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On unstable modules over the Dickson algebras, the Singer functors R_s and the functors Fix_s

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The category D_s–$\mathcal{U}$ of unstable modules over the Steenrod algebra equipped with a compatible module structure over the Dickson algebra D_s is studied at the prime 2, with applications to the Singer functor R_s, considered as a functor from unstable modules $\mathcal{U}$ to D_s–$\mathcal{U}$. An explicit copresentation of $R_s M$ is given using Lannes' T–functor when M is a reduced unstable module; applying Lannes' functor Fix_s, this is used to show that R_s gives a fully-faithful embedding of $\mathcal{U}$ in D_s–$\mathcal{U}$. In addition, the right adjoint $\mathfrak{Z}_s$ to R_s is introduced and is related to the indecomposables functor and the functor Fix_s.

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1 Introduction

The Dickson algebras over the field with two elements, $\mathbb{F}$, play an important role in the theory of unstable algebras over the mod–2 Steenrod algebra; the Dickson algebra D_s is the algebra of invariants $H^*V_s^{\mathrm{Aut}(V_s)}$, where H^*V_s denotes the group cohomology of the rank s elementary abelian 2–group V_s. The category D_s–$\mathcal{U}$ of D_s–modules in the category $\mathcal{U}$ of unstable modules arises naturally; for example, Singer introduced the functors R_s in his work on the homology of the Steenrod algebra ([22] and related work), where R_s can be considered as a functor from the category $\mathcal{U}$ to D_s–$\mathcal{U}$. One of the aims of this paper is to study the functor R_s, considered both as a functor to D_s–$\mathcal{U}$ and as a functor to $\mathcal{U}$, from the viewpoint of modern unstable module theory.

The functors R_s can be applied in calculating the E_2–term of the Adams spectral sequence: Lannes and Zarati [11] related them to the derived functors of destabilization (the left adjoint to the inclusion of the category $\mathcal{U}$ in the category of graded modules over the Steenrod algebra $\mathcal{A}$) and these derived functors appear, via a Grothendieck spectral sequence, in the calculation of Ext groups in the category of $\mathcal{A}$–modules. The role of the Singer functors in calculating these derived functors on general modules over the Steenrod algebra has been clarified (at odd primes) by the author in [17], establishing the relationship between the approach of Lannes and Zarati [11] and that of Singer.

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For these applications, it is important to understand the behaviour of the functor $\mathrm{Hom}_{\mathcal{U}}(R_s M, -)$, for $M \in \mathrm{Ob}\,\mathcal{U}$. Restricting to the full subcategory of $\mathcal{U}$ with objects of the form H^*V, this is equivalent to understanding $R_s M$ in the category $\mathcal{U}/\mathcal{N}il$ of unstable modules localized away from the nilpotent unstable modules, using the work of Henn, Lannes and Schwartz [8].

The paper exploits the structure of the module categories D_s–$\mathcal{U}$, their relation with the categories H^*V_s–$\mathcal{U}$ of unstable modules over H^*V_s and the localized categories D_s–$\mathcal{U}/\mathcal{N}il$; important tools are the functor $\mathrm{Fix}_s\colon H^*V_s\text{–}\mathcal{U} \to \mathcal{U}$ (see Lannes [10]) and the study of ω_s–torsion for unstable modules over H^*V_s, which was initiated by Dwyer and Wilkerson [3; 4] and developed by Lannes and Zarati [12]. The latter leads to the notion of ω_s–closure: an unstable D_s–module M is ω_s–closed if it is ω_s–torsion-free and is maximal with this property in the equivalence class up to ω_s–torsion.

A key new ingredient in studying the functor R_s is an approximation $\widetilde{R}_s\colon \mathcal{U} \to D_s\text{–}\mathcal{U}$ which is defined as the equalizer of a diagram

$$D_s \otimes M \underset{\tau_M}{\overset{\sigma_M}{\rightrightarrows}} H^*V_s \otimes T_{V_s} M$$

in the category D_s–$\mathcal{U}$, where T_{V_s} is Lannes' T–functor.

Theorem 1 *For $s \in \mathbb{N}$, there is a natural monomorphism $\gamma_s\colon R_s \hookrightarrow \widetilde{R}_s$ of functors $\mathcal{U} \to D_s\text{–}\mathcal{U}$ such that, for an unstable module M,*

(1) *$\widetilde{R}_s M$ is the ω_s–closure of $R_s M$;*

(2) *the morphism $\gamma_s\colon R_s M \to \widetilde{R}_s M$ is an isomorphism if M is reduced.*

This is derived from a model for the Singer functor modulo nilpotent unstable modules. The category $\mathcal{U}/\mathcal{N}il$ embeds in the category $\mathcal{F}$ of functors from the category $\mathcal{V}^f$ of finite-dimensional $\mathbb{F}$–vector spaces to $\mathbb{F}$–vector spaces and the nillocalization of D_s–$\mathcal{U}$ embeds in a functor category $\mathcal{F}^{\mathfrak{g}(D_s)}$, for which $\mathcal{V}^f$ is replaced by a category with objects (V, W), where $W \le V$ is a subspace of codimension at most s in $V \in \mathrm{Ob}\,\mathcal{V}^f$. The functor R_s corresponds to the functor $\kappa_s\colon \mathcal{F} \to \mathcal{F}^{\mathfrak{g}(D_s)}$ given on $F \in \mathrm{Ob}\,\mathcal{F}$ by $\kappa_s F(V, W) := F(W)$. The above diagram corresponds to a copresentation of the functor κ_s; throughout the paper, the comparison with the behaviour after nillocalization is a guiding principle.

Theorem 1 provides a model for $R_s M$ (considered either in $\mathcal{U}$ or in D_s–$\mathcal{U}$) when M is a reduced unstable module. In this case, the calculation of $T_V R_s M$ is accessible by using standard techniques of unstable module theory; as such, it provides a way of approaching the calculation of the functor $\mathrm{Hom}_{\mathcal{U}}(R_s M, -)$.

The functor $\widetilde{R}_s$ leads to a proof of the following result, where the natural transformation $\mathrm{Fix}_s(H^* V_s \otimes_{D_s} R_s(-)) \to 1_{\mathcal{U}}$ is constructed by adjunction from the canonical monomorphism $R_s M \hookrightarrow D_s \otimes M$.

Theorem 2 *For $s \in \mathbb{N}$, the natural transformation $\mathrm{Fix}_s(H^* V_s \otimes_{D_s} R_s(-)) \to 1_{\mathcal{U}}$ of functors on $\mathcal{U}$ is an isomorphism.*

This result is striking: $\mathrm{Fix}_s(H^* V_s \otimes_{D_s} -)$ is equipped with a natural $\mathrm{Aut} V_s$ action; when composed with R_s, the action is trivial. For instance, applying the functor $\mathrm{Fix}_s(H^* V_s \otimes_{D_s} -)$ to the natural inclusion $R_s M \hookrightarrow D_s \otimes M$ yields the natural inclusion $M \hookrightarrow T_{V_s} M$, where $T_{V_s} M$ is an $\mathrm{Aut}(V_s)$–module by functoriality of T_-.

The functor $\mathrm{Fix}_s(H^* V_s \otimes_{D_s} (-))\colon D_s\text{–}\mathcal{U} \to \mathcal{U}$ can be identified in the nillocalized situation, where it corresponds to the composite functor $\Psi_s \mathrm{Ind}_s\colon \mathcal{F}^{\mathfrak{g}(D_s)} \to \mathcal{F}$, which is given on an object $G \in \mathrm{Ob}\,\mathcal{F}^{\mathfrak{g}(D_s)}$ by $\Psi_s \mathrm{Ind}_s G(V) := G(V \oplus \mathbb{F}^s, V)$. In this setting, Theorem 2 corresponds to the natural isomorphism $\Psi_s \mathrm{Ind}_s \kappa_s \cong 1_{\mathcal{F}}$ (see Lemma 7.1.6); the force of the theorem is that this lifts to unstable modules. Since the functor Fix_s does not see ω_s–torsion, in the proof of Theorem 2, R_s can be replaced by the model $\widetilde{R}_s$ of Theorem 1, which leads to the result.

As a consequence, one obtains the following.

Corollary 3 *The functor R_s induces a fully-faithful embedding $R_s\colon \mathcal{U} \hookrightarrow D_s\text{–}\mathcal{U}$, for $s \in \mathbb{N}$.*

The Singer functor $R_s\colon \mathcal{U} \to D_s\text{–}\mathcal{U}$ admits a right adjoint $\mathfrak{Z}_s$. The functor $\mathfrak{Z}_s$ leads to a stronger conclusion (Theorem 8.3.1); Corollary 3 corresponds to the fact that the adjunction unit $1_{\mathcal{U}} \to \mathfrak{Z}_s R_s$ is an isomorphism.

The functor $\mathfrak{Z}_s$ is of independent interest; there is a natural transformation $Q_s \to \mathfrak{Z}_s$ of functors $D_s\text{–}\mathcal{U} \to \mathcal{U}$, where Q_s is the indecomposables functor, and this is an isomorphism up to nilpotent unstable modules. After nillocalization, Q_s corresponds to $\mathcal{R}^0_s\colon \mathcal{F}^{\mathfrak{g}(D_s)} \to \mathcal{F}$ given by $\mathcal{R}^0_s G(V) = G(V, V)$; in particular the functor Q_s becomes exact upon nillocalization.

The functor $\mathfrak{Z}_s$ is also related to the functor Fix_s, via a natural transformation

$$\mathfrak{Z}_s \to \mathrm{Fix}_s(H^* V_s \otimes_{D_s} (-)).$$

In the case $s = 1$, this leads to a criterion (see Theorem 9.3.3) for an object of $D_1\text{–}\mathcal{U}$ to be in the image of R_1.

Organization of the paper Sections 2, 3 and 4 set the stage, providing background, introducing the categories of unstable modules over the Dickson algebras and the Singer functors respectively. Sections 5 and 6 introduce the tools of nillocalization as they apply to unstable modules over Dickson algebras and Section 7 gives the model for the Singer functors viewed through the filter of nillocalization.

The main results of the paper are proved in Sections 8 and 9.

2 Background

2.1 Unstable modules and unstable algebras

Throughout the paper, $\mathbb{F}$ is the field with two elements and $\mathcal{A}$ is the mod–2 Steenrod algebra (see Schwartz [19] for the basics of the theory of unstable modules over the Steenrod algebra). The category of graded $\mathcal{A}$–modules is denoted by $\mathcal{M}$ and the full subcategory of unstable modules $\mathcal{U}$; these are equipped with the usual tensor product. A commutative algebra B in $\mathcal{M}$ is unstable if the underlying module is unstable and satisfies $\mathrm{Sq}_0 x = x^2$, where Sq_0 denotes the top Steenrod operation; the category of unstable algebras and algebra morphisms is denoted by $\mathcal{K}$. Observe that the degree zero part of an unstable algebra is a Boolean algebra. An unstable algebra is Noetherian if the underlying commutative algebra is finitely-generated.

The category of B–modules in $\mathcal{M}$ is denoted by B–$\mathcal{M}$ and, if K is an unstable algebra, the category of K–modules in $\mathcal{U}$ is denoted by K–$\mathcal{U}$. If $K \to L$ is a morphism of unstable algebras, there is an adjunction

$$L \otimes_K -\colon K\text{–}\mathcal{U} \rightleftarrows L\text{–}\mathcal{U} : \mathrm{Restrict}_K^L,$$

where $L \otimes_K -$ is the induction functor, left adjoint to the exact restriction functor.

