Algebraic & Geometric Topology 12 (2012) 2245–2258

Bourgin–Yang version of the Borsuk–Ulam theorem for $\mathbb{Z}_{p^k}$–equivariant maps

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Let $G = \mathbb{Z}_{p^k}$ be a cyclic group of prime power order and let V and W be orthogonal representations of G with $V^G = W^G = \{0\}$. Let $S(V)$ be the sphere of V and suppose $f\colon S(V) \to W$ is a G–equivariant mapping. We give an estimate for the dimension of the set $f^{-1}\{0\}$ in terms of V and W. This extends the Bourgin–Yang version of the Borsuk–Ulam theorem to this class of groups. Using this estimate, we also estimate the size of the G–coincidences set of a continuous map from $S(V)$ into a real vector space W'.

55M20; 55M35, 55N91, 57S17

1 Introduction

In 1954 and 1955 C T Yang [15; 16] and (independently) D G Bourgin [4] proved a theorem on $\mathbb{Z}_2$–equivariant mappings f from the unit sphere $S(\mathbb{R}^n)$ in $\mathbb{R}^n$ into $\mathbb{R}^m$, where the Euclidean spaces are considered as representations of $\mathbb{Z}_2$ with the antipodal action. They showed that there is an estimate

$$\dim Z_f \geq n - m - 1,$$

where $Z_f := f^{-1}(0)$ and dim means covering dimension and as a consequence generalized the classical Borsuk–Ulam theorem. Munkholm in [9; 10] extended the Borsuk–Ulam theorem to the case of continuous maps $f\colon S^{2n-1} \to \mathbb{R}^m$ and free actions of a cyclic group $G = \mathbb{Z}_{p^k}$ (p prime, $k \geq 1$) on S^{2n-1}, giving an estimate for the covering dimension of the set $A(f) = \{x \in S^{2n-1} \mid f(x) = f(gx) \text{ for all } g \in G\}$.

In [6], Dold extended the Bourgin–Yang problem to a fiberwise setting, giving an estimate for the dimension of $Z_f = f^{-1}(0)$, where $\pi\colon E \to B$ and $\pi'\colon E' \to B$ are vector bundles and $f\colon S(E) \subset E \to E'$ is a $\mathbb{Z}_2$–map, which preserves fibers ($\pi' \circ f = \pi$). Izydorek and Rybicki [7] and Nakaoka [11] considered this problem for the case of the cyclic group $G = \mathbb{Z}_p$ (p prime), and in [8] the last two authors

Published: 5 January 2013 DOI: 10.2140/agt.2012.12.2245

of this paper considered the problem for bundles $E \to B$ whose fiber has the same cohomology (mod p) as a product of spheres. In all these cases, if B is a single point, one obtains Bourgin–Yang versions of the Borsuk–Ulam theorem for $G = \mathbb{Z}_p$, with p prime.

Here we study the Bourgin–Yang problem when G is a cyclic group of a prime power order, $G = \mathbb{Z}_{p^k}$, $k \geq 1$.

Let V, W be two orthogonal representations of G such that $V^G = W^G = \{0\}$ for the sets of fixed points of G. Let $f\colon S(V) \to W$ be a G–equivariant mapping. We denote $Z_f := \{v \in S(V) \mid f(v) = 0\}$.

For $G = \mathbb{Z}_{p^k}$, with p odd, every nontrivial irreducible orthogonal representation is even dimensional and admits a complex structure (see Serre [14]), thus V and W admit one too. Put $d(V) = \dim_{\mathbb{C}} V = \frac{1}{2} \dim_{\mathbb{R}} V$ and $d(W) = \dim_{\mathbb{C}} W = \frac{1}{2} \dim_{\mathbb{R}} W$; these are integral numerical invariants of V and W, respectively. If $G = \mathbb{Z}_{2^k}$ and V, W are orthogonal representations of G, then we put $d(V) = \dim_{\mathbb{R}} V$, and respectively, $d(W) = \dim_{\mathbb{R}} W$.

Our main result says:

Theorem 1.1 *Let V, W be two orthogonal representations of the cyclic group $\mathbb{Z}_{p^k}$ and $f\colon S(V) \to W$ an equivariant map.*

Then the covering dimension $\dim Z_f = \dim(Z_f/G) \geq \phi(V, W)$, *where ϕ is a function depending on $d(V)$, $d(W)$ and the orders of the orbits of actions on $S(V)$ and $S(W)$, which we describe later (see Theorems 3.6 and 3.9).*

In particular, if $d(W) < d(V)/p^{k-1}$, then $\phi(V, W) \geq 0$, which means that there is no G–equivariant map from $S(V)$ into $S(W)$.

