Virtual amalgamation of relatively quasiconvex subgroups

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For relatively hyperbolic groups, we investigate conditions guaranteeing that the subgroup generated by two relatively quasiconvex subgroups $Q_1$ and $Q_2$ is relatively quasiconvex and isomorphic to $Q_1 \ast_{Q_1 \cap Q_2} Q_2$. The main theorem extends results for quasiconvex subgroups of word-hyperbolic groups, and results for discrete subgroups of isometries of hyperbolic spaces. An application on separability of double cosets of quasiconvex subgroups is included.

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1 Introduction

This paper continues the work started by the first author in [10] motivated by the following question:

Problem 1 Suppose $G$ is a relatively hyperbolic group and $Q_1$ and $Q_2$ are relatively quasiconvex subgroups of $G$. Investigate conditions guaranteeing that the natural homomorphism

$$Q_1 \ast_{Q_1 \cap Q_2} Q_2 \longrightarrow G$$

is injective and that its image $\langle Q_1 \cup Q_2 \rangle$ is relatively quasiconvex.

Let $G$ be a group hyperbolic relative to a finite collection of subgroups $\mathbb{P}$, and let $\text{dist}$ be a proper left invariant metric on $G$.

Definition 1 Two subgroups $Q$ and $R$ of $G$ have compatible parabolic subgroups if for any maximal parabolic subgroup $P$ of $G$ either $Q \cap P < R \cap P$ or $R \cap P < Q \cap P$.

Theorem 2 For any pair of relatively quasiconvex subgroups $Q$ and $R$ of $G$ with compatible parabolic subgroups, and any finite index subgroup $H$ of $Q \cap R$, there is a constant $M = M(Q, R, H, \text{dist}) \geq 0$ with the following property. Suppose that $Q' < Q$ and $R' < R$ are subgroups such that:

1. $H = Q' \cap R'$;
2. $\text{dist}(1, g) \geq M$ for any $g$ in $Q' \setminus Q' \cap R'$ or $R' \setminus Q' \cap R'$.
Then the subgroup \( \langle Q' \cup R' \rangle \) of \( G \) satisfies:

1. The natural homomorphism

\[
Q' * Q' \cap R' \to \langle Q' \cup R' \rangle
\]

is an isomorphism.

2. If \( Q' \) and \( R' \) are relatively quasiconvex, then so is \( \langle Q' \cup R' \rangle \).

Theorem 2 extends results by Gitik [6, Theorem 1] for word-hyperbolic groups and by the first author [10, Theorem 1.1] for relatively hyperbolic groups. Yang recently obtained a similar combination results requiring stronger conditions [14]. His results include a combination result for HNN extensions and some applications to subgroup separability.

**Definition 3** Two subgroups \( Q \) and \( R \) of a group \( G \) can be **virtually amalgamated** if there are finite index subgroups \( Q_0 < Q \) and \( R_0 < R \) such that the natural map \( Q_0 \cap R_0 \to G \) is injective.

Let \( Q \) and \( R \) be relatively quasiconvex subgroups of \( G \) with compatible parabolic subgroups and let \( M = M(Q, R, Q \cap R) \) be the constant provided by Theorem 2. If \( Q \cap R \) is a separable subgroup of \( G \), then there is a finite index subgroup \( G' \) of \( G \) containing \( Q \cap R \) such that \( \text{dist}(1, g) > M \) for every \( g \in G \) with \( g \not\in Q \cap R \). In this case, the subgroups \( Q' = G' \cap Q \) and \( R' = G' \cap R \) satisfy the hypothesis of Theorem 2; hence they have a quasiconvex virtual amalgam.

**Corollary 4** (Virtual Quasiconvex Amalgam Theorem) Let \( Q \) and \( R \) be quasiconvex subgroups of \( G \) with compatible parabolic subgroups, and suppose that \( Q \cap R \) is separable. Then \( Q \) and \( R \) can be virtually amalgamated in \( G \).

It is known that many (relatively) hyperbolic groups have the property that all quasiconvex or all finitely generated subgroups are separable; see Agol, Long and Reid [2], Long and Reid [8; 9], Wise [12; 13], and Agol, Groves and Manning [1]. Still, it is a natural question to ask whether the corollary above holds under the hypothesis that \( G \) is residually finite.