The degree-doubling functor $\Phi\colon \mathcal{M} \to \mathcal{M}$ restricts to a functor $\Phi\colon \mathcal{U} \to \mathcal{U}$ and Sq_0 induces a natural transformation $\lambda\colon \Phi \to 1_{\mathcal{U}}$ (see [19, Section 1.7]). An unstable module M is reduced if λ_M is a monomorphism (equivalently, if M does not contain a nilpotent submodule, where an unstable module N is nilpotent if Sq_0 acts locally nilpotently); M is nilclosed if $\mathrm{Ext}^{\varepsilon}_{\mathcal{U}}(N, M) = 0$, for every $\varepsilon \in \{0, 1\}$ and nilpotent N. The full subcategory of nilpotent unstable modules $\mathcal{N}il \subset \mathcal{U}$ is a localizing subcategory (see Gabriel [6] for generalities on localization of abelian categories).

An unstable algebra is reduced (resp. nilclosed) if the underlying unstable module has this property, hence an unstable algebra K is reduced if and only if K contains no nilpotent elements. The functor Φ commutes with tensor products, thus restricts to a functor $\Phi\colon \mathcal{K} \to \mathcal{K}$, and Φ induces an exact functor $\Phi\colon K\text{–}\mathcal{U} \to \Phi K\text{–}\mathcal{U}$. If K

is reduced, ΦK identifies via λ_K as the unstable subalgebra of K generated by the squares of elements of K.

The functor $\widetilde{\Phi}\colon \mathcal{U} \to \mathcal{U}$ is the right adjoint to Φ (see [19, Examples 2.2.3]) and the adjunction unit $M \to \widetilde{\Phi}\Phi M$ is a natural isomorphism. Proposition A.1.1 shows that $\widetilde{\Phi}$ induces a right adjoint to $\Phi\colon K\text{–}\mathcal{U} \to \Phi K\text{–}\mathcal{U}$.

2.2 Lannes' T–functor

For V an elementary abelian 2–group (V_s will be written to denote an elementary abelian 2–group of rank s), H^*V denotes the group cohomology of V with $\mathbb{F}$–coefficients, which is isomorphic to the symmetric algebra $S^*(V^*)$ on the dual of V; the underlying unstable module of H^*V is injective in $\mathcal{U}$ (see [19, Chapter 3]).

Lannes' T–functor $T_V\colon \mathcal{U} \to \mathcal{U}$ is the left adjoint to $H^*V \otimes -\colon \mathcal{U} \to \mathcal{U}$; it is exact and commutes with tensor products. Moreover, T_V restricts to a functor $T_V\colon \mathcal{K} \to \mathcal{K}$ and, for an unstable algebra K, induces an exact functor $K\text{–}\mathcal{U} \to T_V K\text{–}\mathcal{U}$.

A morphism of unstable algebras $\varphi\colon K \to H^*V$ is adjoint to a morphism $T_V K \to \mathbb{F}$ of unstable algebras, which factors across a morphism of Boolean algebras $\widetilde{\varphi}\colon T_V^0 K \to \mathbb{F}$, where T_V^0 denotes the degree zero part of T_V; $\mathbb{F}$ is a flat $T_V^0 K$–module with respect to this morphism, so that $\mathbb{F} \otimes_{T_V^0 K} -$ is exact on the category of $T_V^0 K$–modules.

Definition 2.2.1 For $\varphi\colon K \to H^*V$ a morphism of unstable algebras, let $T_{(V,\varphi)}K$ denote the unstable algebra $\mathbb{F} \otimes_{T_V^0 K} T_V K$, where $\mathbb{F}$ is a $T_V^0 K$–algebra via $\widetilde{\varphi}$, and let $T_{(V,\varphi)}\colon K\text{–}\mathcal{U} \to T_{(V,\varphi)}K\text{–}\mathcal{U}$ be the exact functor $\mathbb{F} \otimes_{T_V^0 K} T_V(-)$.

The functor T_V is natural in V; in particular, there is a natural inclusion $1_{\mathcal{U}} \cong T_0 \hookrightarrow T_V$, for $V \in \mathrm{Ob}\,\mathcal{V}^f$.

Lemma 2.2.2 *For $K \in \mathrm{Ob}\,\mathcal{K}$, $M \in \mathrm{Ob}\,K\text{–}\mathcal{U}$ and $\varphi\colon K \to H^*V$ a morphism of unstable algebras, there are morphisms of unstable algebras*

$$K \to T_V K \to T_{(V,\varphi)}K \cong \mathbb{F} \otimes_{T_V^0 K} T_V K,$$

with respect to which the natural morphisms of unstable modules

$$M \to T_V M \to T_{(V,\varphi)}M \cong \mathbb{F} \otimes_{T_V^0 K} T_V M$$

are morphisms of K–$\mathcal{U}$.

2.3 The Dickson algebras

The group $\mathrm{Aut}(V)$ of linear automorphisms acts on H^*V on the right by morphisms of unstable algebras; the Dickson algebra $D_V \in \mathrm{Ob}\mathcal{K}$ is the ring of invariants $H^*V^{\mathrm{Aut}(V)}$; D_{V_s} will be denoted by D_s. There is an isomorphism of graded algebras

$$D_s \cong \mathbb{F}[\omega_{s,0}, \dots, \omega_{s,s-1}],$$

where the generator $\omega_{s,i}$ has degree $2^s - 2^i$; see Wilkerson [25]. The top Dickson invariant $\omega_{s,0}$ will be written ω_s and identifies with the product of the elements of $(H^1V_s)\backslash\{0\}$; there are related explicit descriptions of the other generators. The algebra H^*V_s is free as a D_s–module, forgetting the action of the Steenrod algebra (see Neusel and Smith [16] for example), in particular is flat as a D_s–module.

Lemma 2.3.1 [11, Définition-Proposition 4.4.7] *Let $\alpha\colon U \hookrightarrow V$ be the inclusion of a subspace of codimension c. There is a canonical surjection of unstable algebras $D_V \twoheadrightarrow \Phi^c D_U$ which fits into a commutative diagram*

$$\begin{tikzcd}
D_V \arrow[rr, hook] \arrow[d, twoheadrightarrow] & & H^*V \arrow[d, twoheadrightarrow, "H^*\alpha"] \\
\Phi^c D_U \arrow[r, hook, "\lambda^c_{D_U}"] & D_U \arrow[r, hook] & H^*U.
\end{tikzcd}$$

In particular, for $s \in \mathbb{N}$, the kernel of $D_s \twoheadrightarrow \Phi D_{s-1}$ is the prime ideal $\omega_s D_s$, which is invariant under the $\mathcal{A}$–action.

The Boolean algebra $T^0_W H^*V_s$ identifies with $\mathbb{F}^{\mathrm{Hom}(W,V_s)}$, and by [19, Proposition 3.9.8], the subalgebra $T^0_W D_s$ is isomorphic to $\mathbb{F}^{\mathrm{Hom}(W,V_s)/\mathrm{Aut}(V_s)}$. The morphism of Boolean algebras $\tilde{\imath}\colon T^0_{V_s} D_s \cong \mathbb{F}^{\mathrm{Hom}(V_s,V_s)/\mathrm{Aut}(V_s)} \to \mathbb{F}$ associated to the canonical inclusion $i\colon D_s \hookrightarrow H^*V_s$ is induced by evaluation on the element of $\mathrm{Hom}(V_s, V_s)/\mathrm{Aut}(V_s)$ represented by the identity morphism of V_s.

Proposition 2.3.2 *For $s \in \mathbb{N}$,*

1. *the unstable algebra $T_{(V_s,i)} D_s$ is isomorphic to H^*V_s;*
2. *$T_{(V_s,i)}$ induces an exact functor $T_{(V_s,i)}\colon D_s\text{–}\mathcal{U} \to H^*V_s\text{–}\mathcal{U}$.*

Proof The first statement is a case of the calculation of the T–functor on rings of invariants (cf [19, Proposition 3.9.8], Dwyer and Wilkerson [5, proof of 1.4]). The second is an immediate consequence, following from the definition of $T_{(V_s,i)}$. $\square$

2.4 Unstable modules over H^*V_s and Fix_s

The module category D_s–$\mathcal{U}$ is related to the category H^*V_s–$\mathcal{U}$ via the adjunction

$$H^*V_s \otimes_{D_s} -\colon D_s\text{–}\mathcal{U} \rightleftarrows H^*V_s\text{–}\mathcal{U} : \mathrm{Restrict}_{D_s}^{H^*V_s}.$$

The functor $\mathrm{Fix}_s\colon H^*V_s\text{–}\mathcal{U} \to \mathcal{U}$ is the left adjoint to the free H^*V_s–module functor $H^*V_s \otimes -\colon \mathcal{U} \to H^*V_s\text{–}\mathcal{U}$; it commutes with tensor products and restricts to a functor $\mathrm{Fix}_s\colon H^*V_s \downarrow \mathcal{K} \to \mathcal{K}$ which is left adjoint to $H^*V_s \otimes -\colon \mathcal{K} \to H^*V_s \downarrow \mathcal{K}$ (cf [12, Théorème 1.3.3]). See [10] and [12; 13] for further properties of the categories H^*V_s–$\mathcal{U}$ and the functors Fix_s.

Lemma 2.2.2 has the following analogue for the functor Fix_s, using Lannes' description of Fix_s in terms of T_{V_s}.

Proposition 2.4.1 *For $s \in \mathbb{N}$, the natural transformation $1_{\mathcal{U}} \to T_{V_s}$ induces a natural transformation $\mathrm{Forget}_s \to \mathrm{Fix}_s$, where $\mathrm{Forget}_s\colon H^*V_s\text{–}\mathcal{U} \to \mathcal{U}$ is the forgetful functor; this factors naturally*

$$\mathrm{Forget}_s \twoheadrightarrow \mathbb{F} \otimes_{H^*V_s} (-) \rightarrow \mathrm{Fix}_s(-)$$

across the H^*V_s–module indecomposables.

Proof By [10, 4.4.3], $\mathrm{Fix}_s M \cong \mathbb{F} \otimes_{T_{V_s} H^*V_s} T_{V_s} M$, where $T_{V_s} H^*V_s \to \mathbb{F}$ is adjoint to the identity on H^*V_s. The natural morphism $M \to T_{V_s} M$ defines a morphism of H^*V_s–modules, as in Lemma 2.2.2. The H^*V_s–module structure on $\mathrm{Fix}_s M$ is induced by the morphism of unstable algebras $H^*V_s \to \mathrm{Fix}_s H^*V_s \cong \mathbb{F}$. The result follows. □

3 Unstable modules over the Dickson algebras

3.1 Unstable modules over D_s, ω_s–torsion and ω_s–closure

Recall from Lemma 2.3.1 that there is a canonical surjection of unstable algebras $D_s \twoheadrightarrow \Phi D_{s-1}$; this induces the functors in the following standard result.

Proposition 3.1.1 *For $s \in \mathbb{N}$, there are adjunctions*

$$\begin{tikzcd} \Phi D_{s-1}\text{–}\mathcal{U} \arrow[r, "\mathrm{Restrict}_s" description] & D_s\text{–}\mathcal{U}, \arrow[l, bend right, "(-)/\omega_s"', "\perp"] \arrow[l, bend left, "\mathrm{Ann}_{\omega_s}", "\perp"'] \end{tikzcd}$$

where $(-)/\omega_s\colon M \mapsto M/\omega_s M$, for $M \in \mathrm{Ob}\, D_s\text{–}\mathcal{U}$ and $\mathrm{Ann}_{\omega_s} M$ is the submodule of elements x such that $\omega_s x = 0$.

Moreover, the adjunction $\mathrm{Restrict}_s \dashv \mathrm{Ann}_{\omega_s}$ *identifies* ΦD_{s-1}*–*$\mathcal{U}$ *as the full subcategory of* D_s*–*$\mathcal{U}$ *of modules annihilated by* ω_s*.*

Remark 3.1.2 (1) The functor $(-)/\omega_s$ identifies with $\Phi D_{s-1} \otimes_{D_s} -$, the induction functor.

