi-fan of X. But since the Todd genus of X is one by Theorem 5.3, the multi-fan of X is an ordinary complete nonsingular fan by Corollary 3.2 and hence it is the fan of a toric manifold. Finally, we note since Σ_X is the boundary complex of the simplicial polytope dual to the simple polytope Q, it determines Q as a manifold with corners up to homeomorphism. This implies Theorem 1.4 because the equivariant homeomorphism type of X is determined by Q and the elements λ_i as remarked above.

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