As a consequence, we also give an estimate of the covering dimension of the set $A(f)$ of $\mathbb{Z}_{p^k}$–coincidences of a continuous map $f\colon S(V) \to W'$, where W' is a real vector space.

The paper is organized as follows. In Section 2, we recall the definition of a length index in equivariant K–theory, presenting its properties and an estimate for the length index of $S(V)$, given by Bartsch in [3]. In Section 3, we estimate the length index of Z_f and as a consequence, we obtain a Bourgin–Yang version of the Borsuk–Ulam theorem for $\mathbb{Z}_{p^k}$, p prime, $k \geq 1$. The case $p = 2$ is discussed separately. Finally, in Section 4 we provide an estimate the size of the Z_{p^k}–coincidences set of a continuous map from $S(V)$ into a real vector space W'.

2 A length index in the equivariant K–theory

In this section, we briefly recall the notion of an equivariant index based on the cohomology length in a given cohomology theory. It was introduced and described in detail by Thomas Bartsch in [3, Chapter 4]. He presented a very general version of the mentioned index, considering an equivariant map between two pairs of G–spaces and defining an index for this triple. We consider the case when these two pairs are equal and the map is equal to the identity. Moreover, we study this notion taking as an equivariant cohomology theory the equivariant K–theory, denoted by $K_G^*(X)$. This is a theory generated by K_G–theory, ie the equivariant K–theory of G–vector bundles, extended to the next gradation by use of the equivariant Bott periodicity (see Atiyah and Segal [1] and Segal [12; 13]).

Let us fix a set $\mathcal{A}$ of G–spaces. Usually, it is a family of orbits, which is obviously finite, if G is finite.

Definition 2.1 The $(\mathcal{A}, K_G^*)$*–cup length* of a pair (X, X') of G–spaces is the smallest r such that there exist $A_1, A_2, \dots, A_r \in \mathcal{A}$ and G–maps $\beta_i\colon A_i \to X$, $1 \le i \le r$ with the property that for all $\gamma \in K_G^*(X, X')$ and for all $\omega_i \in \ker \beta_i^*$ we have

$$\omega_1 \cup \omega_2 \cup \dots \cup \omega_r \cup \gamma = 0 \in K_G^*(X, X').$$

If there is no such r, we say that the $(\mathcal{A}, K_G^*)$–cup length of (X, X') is ∞. $r = 0$ means that $K_G^*(X, X') = 0$. Moreover, the $(\mathcal{A}, K_G^*)$–cup length of X is by definition the cup length of the pair $(X, \varnothing)$.

Instead of $(\mathcal{A}, K_G^*)$–cup length, one can consider also the notion of $(\mathcal{A}, K_G^*, R)$–length index, defined in a little bit different manner (see Bartsch [3]). Once more we leave out the general situation of [3], presenting only the case necessary for our considerations.

Recall that for the equivariant cohomology theory K_G^* and a G–pair (X, X'), the cohomology $K_G^*(X, X')$ is a module over the coefficient ring $K_G^*(\mathrm{pt})$, via the natural G–map $p_X\colon X \to \mathrm{pt}$. We write

$$\omega \cdot \gamma = p_X^*(\omega) \cup \gamma \quad \text{and} \quad \omega_1 \cdot \omega_2 = \omega_1 \cup \omega_2$$

for $\gamma \in K_G^*(X, X')$ and $\omega_1, \omega_2 \in K_G^*(\mathrm{pt})$.

Taking $R := K_G(\mathrm{pt}) = R(G) \subset K_G^*(\mathrm{pt})$, we obtain the following adjustment of [3, Definition 4.1].

Definition 2.2 The $(\mathcal{A}, K_G^*, R)$*–length index* of a pair (X, X') of G–spaces is the smallest r such that there exist $A_1, A_2, \dots, A_r \in \mathcal{A}$ with the following property:

For all $\gamma \in K_G^*(X,X')$ and all

$$\omega_i \in R \cap \ker\bigl(K_G^*(\mathrm{pt}) \to K_G^*(A_i)\bigr) = \ker\bigl(K_G(\mathrm{pt}) \to K_G(A_i)\bigr), \quad i = 1,2,\dots,r,$$

the product

$$\omega_1 \cdot \omega_2 \cdots \omega_r \cdot \gamma = 0 \in K_G^*(X,X').$$

A comparison of these numerical invariants is given in the following statement (see [3, page 59]).