(2) Localization away from the torsion associated to an invariant ideal of an unstable algebra has been considered by Henn [7, Section 3] and Meyer [15, Chapter 7]. Dwyer and Wilkerson [3; 4] and Lannes and Zarati [12] have considered localization away from ω_s–torsion.

Localization inverting the top Dickson invariant is an important tool. For $s \in \mathbb{N}$, the localized algebras $D_s[\omega_s^{-1}]$ and $H^* V_s[\omega_s^{-1}]$ are commutative algebras in $\mathcal{M}$ (see Singer [21] and Wilkerson [24]); moreover, $D_s[\omega_s^{-1}] \otimes_{D_s} -$ induces an exact functor

$$D_s[\omega_s^{-1}] \otimes_{D_s} -\colon D_s\text{–}\mathcal{U} \to D_s[\omega_s^{-1}]\text{–}\mathcal{M},$$

which will be denoted $M \mapsto M[\omega_s^{-1}]$.

Recall that the inclusion $\mathcal{U} \hookrightarrow \mathcal{M}$ has a right adjoint $\mathrm{Un}\colon \mathcal{M} \to \mathcal{U}$, which gives the largest unstable module of an $\mathcal{A}$–module. If X, Y are $\mathcal{A}$–modules, there is a canonical monomorphism $(\mathrm{Un} X) \otimes (\mathrm{Un} Y) \to \mathrm{Un}(X \otimes Y)$. It follows that, for $M \in \mathrm{Ob}\, D_s$–$\mathcal{U}$, there is a natural morphism $M \to \mathrm{Un}(M[\omega_s^{-1}])$ in D_s–$\mathcal{U}$.

Definition 3.1.3 [7; 15] An unstable D_s–module $M \in \mathrm{Ob}\, D_s$–$\mathcal{U}$ is ω_s–closed if the map $M \to \mathrm{Un}(M[\omega_s^{-1}])$ is an isomorphism. An unstable $H^* V_s$–module is ω_s–closed if the underlying unstable D_s–module is ω_s–closed.

Proposition 3.1.4 *For* $s \in \mathbb{N}$*, an unstable* D_s*–module* M *is* ω_s*–closed if the unstable* $H^* V_s$*–module* $H^* V_s \otimes_{D_s} M$ *is* ω_s*–closed.*

In particular, for $N \in \mathrm{Ob}\,\mathcal{U}$*,* $D_s \otimes N$ *is* ω_s*–closed.*

Proof The unstable D_s–module M embeds in $H^* V_s \otimes_{D_s} M$ as the invariants (in D_s–$\mathcal{U}$) of the action of $\mathrm{Aut}(V_s)$ induced by the action on the left hand factor. This implies that, if $H^* V_s \otimes_{D_s} M$ is ω_s–closed, then M is ω_s–closed, since the kernel of a morphism between ω_s–closed objects is ω_s–closed.

By [11, Proposition 2.5.2], the morphism

$$H^* V_s \otimes N \to \mathrm{Un}(H^* V_s[\omega_s^{-1}] \otimes N)$$

is an isomorphism, for N an unstable module, which shows that $H^* V_s \otimes N$ is ω_s–closed. Since $H^* V_s \otimes N$ is isomorphic to $H^* V_s \otimes_{D_s} (D_s \otimes N)$, it follows that $D_s \otimes N$ is ω_s–closed. □

Remark 3.1.5 The converse is false in general. Consider $H^* V_s$ as a D_s–module (which is ω_s–closed); for $s > 1$ an integer, $H^* V_s \otimes_{D_s} H^* V_s$ is not ω_s–closed.

3.2 The indecomposables functor

The augmentation $D_s \to \mathbb{F}$ induces a functor $\mathrm{triv}_s\colon \mathcal{U} \to D_s$–$\mathcal{U}$ which has left adjoint $Q_s\colon D_s$–$\mathcal{U} \to \mathcal{U}$ given by $M \mapsto M/\overline{D_s} M \cong \mathbb{F} \otimes_{D_s} M$, $\overline{D_s}$ denoting the augmentation ideal, and right adjoint which associates to an unstable D_s–module the largest unstable submodule with trivial D_s–action, thus identifying $\mathcal{U}$ as the full subcategory of D_s–$\mathcal{U}$ of objects with trivial D_s–module structure

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\mathrm{triv}_s" description] & D_s\text{–}\mathcal{U} \arrow[l, bend right, "Q_s"'] \arrow[l, bend left] \end{tikzcd}$$

4 Introducing the Singer functors

4.1 The Singer functor R_s

The definition and properties of the Singer functor $R_s\colon \mathcal{U} \to D_s$–$\mathcal{U}$ are reviewed in this section; for further details, the reader is referred to the original work of Singer [23; 20] (and related work) and Lannes and Zarati's paper [11].

In the following, if I is a sequence of nonnegative integers, Sq^I denotes the Milnor basis element of the Steenrod algebra indexed by I. The linear map St_s introduced below corresponds to the Steenrod total power.

Definition 4.1.1 For $s \geq 1$ an integer and M an unstable module, let

(1) $\mathrm{St}_s\colon \Phi^s M \to D_s \otimes M$ be the linear map defined on a homogeneous element $\Phi^s x$ by
$$\mathrm{St}_s(\Phi^s x) := \sum_{I=(i_1,\dots,i_s)} \omega_{s,0}^{\varepsilon} \omega_{s,1}^{i_1} \dots \omega_{s,s-1}^{i_{s-1}} \otimes Sq^I(x)$$
where $|x| = \varepsilon + i_1 \dots + i_s$;

(2) $R_s M$ denote the sub D_s–module (ignoring the $\mathcal{A}$–action) of $D_s \otimes M$ which is generated by the image of St_s.

Remark 4.1.2 (1) The linear maps St_s can be constructed as iterates of the linear maps St_1; the above definition stresses the intimate relationship between the Dickson algebras and the dual Steenrod algebra.

(2) Proposition 4.1.3 below contains the statement that $R_s M \subset D_s \otimes M$ is stable under the action of the Steenrod algebra, which is not immediately obvious from the definition given.

By convention, R_0 is taken to be the identity functor on $\mathcal{U}$ and R_{-1} to be the zero functor.

Proposition 4.1.3 [11] *For $s \in \mathbb{N}$, R_s defines a functor $R_s\colon \mathcal{U} \to D_s\text{–}\mathcal{U}$, equipped with a monomorphism $R_s(-) \hookrightarrow D_s \otimes -$ in $D_s\text{–}\mathcal{U}$.*

(1) *For M an unstable module, the underlying graded D_s–module of $R_s M$ is free on a vector space isomorphic to $\Phi^s M$. Moreover, there is a natural isomorphism $Q_s R_s M \cong \mathbb{F} \otimes_{D_s} R_s M \cong \Phi^s M$ in $\mathcal{U}$.*

(2) *The functor $R_s\colon \mathcal{U} \to D_s\text{–}\mathcal{U}$ is exact and commutes with limits and colimits.*

(3) *For unstable modules M, N, there is a natural isomorphism $R_s(M \otimes N) \cong R_s M \otimes_{D_s} R_s N$ in $D_s\text{–}\mathcal{U}$.*

(4) *There is a natural surjection $\rho_s\colon R_s M \twoheadrightarrow \Phi R_{s-1} M$ in $D_s\text{–}\mathcal{U}$ which makes the following diagram commute:*

$$\begin{tikzcd}
R_s M \arrow[r, hook] \arrow[d, two heads, "\rho_s"'] & & & D_s \otimes M \arrow[d, two heads] \\
\Phi R_{s-1} M \arrow[r, hook] & \Phi(D_{s-1} \otimes M) \cong \Phi D_{s-1} \otimes \Phi M \arrow[rr, "1\otimes\lambda_M"'] & & (\Phi D_{s-1}) \otimes M,
\end{tikzcd}$$

where the terms of the bottom row are considered as D_s–modules via restriction of their natural ΦD_{s-1}–module structures.
Moreover, there is a natural short exact sequence in $D_s\text{–}\mathcal{U}$,

$$0 \to \omega_s R_s M \to R_s M \to \Phi R_{s-1} M \to 0.$$

(5) *If N is a nilpotent unstable module, then $R_s N$ is nilpotent.*

(6) *If M is a reduced (respectively nilclosed) unstable module, then $R_s M$ is reduced (resp. nilclosed).*

The following result will be strengthened in Section 8.3.

Lemma 4.1.4 *For $s \in \mathbb{N}$ and unstable modules M, N, the functor R_s induces a monomorphism $\mathrm{Hom}_{\mathcal{U}}(M, N) \hookrightarrow \mathrm{Hom}_{D_s\text{–}\mathcal{U}}(R_s M, R_s N)$.*

Proof By Proposition 4.1.3, the composite $Q_s R_s$ is naturally isomorphic to the functor Φ^s, hence gives rise to

$$\mathrm{Hom}_{\mathcal{U}}(M, N) \xrightarrow{R_s} \mathrm{Hom}_{D_s\text{–}\mathcal{U}}(R_s M, R_s N) \xrightarrow{Q_s} \mathrm{Hom}_{\mathcal{U}}(\Phi^s M, \Phi^s N),$$

which identifies with the natural morphism corresponding to the functor Φ^s. The functor Φ^s is fully faithful, hence the first morphism is injective, as required. □

Recall that an unstable module N is locally finite if $\mathcal{A}x \subset N$ is finite, for every element x of N.

Proposition 4.1.5 *For $s \in \mathbb{N}$ and X a locally finite unstable module, the natural monomorphism $R_s X \hookrightarrow D_s \otimes X$ is the ω_s–closure of $R_s X$ in $D_s \otimes X$.*

Proof The module $D_s \otimes X$ is ω_s–closed in $D_s\text{–}\mathcal{U}$, by Proposition 3.1.4, hence it suffices to show that the cokernel of $R_s X \to D_s \otimes X$ is ω_s–torsion. Since both functors commute with colimits, it suffices to consider the case where X is a finite unstable module. This case is established by induction on the total dimension of X, using the cases $X = \Sigma^n \mathbb{F}$, for $n \in \mathbb{N}$, for the inductive step. The monomorphism $R_s \Sigma^n \mathbb{F} \hookrightarrow D_s \otimes \Sigma^n \mathbb{F}$ identifies with the n–iterated suspension of the inclusion $\omega_s^n D_s \hookrightarrow D_s$, so the cokernel is ω_s–torsion. □

Proposition 4.1.6 *The functor $R_s\colon \mathcal{U} \to D_s\text{–}\mathcal{U}$ admits a left adjoint $\mathfrak{A}_s$ and a right adjoint $\mathfrak{Z}_s$,*

$$\begin{tikzcd} D_s\text{–}\mathcal{U} \arrow[r, bend left, "\mathfrak{A}_s"] \arrow[r, bend right, "\mathfrak{Z}_s"'] & \mathcal{U}. \arrow[l, "R_s"' near start] \arrow[l, phantom, shift right=2, "\perp"] \arrow[l, phantom, shift left=2, "\perp"] \end{tikzcd}$$

Moreover,

(1) *the functor $\mathfrak{A}_s$ sends projective objects of $D_s\text{–}\mathcal{U}$ to projectives of $\mathcal{U}$;*

(2) *the functor $\mathfrak{Z}_s$ sends injective (respectively reduced) objects of $D_s\text{–}\mathcal{U}$ to injective (resp. reduced) objects of $\mathcal{U}$.*

Proof The result is a formal consequence of the properties of R_s. For example, for $M \in \mathrm{Ob}\, D_s\text{–}\mathcal{U}$, $\mathfrak{Z}_s M$ is reduced if and only if $\mathrm{Hom}_{\mathcal{U}}(N, \mathfrak{Z}_s M) \cong \mathrm{Hom}_{D_s\text{–}\mathcal{U}}(R_s N, M)$ is trivial for every nilpotent unstable module N. The functor R_s preserves nilpotent unstable modules; hence, if M is reduced, then $\mathrm{Hom}_{D_s\text{–}\mathcal{U}}(R_s N, M) = 0$ for nilpotent N. □

Remark 4.1.7 The category D_s–$\mathcal{U}$ has enough projectives and injectives [12; 15]. A family of projective generators of D_s–$\mathcal{U}$ is given by the family of unstable D_s–modules $D_s \otimes F(n)$, where $F(n)$ denotes the free unstable module on a generator of degree n. Similarly, there is a family of injective cogenerators given by the generalized Brown–Gitler modules $J_{D_s}(n)$, for $n \in \mathbb{N}$, where $J_{D_s}(n)$ corepresents the contravariant functor $M \mapsto (M^n)^*$, for $M \in \mathrm{Ob}\, D_s$–$\mathcal{U}$.