A special case of the Virtual Quasiconvex Amalgam Theorem is the following by Baker and Cooper [3, Theorem 5.3].

**Corollary 5** Let \( G \) be a geometrically finite subgroup of isom\((\mathbb{H}^n)\), and let \( Q \) and \( R \) be geometrically finite subgroups of \( G \) with compatible parabolic subgroups. Suppose \( Q \cap R \) is separable in \( G \). Then \( Q \) and \( R \) have a geometrically finite virtual amalgam.
Separability of quasiconvex subgroups and double cosets of quasiconvex subgroups is of interest in the construction of actions on special cube complexes [13]. The machinery we use to prove the main result also gives the following.

**Corollary 6** (Double cosets are separable) Let $G$ be a relatively hyperbolic group such that all its quasiconvex subgroups are separable. If $Q$ and $R$ are quasiconvex subgroups with compatible parabolic subgroups then the double coset $QR$ is separable.

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# 2 Preliminaries

## 2.1 Gromov-hyperbolic spaces

Let $(X, \text{dist})$ be a proper and geodesic $\delta$–hyperbolic space. Recall that a $(\lambda, \mu)$–quasigeodesic is a curve $\gamma: [a, b] \to X$ parameterized by arc length such that

$$|x - y|/\lambda - \mu \leq \text{dist}(\gamma(x), \gamma(y)) \leq \lambda|x - y| + \mu$$

for all $x, y \in [a, b]$. The curve $\gamma$ is a $k$–local $(\lambda, \mu)$–quasigeodesic if the above condition is required only for $x, y \in [a, b]$ such that $|x - y| \leq k$.

**Lemma 7** Coornaert, Delzant and Papadopoulos [5, Chapter 3, Theorem 1.2] (Morse Lemma) For each $\lambda, \mu, \delta$ there exists $k > 0$ with the following property. In a $\delta$–hyperbolic geodesic space, any $(\lambda, \mu)$–quasigeodesic at $k$–Hausdorff distance from the geodesic between its endpoints.

**Lemma 8** [5, Chapter 3, Theorem 1.4] For each $\lambda, \mu, \delta$ there exist $k', \mu'$ so that any $k$–local $(\lambda, \mu)$–quasigeodesic in a $\delta$–hyperbolic geodesic space is a $(\lambda', \mu')$–quasigeodesic.

Fix a basepoint $x_0 \in X$. If $G$ is a subgroup of $\text{Isom}(X)$, we identify each element $g$ of $G$ with the point $gx_0$ of $X$. For $g_1, g_2 \in G$ denote by $\text{dist}(g_1, g_2)$ the distance $\text{dist}(g_1x_0, g_2x_0)$. Since $X$ is a proper space, if $G$ is a discrete subgroup of $\text{Isom}(X)$, this is a proper and left invariant pseudometric on $G$.
Lemma 9 [10, Lemma 4.2] (Bounded Intersection) Let $G$ be a discrete subgroup of $\text{isom}(X)$, let $Q$ and $R$ be subgroups of $G$, and let $\mu > 0$ be a real number. Then there is a constant $M = M(Q, R, \mu) \geq 0$ so that

$$Q \cap N_\mu(R) \subset N_M(Q \cap R).$$

2.2 Relatively quasiconvex subgroups

We follow the approach to relatively hyperbolic groups as developed by Hruska [7].

Definition 10 (Relative Hyperbolicity) A group $G$ is relatively hyperbolic with respect to a finite collection of subgroups $\mathbb{P}$ if $G$ acts properly discontinuously and by isometries on a proper and geodesic $\delta$–hyperbolic space $X$ with the following property: $X$ has a $G$–equivariant collection of pairwise disjoint horoballs whose union is an open set $U$, $G$ acts cocompactly on $X \setminus U$, and $\mathbb{P}$ is a set of representatives of the conjugacy classes of parabolic subgroups of $G$.