Proposition 2.3 *For any system $\mathcal{A}$ and every pair of G–spaces (X,X') we have*

$$(\mathcal{A},\, K_G^*,\, R)\text{–length index of } (X,X') \le (\mathcal{A},\, K_G^*)\text{–cup length of } (X,X').$$

The $(\mathcal{A}, K_G^*, R)$*–length index* has many properties that are important from the point of view of applications to study critical points of G–invariant functions and functionals (see [3]). We shall use only a few of them.

Following [3], for given two powers $1 \le m \le n \le p^{k-1}$ of p we set

$$\mathcal{A}_{m,n} := \{G/H \mid H \subset G; m \le |H| \le n\}, \tag{1}$$

where $|H|$ is the cardinality of H. Next we put

$$l_n(X,X') = (\mathcal{A}_{m,n},\, K_G^*,\, R)\text{–length index of } (X,X'). \tag{2}$$

Remark 2.4 By [3, Observation 5.5], the index l_n does not depend on m. It says that if $\mathcal{A}' \subseteq \mathcal{A}$ is such that for each $A \in \mathcal{A}$ there exists $A' \in \mathcal{A}'$ and a G–map $A \to A'$ then

$$(\mathcal{A},\, K_G^*,\, R)\text{–length index} = (\mathcal{A}',\, K_G^*,\, R)\text{–length index}.$$

The following theorem is fundamental for our version of the Bourgin–Yang theorem for $G = \mathbb{Z}_{p^k}$ (cf Bartsch [2; 3]). We shall write $\mathcal{A}_X$ for a set of all the G–orbits of X (up to a homeomorphism, thus up to an isomorphism of finite G–sets).

Theorem 2.5 [3, Theorem 5.8] *Let V be an orthogonal representation of $G = \mathbb{Z}_{p^k}$ with $V^G = \{0\}$ and $d = d(V) = \frac{1}{2}\dim_{\mathbb{R}} V$. Fix m,n two powers of p as above. Then*

$$l_n(S(V)) \ge \begin{cases} 1 + \left[\frac{(d-1)m}{n}\right] & \text{if } \mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}, \\ \infty & \text{if } \mathcal{A}_{S(V)} \not\subseteq \mathcal{A}_{1,n}, \end{cases}$$

where $[x]$ denotes the least integer greater than or equal to x. Also, if $\mathcal{A}_{S(V)} \subset \mathcal{A}_{n,n}$, then

$$l_n(S(V)) = d.$$

3 Bourgin–Yang theorem for $\mathbb{Z}_{p^k}$

In this section we prove our main theorem. To do it, we first need to discuss a relation between the $l_n = (\mathcal{A}_{m,n}, K_G^*, R)$–length index of a G–set X and its dimension.

If X is a compact G–space, where G is a compact Lie group, in [12, Proposition 5.3, page 147], G Segal showed that there is an Atiyah–Hirzebruch spectral sequence for equivariant K–theory:

$$E_2^{s,t} = H^s(X/G; \mathcal{K}_G^t) \Rightarrow K_G^*(X),$$

where $\mathcal{K}_G^t$ is the sheaf on X/G associated to the presheaf $V \mapsto K_G^t(\pi^{-1}\overline{V})$ (here $\pi\colon X \to X/G$ is the projection) with the stalk $\mathcal{K}_G^t$ at an orbit $Gx = G/G_x$ equal to $R(G_x)$, if t is even, and $\mathcal{K}_G^t = 0$, if t is odd.

Moreover, there exists an invariant filtration of X such that $K_G^*(X)$ is the associated module of the limit of this spectral sequence with respect to this filtration.

If X is a G–CW-complex, which is filtered by its skeletons $\{X^s\}$, it is customary to define a filtration of $K_G^*(X)$ by setting $K_{G,s}^*(X) = \ker(K_G^*(X) \to K_G^*(X^{s-1}))$. It corresponds to a filtering of X by the G–subspaces $\pi^{-1}(Y^s)$, when the orbit space $Y = X/G$ is a CW-complex and $\{Y^s\}$ its skeletons ($\pi\colon X \to Y$ is the projection).

