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On sections of hyperelliptic Lefschetz fibrations

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We construct a relation among right-handed Dehn twists in the mapping class group of a compact oriented surface of genus g with $4g+4$ boundary components. This relation gives an explicit topological description of $4g+4$ disjoint (-1)–sections of a hyperelliptic Lefschetz fibration of genus g on the manifold $\mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}}^2$.

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1 Introduction

Lefschetz fibrations relate the topology of symplectic 4–manifolds to the combinatorics on relations in Dehn twist generators of mapping class groups of surfaces. It is well-known that a Lefschetz fibration of genus 1 on the manifold $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ constructed by blowing up nine intersections of two generic cubics in $\mathbb{CP}^2$ has twelve singular fibers and nine disjoint (-1)–sections. Korkmaz and Ozbagci [7] constructed a relation among right-handed Dehn twists in the mapping class group of a torus with nine boundary components to locate a set of nine disjoint (-1)–sections in a Kirby diagram of $E(1)$. It is also known to algebraic geometers that a hyperelliptic Lefschetz fibration of genus g on the manifold $X_g = \mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}}^2$ has $8g+4$ singular fibers and $4g+4$ disjoint (-1)–sections for $g \geq 2$ (see Saitō and Sakakibara [10, Section 3] and Kitagawa and Konno [6, Remark 1.1]).

In this paper we construct a relation among right-handed Dehn twists in the mapping class group of a compact oriented surface of genus g with $4g+4$ boundary components to locate a set of $4g+4$ disjoint (-1)–sections in a Kirby diagram of X_g. In the case $g=2$, our relation can be considered as an improvement of Onaran's relations [9] in mapping class groups of surfaces of genus two with at most eight boundary components.

In Section 2 we review basic relations in mapping class groups and produce two relations on a torus with eight boundary components. Combining these relations, we construct a new relation on a surface of genus g with $4g+4$ boundary components in Section 3. In Section 4 we apply the relation to visualize $4g+4$ disjoint (-1)–sections in a Kirby diagram of a hyperelliptic Lefschetz fibration of genus g.

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2 Building blocks

In this section we review basic relations in mapping class groups and produce two relations on a torus with boundary both used in the next section.

2.1 Basic relations in mapping class groups

Let $\Sigma_{g,r}$ be a compact oriented surface of genus g with r boundary components and $\mathrm{Diff}_+\Sigma_{g,r}$ the group of orientation-preserving diffeomorphisms of $\Sigma_{g,r}$ fixing the boundary $\partial\Sigma_{g,r}$ pointwise equipped with the C^∞–topology. The group $\pi_0(\mathrm{Diff}_+\Sigma_{g,r})$ of path-components of $\mathrm{Diff}_+\Sigma_{g,r}$ is called the *mapping class group* of $\Sigma_{g,r}$ and we denote it by $\mathcal{M}_{g,r}$. We denote by $\mathcal{F}_{g,r}$ the free group generated by all isotopy classes $\mathcal{S}_{g,r}$ of simple closed curves in the interior of $\Sigma_{g,r}$. There is a natural epimorphism $\varpi : \mathcal{F}_{g,r} \to \mathcal{M}_{g,r}$ which sends (the isotopy class of) a simple closed curve a in the interior of $\Sigma_{g,r}$ to the right-handed Dehn twist t_a along a. We set $\mathcal{R}_{g,r} := \mathrm{Ker}\,\varpi$.

A word in the generators $\mathcal{S}_{g,r}$ is called *positive* if it includes no negative exponents. We put ${}_W(c) := t_{a_r}^{\varepsilon_r}\cdots t_{a_1}^{\varepsilon_1}(c) \in \mathcal{S}_{g,r}$ for $c \in \mathcal{S}_{g,r}$ and $W = a_r^{\varepsilon_r}\cdots a_1^{\varepsilon_1} \in \mathcal{F}_{g,r}$ ($a_1,\dots,a_r$ in $\mathcal{S}_{g,r}$, $\varepsilon_1,\dots,\varepsilon_r$ in $\{\pm1\}$). We often denote a^{-1} by $\bar{a}$ for an element a of $\mathcal{S}_{g,r}$. For two words $W_1, W_2 \in \mathcal{F}_{g,r}$, we denote $W_1 \equiv W_2$ if $\varpi(W_1) = \varpi(W_2)$. If the relation $W_1 \equiv W_2$ holds for $W_1, W_2 \in \mathcal{F}_{g,r}$, we obtain another relation $V W_1 V^{-1} \equiv V W_2 V^{-1}$, which is called a *conjugate* of $W_1 \equiv W_2$, for every $V \in \mathcal{F}_{g,r}$.

We recall definitions of basic relations in mapping class groups.

Definition 2.1 [3] (1) For disjoint simple closed curves a, b in the interior of $\Sigma_{g,r}$, we have a relation $ab \equiv ba$ in $\mathcal{F}_{g,r}$ called a *commutativity relation*. A regular neighborhood of $a \cup b$ is the disjoint union of two annuli.

(2) For simple closed curves a, b in the interior of $\Sigma_{g,r}$ which intersect transversely at one point, we have a relation $aba \equiv bab$ in $\mathcal{F}_{g,r}$ called a *braid relation*. A regular neighborhood of $a \cup b$ is a torus with one boundary component.

(3) For simple closed curves $\alpha, \sigma, \gamma, \delta_1, \delta_2, \delta_3, \delta_4$ in the interior of $\Sigma_{g,r}$ shown in Figure 1, we have a relation $\delta_1\delta_2\delta_3\delta_4 \equiv \gamma\sigma\alpha$ in $\mathcal{F}_{g,r}$ called a *lantern relation*. The union of $\delta_1, \delta_2, \delta_3, \delta_4$ bounds a sphere with four boundary components in $\Sigma_{g,r}$.

(4) An ordered n–tuple $(c_1,\dots,c_n)$ of simple closed curves in the interior of $\Sigma_{g,r}$ is called a *chain* of length n if c_i and c_{i+1} intersect transversely at one point ($i = 1,\dots,n-1$) and other c_i and c_j never intersect. For a chain $(c_1,\dots,c_{2g+1})$ of length $2g+1$ on $\Sigma_{g,0}$, we have a relation $(c_1\cdots c_{2g+1}c_{2g+1}\cdots c_1)^2 \equiv 1$ in $\mathcal{F}_{g,0}$ called a *hyperelliptic relation* (see Figure 2).

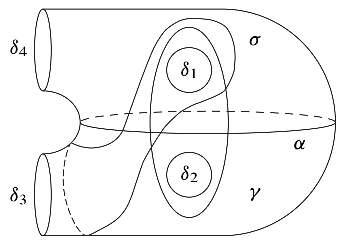

Figure 1: Lantern relation

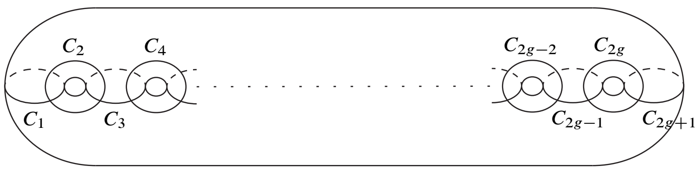

Figure 2: Hyperelliptic relation

Remark 2.2 Let a and b be simple closed curves in the interior of $\Sigma_{g,r}$ and c the simple closed curve $t_b(a) = {}_b(a)$. Then we have the relation $c \equiv ba\bar{b}$ in $\mathcal{F}_{g,r}$. If a and b intersect transversely at one point, we have another relation $b \equiv ac\bar{a}$. This relation together with the relation $c \equiv ba\bar{b}$ yields a braid relation $aba \equiv bab$.

2.2 Two relations on a torus with boundary

In this subsection we construct two relations on a torus with eight boundary components. The first relation is the following.

Proposition 2.3 Relation (A) *For simple closed curves in the interior of $\Sigma_{1,8}$ shown in Figure 3, we have the relation*

$$a_1a_2\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 \equiv a_5a_4b_2\bar{a}_4\sigma_1\sigma_4a_{10}a_3b_2\bar{a}_3\sigma_2a_5a_3a_8b_2\bar{a}_8\bar{a}_3\sigma_3\sigma_5a_{11}.$$

We make use of the five-holed torus relation found by Korkmaz and Ozbagci [7] in order to prove Proposition 2.3.

Lemma 2.4 (Korkmaz–Ozbagci [7]) *For simple closed curves in the interior of $\Sigma_{1,5}$ shown on the right in Figure 4, we have the relation*

$$\delta_2\delta_1a_2\gamma\delta_3 \equiv a_5b_2a_3a_4a_5b_2\sigma_1a_6a_3b_2\sigma_2a_8.$$

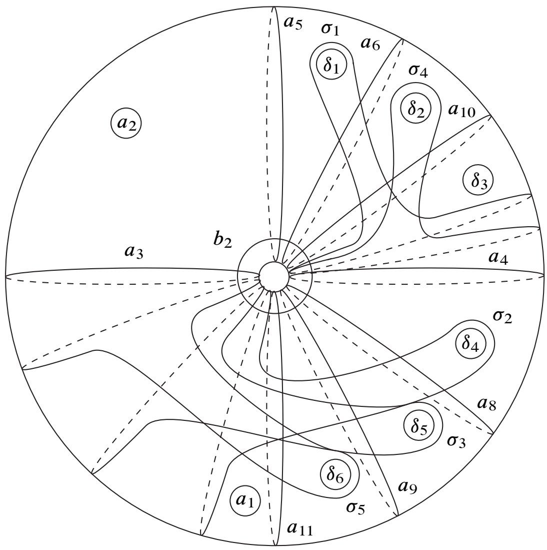

Figure 3: Relation (A)

Remark 2.5 The relation in Lemma 2.4 is deduced from the original five-holed torus relation

$$\delta_2\delta_1 a_2\gamma\delta_3 \equiv a_5 a_3 a_4 b_2 \sigma_1 a_6 a_3 b_2 \sigma_2 a_8 a_5 b_2$$

(see [7, Section 3.5]) by using commutativity relations and conjugations.

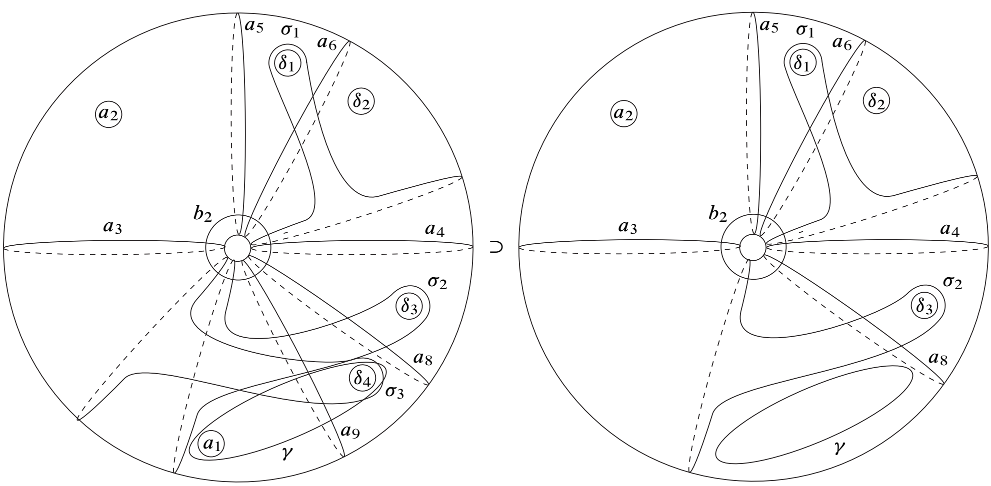

Figure 4: Five-holed torus relation

Proof of Proposition 2.3 Applying commutativity relations and conjugations to the five-holed torus relation in Lemma 2.4, we obtain

$$a_2\delta_1\delta_2\delta_3\gamma \equiv a_5b_2a_3a_4a_5b_2\sigma_1a_6a_3b_2\sigma_2a_8 \equiv a_8b_2\bar{a}_8a_8a_3a_4a_5b_2\sigma_1a_6a_3b_2\sigma_2a_5$$
$$\equiv a_8a_3a_4a_5b_2\sigma_1a_6a_3b_2\sigma_2a_5a_8b_2\bar{a}_8.$$

Multiplying both sides of this relation by $\overline{\gamma}$, we have

$$a_2\delta_1\delta_2\delta_3 \equiv a_8a_3a_4a_5b_2\sigma_1a_6a_3b_2\sigma_2a_5a_8b_2\bar{a}_8\overline{\gamma}.$$

We embed $\Sigma_{1,5}$ into $\Sigma_{1,6}$ and take simple closed curves $a_1, a_9, \delta_4, \sigma_3$ in the interior of $\Sigma_{1,6}$ shown in Figure 4. Then we have a lantern relation

$$\delta_4a_1a_3a_8 \equiv \gamma\sigma_3a_9.$$

Combining these relations and applying commutativity relations, we obtain

$$a_8a_3a_1a_2\delta_1\delta_2\delta_3\delta_4 \equiv a_8a_3a_4a_5b_2\sigma_1a_6a_3b_2\sigma_2a_5a_8b_2\bar{a}_8\overline{\gamma}\gamma\sigma_3a_9$$
$$\equiv a_8a_3a_4a_5b_2\sigma_1a_6a_3b_2\sigma_2a_5a_8b_2\bar{a}_8\sigma_3a_9.$$

Multiplying both sides of this relation by $\bar{a}_3\bar{a}_8$, we have a relation

(A1) $$a_1a_2\delta_1\delta_2\delta_3\delta_4 \equiv a_4a_5b_2\sigma_1a_6a_3b_2\sigma_2a_5a_8b_2\bar{a}_8\sigma_3a_9$$

on $\Sigma_{1,6}$.