Throughout the rest of the paper, $G$ is a relatively hyperbolic group acting on a proper and geodesic $\delta$–hyperbolic space $X$ with a $G$–equivariant collection of horoballs satisfying all conditions of Definition 10. As before, we fix a basepoint $x_0 \in X \setminus U$, identify each element $g$ of $G$ with $gx_0 \in X$ and let $\text{dist}(g_1 x_0, g_2 x_0)$ denote $\text{dist}(g_1, g_2)$ for $g_1, g_2 \in G$.

Lemma 11 Bowditch [4, Lemma 6.4] (Cocompact actions of parabolic subgroups on thick horospheres) Let $B$ be a horoball of $X$ with $G$–stabilizer $P$. For any $M > 0$, $P$ acts cocompactly on $N_M(B) \cap (X \setminus U)$.

Lemma 12 (Parabolic approximation) Let $Q$ be a subgroup of $G$ and let $\mu > 0$ be a real number. There is a constant $M = M(Q, \mu)$ with the following property. If $P$ is a maximal parabolic subgroup of $G$ stabilizing a horoball $B$, and $\{1, q\} \subset Q \cap N_\mu(B)$ then there is $p \in Q \cap P$ such that $\text{dist}(p, q) < M$.

Proof By Lemma 11, $\text{dist}(q, P) < M_1$ for some constant $M_1 = M_1(Q, P)$. Then Lemma 9 implies that $\text{dist}(q, Q \cap P) < M_2$ where $M_2 = N(Q, P, M_1)$. Since $B$ is a horoball at distance less than $\mu$ from $1$, there are only finitely many possibilities for $B$ and hence for the subgroup $P$. Let $M$ the maximum of all $N(Q, P, \mu)$ among the possible $P$.

Definition 13 (Relatively quasiconvex subgroup) A subgroup $Q$ of $G$ is relatively quasiconvex if there is $\mu \geq 0$ such that for any geodesic $c$ in $X$ with endpoints in $Q$, $c \cap (X \setminus U) \subset N_\mu(Q)$.

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The choice of horoballs turns out not to make a difference.

**Proposition 14** [7] If \( Q \) is relatively quasiconvex in \( G \) then for any \( L \geq 0 \) there is \( \mu \geq 0 \) such that for any geodesic \( c \) in \( X \) with endpoints in \( Q \), \( c \cap N_L(X \setminus U) \subseteq N_\mu(Q) \).

### 3 A lemma on Gromov’s inner product

Let \( Q \) and \( R \) be relatively quasiconvex subgroups with compatible parabolic subgroups, and let \( H \) be a finite index subgroup of \( Q \cap R \).

Let \( Q' \) and \( R' \) be subgroups of \( Q \) and \( R \) respectively such that \( Q' \cap R' = H \). Let \( g \in Q' R' \) (or \( g \in R' Q' \)) such that \( g \not\in H \). Suppose \( g = qr \) (or \( g = rq \)) with \( q \in Q' \), \( r \in R' \) and such that dist(1, \( q \)) + dist(1, \( r \)) is minimal among all such products.

**Lemma 15** Suppose that there exists \( a \in H \) and a point \( p \) at distance at most \( A \) from the geodesic segment \([1, g] \) so that dist(\( p, qa \)) \( \leq B \). Then

\[
\text{dist}(1, q) + \text{dist}(1, r) \leq \text{dist}(1, g) + 2A + 2B.
\]

**Proof** Let \( p' \in [1, g] \) be such that dist(\( p, p' \)) < \( A \). Then

\[
\text{dist}(1, qa) + \text{dist}(1, a^{-1}r) \leq \text{dist}(1, p') + \text{dist}(p', qa) + \text{dist}(qa, p') + \text{dist}(p', g) \\
\leq \text{dist}(1, g) + 2A + 2B.
\]

Since \( g \) can be written as \((qa)(a^{-1}r)\), the minimality assumption implies dist(1, \( q \)) + dist(1, \( r \)) \( \leq \) dist(1, \( g \)) + 2A + 2B.