The general case is discussed in [12, Section 5] by use of the nerve of a G–stable closed finite covering of X. To each finite covering $\mathcal{U} = \{U_j\}_{j\in\mathcal{S}}$ of a compact G–space X by G–stable closed sets is associated a compact G–space $\mathcal{W}_{\mathcal{U}}$, with a G–map $w\colon \mathcal{W}_{\mathcal{U}} \to X$ and a filtration by G–subspaces $\mathcal{W}_{\mathcal{U}}^0 \subset \mathcal{W}_{\mathcal{U}}^1 \subset \cdots \subset \mathcal{W}_{\mathcal{U}}^j \subset \cdots \subset \mathcal{W}_{\mathcal{U}}$, so that the following conditions are satisfied:

(i) $w^*\colon K_G^*(X) \to K_G^*(\mathcal{W}_{\mathcal{U}})$ is an isomorphism, and

(ii) when $\mathcal{V}$ is a refinement of $\mathcal{U}$, there is a G–map $\mathcal{W}_{\mathcal{U}} \to \mathcal{W}_{\mathcal{V}}$ defined up to G–homotopy, respecting the filtrations and the projections onto X.

Definition 3.1 We say that an element of $K_G^*(X)$ *is in* $K_{G,s}^*(X)$ if, for some finite covering $\mathcal{U}$, it is in the kernel of $w^*\colon K_G^*(X) \to K_G^*(\mathcal{W}_{\mathcal{U}}^{s-1})$.

For the filtration of $K_G^*(X)$ defined above

$$K_G^*(X) = K_{G,0}^*(X) \supset K_{G,1}^*(X) \supset \cdots \supset K_{G,s}^*(X) \supset \cdots, \tag{3}$$

$K_G^*(X)$ is a filtered ring in the sense that

$$K_{G,s}^*(X) \cdot K_{G,s'}^*(X) \subset K_{G,s+s'}^*(X),$$

thus $K_{G,s}^*(X)$ is an ideal in $K_G^*(X)$ (see [12, pages 145–146]).

Moreover, we have the following:

Proposition 3.2 (i) *An element of $K^*_G(X)$ is in $K^*_{G,1}(X)$ if, and only if, its restriction to each orbit is zero, ie*

$$K^*_{G,1}(X) = \ker\Big(K^*_G(X) \to \prod_{x\in X} K^*_G(G/G_x)\Big) = \bigcap_{x\in X} \ker(K^*_G(X) \to K^*_G(G/G_x)).$$

(ii) *If the subgroups of G are totally ordered and H is the largest isotropic subgroup on X, then*

$$K^*_{G,1}(X) = \ker(K^*_G(X) \to K^*_G(G/H)).$$

Proof (i) [12, Proposition 5.1(i), page 146].

(ii) By (i), ξ belongs to $K^*_{G,1}(X)$ if, and only if, $\xi \in \ker(K^*_G(X) \to K^*_G(G/G_x))$, for all $x \in X$. Then, $K^*_{G,1}(X) \subset \ker(K^*_G(X) \to K^*_G(G/H))$. Conversely, since the subgroups of G are totally ordered and H is the largest isotropic subgroup on X, there exist natural G–maps $G/G_x \to G/H$, for all $x \in X$. Then

$$\ker(K^*_G(X) \to K^*_G(G/H)) \subset \ker(K^*_G(X) \to K^*_G(G/G_x)) \quad \text{for all } x \in X,$$

and consequently, $\ker(K^*_G(X) \to K^*_G(G/H)) \subset K^*_{G,1}(X)$. □

Lemma 3.3 *If X is a compact G–space such that $\dim(X/G) \le 2r-1$, then*

(i) $K_{G,2i}(X) = K^0_{G,2i}(X) = K^0_{G,2i-1}(X) = K_{G,2i-1}(X)$, *for all* $i = 1, 2, 3, \dots,$

(ii) $(K_{G,2}(X))^r = (K^0_{G,2}(X))^r = 0$.

Proof Consider the Atiyah–Hirzebruch spectral sequence in the equivariant K–theory:

$$E_2^{s,t} = H^s(X/G; \mathcal{K}^t_G) \Rightarrow K^*_G(X)$$

as above. We have that $E_2^{s,t} = 0$ if $s < 0$ or if t is odd and this implies that

$$E_2^{s,t} = E_\infty^{s,t} = 0 \quad \text{for } s < 0 \text{ or } t \text{ odd}.$$

It follows from the definition of the filtration of $K_G(X) = K^0_G(X)$,

$$K^0_G(X) = K^0_{G,0}(X) \supset K^0_{G,1}(X) \supset \cdots \supset K^0_{G,s}(X) \supset \cdots,$$

that $K^0_{G,s}(X) = 0$, for all $s > \overline{s}$, where $\overline{s} = \dim(X/G)$. Since by assumption $\overline{s} = \dim(X/G) \le 2r-1$, we have in particular that $K^0_{G,2r}(X) = 0$. Consequently, since $(K^0_{G,2}(X))^r \subset K^0_{G,2r}(X)$ we conclude that $(K^0_{G,2}(X))^r = 0$.