The unstable modules $\mathfrak{A}_s(D_s \otimes F(n))$ and $\mathfrak{Z}_s(J_{D_s}(n))$ are closely related and illuminate the relationship between the Dickson algebras and the Steenrod algebra.

Recall from Section 3 that $\mathrm{triv}_s\colon \mathcal{U} \to D_s$–$\mathcal{U}$ gives an unstable module the trivial D_s–module structure and $\mathrm{Restrict}_s\colon \Phi D_{s-1}$–$\mathcal{U} \to D_s$–$\mathcal{U}$ is induced by the canonical projection $D_s \twoheadrightarrow \Phi D_{s-1}$. The functor $\widetilde{\Phi}\colon \Phi D_{s-1}$–$\mathcal{U} \to D_{s-1}$–$\mathcal{U}$ is provided by Proposition A.1.1.

Proposition 4.1.8 *For $s \in \mathbb{N}$, there are natural isomorphisms*

(1) $\mathfrak{Z}_s \circ \mathrm{Restrict}_s \cong \mathfrak{Z}_{s-1} \circ \widetilde{\Phi}\colon \Phi D_{s-1}$–$\mathcal{U} \to \mathcal{U}$

(2) $\mathfrak{Z}_s \circ \mathrm{triv}_s \cong \widetilde{\Phi}^s\colon \mathcal{U} \to \mathcal{U}$.

Proof For $M \in \mathrm{Ob}\,\mathcal{U}$ and $N \in \mathrm{Ob}\,\Phi D_{s-1}$–$\mathcal{U}$, there are natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{U}}(M, \mathfrak{Z}_s \,\mathrm{Restrict}_s\, N) &\cong \mathrm{Hom}_{D_s\text{–}\mathcal{U}}(R_s M, \mathrm{Restrict}_s\, N)\\ &\cong \mathrm{Hom}_{\Phi D_{s-1}\text{–}\mathcal{U}}((R_s M)/\omega_s, N).\end{aligned}$$

By Proposition 4.1.3, there is a natural isomorphism $(R_s M)/\omega_s \cong \Phi R_{s-1} M$, hence

$$\mathrm{Hom}_{\Phi D_{s-1}\text{–}\mathcal{U}}((R_s M)/\omega_s, N) \cong \mathrm{Hom}_{D_{s-1}\text{–}\mathcal{U}}(R_{s-1} M, \widetilde{\Phi} N) \cong \mathrm{Hom}_{\mathcal{U}}(M, \mathfrak{Z}_{s-1} \widetilde{\Phi} N),$$

by adjunction. The first statement follows.

The second statement can either be proved directly by a similar argument, or deduced by induction from the first, since triv_s is the composite of the functors $\mathrm{Restrict}_i$ for $1 \le i \le s$. □

Further results on the functors $\mathfrak{Z}_s$ are given in Section 9, using deeper properties of the Singer functors R_s.

5 Functor categories and nillocalization

This section reviews the techniques of nillocalization, as they apply to the study of the category of unstable modules over an unstable algebra. This is based on the foundations of Henn, Lannes and Schwartz [8; 9], related to earlier work of Lam, Rector [18] and Adams and Wilkerson [1], and on subsequent work of Djament [2], Henn [7], Lannes and Zarati [12; 13], Mekkia [14] and Meyer [15].

5.1 Nillocalizations

The general theory of localization of abelian categories [6] provides an adjunction $l\colon \mathcal{U} \rightleftarrows \mathcal{U}/\mathcal{N}il : r$ and, moreover, the functor l is exact [19, Chapter 5]. The adjunction unit $M \to rlM$ corresponds to nilclosure: M is reduced (respectively nilclosed) if and only if it is a monomorphism (resp. isomorphism).

Notation 5.1.1 Write $\mathcal{V}^f$ for the full subcategory of finite-dimensional spaces in $\mathcal{V}$, the category of $\mathbb{F}$–vector spaces; the category of functors from $\mathcal{V}^f$ to $\mathcal{V}$ is denoted by $\mathcal{F}$ and the full subcategory of locally finite (or analytic) functors, $\mathcal{F}_\omega$ (see [8; 19]).

The category $\mathcal{U}/\mathcal{N}il$ identifies with the full subcategory $\mathcal{F}_\omega \subset \mathcal{F}$ of analytic functors via $l\colon \mathcal{U} \to \mathcal{F}$, $lM(V) = T_V^0 M$, which has right adjoint equally denoted by r:

$$l\colon \mathcal{U} \rightleftarrows \mathcal{F} : r.$$

Example 5.1.2 For $V \in \operatorname{Ob}\mathcal{V}^f$, we have that the analytic functor lH^*V is the injective $I_V(-) = \mathbb{F}^{\operatorname{Hom}(-,V)}$. The analogue of the functor T_V in the category $\mathcal{F}$ is the shift functor $\Delta_V\colon \mathcal{F} \to \mathcal{F}$, defined by precomposition with $-\oplus V\colon \mathcal{V}^f \to \mathcal{V}^f$.

The functors l and r both commute with tensor products, which is an important fact in considering module structures in the respective categories. Similarly, the functor l sends an unstable algebra to a functor with values in Boolean algebras. The category of Boolean algebras is equivalent to the opposite of the category of profinite sets, via the functor $X \mapsto \mathbb{F}^X$, where $\mathbb{F}^X$ denotes the space of continuous maps from the profinite set X to $\mathbb{F}$.

Notation 5.1.3 Let

(1) $\mathcal{PS}$ denote the category of presheaves of profinite sets on $\mathcal{V}^f$, so that the continuous map functor induces $\mathbb{F}^{(-)}\colon \mathcal{PS}^{\mathrm{op}} \to \mathcal{F}$;

(2) $\mathfrak{g}\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{PS}$ be the functor $\mathfrak{g}(K)\colon V \mapsto \operatorname{Hom}_{\mathcal{K}}(K, H^*V)$.

Lannes' linearization principle fits into this framework via the isomorphism

$$l(K)(V) = T_V^0 K \cong \mathbb{F}^{\mathfrak{g}(K)(V)}.$$

If K is a Noetherian unstable algebra, then $\mathfrak{g}(K)$ takes values in finite sets (cf [8]).

Example 5.1.4 For $s \in \mathbb{N}$,

(1) $\mathfrak{g}(H^* V_s)(W) = \mathrm{Hom}_{\mathcal{V}^f}(W, V_s)$;

(2) $\mathfrak{g}(D_s)(W) = \mathrm{Hom}_{\mathcal{V}^f}(W, V_s)/\mathrm{Aut}(V_s)$, which is equivalent to the set of subspaces of W of codimension at most s, regarded as a contravariant functor by pullback of subspaces.

The inclusion $D_s \overset{i}{\hookrightarrow} H^* V_s$ induces the surjection to coinvariants $\mathfrak{g}(H^* V_s) \twoheadrightarrow \mathfrak{g}(D_s)$.

5.2 Nillocalization of the category of modules over a Noetherian unstable algebra

Let K be a Noetherian unstable algebra; an object of $K{-}\mathcal{U}$ is said to be nilpotent if the underlying unstable module is nilpotent. There is an exact localization functor $K{-}\mathcal{U} \to K{-}\mathcal{U}/\mathcal{N}il$. (This notation should not lead to confusion, since there is a forgetful functor to $\mathcal{U}/\mathcal{N}il$ and the category $K{-}\mathcal{U}/\mathcal{N}il$ only depends on K up to nillocalization.)

An element of $K{-}\mathcal{U}$ is nilclosed if and only if the underlying unstable module is nilclosed; in this case, the unstable K–module structure is the restriction of the induced unstable $rl(K)$–module structure ($rl(K)$ has a canonical unstable algebra structure [8]).

An element $\varphi \in \mathfrak{g}(K)(V)$ can be considered as a morphism of Boolean algebras $T_V^0 K \to \mathbb{F}$ and the functor $T_{(V,\varphi)}$ is defined (cf Definition 2.2.1), which has degree zero part denoted $T^0_{(V,\varphi)}$. The pair (V,φ) can be considered as an object of a comma category, which motivates the following.

Notation 5.2.1 For $\mathfrak{X}$ a presheaf of finite sets on $\mathcal{V}^f$, denote by

(1) $\mathcal{V}^f_{/\mathfrak{X}}$ the comma category, with objects pairs (V,x), where $V \in \mathcal{V}^f$ and $x \in \mathfrak{X}(V)$, and a morphism $(V,x) \to (W,y)$ is a linear map $f\colon V \to W$ such that $\mathfrak{X}(f)y = x$;

(2) $\mathcal{F}^{\mathfrak{X}}$ the category of functors $\mathrm{Funct}(\mathcal{V}^f_{/\mathfrak{X}}, \mathcal{V})$;

(3) $\mathbb{F}^{\mathfrak{X}}{-}\mathcal{F}$ the category of $\mathbb{F}^{\mathfrak{X}}$–modules in $\mathcal{F}$.

Example 5.2.2 (1) The category $\mathcal{V}^f_{/\mathfrak{g}(H^* V_s)}$ is the over-category $\mathcal{V}^f/V_s$.

(2) The category $\mathcal{V}^f_{/\mathfrak{g}(D_s)}$ has objects (V,U), where $U \le V$ is a subspace of codimension at most s; a morphism $(V,U) \to (V',U')$ is a linear map $V \to V'$ sending U to U' and such that the induced map $V/U \to V'/U'$ is a monomorphism.

The category $\mathscr{F}^{\mathfrak{X}}$ is abelian equipped with a tensor product, this structure being inherited from $\mathscr{V}$. Moreover, Yoneda's lemma shows that it has sufficiently many projectives and injectives.

Example 5.2.3 Consider the case $\mathfrak{X} = \mathfrak{g}(D_s)$, for $s \in \mathbb{N}$. If (V, U) is an object of $\mathscr{V}^f_{/\mathfrak{g}(D_s)}$, then

(1) $P_{(V,U)} \in \operatorname{Ob} \mathscr{F}^{\mathfrak{g}(D_s)}$ denotes the projective functor $\mathbb{F}[\operatorname{Hom}((V, U), -)]$,

(2) $I_{(V,U)} \in \operatorname{Ob} \mathscr{F}^{\mathfrak{g}(D_s)}$ denotes the injective functor $\mathbb{F}^{\operatorname{Hom}(-,(V,U))}$,

where Hom is taken in the category $\mathscr{V}^f_{/\mathfrak{g}(D_s)}$. This gives families of projective generators and injective cogenerators respectively, as (V, U) runs over representatives of isomorphism classes of objects of $\mathscr{V}^f_{/\mathfrak{g}(D_s)}$.