**Lemma 16** (Gromov’s inner product is bounded) There is a constant \( K = K(Q, R, H) \) with the following property:

\[
\text{dist}(1, q) + \text{dist}(1, r) \leq \text{dist}(1, g) + K.
\]

**Proof** Constants which depend only on \( Q \), \( R \), \( H \) and \( \delta \) are denoted by \( M_i \), the index counts positive increments of the constant during the proof. Suppose \( g = qr \), the other case being symmetric. The constant \( K \) of the statement corresponds to \( M_{13} \).

Consider a triangle \( \Delta \) with vertices \( 1, q, g \). Let \( p \in [1, q] \) be a center of \( \Delta \), ie the \( \delta \)–neighborhood of \( p \) intersects all sides of \( \Delta \).

Suppose that \( p \in X \setminus U \). Then dist(\( p, Q \), dist(\( p, qR \)) \( \leq M_1 \) by relative quasiconvexity of \( Q \) and \( R \). By Lemma 9, there exists \( a \in Q \cap R \) so that dist(\( p, qa \)) \( \leq M_2 \). Since \( H \)
is a finite index subgroup of $Q \cap R$, there is $b \in H$ such that $\text{dist}(p, qb) \leq M_3$. By Lemma 15, $\text{dist}(1, q) + \text{dist}(1, r) \leq \text{dist}(1, g) + 2M_3 + 2\delta$.

Suppose instead that $p$ is in a horoball $B$, whose stabilizer is $P$. We can assume $\text{dist}(q, B) \leq M_8$. Indeed, let $p_1$ be the entrance point of the geodesic $[q, 1]$ in $B$; then $\text{dist}(p_1, Q) < M_4$ by quasiconvexity of $Q$. Notice that $\text{dist}(p_1, [q, g])$ is at most $2\delta$ since $p$ is a center of $\Delta$ and $p_1 \in [q, p]$ (consider a triangle with vertices $p, q, p'$ for $p' \in [q, g]$ so that $d(p, p') \leq \delta$). By quasiconvexity of $R$, there is $p_2 \in [q, g]$ such that $\text{dist}(p_1, p_2), \text{dist}(p_2, qR) < M_5$. Lemma 9 implies there is $a \in Q \cap R$ such that $\text{dist}(qa, p_1), \text{dist}(qa, p_2) < M_6$. Since $H$ is a finite index subgroup of $Q \cap R$, there is $b \in H$ such that $\text{dist}(qb, p_1), \text{dist}(qb, p_2) < M_7$. Since $g$ can be written as $(qb)(b^{-1}r)$, by minimality we have

\[
\text{dist}(1, p_1) + \text{dist}(p_1, q) + \text{dist}(q, p_2) + \text{dist}(p_2, g) \\
= \text{dist}(1, q) + \text{dist}(1, g) \\
\leq \text{dist}(1, qb) + \text{dist}(1, b^{-1}r) \\
= \text{dist}(1, p_1) + \text{dist}(p_1, qb) + \text{dist}(qb, p_2) + \text{dist}(p_2, g),
\]

and therefore

\[
2 \text{dist}(q, B) = 2 \text{dist}(p_1, q) \\
\leq \text{dist}(p_1, q) + \text{dist}(q, p_2) + \text{dist}(p_1, p_2) \\
\leq \text{dist}(p_1, qb) + \text{dist}(qb, p_2) + \text{dist}(p_1, p_2) \\
\leq 2M_8.
\]
Since $Q$ and $R$ have compatible parabolic subgroups, assume $Q \cap q^{-1} P q \leq R \cap q^{-1} P q$, the other case being symmetric. By quasiconvexity of $Q$, there is $q_1 \in Q$ at distance $M_9$ from the entrance point of $[1, q]$ in $B$. In particular, the distance from $q_1$ to $[1, g]$ is at most $M_{10}$. Applying the parabolic approximation lemma to $\{1, q^{-1}q_1\} \subset Q \cap N_{M_{10}}(q^{-1} B)$, there is an element $a \in Q \cap q^{-1} P q$ such that $\text{dist}(qa, q_1) \leq M_{11}$. Since $Q \cap q^{-1} P q \leq R \cap q^{-1} P q$ it follows that $a \in Q \cap R$. Since $H$ is finite index in $Q \cap R$, by increasing the constant we can assume that $a \in H$ and $\text{dist}(qa, q_1) \leq M_{12}$. Then Lemma 15 implies

$$\text{dist}(1, q) + \text{dist}(1, r) \leq \text{dist}(1, g) + M_{13}.$$ 

\[\square\]

4 Proof of Theorem 2

Let $Q$ and $R$ be relatively quasiconvex subgroups with compatible parabolic subgroups, and let $H$ be a finite index subgroups of $Q \cap R$.