Moreover, for all $s > \overline{s}$ and for all t we have that

$$0 = H^s(X/G; \mathcal{K}^t_G) \cong E_2^{s,t} \cong E_\infty^{s,t}$$

and using the definition of the infinite term $E_\infty^{s,t} \cong K_{G,s}^{s+t}(X)/K_{G,s+1}^{s+t}(X)$ we have the following exact sequences:

$$0 \longrightarrow K^0_{G,1}(X) \longrightarrow K^0_G(X) \longrightarrow E_\infty^{0,0} \longrightarrow 0$$

$$0 \longrightarrow K^0_{G,2}(X) \longrightarrow K^0_{G,1}(X) \longrightarrow E_\infty^{1,-1} \longrightarrow 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0 \longrightarrow K^0_{G,2i}(X) \longrightarrow K^0_{G,2i-1}(X) \longrightarrow E_\infty^{2i-1,-2i+1} \longrightarrow 0$$

$$0 \longrightarrow K^0_{G,2i+1}(X) \longrightarrow K^0_{G,2i}(X) \longrightarrow E_\infty^{2i,-2i} \longrightarrow 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0 \longrightarrow K^0_{G,\bar{s}}(X) \longrightarrow K^0_{G,\bar{s}-1}(X) \longrightarrow E_\infty^{\bar{s}-1,-\bar{s}+1} \longrightarrow 0$$

$$0 \longrightarrow K^0_{G,\bar{s}}(X) \longrightarrow E_\infty^{\bar{s},-\bar{s}} \longrightarrow 0$$

Since $E_\infty^{2i-1,-2i+1} = 0$, it follows that

$$K^0_{G,2i}(X) = K^0_{G,2i-1}(X) \quad \text{for all } i = 1, 2, 3, \dots. \qquad \square$$

Remark 3.4 From Deo and Singh [5, Theorem 1.1] it follows that for an action of a finite group G on a paracompact Hausdorff space X we have $\dim X = \dim(X/G)$. Consequently, in this case, to estimate $\dim X$ it is enough to estimate $\dim(X/G)$.

Theorem 3.5 *Let X be a compact G–space, with $G = \mathbb{Z}_{p^k}$, and suppose that $\mathcal{A}_X \subset \mathcal{A}_{m,n}$. If $l_n(X) \geq r+1$, then* $\dim X = \dim(X/G) \geq 2r$.

Proof First, we observe that to compute $l_n = (\mathcal{A}_{m,n}, K^*_G(X), R)$–length index with

$$\mathcal{A}_{m,n} = \{G/H \mid H \subset G; m \leq |H| \leq n\},$$

there is no loss of generality in assuming that $\mathcal{A}_{m,n}$ consists of just one element G/H, because the subgroups of G are totally ordered (see Remark 2.4, [3, page 76], and (4) below). Take $H \in \mathcal{A}_{m,n}$ such that $|H| = n$. Once more, if $K \subset H$ then there is a G–map $\phi\colon G/K \to G/H$, and consequently

$$\ker(K_G(\mathrm{pt}) \to K_G(G/H)) \subset \ker(K_G(\mathrm{pt}) \to K_G(G/K)). \tag{4}$$

Let $Gx \simeq G/G_x \in \mathcal{A}_{m,n}$ be an orbit of $x \in X$ with biggest isotropy subgroup $G_x \subset H$. If $\omega \in \ker(K_G(\mathrm{pt}) \to K_G(G/G_x))$, then $p_X^*(\omega) \in \ker(\beta^*\colon K_G(X) \to K_G(G/G_x))$, for any G–map $\beta\colon G/G_x \to X$, in particular for the inclusion $Gx \subset X$ (see [3, page 59]). Consequently, if $\omega \in \ker(K_G(\mathrm{pt}) \to K_G(G/H))$, then

$$p_X^*(\omega) \in K^0_{G,1}(X) = K^0_{G,2}(X),$$

by Equation (4), Proposition 3.2(ii), and Lemma 3.3(i). Therefore, for every ω_i in $\ker(K_G(\mathrm{pt}) \to K_G(G/H))$, $i = 1, 2, \dots, r$, we have

$$\omega_1 \cdot \omega_2 \cdots \omega_r = p_X^*(\omega_1) \cup p_X^*(\omega_2) \cup \cdots \cup p_X^*(\omega_r) \in (K^0_{G,2}(X))^r \subset K_G(X). \tag{5}$$

If $\dim(X/G) \le 2r - 1$, by Lemma 3.3(ii), $(K^0_{G,2}(X))^r = 0$ and it follows from (5) and from definition of the length index that $l_n(X) \le r$. □

We are in a position to prove our main theorem.