The functor $I_{(V_s,0)}$ plays an important role; it can be identified as follows. There is a natural isomorphism

$$\operatorname{Hom}_{\mathscr{V}^f_{/\mathfrak{g}(D_s)}}((V, U), (V_s, 0)) \cong \operatorname{Inj}(V/U, V_s),$$

where the right hand side is the set of injective linear maps. Hence we have that $I_{(V_s,0)}(V, U) \cong \mathbb{F}^{\operatorname{Inj}(V/U,V_s)}$.

The functors $T^0_{(V,\varphi)}$ of Definition 2.2.1 are constructed using the splitting associated to the canonical idempotents of the finite-dimensional Boolean algebra $T^0_V K$, which gives an isomorphism for $M \in \operatorname{Ob} K{-}\mathscr{U}$, namely

$$T^0_V M \cong \bigoplus_{\varphi \in \mathfrak{g}(K)(V)} T^0_{(V,\varphi)} M.$$

This corresponds to a functor defined in the general framework which was introduced in Notation 5.2.1 (see [2, Chapitre 3]). Namely the forgetful functor $\mathscr{V}^f_{/\mathfrak{X}} \to \mathscr{V}^f$ induces a functor $\iota^{\mathfrak{X}} \colon \mathscr{F} \to \mathscr{F}^{\mathfrak{X}}$ by precomposition, given explicitly by $\iota^{\mathfrak{X}} F(V, x) = F(V)$. This admits a right adjoint $\Omega^{\mathfrak{X}}$ given by $\Omega^{\mathfrak{X}} G(V) = \bigoplus_{x \in \mathfrak{X}(V)} G(V, x)$.

The importance of this adjunction is through the following.

Proposition 5.2.4 [2, Proposition 3.3.10] *For $\mathfrak{X}$ a presheaf of finite sets on $\mathscr{V}^f$, the adjunction $\iota^{\mathfrak{X}} \dashv \Omega^{\mathfrak{X}}$ induces an equivalence between $\mathscr{F}^{\mathfrak{X}}$ and the category $\mathbb{F}^{\mathfrak{X}}{-}\mathscr{F}$ of $\mathbb{F}^{\mathfrak{X}}$–modules in $\mathscr{F}$.*

There is a relative version of the above construction [2, Définition et Proposition 3.3.4]. For $\alpha \colon \mathfrak{X} \to \mathfrak{Y}$ a morphism of presheaves of finite sets, $\mathscr{V}^f_{/\alpha}$ induces a functor

$\alpha^!\colon \mathscr{F}^{\mathfrak{Y}} \to \mathscr{F}^{\mathfrak{X}}$ by precomposition, which admits a right adjoint $\alpha_!\colon \mathscr{F}^{\mathfrak{X}} \to \mathscr{F}^{\mathfrak{Y}}$ given on $F \in \mathrm{Ob}\mathscr{F}^{\mathfrak{X}}$ by

$$(\alpha_! F)(V, y) = \bigoplus_{x \in \alpha_V^{-1} y} F(V, x).$$

Consult [2, Chapitre 3] for the general properties.

Remark 5.2.5 For $\mathfrak{X}$ a presheaf of finite sets, $\iota^{\mathfrak{X}} = (\mathfrak{X} \to *)^!$ and $\Omega^{\mathfrak{X}} = (\mathfrak{X} \to *)_!$, where $*$ is the terminal presheaf.

Example 5.2.6 For $m\colon K \to L$ a morphism of Noetherian unstable algebras, we have that $\mathfrak{g}(m)\colon \mathfrak{g}(L) \to \mathfrak{g}(K)$ induces an adjunction

$$\mathfrak{g}(m)^!\colon \mathscr{F}^{\mathfrak{g}(K)} \rightleftarrows \mathscr{F}^{\mathfrak{g}(L)} : \mathfrak{g}(m)_!.$$

This is related to the induction-restriction adjunction $L \otimes_K -\colon K\text{–}\mathscr{U} \rightleftarrows L\text{–}\mathscr{U} : \mathrm{Restrict}_K^L$ in Theorem 5.2.8.

Definition 5.2.7 A functor G of $\mathscr{F}^{\mathfrak{X}}$ is analytic if $\Omega^{\mathfrak{X}} G \in \mathrm{Ob}\mathscr{F}$ is analytic; the full subcategory of analytic functors in $\mathscr{F}^{\mathfrak{X}}$ is denoted $\mathscr{F}^{\mathfrak{X}}_{\omega}$.

Theorem 5.2.8 *For K a Noetherian unstable algebra, the adjunction $l\colon \mathscr{U} \rightleftarrows \mathscr{F} : r$ induces an adjunction*

$$\mathfrak{l}_K\colon K\text{–}\mathscr{U} \rightleftarrows \mathscr{F}^{\mathfrak{g}(K)} : \mathfrak{r}_K,$$

where $(\mathfrak{l}_K M)(V, \varphi) = T^0_{(V,\varphi)} M$ and the underlying functor $\mathfrak{r}_K\colon \mathscr{F}^{\mathfrak{g}(K)} \to \mathscr{U}$ is the composite $r\Omega^{\mathfrak{g}(K)}$.

The functor $\mathfrak{l}_K$ is exact and commutes with tensor products. Moreover, $\mathfrak{l}_K$ induces an equivalence of categories

$$K\text{–}\mathscr{U}/\mathscr{N}il \xrightarrow{\cong} \mathscr{F}^{\mathfrak{g}(K)}_{\omega}.$$

For $m\colon K \to L$ a morphism of Noetherian unstable algebras,

1. *$\mathfrak{l}_L(L \otimes_K -)\colon K\text{–}\mathscr{U} \to \mathscr{F}^{\mathfrak{g}(L)}$ is naturally equivalent to $\mathfrak{g}(m)^! \mathfrak{l}_K$;*
2. *$\mathrm{Restrict}_K^L \mathfrak{r}_L\colon \mathscr{F}^{\mathfrak{g}(L)} \to K\text{–}\mathscr{U}$ is naturally equivalent to $\mathfrak{r}_K \mathfrak{g}(m)_!$.*

Proof The functor $l\colon \mathscr{U} \to \mathscr{F}$ commutes with tensor products, hence induces a functor $l\colon K\text{–}\mathscr{U} \to \mathbb{F}^{\mathfrak{g}(K)}\text{–}\mathscr{F}$. The category $\mathbb{F}^{\mathfrak{g}(K)}\text{–}\mathscr{F}$ is equivalent to $\mathscr{F}^{\mathfrak{g}(K)}$ by Proposition 5.2.4, and this yields the functor $\mathfrak{l}_K$. Likewise, the composite $r\Omega^{\mathfrak{g}(K)}$ induces a functor to $rl(K)\text{–}\mathscr{U}$; restriction along the adjunction unit $K \to rl(K)$, which is a morphism of unstable algebras, gives $\mathfrak{r}_K$. That these functors are adjoint is formal and the basic properties follow from the general theory of nillocalization [8].

Consider the morphism $m\colon K \to L$. Statement (1) can be verified directly by using the explicit form of $\mathfrak{l}_K$ and $\mathfrak{l}_L$, as follows. Consider $M \in \mathrm{Ob}\, K{-}\mathcal{U}$ and an element $\psi \in \mathfrak{g}(L)(V)$; there are natural isomorphisms

$$T^0_{(V,\psi)}(L \otimes_K M) \cong \mathbb{F} \otimes_{T^0_V L} (T^0_V L \otimes_{T^0_V K} T^0_V M) \cong \mathbb{F} \otimes_{T^0_V K} T^0_V M,$$

where the latter tensor product is formed with respect to $\mathfrak{g}(m)\psi \in \mathfrak{g}(K)(V)$. This establishes the first identification. Statement (2) follows by adjunction from (1). □

Example 5.2.9 For $s \in \mathbb{N}$,

1. $D_s{-}\mathcal{U}/\mathcal{N}il$ is equivalent to the category $\mathcal{F}_\omega^{\mathfrak{g}(D_s)}$, embedded as a full subcategory of $\mathcal{F}^{\mathfrak{g}(D_s)}$;
2. $H^*V_s{-}\mathcal{U}/\mathcal{N}il$ is equivalent to $\mathcal{F}_\omega^{\mathfrak{g}(H^*V_s)}$.

6 Nillocalization of unstable modules over the Dickson algebras

The results of Section 5.2 are applied to the categories $D_s{-}\mathcal{U}$ to obtain the analogues of the structures considered in Section 3. The reader is referred to [2] for further results; in particular, the adjunctions considered here fit into *recollement* diagrams of abelian categories.

Throughout the section, the identification of $\mathcal{V}^f_{/\mathfrak{g}(D_s)}$ given in Example 5.2.2 is used without further comment.

6.1 Restriction

For $0 < s \in \mathbb{Z}$, the surjection $D_s \twoheadrightarrow \Phi D_{s-1}$ of Lemma 2.3.1 induces an inclusion $\mathfrak{g}(D_{s-1}) \cong \mathfrak{g}(\Phi D_{s-1}) \hookrightarrow \mathfrak{g}(D_s)$. As in Section 3.1 in the setting of modules over the Dickson algebras, there are associated adjunctions.

Proposition 6.1.1 *For $1 \le s \in \mathbb{Z}$, there is an adjunction*

$$\mathcal{R}_s\colon \mathcal{F}^{\mathfrak{g}(D_s)} \rightleftarrows \mathcal{F}^{\mathfrak{g}(D_{s-1})} : \mathcal{P}_s,$$

in which $\mathcal{R}_s = \mathfrak{g}(\, D_s \twoheadrightarrow \Phi D_{s-1})$ is restriction and $\mathcal{P}_s = \mathfrak{g}(D_s \twoheadrightarrow \Phi D_{s-1})_!$ is extension by zero.

Moreover, there are natural equivalences of exact functors

1. $\mathcal{R}_s \mathfrak{l}_{D_s} \cong \mathfrak{l}_{D_{s-1}}\big((-)/\omega_s\big)\colon D_s{-}\mathcal{U} \to \mathcal{F}^{\mathfrak{g}(D_{s-1})}$;
2. $\mathcal{P}_s \mathfrak{l}_{D_{s-1}} \cong \mathfrak{l}_{D_s} \circ \mathrm{Restrict}_s\colon \Phi D_{s-1}{-}\mathcal{U} \to \mathcal{F}^{\mathfrak{g}(D_s)}$.

Proof The identification of the functors is straightforward (cf the general results of [2, Appendice C.6]); the final statement follows from Theorem 5.2.8. □

6.2 Full restriction

The augmentation $D_s \to \mathbb{F}$ gives rise to an adjunction, as in Proposition 6.1.1. In the following statement, $F \in \mathrm{Ob}\mathscr{F}$ and $G \in \mathrm{Ob}\mathscr{F}^{\mathfrak{g}(D_s)}$.

Proposition 6.2.1 *For $s \in \mathbb{N}$, there is an adjunction*

$$\mathcal{R}^0_s\colon \mathscr{F}^{\mathfrak{g}(D_s)} \rightleftarrows \mathscr{F} :\mathcal{P}^0_s,$$

where $\mathcal{R}^0_s = \mathfrak{g}(D_s \to \mathbb{F})^!$ and $\mathcal{P}^0_s = \mathfrak{g}(D_s \to \mathbb{F})_!$.