Let $K = K(Q, R, H)$ be the constant of Lemma 16. Let $M$ be large enough so that $M > k, \lambda, \lambda'$ and $\mu, \mu'$ are as in Lemma 8 for $\lambda = 1, \mu = K$.

Let $Q'$ and $R'$ be subgroups satisfying the hypothesis of the theorem, in particular $Q' \cap R' = H$. Consider $1 \neq g \in Q' \ast Q' \cap R' R'$ and suppose that $g \not\in Q' \cap R'$. Then $g = g_1 \ldots g_n$ where the $g_i$'s are alternatively elements of $Q' \setminus Q' \cap R'$ and $R' \setminus Q' \cap R'$. Moreover, assume that this product is minimal in the sense that $\sum \text{dist}(1, g_i)$ is minimal among all such products describing $g$.

Lemma 17 For each $i$, let $h_i = g_1 \ldots g_i$. Then the concatenation $\alpha = \alpha_1 \ldots \alpha_{n-1}$ of geodesics $\alpha_i$ from $h_i$ to $h_{i+1}$ is an $M$–local $(1, K)$–quasigeodesic.

Proof By the choice of $Q'$ and $R'$ each segment $\alpha_i$ has length at least $M$. Let $x \in [h_{i-1}, h_i]$ and $y \in [h_i, h_{i+1}]$. By Lemma 16, we have

$$\text{dist}(h_{i-1}, x) + \text{dist}(x, y) + \text{dist}(y, h_{i+1}) \geq \text{dist}(h_{i-1}, h_{i+1})$$

$$\geq \text{dist}(h_{i-1}, h_i) + \text{dist}(h_i, h_{i+1}) - K$$

$$= \text{dist}(h_{i-1}, x) + \text{dist}(x, h_i) + \text{dist}(h_i, y) + \text{dist}(y, h_{i+1}) - K.$$

Therefore $\text{dist}(x, y) + K \geq \text{dist}(x, h_i) + \text{dist}(h_i, y)$.

Since $M > k$, Lemma 8 implies that $\alpha$ is a $(\lambda', \mu')$–quasigeodesic. Since $M > \lambda' \mu'$, it follows that $\alpha$ has different endpoints. Therefore we have shown that the map $Q' \ast Q' \cap R' R' \to G$ is injective.
It is left to prove that if $Q'$ and $R'$ are relatively quasiconvex, then $\langle Q', R' \rangle$ is relatively quasiconvex. Let $g \in \langle Q \cap R \rangle$ and let $\gamma$ be a geodesic from 1 to $g$. Since $H$ is quasiconvex, if $g \in H$ then $\gamma \cap (X \setminus U)$ is uniformly close to $H$ and hence to $\langle Q \cap R \rangle$. Suppose that $g \not\in H$. By Lemma 7 (Morse Lemma), any $(\lambda', \mu')$–quasigeodesic is at Hausdorff distance at most $L$ from any geodesic between its endpoints. In particular, $\gamma \cap (X \setminus U) \subseteq N_L(\alpha) \cap (X \setminus U)$ where $\alpha$ is the quasigeodesic constructed above. It is enough to show that $\alpha \cap N_L(X \setminus U)$ is contained in $N_{\mu}((Q' \cup R'))$. Let $p \in \alpha \cap N_L(X \setminus U)$ and let $i$ be so that $p \in [h_i, h_{i+1}] \cap N_L(X \setminus U)$. Assume $g_{i+1} \in Q'$, the other case being symmetric. As $Q'$ is relatively quasiconvex and in view of Proposition 14, there is a constant $\mu$ so that $p \in N_{\mu}(h_i Q') \subseteq N_{\mu}((Q' \cup R'))$ (as $h_i \in \langle Q' \cup R' \rangle$).