We change the name δ_2 of a curve in relation (A1) into γ (shown on the right in Figure 5) and apply commutativity relations and conjugations to it to obtain

$$a_1a_2\delta_1\delta_3\delta_4\gamma \equiv a_4a_5b_2\sigma_1a_6a_3b_2\sigma_2a_5a_8b_2\bar{a}_8\sigma_3a_9$$
$$\equiv a_5a_4b_2\bar{a}_4a_4\sigma_1a_6a_3b_2\sigma_2a_5a_8b_2\bar{a}_8\sigma_3a_9$$
$$\equiv a_4a_6a_3b_2\sigma_2a_5a_8b_2\bar{a}_8\sigma_3a_9a_5a_4b_2\bar{a}_4\sigma_1.$$

Multiplying both sides of this relation by $\overline{\gamma}$, we have

$$a_1a_2\delta_1\delta_3\delta_4 \equiv a_4a_6a_3b_2\sigma_2a_5a_8b_2\bar{a}_8\sigma_3a_9a_5a_4b_2\bar{a}_4\sigma_1\overline{\gamma}.$$

We embed $\Sigma_{1,6}$ into $\Sigma_{1,7}$ and take simple closed curves $a_{10}, \delta_2, \delta_5, \sigma_4$ in the interior of $\Sigma_{1,7}$ shown in Figure 5. Then we have a lantern relation

$$\delta_2\delta_5a_4a_6 \equiv \gamma\sigma_4a_{10}.$$

Combining these relations and applying commutativity relations, we obtain

$$a_4a_6a_1a_2\delta_1\delta_2\delta_3\delta_4\delta_5 \equiv a_4a_6a_3b_2\sigma_2a_5a_8b_2\bar{a}_8\sigma_3a_9a_5a_4b_2\bar{a}_4\sigma_1\overline{\gamma}\gamma\sigma_4a_{10}$$
$$\equiv a_4a_6a_3b_2\sigma_2a_5a_8b_2\bar{a}_8\sigma_3a_9a_5a_4b_2\bar{a}_4\sigma_1\sigma_4a_{10}.$$

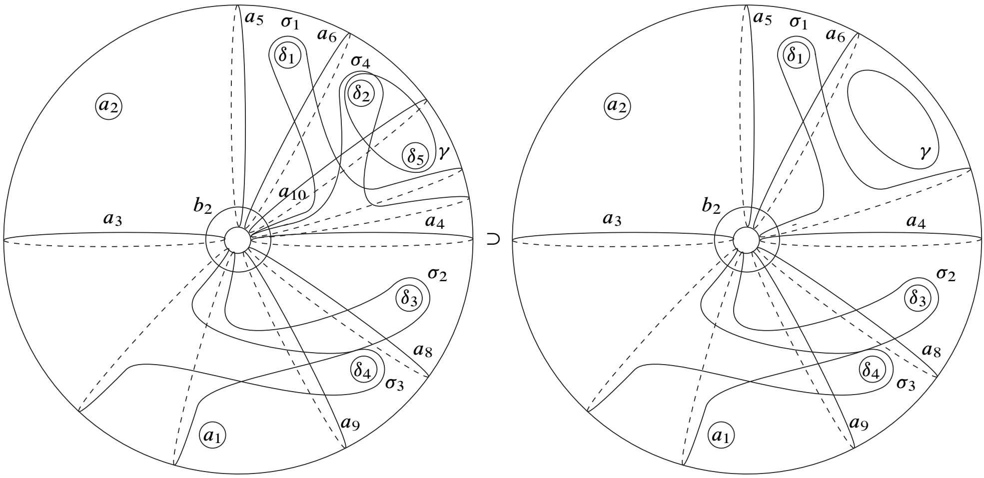

Figure 5: Embedding of $\Sigma_{1,6}$ into $\Sigma_{1,7}$ (I)

Multiplying both sides of this relation by $\bar{a}_6\bar{a}_4$, we have a relation

(A2) $$a_1a_2\delta_1\delta_2\delta_3\delta_4\delta_5 \equiv a_3b_2\sigma_2a_5a_8b_2\bar{a}_8\sigma_3a_9a_5a_4b_2\bar{a}_4\sigma_1\sigma_4a_{10}$$

on $\Sigma_{1,7}$.

We change the name a_1 of a curve in relation (A2) into γ (shown on the right in Figure 6) and apply commutativity relations and conjugations to it to obtain

$$\begin{aligned} a_2\delta_1\delta_2\delta_3\delta_4\delta_5\gamma &\equiv a_3b_2\sigma_2a_5a_8b_2\bar{a}_8\sigma_3a_9a_5a_4b_2\bar{a}_4\sigma_1\sigma_4a_{10} \\ &\equiv a_3b_2\bar{a}_3\sigma_2a_5a_3a_8b_2\bar{a}_8\sigma_3a_9a_5a_4b_2\bar{a}_4\sigma_1\sigma_4a_{10} \\ &\equiv a_3b_2\bar{a}_3\sigma_2a_5a_3a_8b_2\bar{a}_8\bar{a}_3\sigma_3a_3a_9a_5a_4b_2\bar{a}_4\sigma_1\sigma_4a_{10} \\ &\equiv a_3a_9a_5a_4b_2\bar{a}_4\sigma_1\sigma_4a_{10}a_3b_2\bar{a}_3\sigma_2a_5a_3a_8b_2\bar{a}_8\bar{a}_3\sigma_3. \end{aligned}$$

Multiplying both sides of this relation by $\overline{\gamma}$, we have

$$a_2\delta_1\delta_2\delta_3\delta_4\delta_5 \equiv a_3a_9a_5a_4b_2\bar{a}_4\sigma_1\sigma_4a_{10}a_3b_2\bar{a}_3\sigma_2a_5a_3a_8b_2\bar{a}_8\bar{a}_3\sigma_3\overline{\gamma}.$$

We embed $\Sigma_{1,7}$ into $\Sigma_{1,8}$ and take simple closed curves $a_1, a_{11}, \delta_6, \sigma_5$ in the interior of $\Sigma_{1,8}$ shown in Figure 6. Then we have a lantern relation

$$\delta_6a_1a_3a_9 \equiv \gamma\sigma_5a_{11}.$$

Combining these relations and applying commutativity relations, we obtain

$$\begin{aligned} a_3a_9a_1a_2\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 &\equiv a_3a_9a_5a_4b_2\bar{a}_4\sigma_1\sigma_4a_{10}a_3b_2\bar{a}_3\sigma_2a_5a_3a_8b_2\bar{a}_8\bar{a}_3\sigma_3\overline{\gamma}\gamma\sigma_5a_{11} \\ &\equiv a_3a_9a_5a_4b_2\bar{a}_4\sigma_1\sigma_4a_{10}a_3b_2\bar{a}_3\sigma_2a_5a_3a_8b_2\bar{a}_8\bar{a}_3\sigma_3\sigma_5a_{11}. \end{aligned}$$

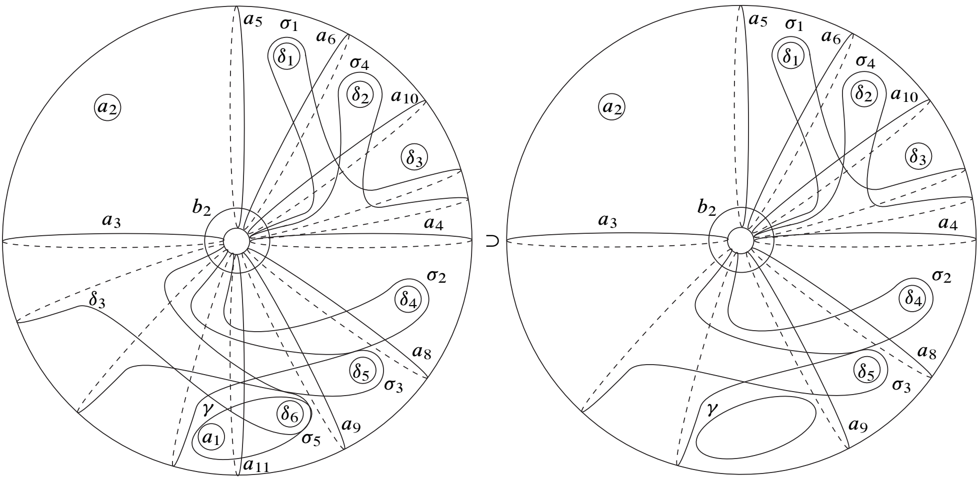

Figure 6: Embedding of $\Sigma_{1,7}$ into $\Sigma_{1,8}$ (I)

Multiplying both sides of this relation by $\bar{a}_9\bar{a}_3$, we finally obtain relation (A). This completes the proof of Proposition 2.3. □

The second relation constructed in this subsection is the following.

Proposition 2.6 Relation (B) *For simple closed curves in the interior of $\Sigma_{1,8}$ shown in Figure 7, we have the relation*

$$a_1a_2a_7a_8\delta_1\delta_2\delta_3\delta_4 \equiv a_4a_5''\bar{a}_6b_2a_6a_3b_2\bar{a}_3\tau'\tau'''a_5a_4''\bar{a}_3b_2a_3a_6b_2\bar{a}_6\tau\tau''.$$

We make use of the four-holed torus relation found by Korkmaz and Ozbagci [7] in order to prove Proposition 2.6.

Lemma 2.7 (Korkmaz–Ozbagci [7]) *For simple closed curves in the interior of $\Sigma_{1,4}$ shown on the left in Figure 8, we have the relation*

$$a_2a_1a_7\gamma \equiv (a_3a_6b_2a_4a_5b_2)^2.$$

Remark 2.8 The relation in Lemma 2.7 is not the exact four-holed torus relation but the relation written in a more symmetric form (see [7, Section 3.4, Remark]).