Theorem 3.6 *Let V, W be two complex orthogonal representations of the cyclic group $G = \mathbb{Z}_{p^k}$, p prime, $k \ge 1$, such that $V^G = W^G = \{0\}$. Let*

$$f\colon S(V) \xrightarrow{G} W$$

be an equivariant map and $Z_f = f^{-1}(0)$. Suppose $\mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}$ and $\mathcal{A}_{S(W)} \subset \mathcal{A}_{m,n}$. Then

$$l_n(Z_f) \ge 1 + \left[\frac{(d(V)-1)m}{n}\right] - d(W).$$

Consequently,

$$\dim Z_f \ge 2\left(\left[\frac{(d(V)-1)m}{n}\right] - d(W)\right) := \phi(V, W).$$

In particular, if $d(W) < d(V)/p^{k-1}$, then $\phi(V, W) \ge 0$, which means that there is no G–equivariant map from $S(V)$ into $S(W)$.

Proof Denote $\mathcal{U} = S(V) \setminus Z_f$, which is an open and invariant set. From the continuity of the equivariant K–theory, it follows that there exists an open invariant set $\mathcal{V} \subset S(V)$ such that

$$Z_f \subset \mathcal{V} \quad \text{and} \quad K_G^*(\mathcal{V}) = K_G^*(Z_f).$$

This yields $l_n(Z_f) = l_n(\mathcal{V})$. Moreover,

$$K_G^*(W \setminus \{0\}) = K_G^*(S(W)),$$

by the equivariant deformation argument and then $l_n(W \setminus \{0\}) = l_n(S(W))$. Since f maps $\mathcal{U}$ equivariantly into $W \setminus \{0\}$ we have

$$l_n(\mathcal{U}) \le l_n(W \setminus \{0\}) = l_n(S(W))$$

by the corresponding monotonicity property of the length index (see [3, Theorem 4.6]). Obviously, $\mathcal{U} \cup \mathcal{V} = S(V)$. It follows by the additivity property of the length index (see [3, Theorem 4.6]) that

$$l_n(S(V)) \le l_n(\mathcal{V}) + l_n(\mathcal{U}),$$

which gives

$$l_n(Z_f) \geq l_n(S(V)) - l_n(S(W)).$$

By Theorem 3.5, we have $l_n(S(W)) \leq d(W)$. Further, by assumption $\mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}$ and it follows from Theorem 2.5 that

$$l_n(Z_f) \geq 1 + \left[\frac{(d(V)-1)m}{n}\right] - d(W).$$

Consequently, from Theorem 3.5,

$$\dim Z_f \geq 2\left(\left[\frac{(d(V)-1)m}{n}\right] - d(W)\right) := \phi(V,W). \qquad \square$$

Now, let us consider V, W real orthogonal representations of the group $G = \mathbb{Z}_{2^k}$ and let $f\colon S(V) \to W$ be an equivariant map. Since Bartsch's computation works for complex orthogonal representations we consider

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} \quad \text{and} \quad W_{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}.$$

Note that $V_{\mathbb{C}}$ is a complex space of complex dimension $d(V_{\mathbb{C}}) = \dim_{\mathbb{C}} V_{\mathbb{C}} = \dim_{\mathbb{R}} V = d(V)$, and analogously $d(W_{\mathbb{C}}) = \dim_{\mathbb{C}} W_{\mathbb{C}} = \dim_{\mathbb{R}} W = d(W)$.

Moreover, $V_{\mathbb{C}} \simeq V \oplus V$ is a real representation of the group $G = \mathbb{Z}_{2^k}$, and has a natural structure of a unitary representation of G given by $g(x\cdot 1 + y\cdot \imath) = gx\cdot 1 + gy\cdot \imath$. The same is true for $W_{\mathbb{C}}$. We know that $S(V \oplus V) = S(V) * S(V)$, as a G–space.