Explicitly, $\mathcal{R}^0_s\colon \mathscr{F}^{\mathfrak{g}(D_s)} \to \mathscr{F}$ is the restriction functor defined by $\mathcal{R}^0_s G(V) := G(V,V)$ and $\mathcal{P}^0_s\colon \mathscr{F} \to \mathscr{F}^{\mathfrak{g}(D_s)}$ is extension by zero $\mathcal{P}^0_s F(V,U) = 0$ unless $V = U$, when $\mathcal{P}^0_s F(V,V) = F(V)$. In particular, the composite $\mathcal{R}^0_s\mathcal{P}^0_s$ is naturally equivalent to $1_{\mathscr{F}}$.

Moreover, there are natural equivalences of exact functors:

1. $\mathcal{R}^0_s \mathfrak{l}_{D_s} \cong lQ_s\colon D_s\text{–}\mathscr{U} \to \mathscr{F}$;
2. $\mathcal{P}^0_s l \cong \mathfrak{l}_{D_s} \circ \mathrm{triv}_s\colon \mathscr{U} \to \mathscr{F}^{\mathfrak{g}(D_s)}$.

6.3 Induction and restriction

The canonical inclusion $D_s \overset{i}{\hookrightarrow} H^*V_s$ induces the functor $\mathscr{V}^f_{/\mathfrak{g}(H^*V_s)} \to \mathscr{V}^f_{/\mathfrak{g}(D_s)}$, which sends an object $f\colon V \to V_s$ of $\mathscr{V}^f/V_s$ to $(V, \ker f)$. As above, one has the following.

Proposition 6.3.1 *For $s \in \mathbb{N}$, there is an adjunction*

$$\mathrm{Ind}_s\colon \mathscr{F}^{\mathfrak{g}(D_s)} \rightleftarrows \mathscr{F}^{\mathfrak{g}(H^*V_s)} :\mathrm{Res}_s,$$

where $\mathrm{Ind}_s = \mathfrak{g}(D_s \overset{i}{\hookrightarrow} H^*V_s)^!$ and $\mathrm{Res}_s = \mathfrak{g}(D_s \overset{i}{\hookrightarrow} H^*V_s)_!$. The functors Ind_s and Res_s are exact and Ind_s commutes with tensor products.

The induction functor is given explicitly by $(\mathrm{Ind}_s G)(V \xrightarrow{f} V_s) = G(V, \ker f)$, for $G \in \mathrm{Ob}\mathscr{F}^{\mathfrak{g}(D_s)}$.

Recall from Section 5.2 that there is an adjunction

$$\iota^{\mathfrak{g}(D_s)}\colon \mathscr{F} \rightleftarrows \mathscr{F}^{\mathfrak{g}(D_s)} :\Omega^{\mathfrak{g}(D_s)}.$$

In the following, $F \in \mathrm{Ob}\mathscr{F}$, $G \in \mathrm{Ob}\mathscr{F}^{\mathfrak{g}(D_s)}$ and $H \in \mathrm{Ob}\mathscr{F}^{\mathfrak{g}(H^*V_s)}$. Moreover, $I_{(V_s,0)}$ denotes the injective cogenerator of $\mathscr{F}^{\mathfrak{g}(D_s)}$ introduced in Example 5.2.3.

Proposition 6.3.2 *For $s \in \mathbb{N}$,*

(1) *there are adjunctions*

$$\begin{tikzcd} \mathscr{F}^{\mathfrak{g}(H^*V_s)} \arrow[rr, bend left, "\Psi_s"] \arrow[rr, bend right, "\Omega^{\mathfrak{g}(H^*V_s)}"'] & \arrow[u, phantom, "\perp"] \arrow[d, phantom, "\perp"] & \mathscr{F} \arrow[ll, "\iota^{\mathfrak{g}(H^*V_s)}"'] \end{tikzcd},$$

where $\Psi_s H(V) = H(V \oplus V_s \twoheadrightarrow V_s)$ and the functor Ψ_s is exact and commutes with tensor products;

(2) $\mathrm{Ind}_s \iota^{\mathfrak{g}(D_s)}$ *is naturally isomorphic to* $\iota^{\mathfrak{g}(H^*V_s)}$ *and* $\Omega^{\mathfrak{g}(D_s)}\mathrm{Res}_s$ *is naturally isomorphic to* $\Omega^{\mathfrak{g}(H^*V_s)}$*;*

(3) *the map* $\Psi_s \mathrm{Ind}_s\colon \mathscr{F}^{\mathfrak{g}(D_s)} \to \mathscr{F}$ *is determined by* $\Psi_s \mathrm{Ind}_s G(V) = G(V \oplus V_s, V)$*; it is exact, commutes with tensor products and is left adjoint to the functor* $\mathrm{Res}_s \iota^{\mathfrak{g}(H^*V_s)}\colon \mathscr{F} \to \mathscr{F}^{\mathfrak{g}(D_s)}$*;*

(4) $\mathrm{Res}_s \iota^{\mathfrak{g}(H^*V_s)}\colon \mathscr{F} \to \mathscr{F}^{\mathfrak{g}(D_s)}$ *is exact and is naturally equivalent to the functor*

$$F \mapsto \iota^{\mathfrak{g}(D_s)} F \otimes I_{(V_s,0)}.$$

Moreover, there are natural isomorphisms

$$l\mathrm{Fix}_s \cong \Psi_s \mathfrak{l}_{H^*V_s}\colon H^*V_s\text{--}\mathcal{U} \to \mathscr{F},$$
$$l\mathrm{Fix}_s(H^*V_s \otimes_{D_s} -) \cong \Psi_s \mathrm{Ind}_s \mathfrak{l}_{D_s}\colon D_s\text{--}\mathcal{U} \to \mathscr{F}.$$

Proof The first statement follows from the natural isomorphism

$$\mathrm{Hom}_{\mathscr{V}^f/V_s}((A \to V_s), (V \oplus V_s \xrightarrow{\mathrm{pr}} V_s)) \cong \mathrm{Hom}_{\mathscr{V}^f}(A, V).$$

The remaining numbered statements are straightforward and follow from [2, Définition et Proposition 3.3.4].

The functor $l\ \mathrm{Fix}_s$ is left adjoint to the functor

$$F \mapsto H^*V_s \otimes rF \cong r(I_{V_s} \otimes F),$$

for $F \in \mathrm{Ob}\mathscr{F}$. The latter is isomorphic to $\mathfrak{r}_{H^*V_s}\iota^{\mathfrak{g}(H^*V_s)}F$, since the underlying functor of $\mathfrak{r}_{H^*V_s}$ is $r\Omega^{\mathfrak{g}(H^*V_s)}$ and $\Omega^{\mathfrak{g}(H^*V_s)}\iota^{\mathfrak{g}(H^*V_s)}F \cong I_{V_s} \otimes F$ (cf [2, Définition et Proposition 3.3.4]). Since $\mathfrak{l}_{H^*V_s}$ is left adjoint to $\mathfrak{r}_{H^*V_s}$ and Ψ_s is left adjoint to $\iota^{\mathfrak{g}(H^*V_s)}$, it follows by unicity of adjoints that $l\mathrm{Fix}_s$ is equivalent to $\Psi_s \mathfrak{l}_{H^*V_s}$.

The final identification follows from the natural isomorphism given by Theorem 5.2.8, $\mathfrak{l}_{H^*V_s}(H^*V_s \otimes_{D_s} -) \cong \mathrm{Ind}_s \mathfrak{l}_{D_s}$. □

Example 6.3.3 Proposition 6.3.2(4) contains the isomorphism

$$\mathrm{Res}_s i^{\mathfrak{g}(H^*V_s)}\mathbb{F} \cong I_{(V_s,0)}.$$

The importance of $I_{(V_s,0)}$ is shown by the isomorphism $\mathfrak{r}_{D_s} I_{(V_s,0)} \cong \mathrm{Restrict}_{D_s}^{H^*V_s} H^*V_s$ in $D_s{-}\mathcal{U}$, which follows from Theorem 5.2.8.

7 The Singer functors up to nilpotent unstable modules

This section introduces the functors $\kappa_s\colon \mathcal{F} \to \mathcal{F}^{\mathfrak{g}(D_s)}$ which model the Singer functors up to nilpotent unstable modules. The proof that these correspond to the functors R_s is postponed until Section 8.

7.1 Avatars of the Singer functors

Some of the material of this section is available in [2] under a dual formulation using comodules over Boolean coalgebras.

Definition 7.1.1 For $s \in \mathbb{N}$, let $\kappa_s\colon \mathcal{F} \to \mathcal{F}^{\mathfrak{g}(D_s)}$ denote the functor defined on $F \in \mathrm{Ob}\,\mathcal{F}$ by

$$(\kappa_s F)(V, W) := F(W).$$

Notation 7.1.2 For $s \in \mathbb{N}$, $V \in \mathrm{Ob}\,\mathcal{V}^f$, let $\mathrm{Stab}(V, V \oplus V_s) \subset \mathrm{Aut}(V \oplus V_s)$ denote the pointwise stabilizer of V.

The following is clear.

Lemma 7.1.3 *Let $V, W \in \mathrm{Ob}\,\mathcal{V}^f$ and $f\colon V \to W$ be a linear morphism. Then*

(1) $\mathrm{Stab}(V, V \oplus V_s)$ *is isomorphic to the semidirect product* $\mathrm{Hom}(V_s, V) \rtimes \mathrm{Aut}(V_s)$, *where* $\mathrm{Aut}(V_s)$ *acts on the right on* $\mathrm{Hom}(V_s, V)$ *by precomposition;*

(2) *the action of* $\mathrm{Aut}(V \oplus V_s)$ *on* $V \oplus V_s$ *induces an action of* $\mathrm{Stab}(V, V \oplus V_s)$ *on* $(V \oplus V_s, V) \in \mathrm{Ob}\,\mathcal{V}^f_{/\mathfrak{g}(D_s)}$;

(3) $f\colon V \to W$ *induces a group morphism* $\mathrm{Stab}(V, V \oplus V_s) \to \mathrm{Stab}(W, W \oplus V_s)$ *which, with respect to the semidirect product decomposition, is induced by* $\mathrm{Hom}(V_s, f)\colon \mathrm{Hom}(V_s, V) \to \mathrm{Hom}(V_s, W)$.

In the following statement, $G \in \mathrm{Ob}\,\mathcal{F}^{\mathfrak{g}(D_s)}$, $V \in \mathrm{Ob}\,\mathcal{V}^f$ and $(V, U) \in \mathrm{Ob}\,\mathcal{V}^f_{/\mathfrak{g}(D_s)}$.

Proposition 7.1.4 *Let s be a natural number.*

(1) *The functor $\kappa_s\colon \mathscr{F} \to \mathscr{F}^{\mathfrak{g}(D_s)}$ is exact and commutes with tensor products, limits and colimits.*

(2) *There is a natural monomorphism $\kappa_s \hookrightarrow \iota^{\mathfrak{g}(D_s)}$ of functors from $\mathscr{F}$ to $\mathscr{F}^{\mathfrak{g}(D_s)}$.*

(3) *The functor κ_s is left adjoint to $\mathcal{R}^0_s$ and admits a left adjoint $\mathfrak{a}_s\colon \mathscr{F}^{\mathfrak{g}(D_s)} \to \mathscr{F}$ given by $\mathfrak{a}_s G(V) \cong G(V \oplus V_s, V)/\mathrm{Stab}(V, V \oplus V_s)$.*

(4) *The adjunction counit $\mathfrak{a}_s\kappa_s \to 1_{\mathscr{F}}$ is an isomorphism.*

(5) *The adjunction unit $1_{\mathscr{F}} \to \mathcal{R}^0_s\kappa_s$ is an isomorphism.*

Proof The first statement is clear; for the second, the inclusion $U \subset V$ induces a natural morphism $F(U) \to F(V)$.