Proof of Proposition 2.6 We consider the four-holed torus relation reviewed in Lemma 2.7. We then embed $\Sigma_{1,4}$ into $\Sigma_{1,5}$ and take simple closed curves $a_5', a_8, \delta_1, \tau$ in the interior of $\Sigma_{1,5}$ shown in Figure 8. Then we have a lantern relation

$$\delta_1a_8a_6a_5 \equiv \gamma\tau a_5'.$$

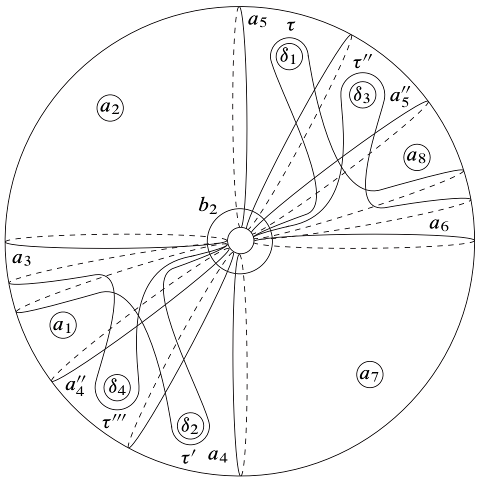

Figure 7: Relation (B)

Combining this relations with the four-holed torus relation, and applying commutativity relations and conjugations, we obtain a relation

$$\text{(B1)} \qquad \begin{aligned} a_1a_2a_7a_8\delta_1 &\equiv \bar{a}_5a_4a_5b_2a_3a_6b_2a_4a_5b_2a_3a_6b_2\overline{\gamma}\cdot\bar{a}_6\gamma\tau a_5' \\ &\equiv a_4b_2a_3a_6b_2a_4a_5b_2a_6a_3b_2\bar{a}_6\tau a_5' \\ &\equiv a_4a_5b_2a_6a_3b_2\bar{a}_6\tau a_5'a_4b_2a_3a_6b_2 \end{aligned}$$

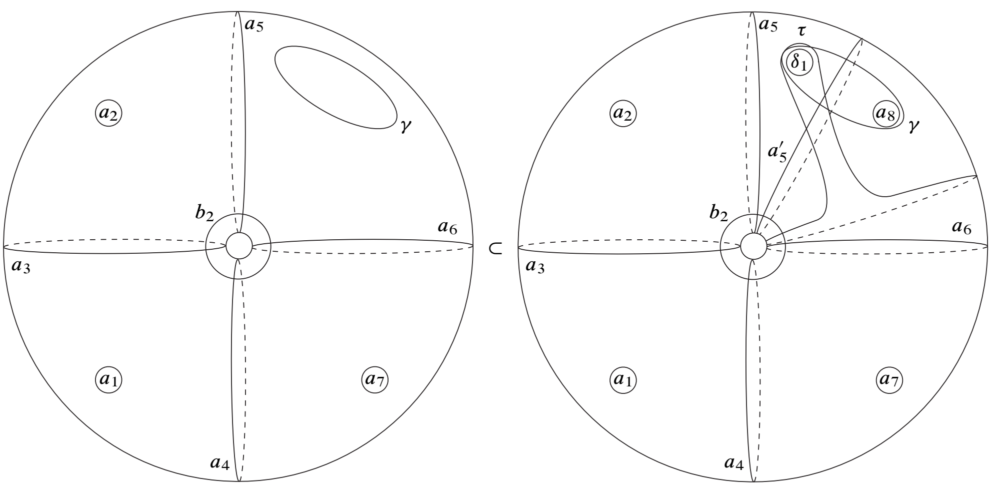

Figure 8: Four-holed torus relation

on $\Sigma_{1,5}$.

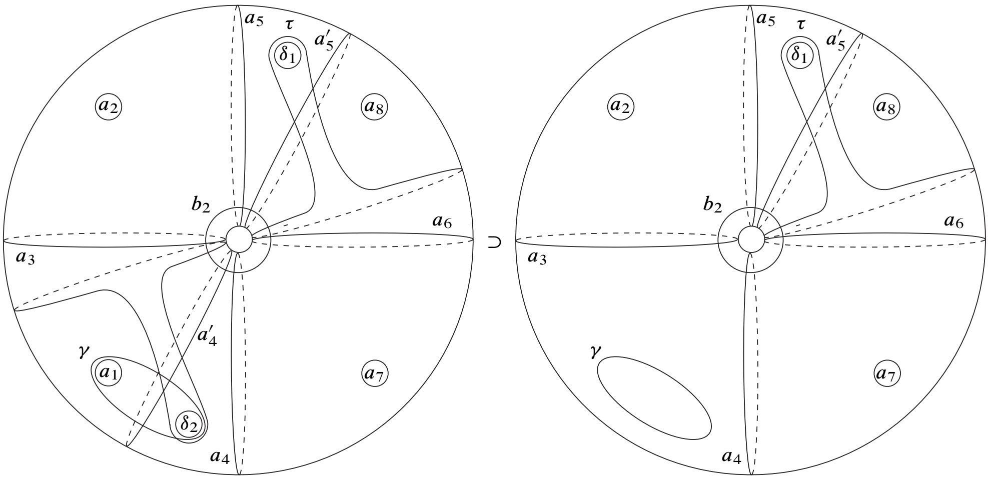

Figure 9: Embedding of $\Sigma_{1,5}$ into $\Sigma_{1,6}$

We change the name a_1 of a curve in relation (B1) into γ (shown on the right in Figure 9) to obtain

$$\gamma a_2 a_7 a_8 \delta_1 \equiv a_4 a_5 b_2 a_6 a_3 b_2 \bar{a}_6 \tau a_5' a_4 b_2 a_3 a_6 b_2.$$

We embed $\Sigma_{1,5}$ into $\Sigma_{1,6}$ and take simple closed curves $a_1, a_4', \delta_2, \tau'$ in the interior of $\Sigma_{1,6}$ shown in Figure 9. Then we have a lantern relation

$$a_4 a_3 a_1 \delta_2 \equiv \gamma \tau' a_4'.$$

Combining these relations and applying commutativity relations and conjugations, we obtain a relation

$$\text{(B2)} \qquad \begin{aligned} a_1 a_2 a_7 a_8 \delta_1 \delta_2 &\equiv \bar{a}_4 a_4 a_5 b_2 a_6 a_3 b_2 \bar{a}_6 \tau a_5' a_4 b_2 a_3 a_6 b_2 \overline{\gamma} \cdot \bar{a}_3 \gamma \tau' a_4' \\ &\equiv a_5 b_2 a_6 a_3 b_2 \bar{a}_6 \tau a_5' a_4 b_2 a_6 a_3 b_2 \bar{a}_3 \tau' a_4' \\ &\equiv b_2 a_6 a_3 b_2 \bar{a}_3 \tau' a_4' a_5 b_2 a_6 a_3 b_2 \bar{a}_6 \tau a_5' a_4. \end{aligned}$$

on $\Sigma_{1,6}$.

We change the name a_8 of a curve in relation (B2) into γ (shown on the left in Figure 10) to obtain

$$a_1 a_2 a_7 \gamma \delta_1 \delta_2 \equiv b_2 a_6 a_3 b_2 \bar{a}_3 \tau' a_4' a_5 b_2 a_6 a_3 b_2 \bar{a}_6 \tau a_5' a_4.$$

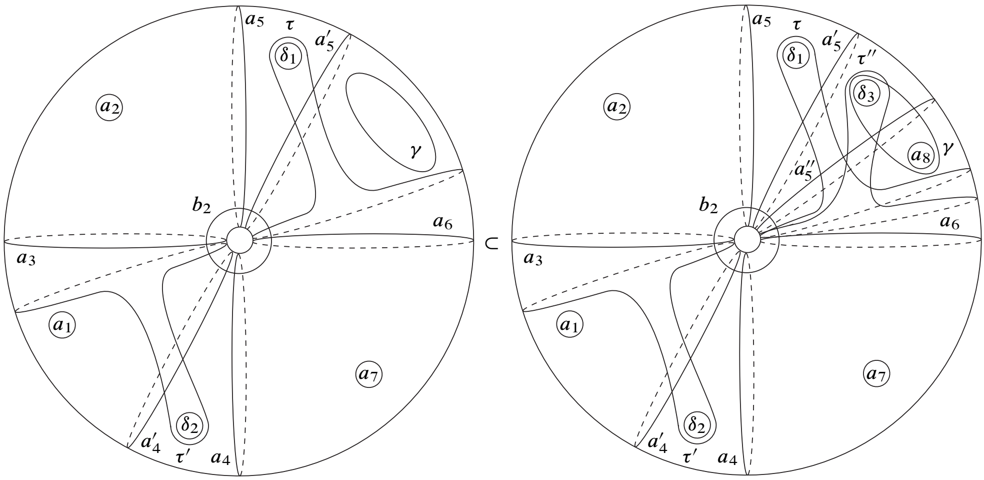

Figure 10: Embedding of $\Sigma_{1,6}$ into $\Sigma_{1,7}$ (II)

We embed $\Sigma_{1,6}$ into $\Sigma_{1,7}$ and take simple closed curves $a''_5, a_8, \delta_3, \tau''$ in the interior of $\Sigma_{1,7}$ shown in Figure 10. Then we have a lantern relation

$$\delta_3 a_8 a_6 a'_5 \equiv \gamma \tau'' a''_5.$$

Combining these relations and applying commutativity relations and conjugations, we obtain a relation

(B3)
$$\begin{aligned} a_1a_2a_7a_8\delta_1\delta_2\delta_3 &\equiv \bar{a}_6 b_2 a_6 a_3 b_2 \bar{a}_3 \tau' a'_4 a_5 b_2 a_6 a_3 b_2 \bar{a}_6 \tau a'_5 a_4 \bar{\gamma} \cdot \bar{a}'_5 \gamma \tau'' a''_5 \\ &\equiv \bar{a}_6 b_2 a_6 a_3 b_2 \bar{a}_3 \tau' a'_4 a_5 b_2 a_6 a_3 b_2 \bar{a}_6 \tau \tau'' a_4 a''_5 \\ &\equiv b_2 a_3 a_6 b_2 \bar{a}_6 \tau \tau'' a_4 a''_5 \bar{a}_6 b_2 a_6 a_3 b_2 \bar{a}_3 \tau' a'_4 a_5. \end{aligned}$$

We change the name a_1 of a curve in relation (B3) into γ (shown on the right in Figure 11) to obtain

$$\gamma a_2 a_7 a_8 \delta_1 \delta_2 \delta_3 \equiv b_2 a_3 a_6 b_2 \bar{a}_6 \tau \tau'' a_4 a''_5 \bar{a}_6 b_2 a_6 a_3 b_2 \bar{a}_3 \tau' a'_4 a_5.$$

We embed $\Sigma_{1,7}$ into $\Sigma_{1,8}$ and take simple closed curves $a_1, a''_4, \delta_4, \tau'''$ in the interior of $\Sigma_{1,8}$ shown in Figure 11. Then we have a lantern relation

$$\delta_4 a_1 a_3 a'_4 \equiv \gamma \tau''' a''_4.$$

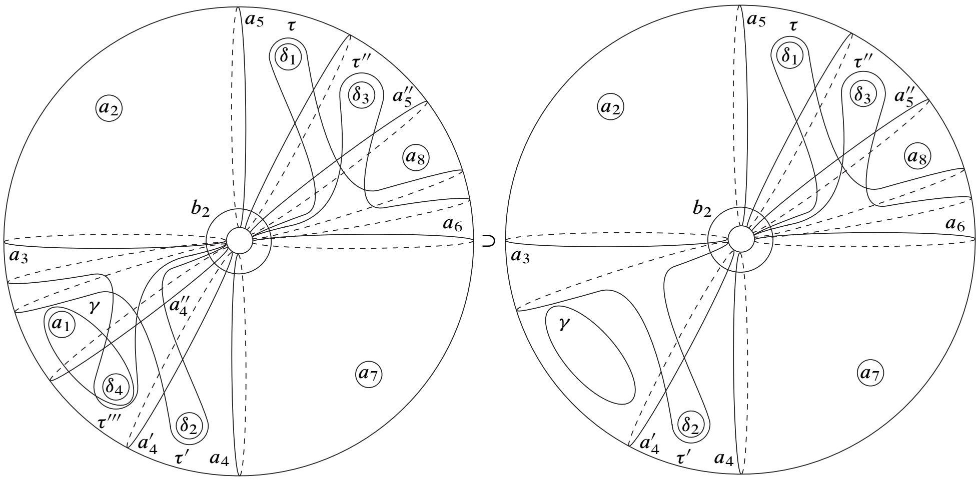

Figure 11: Embedding of $\Sigma_{1,7}$ into $\Sigma_{1,8}$ (II)

Combining these relations and applying commutativity relations and conjugations, we finally obtain relation (B):

$$\begin{aligned} a_1a_2a_7a_8\delta_1\delta_2\delta_3\delta_4 &\equiv \bar{a}_3b_2a_3a_6b_2\bar{a}_6\tau\tau''a_4a_5''\bar{a}_6b_2a_6a_3b_2\bar{a}_3\tau'a_4'a_5\overline{\gamma}\cdot\bar{a}_4'\gamma\tau'''a_4'' \\ &\equiv \bar{a}_3b_2a_3a_6b_2\bar{a}_6\tau\tau''a_4a_5''\bar{a}_6b_2a_6a_3b_2\bar{a}_3\tau'\tau'''a_5a_4'' \\ &\equiv a_4a_5''\bar{a}_6b_2a_6a_3b_2\bar{a}_3\tau'\tau'''a_5a_4''\bar{a}_3b_2a_3a_6b_2\bar{a}_6\tau\tau''. \end{aligned}$$

This completes the proof of Proposition 2.6. □

Remark 2.9 Both of relations (A) and (B) are different from the eight-holed torus relation of Korkmaz and Ozbagci [7] though the constructions are similar.