5 Separability of double cosets

We now show Corollary 6. Suppose that all quasiconvex subgroups of $G$ are separable. Let $Q$ and $R$ be quasiconvex subgroups with compatible parabolic subgroups. Let $g \in G$ and suppose that $g \not\in QR$. We follow an argument described in Minasyan [11] and Yang [14].

Let $K = K(Q, R, Q \cap R)$ be the constant of Lemma 16. As in the proof of Theorem 2, let $M$ be large enough so that $M > k, \lambda' \mu'$ where $k, \lambda'$ and $\mu'$ are as in Lemma 8 for $\lambda = 1, \mu = K$. In addition, assume that

$$
(1) \quad M > \lambda' \text{dist}(1, g) + \lambda' \mu'.
$$

**Lemma 18** There are finite index subgroups $Q'$ and $R'$ of $Q$ and $R$ respectively such that $g \not\in Q \langle Q', R' \rangle R$.

**Proof** Since $Q \cap R$ is separable, there are finite index subgroups $Q'$ and $R'$ of $Q$ and $R$ respectively, such that $Q' \cap R' = Q \cap R$ and dist$(1, f) \geq 2M$ for any $f$ in $Q' \setminus Q' \cap R'$ or $R' \setminus Q' \cap R'$. By Theorem 2 $\langle Q' \cup R' \rangle$ is a quasiconvex subgroup of $G$ isomorphic to $Q' \ast_{Q \cap R} R'$.

Suppose that $g \in Q \langle Q', R' \rangle R$. Since $g \not\in QR$ it follows that $g = g_1 \cdots g_{2n}$ where $g_1 \in Q$, $g_{2i} \in R$, $g_{2i+1} \in Q' \setminus Q \cap R$, $g_{2i} \in R' \setminus Q \cap R$, and $n \geq 2$. Assume that this product is minimal in the sense that $\sum \text{dist}(1, g_i)$ is minimal among all such products describing $g$.

For each $i$, let $h_i = g_1 \cdots g_i$; let $\alpha_i$ be a geodesic from $h_i$ to $h_{i+1}$. By the choice of $Q'$ and $R'$ each segment $\alpha_i$ has length at least $2M$ except $\alpha_1$ and $\alpha_{2n-1}$.
Notice that $g_2 \cdots g_{2n-1}$ represents an element of $Q' *_{Q \cap R} R'$ and such product is minimal in the sense of the previous section, so that by Lemma 17 the concatenation $\alpha_2 \cdots \alpha_{2n-1}$ is an $M$–local $(1, K)$–quasigeodesic. Minimality of $g_1 \cdots g_{2n}$ and Lemma 16 imply that the concatenations $\alpha_1 \alpha_2$ and $\alpha_{2n-1} \alpha_{2n}$ are $M$–local $(1, K)$–quasigeodesics. Since $\alpha_2$ and $\alpha_{2n-1}$ have both length at least $2M$, it follows that the concatenation $\alpha = \alpha_1 \cdots \alpha_{2n}$ an $M$–local $(1, K)$–quasigeodesic.

By Lemma 8, it follows that $\alpha$ is a $(\lambda', \mu')$–quasigeodesic between 1 and $g$. It follows that $\operatorname{dist}(1, g) \geq 4M/\lambda' - \mu'$; this is a contradiction with Equation (1) above.

Since $Q'$ and $R'$ are of finite index, there are $q_1, \ldots, q_k \in Q$ and $r_1, \ldots, r_m \in R$ such that

$$Q \langle Q', R' \rangle R = \bigcup_{q_i, r_j} q_i \langle Q', R' \rangle r_j.$$  

Since $\langle Q', R' \rangle$ is quasiconvex, it is closed in the profinite topology. It follows that $Q \langle Q', R' \rangle R$ is a finite union of closed sets. Therefore $Q \langle Q', R' \rangle R$ is a closed set in the profinite topology containing $QR$ and such that $g \not\in Q \langle Q', R' \rangle R$. Since $g$ was an arbitrary element of $g \in G$ not in $QR$, it follows that $QR$ is closed in the profinite topology of $G$.

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