The fact that $\mathcal{R}^0_s$ is right adjoint to κ_s follows from the isomorphism

$$\mathrm{Hom}_{\mathscr{V}^f_{/\mathfrak{g}(D_s)}}((V,V),(A,B)) \cong \mathrm{Hom}_{\mathscr{V}^f}(V,B),$$

for $V \in \mathrm{Ob}\,\mathscr{V}^f$ and $(A,B) \in \mathrm{Ob}\,\mathscr{V}^f_{/\mathfrak{g}(D_s)}$.

The left adjoint $\mathfrak{a}_s$ exists for formal reasons and $\mathfrak{a}_s$ is a right exact functor which preserves projective objects. Lemma 7.1.3 implies that the association given by $V \mapsto G(V \oplus V_s, V)/\mathrm{Stab}(V, V \oplus V_s)$ defines a right exact functor. Hence, since $\mathscr{F}^{\mathfrak{g}(D_s)}$ has enough projectives (see Example 5.2.3), it suffices to check that this coincides with $\mathfrak{a}_s$ on the full subcategory of projective objects in $\mathscr{F}^{\mathfrak{g}(D_s)}$. Let (A,B) and V be as above, then there is a natural isomorphism

$$\mathrm{Hom}_{\mathscr{V}^f}(A,B) \cong \mathrm{Hom}_{\mathscr{V}^f_{/\mathfrak{g}(D_s)}}((A,B),(V \oplus V_s, V))/\mathrm{Stab}(V, V \oplus V_s).$$

It follows that there is a natural isomorphism

$$\mathfrak{a}_s P_{(A,B)}(V) \cong P_{(A,B)}(V \oplus V_s, V)/\mathrm{Stab}(V, V \oplus V_s),$$

as required. The identification of the adjunction morphisms is clear. □

Recall from Example 5.1.2 that, for $U \in \mathrm{Ob}\,\mathscr{V}^f$, I_U is the injective functor $\mathbb{F}^{\mathrm{Hom}_{\mathscr{V}^f}(-,U)}$ of $\mathscr{F}$, which is contravariantly functorial in U.

The composite functor $\Omega^{\mathfrak{g}(D_s)}\kappa_s\colon \mathscr{F} \to \mathscr{F}$ is of particular interest; the following result is required to relate the functor κ_s to the Singer functor R_s.

Proposition 7.1.5 *For $U \in \mathrm{Ob}\,\mathscr{V}^f$, there is a natural isomorphism*

$$\Omega^{\mathfrak{g}(D_s)}\kappa_s I_U \cong I_{U \oplus V_s}^{\mathrm{Stab}(U, U \oplus V_s)}.$$

Proof The functor $\Omega^{\mathfrak{g}(D_s)}\kappa_s$ is right adjoint to the functor $\mathfrak{a}_s\iota^{\mathfrak{g}(D_s)}$, which identifies with the composite functor $F \mapsto (\Delta_{V_s} F)/\mathrm{Stab}(-, -\oplus V_s)$, by Proposition 7.1.4. The functor Δ_{V_s} is left adjoint to the functor $-\otimes I_{V_s}$ (see Example 5.1.2), hence it follows that there is a natural isomorphism

$$\mathrm{Hom}_{\mathscr{F}}((\Delta_{V_s} F)/\mathrm{Stab}(-, -\oplus V_s), I_U) \cong \mathrm{Hom}_{\mathscr{F}}(F, (I_U \otimes I_{V_s})^{\mathrm{Stab}(U, U\oplus V_s)})$$

and the group action is induced by the natural right action on $I_U \otimes I_{V_s} \cong I_{U\oplus V_s}$. The result follows. □

We record the following, which is clear.

Lemma 7.1.6 *The composite functor $\Psi_s \mathrm{Ind}_s \kappa_s$, considered as a functor $\mathscr{F} \to \mathscr{F}$, is naturally equivalent to the identity functor.*

7.2 Copresentations of κ_s

In order to understand the underlying object of $\Omega^{\mathfrak{g}(D_s)}\kappa_s$, an alternative description is used; this is obtained by giving a copresentation of κ_s via an equalizer diagram.

Recall that a diagram

$$\begin{tikzcd} X \arrow[r, "d"'] & Y \arrow[l, bend right, dotted, "g"'] \arrow[r, shift left, "e"] \arrow[r, shift right, "f"'] & Z \arrow[l, bend right, dotted, "h"'] \end{tikzcd}$$

is a split equalizer if there exist morphisms $g\colon Y \to X$ and $h\colon Z \to Y$ such that $gd = 1_X$, $hf = 1_Y$ and $dg = he\colon Y \to Y$. A split equalizer is, in particular, an equalizer diagram.

Notation 7.2.1 For $s \in \mathbb{N}$, let $\delta_s\colon \mathscr{V}^f_{/\mathfrak{g}(D_s)} \to \mathscr{V}^f$ denote the functor $(V, U) \mapsto V \oplus V/U$.

Lemma 7.2.2 *For $(V, U) \in \mathrm{Ob}\,\mathscr{V}^f_{/\mathfrak{g}(D_s)}$, there is a natural equalizer diagram in $\mathscr{V}^f$*

$$U \longrightarrow V \underset{1\prod q}{\overset{1\prod 0}{\rightrightarrows}} V \oplus V/U = \delta_s(V, U),$$

where $q\colon V \to V/U$ is the quotient morphism. If $s\colon V/U \to V$ is a section of q, then the equalizer is split by the morphisms $1 \coprod -s\colon V \oplus V/U \to V$ and the induced projection $V \to U$.

In the following, the notation introduced in Example 5.2.3 for the injective cogenerators of $\mathscr{F}^{\mathfrak{g}(D_s)}$ is used.

Proposition 7.2.3 *For $s \in \mathbb{N}$,*

(1) *there is a monomorphism*

$$(\delta_s)^!(-) \hookrightarrow I_{(V_s,0)} \otimes \iota^{\mathfrak{g}(D_s)} \Delta_{V_s}(-)$$

of functors from $\mathscr{F}$ to $\mathscr{F}^{\mathfrak{g}(D_s)}$;

(2) *the equalizer of Lemma 7.2.2 induces an equalizer diagram of functors from $\mathscr{F}$ to $\mathscr{F}^{\mathfrak{g}(D_s)}$*

$$\kappa_s \longrightarrow \iota^{\mathfrak{g}(D_s)} \rightrightarrows (\delta_s)^!$$

and hence an equalizer diagram

$$\kappa_s \longrightarrow \iota^{\mathfrak{g}(D_s)} \rightrightarrows I_{(V_s,0)} \otimes \iota^{\mathfrak{g}(D_s)} \Delta_{V_s}(-); \tag{1}$$

(3) *applying the functor $\Omega^{\mathfrak{g}(D_s)}$ to the equalizer (1) gives the equalizer diagram of functors from $\mathscr{F}$ to $\mathbb{F}^{\mathfrak{g}(D_s)}$–$\mathscr{F}$*

$$\Omega^{\mathfrak{g}(D_s)} \kappa_s \longrightarrow \mathbb{F}^{\mathfrak{g}(D_s)} \otimes (-) \underset{\tau}{\overset{\sigma}{\rightrightarrows}} I_{V_s} \otimes \Delta_{V_s}(-), \tag{2}$$

where σ, τ are induced by the natural morphisms of $\mathscr{F}$

$$F \underset{\tau'_F}{\overset{\sigma'_F}{\rightrightarrows}} I_{V_s} \otimes \Delta_{V_s} F,$$

where σ'_F is the tensor product of the unit $\mathbb{F} \to I_{V_s}$ with the natural inclusion $F \cong \Delta_0 F \hookrightarrow \Delta_{V_s} F$ and τ'_F is the adjunction unit for $\Delta_{V_s} \dashv (I_{V_s} \otimes -)$.

Proof The first statement can be established by an adjunction argument or be seen as follows. Recall from Example 5.2.3 there is an identification $I_{(V_s,0)}(V,U) \cong \mathbb{F}^{\mathrm{Inj}(V/U,V_s)}$. Consider $F \in \mathrm{Ob}\,\mathscr{F}$; the natural monomorphism

$$(\delta_s)^! F(V,U) \to I_{(V_s,0)}(V,U) \otimes \iota^{\mathfrak{g}(D_s)} \Delta_{V_s} F(V,U) \cong \mathbb{F}^{\mathrm{Inj}(V/U,V_s)} \otimes F(V \oplus V_s)$$

has component indexed by a monomorphism $V/U \hookrightarrow V_s$ given by the induced morphism $F(V \oplus V/U) \to F(V \oplus V_s)$.

The first diagram of the second statement is obtained by precomposition with the natural diagram of Lemma 7.2.2. Since limits are computed in $\mathscr{F}^{\mathfrak{g}(D_s)}$ pointwise, it suffices to show that, for $F \in \mathrm{Ob}\,\mathscr{F}$ and $(V,U) \in \mathrm{Ob}\,\mathscr{V}^f_{/\mathfrak{g}(D_s)}$, the diagram in $\mathscr{V}$

$$F(U) \to F(V) \rightrightarrows F(V \oplus V/U)$$

is a split equalizer; this follows from Lemma 7.2.2, since split equalizers are preserved by functors. Composing with the monomorphism of the first statement gives the second equalizer diagram.

The third statement follows by applying the exact functor $\Omega^{\mathfrak{g}(D_s)}$ to the previous equalizer diagram, using [2, Définition et Proposition 3.3.4] to identify the functors. Namely, for $F \in \mathrm{Ob}\mathcal{F}$, there is a natural isomorphism $\Omega^{\mathfrak{g}(D_s)}\iota^{\mathfrak{g}(D_s)}F \cong \mathbb{F}^{\mathfrak{g}(D_s)} \otimes F$ in $\mathbb{F}^{\mathfrak{g}(D_s)}-\mathcal{F}$ and there are natural isomorphisms $\Omega^{\mathfrak{g}(D_s)}(I_{(V_s,0)} \otimes \iota^{\mathfrak{g}(D_s)}\Delta_{V_s}F) \cong (\Omega^{\mathfrak{g}(D_s)}I_{(V_s,0)}) \otimes \Delta_{V_s}F \cong I_{V_s} \otimes \Delta_{V_s}F$, where the second isomorphism follows from $\Omega^{\mathfrak{g}(D_s)}I_{(V_s,0)} \cong I_{V_s}$, which is a formal consequence of the adjunction $\iota^{\mathfrak{g}(D_s)} \dashv \Omega^{\mathfrak{g}(D_s)}$.

The identification of the natural transformations σ, τ follows by unravelling the definitions. □

Remark 7.2.4 Lemma 7.1.6 shows that $\Psi_s\mathrm{Ind}_s\kappa_s$ is naturally equivalent to the identity functor. It is instructive to see how this can be recovered from the copresentation of κ_s given in Proposition 7.2.3; this is a guiding principle in the proof of Theorem 8.2.2.

The functor $\Psi_s\mathrm{Ind}_s$ is exact and an explicit description is given in Proposition 6.3.2. Applying $\Psi_s\mathrm{Ind}_s$ to the parallel arrows of (1), evaluated on $F \in \mathrm{Ob}\mathcal{F}$, gives the diagram

$$F(- \oplus \mathbb{F}_1^s) \rightrightarrows \mathbb{F}^{\mathrm{Inj}(\mathbb{F}_1^s,\mathbb{F}_2^s)} \otimes F(- \oplus \mathbb{F}_1^s \oplus \mathbb{F}_2^s),$$

where the suffixes are used to distinguish the direct factors. Fixing an element $\alpha \in \mathrm{Inj}(\mathbb{F}_1^s, \mathbb{F}_2^s) \cong \mathrm{Aut}(\mathbb{F}^s)$, the associated components evaluated on $V \in \mathrm{Ob}\mathcal{V}^f$ are

$$F(V \oplus \mathbb{F}_1^s) \underset{F(1_V \prod 1_{\mathbb{F}^s} \prod \alpha)}{\overset{F(1_V \prod 1_{\mathbb{F}^s} \prod 0)}{\rightrightarrows}} F(V \oplus \mathbb{F}_1^s \oplus \mathbb{F}_2^s). \tag{3}$$

It is clear that

(1) the natural inclusion $F(V) \hookrightarrow F(V \oplus \mathbb{F}_1^s)$ equalizes the parallel arrows,

(2) for $\alpha = 1_{\mathbb{F}^s}$, the equalizer of (3) is $F(V)$,

where the second point is seen by applying Lemma 7.2.2 to $(V \oplus \mathbb{F}_1^s \oplus \mathbb{F}_2^s, V \oplus \mathbb{F}_1^s)$.