3 Constructions

In this section we construct a new relation on a compact oriented surface of genus g with $4g+4$ boundary components by combining copies of relations (A) and (B) obtained in the previous section.

3.1 Higher genus

We assume $g \geq 3$. For integers m, n $(0 < m \leq n)$ and words $W_m, W_{m+1}, \ldots, W_n$ in $\mathcal{F}_{g,r}$, we denote the product $W_mW_{m+1}\cdots W_n$ (respectively $W_n \cdots W_{m+1}W_m$) by $\prod_{i=m}^{n} W_i$ (respectively $\prod_{i=n}^{m} W_i$).

Theorem 3.1 Relation (H_g) *For simple closed curves in the interior of* $\Sigma_{g,4g+4}$ *shown in Figure 12, we have the relation*

$$\delta_1\delta_2\cdots\delta_{4g+3}\delta_{4g+4} \equiv \prod_{i=g-1}^{2} \beta_i'''\beta_i\tau_{i-1}'\tau_{i-1}''' \cdot \beta_1\sigma_1'\sigma_4' a_{3g+3}\beta_1'\sigma_2' a_1\beta_1''\sigma_3'\sigma_5' \times \prod_{i=2}^{g-1} \beta_i''\beta_i'\tau_{i-1}\tau_{i-1}'' \cdot \beta_g\sigma_1\sigma_4 a_{3g}\beta_g'\sigma_2 a_{3g-1}\beta_g''\sigma_3\sigma_5$$

in $\mathcal{M}_{g,4g+4}$*, where*

$$\begin{aligned}
&\beta_1 := {}_{a_{3g+4}}(b_1), && \beta_1' := {}_{a_3}(b_1), && \beta_1'' := {}_{a_{3g+5}a_3}(b_1),\\
&\beta_g := {}_{a_{3g+1}}(b_g), && \beta_g' := {}_{a_{3g-3}}(b_g), && \beta_g'' := {}_{a_{3g-3}a_{3g+2}}(b_g),\\
&\beta_i := {}_{a_{3i-3}}(b_i), && \beta_i' := {}_{a_{3i}}(b_i), && \beta_i'' := {}_{\bar{a}_{3i-3}}(b_i), && \beta_i''' := {}_{\bar{a}_{3i}}(b_i),
\end{aligned}$$

and $i = 2, \dots, g-1$.

Proof We combine two copies of relation (A) and $g-2$ copies of relation (B) to obtain the desired relation. We first consider two relations for simple closed curves shown in Figure 13. One is a copy of relation (A):

$$a_5a_4'\delta_1\delta_4\delta_6\delta_2\delta_3\delta_5 \equiv a_1a_{3g+4}b_1\bar{a}_{3g+4}\sigma_1'\sigma_4'a_{3g+3}a_3b_1\bar{a}_3\sigma_2'a_1a_3a_{3g+5}b_1\bar{a}_{3g+5}\bar{a}_3\sigma_3'\sigma_5'a_2$$

Applying commutativity relations and conjugations, we obtain a relation

$$\begin{aligned}
a_4'a_5\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 &\equiv a_1a_2a_{3g+4}b_1\bar{a}_{3g+4}\sigma_1'\sigma_4'a_{3g+3}a_3b_1\bar{a}_3\sigma_2'a_1a_3a_{3g+5}b_1\bar{a}_{3g+5}\bar{a}_3\sigma_3'\sigma_5'\\
&\equiv a_1a_2\beta_1\sigma_1'\sigma_4'a_{3g+3}\beta_1'\sigma_2'a_1\beta_1''\sigma_3'\sigma_5'.
\end{aligned}$$

Note that $\beta_1 \equiv a_{3g+4}b_1\bar{a}_{3g+4}$, $\beta_1' \equiv a_3b_1\bar{a}_3$ and $\beta_1'' \equiv a_{3g+5}a_3b_1\bar{a}_3\bar{a}_{3g+5}$ by Remark 2.2. The other is a copy of relation (B):

$$a_1a_2a_7'a_8\delta_7\delta_8\delta_9\delta_{10} \equiv a_4a_5'\bar{a}_6b_2a_6a_3b_2\bar{a}_3\tau_1'\tau_1'''a_5a_4'\bar{a}_3b_2a_3a_6b_2\bar{a}_6\tau_1\tau_1''$$

Applying commutativity relations and conjugations, we obtain a relation

$$a_1a_2a_7'a_8\delta_7\delta_8\delta_9\delta_{10} \equiv a_5a_4'\bar{a}_3b_2a_3a_6b_2\bar{a}_6\tau_1\tau_1''a_4a_5'\bar{a}_6b_2a_6a_3b_2\bar{a}_3\tau_1'\tau_1'''.$$

We embed two copies of $\Sigma_{1,8}$ in Figure 13 into $\Sigma_{2,12}$ as shown in Figure 14.

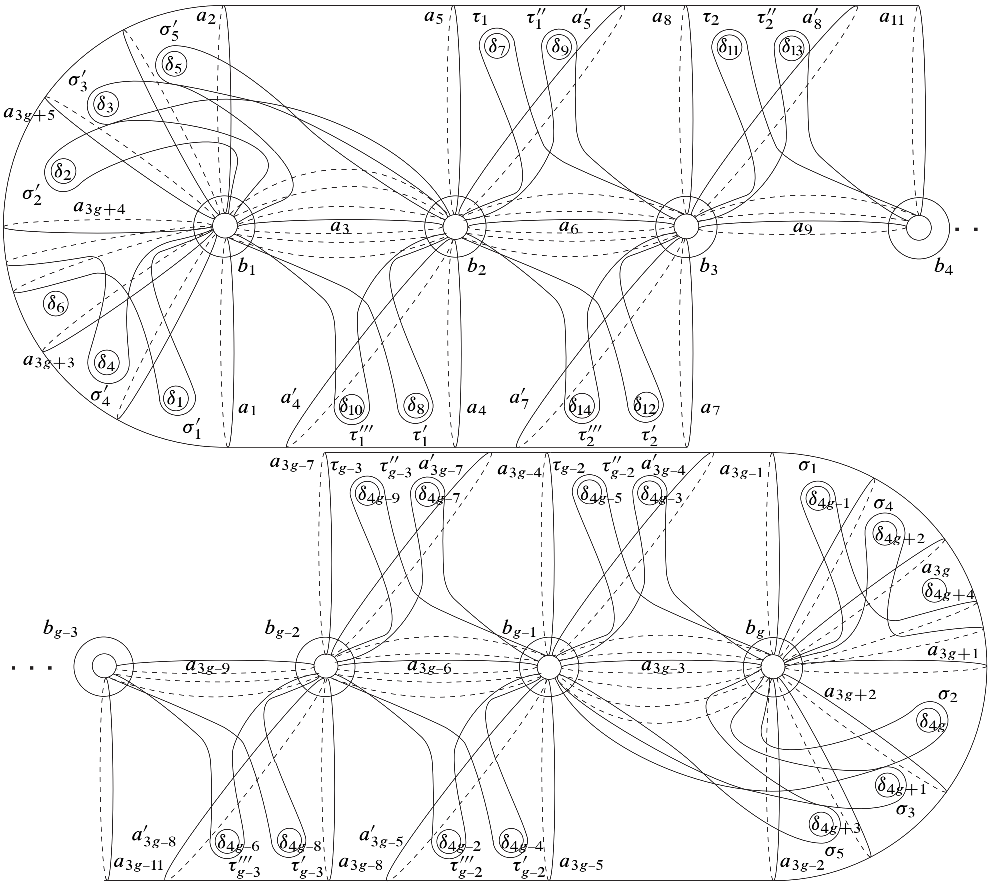

Figure 12: Relation (H_g) for $g \geq 3$

Combining these relations and applying commutativity relations and conjugations, we obtain a relation

$$
\text{(C2)} \quad a'_7 a_8 \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 \delta_9 \delta_{10}
\begin{aligned}[t]
&\equiv \bar{a}_5 \bar{a}'_4 a_5 a'_4 \bar{a}_3 b_2 a_3 a_6 b_2 \bar{a}_6 \tau_1 \tau''_1 a_4 a'_5 \bar{a}_6 b_2 a_6 a_3 b_2 \bar{a}_3 \tau'_1 \tau'''_1 \bar{a}_1 \bar{a}_2 \\
&\qquad \cdot a_1 a_2 \beta_1 \sigma'_1 \sigma'_4 a_{3g+3} \beta'_1 \sigma'_2 a_1 \beta''_1 \sigma'_3 \sigma'_5 \\
&\equiv \bar{a}_3 b_2 a_3 a_6 b_2 \bar{a}_6 \tau_1 \tau''_1 a_4 a'_5 \bar{a}_6 b_2 a_6 a_3 b_2 \bar{a}_3 \tau'_1 \tau'''_1 \\
&\qquad \cdot \beta_1 \sigma'_1 \sigma'_4 a_{3g+3} \beta'_1 \sigma'_2 a_1 \beta''_1 \sigma'_3 \sigma'_5 \\
&\equiv a_4 a'_5 \bar{a}_6 b_2 a_6 a_3 b_2 \bar{a}_3 \tau'_1 \tau'''_1 \cdot \beta_1 \sigma'_1 \sigma'_4 a_{3g+3} \beta'_1 \sigma'_2 a_1 \beta''_1 \sigma'_3 \sigma'_5 \\
&\qquad \cdot \bar{a}_3 b_2 a_3 a_6 b_2 \bar{a}_6 \tau_1 \tau''_1.
\end{aligned}
$$

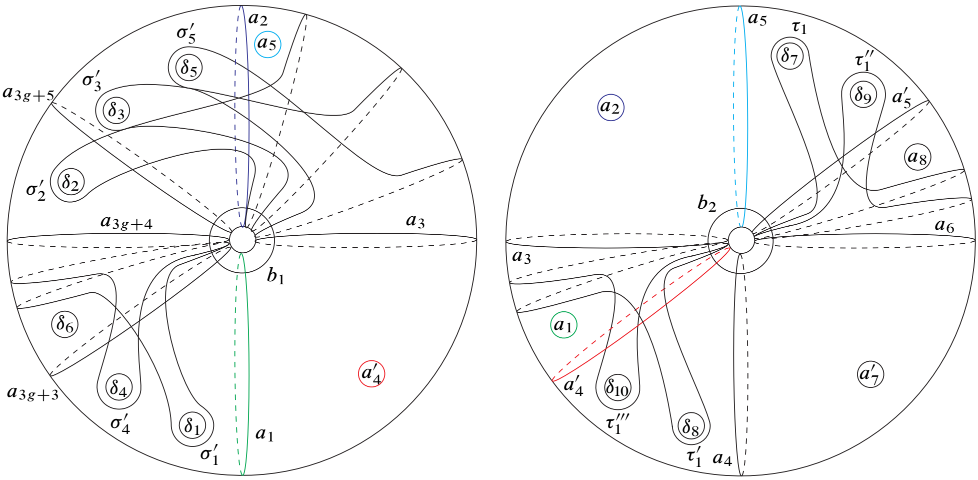

Figure 13: Relations (A) and (B)

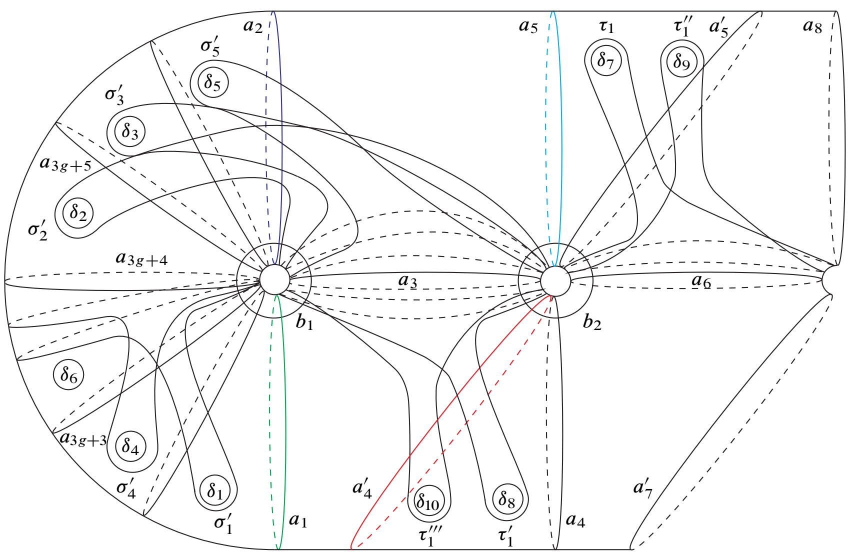

Figure 14: Embeddings of two copies of $\Sigma_{1,8}$ into $\Sigma_{2,12}$ (I)

We next consider relation (C2) and another copy of relation (B) for simple closed curves shown in Figure 15:

$$a_4 a'_5 a'_{10} a_{11} \delta_{11} \delta_{12} \delta_{13} \delta_{14} \equiv a_8 a'_7 \bar{a}_6 b_3 a_6 a_9 b_3 \bar{a}_9 \tau_2 \tau''_2 a_7 a'_8 \bar{a}_9 b_3 a_9 a_6 b_3 \bar{a}_6 \tau'_2 \tau'''_2$$

We embed $\Sigma_{2,12}$ in Figure 14 and $\Sigma_{1,8}$ in Figure 15 into $\Sigma_{3,16}$ as shown in Figure 16.