Now, we define an equivariant extension $\tilde{f}\colon S(V_{\mathbb{C}}) \to W_{\mathbb{C}}$ of the map f by the formula

(6) $$\tilde{f}((x,t,y)) = tf(x)\cdot 1 + (1-t)f(y)\cdot \imath.$$

Lemma 3.7 *The map $\tilde{f}$ is G–equivariant and we have*

$$Z_{\tilde{f}} = Z_f * Z_f.$$

Proof Surely $\tilde{f}$ is equivariant. Note that the vectors $tf(x)\cdot 1$ and $(1-t)f(y)\cdot \imath$ are in perpendicular subspaces (orthogonal subrepresentations). Consequently, their sum is equal to 0 if, and only if, both are equal to 0. This shows that a point (x,t,y) in $S(V) * S(V)$ is mapped by $\tilde{f}$ onto 0 if and only if it belongs to $Z_f * Z_f$, which proves the lemma. $\square$

Corollary 3.8

$$2l_n(Z_f) \geq l_n(Z_{\tilde{f}}) \quad \textit{and} \quad \dim Z_{\tilde{f}} = 2\dim Z_f + 1.$$

Proof The inequality of the statement follows from Lemma 3.7 and [3, Corollary 4.10]. The equality is a direct consequence of Lemma 3.7 and of a well-known fact about the dimension of the join. □

As a consequence of the previous results we obtain the following Bourgin–Yang version of the Borsuk–Ulam theorem for real orthogonal representations of $G = \mathbb{Z}_{2^k}$, $k \geq 1$.

Theorem 3.9 *Let V, W be two real orthogonal representations of the cyclic group $G = \mathbb{Z}_{2^k}$, $k \geq 1$, such that $V^G = W^G = \{0\}$. Let*

$$f\colon S(V) \xrightarrow{G} W$$

be an equivariant map and $Z_f = f^{-1}(0)$. Suppose $\mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}$ and $\mathcal{A}_{S(W)} \subset \mathcal{A}_{m,n}$. Then

$$2l_n(Z_f) \geq 1 + \left[\frac{(d(V)-1)m}{n}\right] - d(W)$$

and

$$\dim Z_f \geq \left[\frac{(d(V)-1)m}{n}\right] - d(W) = \phi(V, W).$$

In particular, if $d(W) < d(V)/2^{k-1}$, then $\phi(V, W) \geq 0$, which means that there is no G–equivariant map from $S(V)$ into $S(W)$.

Proof Let us consider the equivariant extension $\tilde{f}\colon S(V_{\mathbb{C}}) \to W_{\mathbb{C}}$ of f as defined in (6). Applying Theorem 3.6 and using Corollary 3.8, we obtain the desired result. □

Remark 3.10 In particular, if $\mathcal{A}_{S(V)} \subset \mathcal{A}_{n,n}$ and $\mathcal{A}_{S(W)} \subset \mathcal{A}_{n,n}$ by Theorem 3.6 we have

$$\dim Z_f \geq 2(d(V) - d(W)) - 2.$$

We note that, if we consider $G = \mathbb{Z}_p$ (p prime), that is, $k = 1$, then by using the equivariant Borel cohomology it is possible to show that

$$\dim Z_f \geq 2(d(V) - d(W)) - 1,$$

which is the same estimate given by Izydorek and Rybicki by considering $B = \mathrm{pt}$ in the parametrized Borsuk–Ulam problem [7, Corollary 2.1].

Remark 3.11 We observe that Theorem 3.9 extends the Bourgin–Yang theorem for $G = \mathbb{Z}_{2^k}$, since for $k = 1$ we have

$$\dim Z_f \geq d(V) - d(W) - 1.$$

Remark 3.12 We note that opposite to the case $G = \mathbb{Z}_p$, for $G = \mathbb{Z}_{p^k}$ the classical formulation of the Borsuk–Ulam theorem does not hold. Bartsch in [3, Theorem 3.22] showed that for any p–group G which contains an element g of order p^2, there exists an equivariant map $f\colon S(V) \to S(W)$ between two complex representations V, W such that $\dim V > \dim W$.

4 Estimating the size of the $\mathbb{Z}_{p^k}$–coincidences set

Let V be an orthogonal representation of a group G and let W' be a real vector space. Given a continuous map $f\colon S(V) \to W'$, we denote by $A(f)$ the G–coincidences set of f, that is,

$$A(f) = \{v \in S(V) \mid f(gv) = f(v),\ \text{for all } g \in G\}.$$

In this section, we estimate the size of the set $A(f)$, for $G = \mathbb{Z}_{p^k}$, p prime, $k \geq 1$, as follows.

Theorem 4.1 *Let V be an orthogonal representation of the cyclic group $G = \mathbb{Z}_{p^k}$, p prime, $k \geq 1$, such that $V^G = \{0\}$ and let W' be a real vector space. Let $f\colon S(V) \to W'$ be a continuous map. If $\mathcal{A}_{S(V)} \subset \mathcal{A}_{1,p^{k-1}}$, then*

$$\dim A(f) \geq 2\left[\frac{d(V)-1}{p^{k-1}}\right] - (p^k - 1)\dim W'.$$

Proof Let us consider the real vector space $\bigoplus_{i=1}^{p^k} W'$, which is the direct sum of p^k copies of W'. We have that $\bigoplus_{i=1}^{p^k} W'$ admits an action of the cyclic group $G = \mathbb{Z}_{p^k}$, given by

$$g(w_1, w_2, \ldots, w_{p^k}) = (w_2, \ldots, w_{p^k}, w_1),$$

for a fixed generator $g \in G$ and for each $(w_1, \ldots, w_{p^k}) \in \bigoplus_{i=1}^{p^k} W'$.