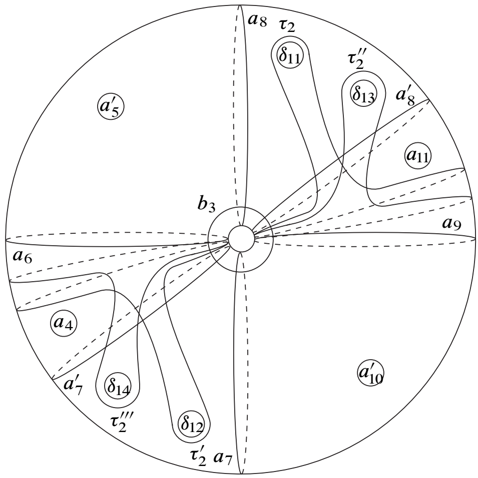

Figure 15: Another relation (B)

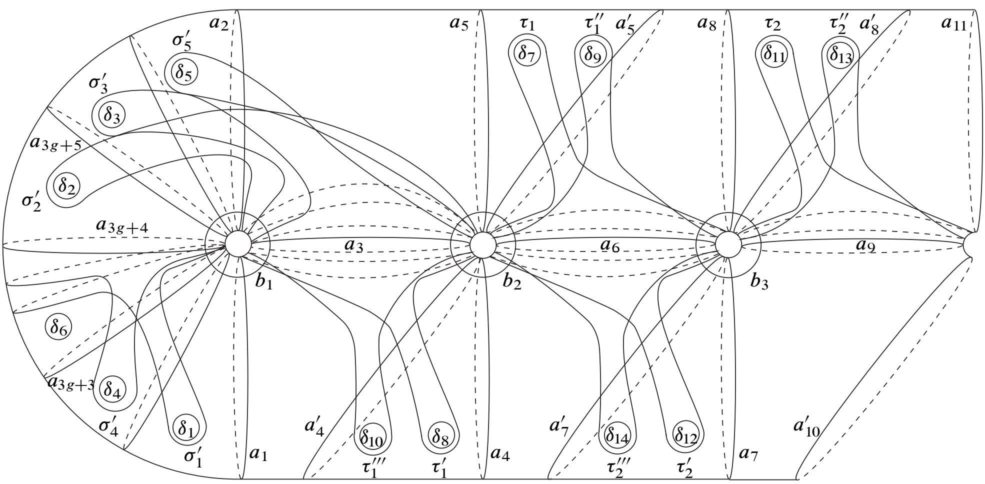

Figure 16: Embeddings of $\Sigma_{2,12}$ and $\Sigma_{1,8}$ into $\Sigma_{3,16}$

Combining these relations and applying commutativity relations and conjugations, we obtain a relation

$$\text{(C3)}\quad a_{10}'a_{11}\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9\delta_{10}\delta_{11}\delta_{12}\delta_{13}\delta_{14}$$
$$\equiv \bar{a}_7'\bar{a}_8a_8a_7'\bar{a}_6b_3a_6a_9b_3\bar{a}_9\tau_2\tau_2''a_7a_8'\bar{a}_9b_3a_9a_6b_3\bar{a}_6\tau_2'\tau_2'''\bar{a}_4\bar{a}_5'$$
$$\cdot a_4a_5'\bar{a}_6b_2a_6a_3b_2\bar{a}_3\tau_1'\tau_1''' \cdot \beta_1\sigma_1'\sigma_4'a_{3g+3}\beta_1'\sigma_2'a_1\beta_1''\sigma_3'\sigma_5'$$
$$\cdot \bar{a}_3b_2a_3a_6b_2\bar{a}_6\tau_1\tau_1''$$

$$\begin{aligned}&\equiv \bar a_6 b_3 a_6 a_9 b_3 \bar a_9 \tau_2 \tau_2'' a_7 a_8' \bar a_9 b_3 a_9 a_6 b_3 \bar a_6 \tau_2' \tau_2''' \cdot \bar a_6 b_2 a_6 a_3 b_2 \bar a_3 \tau_1' \tau_1''' \\ &\qquad \cdot \beta_1 \sigma_1' \sigma_4' a_{3g+3} \beta_1' \sigma_2' a_1 \beta_1'' \sigma_3' \sigma_5' \cdot \bar a_3 b_2 a_3 a_6 b_2 \bar a_6 \tau_1 \tau_1'' \\ &\equiv a_7 a_8' \bar a_9 b_3 a_9 a_6 b_3 \bar a_6 \tau_2' \tau_2''' \cdot \bar a_6 b_2 a_6 a_3 b_2 \bar a_3 \tau_1' \tau_1''' \\ &\qquad \cdot \beta_1 \sigma_1' \sigma_4' a_{3g+3} \beta_1' \sigma_2' a_1 \beta_1'' \sigma_3' \sigma_5' \cdot \bar a_3 b_2 a_3 a_6 b_2 \bar a_6 \tau_1 \tau_1'' \\ &\qquad\qquad \cdot \bar a_6 b_3 a_6 a_9 b_3 \bar a_9 \tau_2 \tau_2''.\end{aligned}$$

We repeat similar procedures by making use of $g-4$ copies of relation (B):

$$\begin{aligned}&a_{3i-5} a_{3i-4}' a_{3i+1} a_{3i+2} \delta_{4i-1} \delta_{4i} \delta_{4i+1} \delta_{4i+2} \\ &\qquad \equiv a_{3i-1} a_{3i-2}' \bar a_{3i-3} b_i a_{3i-3} a_{3i} b_i \bar a_{3i} \tau_{i-1} \tau_{i-1}'' \\ &\qquad\qquad \cdot a_{3i-2} a_{3i-1}' \bar a_{3i} b_i a_{3i} a_{3i-3} b_i \bar a_{3i-3} \tau_{i-1}' \tau_{i-1}'''\end{aligned}$$

for $i = 4, \dots, g-1$ to obtain relations (C4), (C5), … and

$$\begin{aligned}&(\mathrm{C}(g-1)) \quad a_{3g-2} a_{3g-1} \delta_1 \delta_2 \cdots \delta_{4g-3} \delta_{4g-2} \\ &\qquad \equiv a_{3g-5} a_{3g-4}' \prod_{i=g-1}^{2} \bar a_{3i} b_i a_{3i} a_{3i-3} b_i \bar a_{3i-3} \tau_{i-1}' \tau_{i-1}''' \\ &\qquad\qquad \cdot \beta_1 \sigma_1' \sigma_4' a_{3g+3} \beta_1' \sigma_2' a_1 \beta_1'' \sigma_3' \sigma_5' \prod_{i=2}^{g-1} \bar a_{3i-3} b_i a_{3i-3} a_{3i} b_i \bar a_{3i} \tau_{i-1} \tau_{i-1}''\end{aligned}$$

for simple closed curves on $\Sigma_{g-1,4g}$ shown in Figure 17.

We finally consider the other copy of relation (A) for simple closed curves shown in Figure 18:

$$\begin{aligned}&a_{3g-5} a_{3g-4}' \delta_{4g+1} \delta_{4g+2} \delta_{4g+4} \delta_{4g} \delta_{4g+1} \delta_{4g+3} \\ &\qquad \equiv a_{3g-1} a_{3g+1} b_g \bar a_{3g+1} \sigma_1 \sigma_4 a_{3g} a_{3g-3} b_g \bar a_{3g-3} \sigma_2 \\ &\qquad\qquad \cdot a_{3g-1} a_{3g-3} a_{3g+2} b_g \bar a_{3g+2} \bar a_{3g-3} \sigma_3 \sigma_5 a_{3g-2}\end{aligned}$$

Applying commutativity relations and conjugations, we obtain a relation

$$\begin{aligned}&a_{3g-5} a_{3g-4}' \delta_{4g-1} \delta_{4g} \delta_{4g+1} \delta_{4g+2} \delta_{4g+3} \delta_{4g+4} \\ &\qquad \equiv a_{3g-2} a_{3g-1} a_{3g+1} b_g \bar a_{3g+1} \sigma_1 \sigma_4 a_{3g} a_{3g-3} b_g \\ &\qquad\qquad \cdot \bar a_{3g-3} \sigma_2 a_{3g-1} a_{3g-3} a_{3g+2} b_g \bar a_{3g+2} \bar a_{3g-3} \sigma_3 \sigma_5 \\ &\qquad \equiv a_{3g-2} a_{3g-1} \beta_g \sigma_1 \sigma_4 a_{3g} \beta_g' \sigma_2 a_{3g-1} \beta_g'' \sigma_3 \sigma_5.\end{aligned}$$

Note that

$$\beta_g \equiv a_{3g+1} b_g \bar a_{3g+1}, \quad \beta_g' \equiv a_{3g-3} b_g \bar a_{3g-3}, \quad \beta_g'' \equiv a_{3g-3} a_{3g+2} b_g \bar a_{3g+2} \bar a_{3g-3}$$

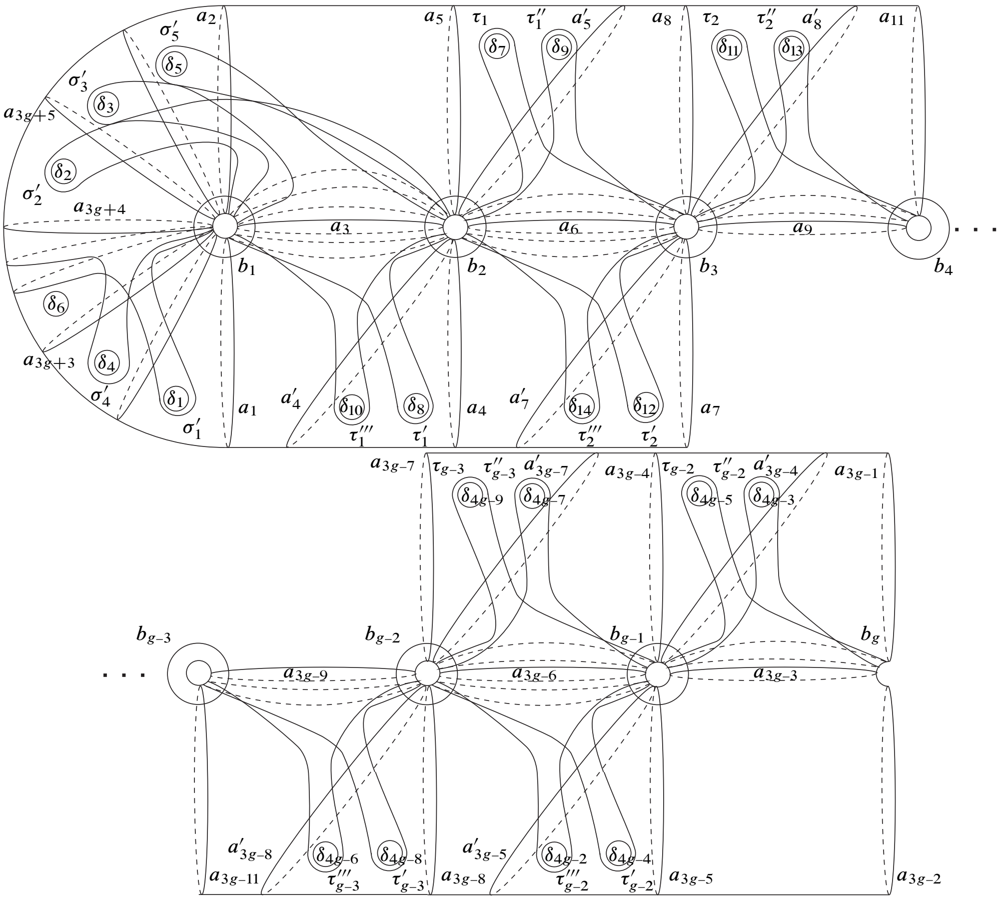

Figure 17: Relation (C($g-1$))

by Remark 2.2. We embed $\Sigma_{g-1,4g}$ in Figure 17 and $\Sigma_{1,8}$ in Figure 18 into $\Sigma_{g,4g+4}$ as shown in Figure 12.

Combining these relations and applying commutativity relations and conjugations, we obtain relation (H$_g$). Note that $\beta_i \equiv a_{3i-3} b_i \bar{a}_{3i-3}$, $\beta'_i \equiv a_{3i} b_i \bar{a}_{3i}$, $\beta''_i \equiv \bar{a}_{3i-3} b_i a_{3i-3}$ and $\beta'''_i \equiv \bar{a}_{3i} b_i a_{3i}$ $(i = 2, \dots, g-1)$ by Remark 2.2. Thus we complete the proof of Theorem 3.1. □

3.2 Genus two

In this subsection we construct a relation on $\Sigma_{2,12}$ similar to relations constructed in the previous subsection.

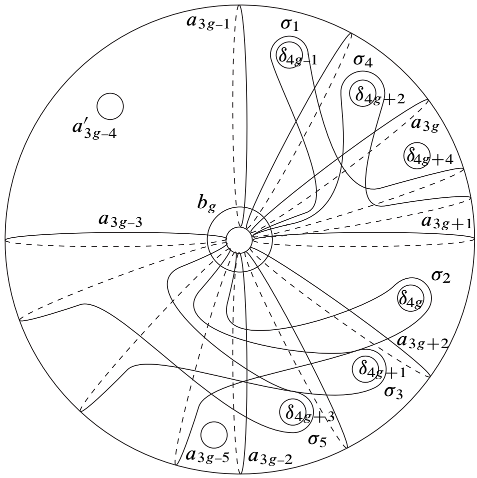

Figure 18: The other relation (A)

Theorem 3.2 Relation (H_2) *For simple closed curves in the interior of $\Sigma_{2,12}$ shown in Figure 19, we have the relation*

$$\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6\delta_7\delta_8\delta_9\delta_{10}\delta_{11}\delta_{12} \equiv \beta_1\sigma'_1\sigma'_4 a_{13}\beta'_1\sigma'_2 a_1\beta''_1\sigma'_3\sigma'_5\beta_2\sigma_1\sigma_4 a_{12}\beta'_2\sigma_2 a_5\beta''_2\sigma_3\sigma_5$$

in $\mathcal{M}_{2,12}$, where

$$\beta_1 := {}_{a_2}(b_1), \quad \beta'_1 := {}_{a_3}(b_1), \quad \beta''_1 := {}_{a_3a_9}(b_1),$$
$$\beta_2 := {}_{a_4}(b_2), \quad \beta'_2 := {}_{a_3}(b_2), \quad \beta''_2 := {}_{a_3a_8}(b_2).$$

Proof We first consider two copies of relation (A) for simple closed curves shown in Figure 20:

$$a_5a_{14}\delta_7\delta_8\delta_9\delta_{10}\delta_{11}\delta_{12} \equiv a_1a_2b_1\bar{a}_2\sigma'_1\sigma'_4a_{13}a_3b_1\bar{a}_3\sigma'_2a_1a_3a_9b_1\bar{a}_9\bar{a}_3\sigma'_3\sigma'_5a_{15}$$
$$a_1a_{15}\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 \equiv a_5a_4b_2\bar{a}_4\sigma_1\sigma_4a_{12}a_3b_2\bar{a}_3\sigma_2a_5a_3a_8b_2\bar{a}_8\bar{a}_3\sigma_3\sigma_5a_{14}.$$

Applying commutativity relations and conjugations, we obtain relations

$$a_5a_{14}\delta_7\delta_8\delta_9\delta_{10}\delta_{11}\delta_{12} \equiv a_{15}a_1\beta_1\sigma'_1\sigma'_4a_{13}\beta'_1\sigma'_2a_1\beta''_1\sigma'_3\sigma'_5$$
$$a_1a_{15}\delta_1\delta_2\delta_3\delta_4\delta_5\delta_6 \equiv a_{14}a_5\beta_2\sigma_1\sigma_4a_{12}\beta'_2\sigma_2a_5\beta''_2\sigma_3\sigma_5.$$

Note that $\beta_1 \equiv a_2b_1\bar{a}_2$, $\beta'_1 \equiv a_3b_1\bar{a}_3$, $\beta''_1 \equiv a_3a_9b_1\bar{a}_9\bar{a}_3$, $\beta_2 \equiv a_4b_2\bar{a}_4$, $\beta'_2 \equiv a_3b_2\bar{a}_3$ and $\beta''_2 \equiv a_3a_8b_2\bar{a}_8\bar{a}_3$ by Remark 2.2.

Combining these relations and applying commutativity relations and conjugations, we obtain relation (H_2). Thus we complete the proof of Theorem 3.2. □

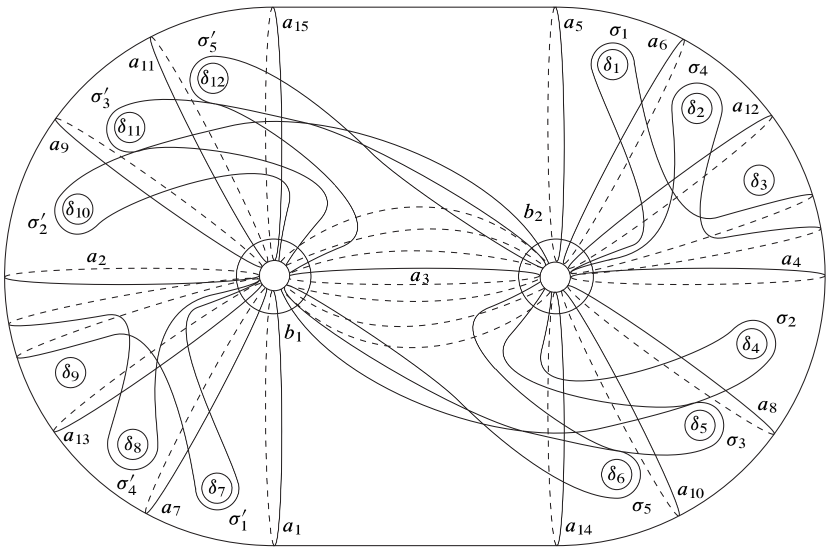

Figure 19: Embeddings of two copies of $\Sigma_{1,8}$ into $\Sigma_{2,12}$ (II)

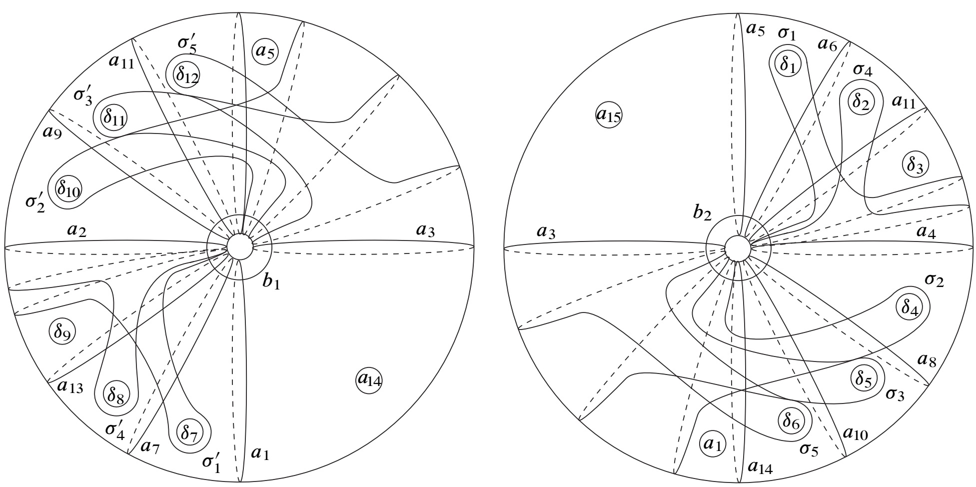

Figure 20: Two copies of relation (A)

4 Sections of Lefschetz fibrations

In this section we show that the relation constructed in the previous section gives an explicit topological description of $4g+4$ disjoint (-1)–sections of a hyperelliptic Lefschetz fibration of genus g on the manifold $\mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}^2}$.

We begin with a definition of Lefschetz fibrations (see [4; 8]).

Definition 4.1 Let M be a closed oriented smooth 4–manifold. A smooth map $f: M \to S^2$ is called a *Lefschetz fibration* of genus g if it satisfies the following conditions:

(i) f has finitely many critical values $b_1, \dots, b_n \in S^2$ and f is a smooth fiber bundle over $S^2 - \{b_1, \dots, b_n\}$ with fiber $\Sigma_{g,0}$

(ii) for each i $(i = 1, \dots, n)$, there exists a unique critical point p_i in the *singular fiber* $f^{-1}(b_i)$ such that f is locally written as $f(z_1, z_2) = z_1^2 + z_2^2$ with respect to some local complex coordinates around p_i and b_i which are compatible with orientations of M and S^2

(iii) no fiber contains a (-1)–sphere.

Remark 4.2 We always assume that a Lefschetz fibration is relatively minimal, it has at most one critical point on each fiber, and the genus of the base is equal to zero. A more general definition can be found in [4, Chapter 8].

Suppose that $g \geq 2$. According to theorems of Kas and Matsumoto, there exists a one-to-one correspondence between the isomorphism classes of Lefschetz fibrations and the equivalence classes of positive relators modulo simultaneous conjugations

$$c_1 \cdots c_n \sim {}_W(c_1) \cdots {}_W(c_n),$$

and elementary transformations

$$c_1 \cdots c_i \cdot c_{i+1} \cdots c_n \sim c_1 \cdots c_{i+1} \cdot {}_{c_{i+1}^{-1}}(c_i) \cdots c_n,$$

$$c_1 \cdots c_i \cdot c_{i+1} \cdots c_n \sim c_1 \cdots {}_{c_i}(c_{i+1}) \cdot c_i \cdots c_n,$$

where $c_1 \cdots c_n \in \mathcal{R}_{g,0}$ is a *positive relator* in the generators $\mathcal{S}_{g,0}$ and $W \in \mathcal{F}_{g,0}$. This correspondence is described by using the holonomy (or monodromy) homomorphism induced by the classifying map of f restricted on $S^2 - \{b_1, \dots, b_n\}$ (see [4; 8]).

Definition 4.3 Let $f: M \to S^2$ be a Lefschetz fibration of genus g. A smooth map $s: S^2 \to M$ is called a *section* of f if it satisfies $f \circ s = \mathrm{id}_{S^2}$. A section s of f is an embedding of S^2 into M. The self-intersection number of the homology class $s_*([S^2]) \in H_2(M; \mathbb{Z})$ is called the *self-intersection number* of s. A section of f with self-intersection number k is often called a *k–section*.