Let us denote by Δ the vector subspace of $\bigoplus_{i=1}^{p^k} W'$ consisting of all points

$$(w_1, w_2, \ldots, w_{p^k}) \quad \text{such that } w_1 = \cdots = w_{p^k}.$$

We have $\bigoplus_{i=1}^{p^k} W' = \Delta \oplus \Delta^\perp$, where $\Delta^\perp$ is the orthogonal complement of Δ. Since Δ is a $\dim W'$–dimensional G–subspace of $\bigoplus_{i=1}^{p^k} W'$, let us observe that $\Delta^\perp$ is a $(p^k - 1)\dim W'$–dimensional G–subrepresentation of $\bigoplus_{i=1}^{p^k} W'$, for which $(\Delta^\perp)^G = \{0\}$. Consequently, by the same argument as for Theorem 1.1, it has a complex structure compatible with the action of G.

Consider the G–equivariant map $F\colon S(V) \to \Delta \oplus \Delta^{\perp}$ defined by

$$F(v) = \big(f(v), f(gv), \dots, f(g^{p^k-1}v)\big).$$

The linear orthogonal projection along the diagonal Δ defines a G–equivariant map $r\colon \Delta \oplus \Delta^{\perp} \to \Delta^{\perp}$. Let us denote by h the composition

$$S(V) \xrightarrow{F} \Delta \oplus \Delta^{\perp} \xrightarrow{r} \Delta^{\perp},$$

with $Z_h = h^{-1}(0) = (r \circ F)^{-1}(0) = F^{-1}(\Delta) = A(f)$. Since $h\colon S(V) \to \Delta^{\perp}$ is a G–equivariant map and $\mathcal{A}_{S(\Delta^{\perp})} \subset \mathcal{A}_{1,p^{k-1}}$, it follows from Theorem 3.6 that

$$\dim A(f) = \dim Z_h \geq 2\left[\frac{d(V)-1}{p^{k-1}}\right] - (p^k - 1)\dim W'. \qquad \square$$

For $G = \mathbb{Z}_{2^k}$, $k \geq 1$, by using the same steps as in the proof of Theorem 4.1 and applying Theorem 3.9 we have the following:

Theorem 4.2 *Let V be a real orthogonal representation of the cyclic group $G = \mathbb{Z}_{2^k}$, $k \geq 1$, such that $V^G = \{0\}$ and let W' be a real vector space. Let $f\colon S(V) \to W'$ be a continuous map. If $\mathcal{A}_{S(V)} \subset \mathcal{A}_{1,2^{k-1}}$, then*

$$\dim A(f) \geq \left[\frac{d(V)-1}{2^{k-1}}\right] - (2^k - 1)\dim W'. \qquad \square$$

Remark 4.3 In [10], for the group $G = \mathbb{Z}_{p^k}$, p an odd prime, $k \geq 1$, Munkholm studied the dimension of $A(f)$ under the assumption that the action on $S(V)$ is free by another approach. His formula for an estimate of $\dim A(f)$ from below is of a different form than the one presented here, and there is not a direct comparison between them. He studied the case of a free action, and then his estimate seems to be better than ours. On the other hand, our formula holds for every representation V with $S(V)^G = \varnothing$.

Acknowledgements The authors would like to express their thanks to the referee for pointing out an essential mistake in argument of the proof of main theorem in the first version of this article and for his careful reading and suggestions which led to this present version.

Wacław Marzantowicz was supported by the Polish Research Grant number NCN 2011/03/B/ST1/04533 and FAPESP of Brazil, Grant number 2010/51910-9, for a stay at Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, São Carlos, Brazil. Denise de Mattos was supported by FAPESP of Brazil, grant number 2011/18758-1, for a stay at Faculty of Mathematics and Computer Sciences, Adam

Mickiewicz University of Poznań, Poland. Edivaldo L dos Santos was supported by FAPESP of Brazil, grant number 2011/18761-2, for a stay at Faculty of Mathematics and Computer Sciences, Adam Mickiewicz University of Poznań, Poland.

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Received: 30 April 2012 Revised: 14 August 2012