For a positive integer r, we attach r disks to the boundary components of $\Sigma_{g,r}$ to obtain a closed surface $\Sigma_{g,0}$ and an embedding $\Sigma_{g,r} \hookrightarrow \Sigma_{g,0}$. This embedding induces a natural commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{R}_{g,r} & \longrightarrow & \mathcal{F}_{g,r} & \xrightarrow{\varpi} & \mathcal{M}_{g,r} & \longrightarrow & 1 \\ & & \lambda\big\downarrow & & \lambda\big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \mathcal{R}_{g,0} & \longrightarrow & \mathcal{F}_{g,0} & \xrightarrow{\varpi} & \mathcal{M}_{g,0} & \longrightarrow & 1, \end{array}$$

where the two horizontal sequences are exact. If two words W_1 and W_2 in $\mathcal{F}_{g,r}$ satisfy $W_1 \equiv W_2$, then we have $\lambda(W_1) \equiv \lambda(W_2)$ in $\mathcal{F}_{g,0}$. In this case we call the relation $W_1 \equiv W_2$ a *lift* of the relation $\lambda(W_1) \equiv \lambda(W_2)$.

Lemma 4.4 [1; 2; 11] *Let $f : M \to S^2$ be a Lefschetz fibration of genus g and $c_1 \cdots c_n \in \mathcal{R}_{g,0}$ a positive relator corresponding to f. Suppose that there exists a relation $a_1 \cdots a_n \equiv \delta_1^{k_1} \cdots \delta_r^{k_r}$ ($a_1, \dots, a_n \in \mathcal{S}_{g,r}$, $k_1, \dots, k_r > 0$) in $\mathcal{F}_{g,r}$ which is a lift of the relation $c_1 \cdots c_n \equiv 1$ in $\mathcal{F}_{g,0}$, where $\delta_1, \dots, \delta_r$ are simple closed curves parallel to the boundary components of $\Sigma_{g,r}$. Then f admits disjoint r sections $s_1, \dots, s_r : S^2 \to M$ with self-intersection number $-k_1, \dots, -k_r$, respectively.*

For a chain $(c_1, \dots, c_{2g+1})$ of length $2g+1$ on $\Sigma_{g,0}$, we obtain a Lefschetz fibration $X_g \to S^2$ of genus g associated to the hyperelliptic relation

$$(c_1 \cdots c_{2g+1} c_{2g+1} \cdots c_1)^2 \equiv 1 \quad \text{in } \mathcal{F}_{g,0}.$$

The total space X_g of this fibration is known to be diffeomorphic to $\mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}}^2$ (see [4; 5]).

We denote the positive word on the right-hand side of relation (H_g) by U_g for $g \geq 2$. We consider the above embedding $\Sigma_{g,r} \hookrightarrow \Sigma_{g,0}$ and the commutative diagram for $r = 4g+4$. By Theorems 3.1 and 3.2, relation (H_g): $U_g \equiv \delta_1 \delta_2 \cdots \delta_{4g+3} \delta_{4g+4}$ in $\mathcal{F}_{g,4g+4}$ is a lift of the relation $\lambda(U_g) \equiv 1$ in $\mathcal{F}_{g,0}$. This implies that the Lefschetz fibration $Y_g \to S^2$ of genus g associated to the relation $\lambda(U_g) \equiv 1$ admits disjoint $4g+4$ sections with self-intersection number -1 by virtue of Lemma 4.4.

Theorem 4.5 *The two Lefschetz fibrations X_g and Y_g are isomorphic to each other.*

Proof Suppose that $g \geq 3$. We set

$$\begin{aligned} c_1 &:= \lambda(a_1), & c_{2i} &:= \lambda(b_i) \quad (i = 1, \dots, g), \\ c_{2g+1} &:= \lambda(a_{3g-1}), & c_{2i+1} &:= \lambda(a_{3i}) \quad (i = 1, \dots, g-1). \end{aligned}$$

Since $(a_1, b_1, a_3, b_2, \dots, a_{3g-3}, b_g, a_{3g-1})$ is a chain of length $2g+1$ on $\Sigma_{g,4g+4}$, $(c_1, c_2, c_3, c_4, \dots, c_{2g-1}, c_{2g}, c_{2g+1})$ is a chain of length $2g+1$ on $\Sigma_{g,0}$. It is easily seen from Figure 12 that

$$\begin{aligned}
\lambda(a_{3g+3}) = \lambda(a_{3g+4}) = \lambda(a_{3g+5}) = \lambda(\sigma_1') = \lambda(\sigma_4') = c_1,\\
\lambda(a_{3g}) = \lambda(a_{3g+1}) = \lambda(a_{3g+2}) = \lambda(\sigma_1) = \lambda(\sigma_4) = c_{2g+1},\\
\lambda(\sigma_2') = \lambda(\sigma_3') = \lambda(\sigma_5') = c_3,\\
\lambda(\sigma_2) = \lambda(\sigma_3) = \lambda(\sigma_5) = c_{2g-1},\\
\lambda(\tau_{i-1}') = \lambda(\tau_{i-1}''') = c_{2i-1},\\
\lambda(\tau_{i-1}) = \lambda(\tau_{i-1}'') = c_{2i+1},
\end{aligned}$$

for $i = 2, \dots, g-1$. Hence we obtain

$$\begin{aligned}
\lambda(U_g) = \prod_{i=g-1}^{2} ({}_{\bar{c}_{2i+1}}(c_{2i})_{c_{2i-1}}(c_{2i}) \cdot c_{2i-1}^2) \cdot {}_{c_1}(c_2) \cdot c_1^3 \cdot {}_{c_3}(c_2) \cdot c_3 c_1 \cdot {}_{c_1 c_3}(c_2) \cdot c_3^2 \\
\cdot \prod_{i=2}^{g-1} ({}_{\bar{c}_{2i-1}}(c_{2i})_{c_{2i+1}}(c_{2i}) \cdot c_{2i+1}^2) \cdot {}_{c_{2g+1}}(c_{2g}) \cdot c_{2g+1}^3 \cdot {}_{c_{2g-1}}(c_{2g}) \\
\cdot c_{2g-1} c_{2g+1} \cdot {}_{c_{2g-1} c_{2g+1}}(c_{2g}) \cdot c_{2g-1}^2.
\end{aligned}$$

We now prove that $\lambda(U_g) \sim (c_1 \cdots c_{2g+1} c_{2g+1} \cdots c_1)^2$ for $g \geq 3$. Applying elementary transformations (including cyclic permutations), we obtain the following sequence of equivalences:

$$\begin{aligned}
\lambda(U_g) &\sim c_{2g-1} \cdot \prod_{i=g-1}^{2} ({}_{\bar{c}_{2i+1}}(c_{2i}) \cdot c_{2i-1} c_{2i} c_{2i-1}) \cdot c_1 c_2 c_1^2 c_3 c_2 c_1 c_3 \cdot {}_{c_1}(c_2) \cdot c_3 \\
&\qquad \cdot \prod_{i=2}^{g-1} ({}_{\bar{c}_{2i-1}}(c_{2i}) \cdot c_{2i+1} c_{2i} c_{2i+1}) \\
&\qquad \cdot c_{2g+1} c_{2g} c_{2g+1}^2 c_{2g-1} c_{2g} c_{2g+1} c_{2g-1} \cdot {}_{c_{2g+1}}(c_{2g}) \\
&\sim \prod_{i=g-1}^{2} (c_{2i+1} \cdot {}_{\bar{c}_{2i+1}}(c_{2i}) \cdot c_{2i-1} c_{2i}) \cdot c_3 c_1 c_2 c_1^2 c_3 c_2 c_1 c_3 \cdot {}_{c_1}(c_2) \\
&\qquad \cdot \prod_{i=2}^{g-1} (c_{2i-1} \cdot {}_{\bar{c}_{2i-1}}(c_{2i}) \cdot c_{2i+1} c_{2i}) \\
&\qquad \cdot c_{2g-1} c_{2g+1} c_{2g} c_{2g+1}^2 c_{2g-1} c_{2g} c_{2g+1} c_{2g-1} \cdot {}_{c_{2g+1}}(c_{2g}) \\
&\sim c_1 c_{2g+1} \cdot \prod_{i=g-1}^{2} c_{2i} c_{2i+1} c_{2i-1} c_{2i} \cdot c_3 c_2 c_1^2 c_3 c_2 c_1 c_3 \cdot {}_{c_1}(c_2)
\end{aligned}$$

$$
\begin{aligned}
&\qquad \cdot \prod_{i=2}^{g-1} c_{2i}c_{2i-1}c_{2i+1}c_{2i} \cdot c_{2g-1}c_{2g}c_{2g+1}^2 c_{2g-1}c_{2g}c_{2g+1}c_{2g-1} \cdot {}_{c_{2g+1}}(c_{2g}) \\
&\sim \prod_{i=2g-2}^{3} c_i c_{i+1} \cdot c_3 c_2 c_1^2 c_3 c_2 c_1 c_3 \cdot {}_{c_1}(c_2) \cdot c_1 \\
&\qquad \cdot \prod_{i=3}^{2g-2} c_{i+1}c_i \cdot c_{2g-1}c_{2g}c_{2g+1}^2 c_{2g-1}c_{2g}c_{2g+1}c_{2g-1} \cdot {}_{c_{2g+1}}(c_{2g}) \cdot c_{2g+1} \\
&\sim \prod_{i=2g-2}^{3} c_i c_{i+1} \cdot c_3 c_2 c_3 c_1^2 c_2 c_1^2 c_3 c_2 \\
&\qquad \cdot \prod_{i=3}^{2g-2} c_{i+1}c_i \cdot c_{2g-1}c_{2g}c_{2g-1}c_{2g+1}^2 c_{2g}c_{2g+1}c_{2g-1}c_{2g+1}c_{2g} \\
&\sim \prod_{i=2g-2}^{3} c_i c_{i+1} \cdot c_2 c_3 c_2 c_1 c_2 c_1 c_2 c_1 c_3 c_2 \\
&\qquad \cdot \prod_{i=3}^{2g-2} c_{i+1}c_i \cdot c_{2g}c_{2g-1}c_{2g}c_{2g+1}c_{2g}c_{2g+1}c_{2g}c_{2g+1}c_{2g-1}c_{2g} \\
&\sim \prod_{i=2g-2}^{2} c_i c_{i+1} \cdot c_1 c_2 c_1 c_1 c_2 c_1 \\
&\qquad \cdot \prod_{i=2}^{2g-1} c_{i+1}c_i \cdot c_{2g+1}c_{2g}c_{2g+1}c_{2g+1}c_{2g}c_{2g+1}c_{2g-1}c_{2g} \\
&\sim c_{2g}c_{2g+1}c_{2g-1}c_{2g} \cdot \prod_{i=2g-2}^{1} c_i c_{i+1} \cdot c_1 c_1 \cdot \prod_{i=1}^{2g} c_{i+1}c_i \cdot c_{2g+1}c_{2g+1} \\
&\sim \prod_{i=2g}^{1} c_i c_{i+1} \cdot c_1 c_1 \cdot \prod_{i=1}^{2g} c_{i+1}c_i \cdot c_{2g+1}c_{2g+1} \\
&\sim \prod_{i=2g}^{1} c_i \cdot \prod_{i=2g+1}^{2} c_i \cdot c_1 c_1 \cdot \prod_{i=2}^{2g+1} c_i \cdot \prod_{i=1}^{2g} c_i \cdot c_{2g+1}c_{2g+1} \\
&= \prod_{i=2g}^{1} c_i \cdot \prod_{i=2g+1}^{1} c_i \cdot \prod_{i=1}^{2g+1} c_i \cdot \prod_{i=1}^{2g+1} c_i \cdot c_{2g+1} \\
&\sim \prod_{i=2g+1}^{1} c_i \cdot \prod_{i=2g+1}^{1} c_i \cdot \prod_{i=1}^{2g+1} c_i \cdot \prod_{i=1}^{2g+1} c_i \\
&\sim \prod_{i=1}^{2g+1} c_i \cdot \prod_{i=2g+1}^{1} c_i \cdot \prod_{i=1}^{2g+1} c_i \cdot \prod_{i=2g+1}^{1} c_i = (c_1 \cdots c_{2g+1}c_{2g+1} \cdots c_1)^2.
\end{aligned}
$$

Suppose that $g=2$. We set

$$c_1 := \lambda(a_1), \quad c_2 := \lambda(b_1), \quad c_3 := \lambda(a_3), \quad c_4 := \lambda(b_2), \quad c_5 := \lambda(a_5).$$

Since (a_1,b_1,a_3,b_2,a_5) is a chain of length 5 on $\Sigma_{2,12}$, (c_1,c_2,c_3,c_4,c_5) is a chain of length 5 on $\Sigma_{2,0}$. It is easily seen from Figure 19 that

$$\begin{aligned}\lambda(a_2)=\lambda(a_9)=\lambda(a_{13})=\lambda(\sigma_1')=\lambda(\sigma_4')=c_1,\\ \lambda(a_4)=\lambda(a_8)=\lambda(a_{12})=\lambda(\sigma_1)=\lambda(\sigma_4)=c_5,\\ \lambda(\sigma_2')=\lambda(\sigma_3')=\lambda(\sigma_5')=\lambda(\sigma_2)=\lambda(\sigma_3)=\lambda(\sigma_5)=c_3.\end{aligned}$$

Hence we obtain

$$\lambda(U_2) = {}_{c_1}(c_2)\cdot c_1^3\cdot {}_{c_3}(c_2)\cdot c_3c_1\cdot {}_{c_3c_1}(c_2)\cdot c_3^2\cdot {}_{c_5}(c_4)\cdot c_5^3\cdot {}_{c_3}(c_4)\cdot c_3c_5\cdot {}_{c_3c_5}(c_4)\cdot c_3^2.$$

We now prove that $\lambda(U_2)\sim(c_1c_2c_3c_4c_5c_5c_4c_3c_2c_1)^2$. Applying elementary transformations (including cyclic permutations), we obtain the following sequence of equivalences:

$$\begin{aligned}\lambda(U_2) &\sim c_1c_2c_1^2c_3c_2c_1c_3\cdot {}_{c_1}(c_2)\cdot c_3\cdot c_5c_4c_5^2c_3c_4c_5c_3\cdot {}_{c_5}(c_4)\cdot c_3\\ &\sim c_5c_2c_1^2c_3c_2c_1c_3\cdot {}_{c_1}(c_2)\cdot c_3\cdot c_4c_5^2c_3c_4c_5c_3\cdot {}_{c_5}(c_4)\cdot c_3c_1\\ &\sim c_2c_1^2c_3c_2c_1c_3\cdot {}_{c_1}(c_2)\cdot c_1c_3\cdot c_4c_5^2c_3c_4c_5c_3\cdot {}_{c_5}(c_4)\cdot c_3c_5\\ &\sim c_2c_1^2c_3c_2c_1c_3c_1c_2c_3\cdot c_4c_5^2c_3c_4c_5c_3\cdot {}_{c_5}(c_4)\cdot c_5c_3\\ &\sim c_2c_1^2c_3c_2c_1c_3c_1c_2c_3c_4c_5^2c_3c_4c_5c_3c_5c_4c_3\\ &\sim c_1c_3c_2c_3c_1c_1c_2c_3c_4c_3c_5^2c_4c_5c_3c_5c_4c_3c_2c_1\\ &\sim c_1c_2c_3c_2c_1c_1c_2c_4c_3c_4c_5^2c_4c_5c_3c_5c_4c_3c_2c_1\\ &\sim c_1c_2c_3c_4c_2c_1c_1c_2c_3c_4c_5c_4c_5c_4c_3c_5c_4c_3c_2c_1\\ &\sim c_1c_2c_3c_4c_2c_1c_1c_2c_3c_5c_4c_5c_5c_4c_3c_5c_4c_3c_2c_1\\ &\sim c_1c_2c_3c_4c_5c_2c_1c_1c_2c_3c_4c_5c_5c_4c_3c_5c_4c_3c_2c_1\\ &\sim c_1c_2c_3c_4c_5c_5c_4c_3c_5c_4c_3c_2c_1c_1c_2c_3c_4c_5c_2c_1\\ &\sim c_1c_2c_3c_4c_5c_5c_4c_3c_2c_1c_1c_2c_3c_4c_5c_5c_4c_3c_2c_1\\ &= (c_1c_2c_3c_4c_5c_5c_4c_3c_2c_1)^2.\end{aligned}$$

This completes the proof of Theorem 4.5. □

The next corollary immediately follows from the theorem.

Corollary 4.6 *The Lefschetz fibration $X_g\to S^2$ of genus g associated to the hyperelliptic relation admits disjoint $4g+4$ sections with self-intersection number -1.*

By virtue of Theorem 3.2, we can even depict disjoint twelve sections of the Lefschetz fibration $Y_2 \to S^2$ in a Kirby diagram of $Y_2 - \nu F$, where νF is an open fibered neighborhood of a regular fiber of Y_2 (see [7, Section 4]). We first construct a handle decomposition of $\Sigma_{2,0} \times D^2$ with one 0–handle, four 1–handles, and one 2–handle with framing 0 from a fixed handle decomposition of $\Sigma_{2,0}$. We then attach twenty 2–handles to $\Sigma_{2,0} \times D^2$ along the simple closed curves β_1, σ_1', σ_4', a_{13}, β_1', σ_2', a_1, β_1'', σ_3', σ_5', β_2, σ_1, σ_4, a_{12}, β_2', σ_2, a_5, β_2'', σ_3, σ_5 (see Figure 19) on different fibers of $\Sigma_{2,0} \times S^1 \to S^1$ with framing one less than the product framing of $\Sigma_{2,0} \times S^1$ to obtain a handle decomposition of $Y_2 - \nu F$. Thus we have a Kirby diagram of $Y_2 - \nu F$ shown in Figure 21. The framing coefficient of every component of the link but one with framing 0 is equal to -1. Twelve disjoint sections coming from the simple closed curves $\delta_1, \dots, \delta_{12}$ are represented by twelve unknots transverse to each fiber of the fibration $\Sigma_{2,0} \times S^1 \to S^1$ and meeting a fiber at twelve points indicated by encircled numbers $1, \dots, 12$ in Figure 21. Attaching a 2–handle with framing -1 along any one of the twelve unknots together with four 3–handles and a 4–handle to $Y_2 - \nu F$, we have a handle decomposition of the closed manifold Y_2.

By virtue of Theorem 3.1, we can also depict disjoint $4g+4$ sections of the Lefschetz fibration $Y_g \to S^2$ in a Kirby diagram of $Y_g - \nu F$ for $g \geq 3$ in a similar way.

The following proposition implies that the largest possible number of disjoint (-1)–sections of $X_g \to S^2$ is equal to $4g+4$ for most g.

Proposition 4.7 *If g is not equal to k^2+k-1 for any positive integer k, then the Lefschetz fibration $X_g \to S^2$ cannot admit disjoint $4g+5$ sections with self-intersection number -1.*

Proof Suppose that the Lefschetz fibration $X_g \to S^2$ admits disjoint $4g+5$ sections $s_1, \dots, s_{4g+5}$ with self-intersection number -1. The orientation of S^2 induces that of $S_i := s_i(S^2)$ for $i = 1, \dots, 4g+5$. We orient a regular fiber F of X_g so that it satisfies $[F] \cdot [S_i] = +1$ for $i = 1, \dots, 4g+5$. Blowing down the (-1)–spheres $S_1, \dots, S_{4g+5}$ in X_g, we obtain a 4–manifold X' and the image F' of F under the projection $X_g \to X'$. Since

$$[F] = [F'] - [S_1] - \cdots - [S_{4g+5}] \quad \text{in } H_2(X_g;\mathbb{Z}) \cong H_2(X';\mathbb{Z}) \oplus (4g+5) H_2(\overline{\mathbb{CP}}^2;\mathbb{Z})$$

and $[F]^2 = 0$, we have $[F']^2 = 4g+5$. On the other hand, $[F']^2$ must be the square of an integer because $[F']$ is a multiple of a generator of $H_2(X';\mathbb{Z}) \cong \mathbb{Z}$. It is easy to see that $4g+5$ is the square of an integer if and only if g is equal to k^2+k-1 for some positive integer k. □

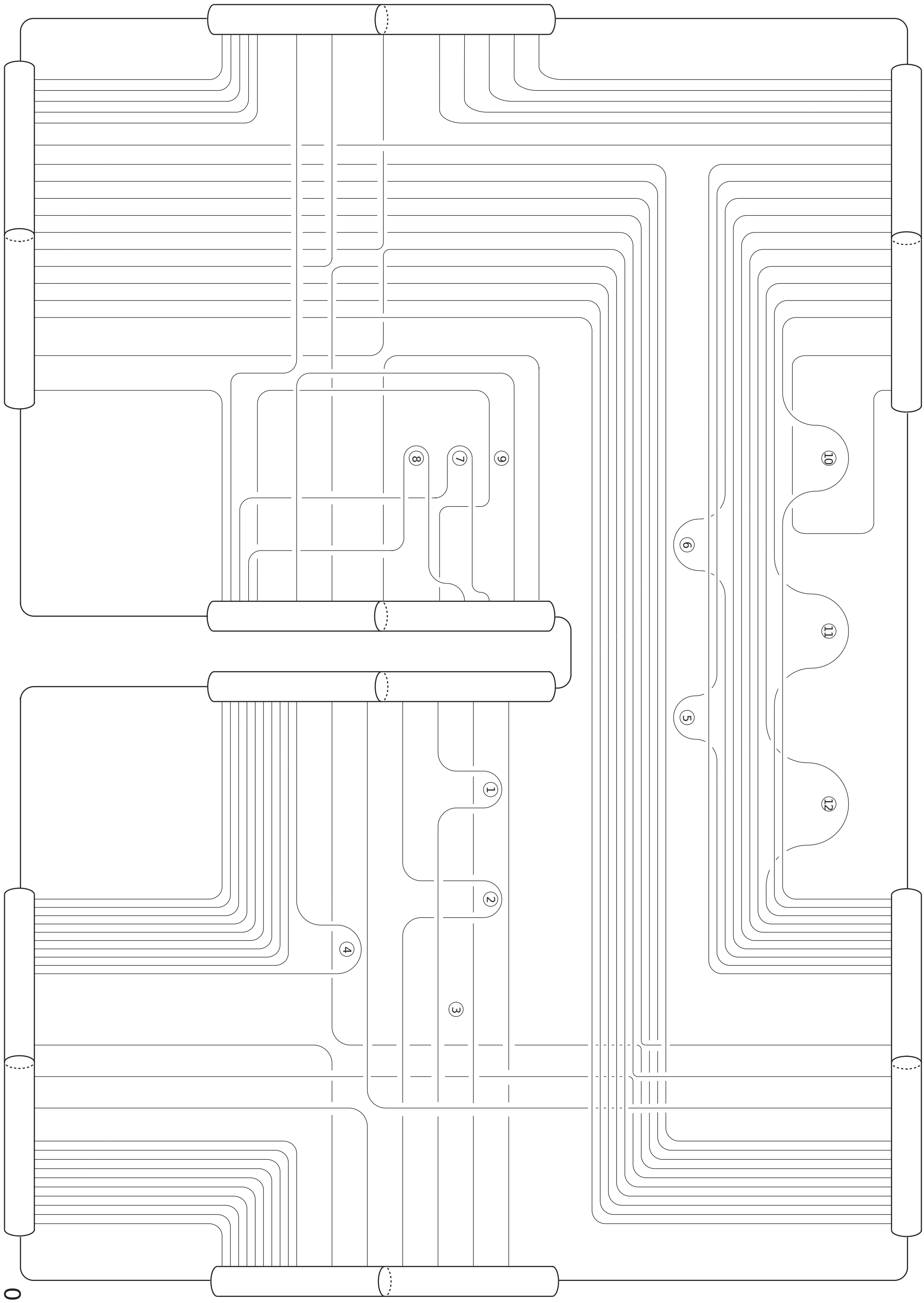

Figure 21: A Kirby diagram of $Y_2 - \nu F$

Remark 4.8 Two generic degree d curves in $\mathbb{CP}^2$ induce a Lefschetz pencil of genus $(d-1)(d-2)/2$. Blowing up the base locus, we obtain a Lefschetz fibration $M_d \to S^2$ of the same genus. This fibration has d^2 sections with self-intersection number -1 and the total space M_d is diffeomorphic to $\mathbb{CP}^2 \# d^2\overline{\mathbb{CP}^2}$. It is well-known that the fibration $M_3 \to S^2$ is isomorphic to $X_1 \to S^2$, whereas the fibration $M_d \to S^2$ for $d \geq 4$ cannot be isomorphic to $X_g \to S^2$ for any g.

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