It follows formally that there is a natural isomorphism $\Psi_s\mathrm{Ind}_s\kappa_s F(V) \cong F(V)$, as expected.

7.3 Compatibility

The functors κ_s introduced above are related under restriction and induction, via the adjunctions $\mathcal{R}_s\colon \mathscr{F}^{\mathfrak{g}(D_s)} \rightleftarrows \mathscr{F}^{\mathfrak{g}(D_{s-1})} : \mathcal{P}_s$ of Proposition 6.1.1.

Proposition 7.3.1 *Let s be a positive integer.*

1. *There is a natural isomorphism $\mathcal{R}_s\iota^{\mathfrak{g}(D_s)} \to \iota^{\mathfrak{g}(D_{s-1})}$ and the adjoint $\iota^{\mathfrak{g}(D_s)} \twoheadrightarrow \mathcal{P}_s\iota^{\mathfrak{g}(D_{s-1})}$ is a surjection with kernel $\mathcal{R}^s\mathcal{P}^s\iota^{\mathfrak{g}(D_s)}$.*
2. *There is a natural isomorphism $\mathcal{R}_s\kappa_s \to \kappa_{s-1}$ and the adjoint $\kappa_s \twoheadrightarrow \mathcal{P}_s\kappa_{s-1}$ is a surjection with kernel $\mathcal{R}^s\mathcal{P}^s\kappa_s$.*
3. *There is a commutative diagram of natural transformations*

$$\begin{tikzcd}
\mathcal{R}^s\mathcal{P}^s\kappa_s \arrow[r, hook] \arrow[d, hook] & \kappa_s \arrow[r, two heads] \arrow[d, hook] & \mathcal{P}_s\kappa_{s-1} \arrow[d, hook] \\
\mathcal{R}^s\mathcal{P}^s\iota^{\mathfrak{g}(D_s)} \arrow[r, hook] & \iota^{\mathfrak{g}(D_s)} \arrow[r, two heads] & \mathcal{P}_s\iota^{\mathfrak{g}(D_{s-1})}
\end{tikzcd}$$

in which the rows are short exact sequences.

Proof Straightforward. □

Remark 7.3.2 For $F \in \mathrm{Ob}\,\mathscr{F}$, the short exact sequence $\mathcal{R}^s\mathcal{P}^s\kappa_s F \to \kappa_s F \to \mathcal{P}_s\kappa_{s-1} F$ is the analogue of the short exact sequence $\omega_s R_s M \to R_s M \to \Phi R_{s-1} M$ from Proposition 4.1.3.

8 Deeper properties of the Singer functors

This section introduces an approximation $\widetilde{R}_s$ to the functor R_s, by lifting the copresentation of κ_s of Section 7.2 to the category D_s–$\mathscr{U}$. The functors $\widetilde{R}_s$ and R_s are shown to coincide on reduced unstable modules; in general, $\widetilde{R}_s$ is the ω_s–closure of R_s.

In Section 8.2, the composite of the functor Fix_s with $H^*V_s \otimes_{D_s} R_s(-)$ is shown to be naturally equivalent to the identity. This is used to deduce that the functor R_s defines a fully-faithful embedding of $\mathscr{U}$ in D_s–$\mathscr{U}$.

8.1 Lifting the functor $\Omega^{\mathfrak{g}(D_s)}\kappa_s$ to $\mathcal{U}$

The parallel arrows of diagram (2) of Proposition 7.2.3 lift to a natural diagram in D_s–$\mathcal{U}$:

$$D_s \otimes M \underset{\tau_M}{\overset{\sigma_M}{\rightrightarrows}} H^* V_s \otimes T_{V_s} M,$$

for $M \in \mathrm{Ob}\,\mathcal{U}$, where

(1) σ_M is the tensor product of the inclusions $M \cong T_0 M \hookrightarrow T_{V_s} M$ and $D_s \hookrightarrow H^* V_s$;

(2) τ_M is the morphism of D_s–modules induced by the adjunction unit (in $\mathcal{U}$) $M \to H^* V_s \otimes T_{V_s} M$.

Remark 8.1.1 The context should ensure that there is no ambiguity with the notation used in Section 7.2.

Definition 8.1.2 For $s \in \mathbb{N}$, let $\widetilde{R}_s\colon \mathcal{U} \to D_s$–$\mathcal{U}$ be the functor determined on an unstable module M by

$$\widetilde{R}_s M := \ker\Big\{ D_s \otimes M \underset{\tau_M}{\overset{\sigma_M}{\rightrightarrows}} H^* V_s \otimes T_{V_s} M \Big\}.$$

Recall that an unstable module N is locally finite if and only if the natural monomorphism $N \hookrightarrow TN$ is an isomorphism [19, Theorem 6.2.1].

Proposition 8.1.3 *For $s \in \mathbb{N}$,*

(1) *there is a natural monomorphism $\widetilde{R}_s \hookrightarrow D_s \otimes -$;*

(2) *$\widetilde{R}_s\colon \mathcal{U} \to D_s$–$\mathcal{U}$ is left exact and commutes with coproducts;*

(3) *$\widetilde{R}_s$, considered as a functor with values in $\mathcal{U}$, preserves the class of reduced (respectively nilclosed) unstable modules;*

(4) *$\widetilde{R}_s$ takes values in the class of ω_s–closed objects of D_s–$\mathcal{U}$;*

(5) *for M, X unstable modules, with X locally finite, there is a natural isomorphism $\widetilde{R}_s(M \otimes X) \cong (\widetilde{R}_s M) \otimes X$; in particular, $\widetilde{R}_s$ commutes with suspension.*

Proof The first three statements are straightforward. The fact that $\widetilde{R}_s$ takes values in the category of ω_s–closed unstable D_s–modules follows from Proposition 3.1.4.

For the final statement, since X is locally finite, the natural inclusion $X \hookrightarrow T_{V_s} X$ is an isomorphism, hence there is a natural isomorphism $T_{V_s}(M \otimes X) \cong (T_{V_s} M) \otimes X$. It is straightforward to check that, via this isomorphism, there are identifications $\sigma_{M \otimes X} = \sigma_M \otimes 1_X$ and $\tau_{M \otimes X} = \tau_M \otimes 1_X$, which implies the result. □

Proposition 8.1.4 *For $s \in \mathbb{N}$, the restrictions of the functors R_s and $\widetilde{R}_s$ to the full subcategory of nilclosed unstable modules are naturally isomorphic.*

Proof By Proposition 4.1.3, R_s commutes with coproducts, preserves the class of nilpotent unstable modules and sends nilclosed unstable modules to nilclosed unstable modules; Proposition 8.1.3 establishes the analogous properties for $\widetilde{R}_s$. It follows from Lemma A.2.2 that it is sufficient to show that the two functors coincide on the full subcategory of $\mathcal{U}$ with objects $\{H^*V_s | s \in \mathbb{N}\}$. Since there are natural monomorphisms $R_s \hookrightarrow D_s \otimes -$ and $\widetilde{R}_s \hookrightarrow D_s \otimes -$, by composing with the natural monomorphism $i \otimes -\colon D_s \otimes - \hookrightarrow H^*V_s \otimes -$, it is sufficient to show that the images of $R_s H^*V$ and $\widetilde{R}_s H^*V$ in $H^*V_s \otimes H^*V$ coincide, for every $V \in \mathrm{Ob}\mathcal{V}^f$.

Lannes and Zarati prove that $R_s H^*V$ is isomorphic to $H^*(V \oplus V_s)^{\mathrm{Stab}(V, V \oplus V_s)}$ in [11, Section 5.4.7.5]; Proposition 7.1.5 implies that this is isomorphic to $\widetilde{R}_s H^*V$. The result follows. □

Theorem 8.1.5 *For $s \in \mathbb{N}$, there is a natural monomorphism*

$$\gamma_s\colon R_s \hookrightarrow \widetilde{R}_s$$

of functors $\mathcal{U} \to D_s$–$\mathcal{U}$ such that

1. γ_s *identifies* $\widetilde{R}_s$ *as the* ω_s*–closure of* R_s;
2. *the morphism* $\gamma_s\colon R_s M \to \widetilde{R}_s M$ *is an isomorphism if* M *is reduced.*

Proof The construction of the natural monomorphism γ_s generalizes the argument employed in the proof of Lemma A.2.2. Recall (cf [19, Section 3.11]) that the set of objects $H^*V_m \otimes J(n)$ (where $J(n)$ denotes the n–th Brown–Gitler module), indexed over nonnegative integers m, n, forms a set of injective cogenerators of $\mathcal{U}$ and that, since $\mathcal{U}$ is locally Noetherian, any unstable module M admits a copresentation of the form $0 \to M \to I^0 \to I^1$, where each I^j is a coproduct of objects of this form. Hence, writing W for V_m, it suffices to show that there is a factorization

$$\begin{tikzcd} & R_s(H^*W \otimes J(n)) \arrow[dl, hook, dotted] \arrow[d, hook] \\ \widetilde{R}_s(H^*W \otimes J(n)) \arrow[r, hook] & D_s \otimes H^*W \otimes J(n). \end{tikzcd}$$

Now we have that $R_s(H^*W \otimes J(n)) \cong R_s H^*W \otimes_{D_s} R_s J(n)$, $D_s \otimes H^*W \otimes J(n) \cong (D_s \otimes H^*W) \otimes_{D_s} (D_s \otimes J(n))$ and the vertical inclusion is the tensor product over D_s of $R_s H^*W \hookrightarrow D_s \otimes H^*W$ and $R_s J(n) \hookrightarrow D_s \otimes J(n)$; the latter is the ω_s–closure of $R_s J(n)$, by Proposition 4.1.5.

Similarly, by Proposition 8.1.3, the horizontal inclusion identifies with the tensor product over D_s of $\widetilde{R}_s H^*W \hookrightarrow D_s \otimes H^*W$ and the isomorphism $\widetilde{R}_s J(n) \to D_s \otimes J(n)$. Since the images of $R_s H^*W$ and $\widetilde{R}_s H^*W$ in $D_s \otimes H^*W$ coincide, by Proposition 8.1.3, this provides the required factorization.

The cokernel of $R_s(H^*W \otimes J(n)) \hookrightarrow \widetilde{R}_s(H^*W \otimes J(n))$ is ω_s–torsion, by the above discussion. It follows that the cokernel of $R_s M \to \widetilde{R}_s M$ is ω_s–torsion, for any unstable module M; since, $\widetilde{R}_s M$ is ω_s–closed (by Proposition 8.1.3), this exhibits $\widetilde{R}_s M$ as the ω_s–closure of $R_s M$.

To prove the final statement, one can forget the D_s–module structure. The result follows from Lemma A.3.1, since γ_s is an isomorphism on nilclosed unstable modules, by Proposition 8.1.4. □

Corollary 8.1.6 *For $s \in \mathbb{N}$, there is a natural isomorphism $\mathfrak{l}_{D_s} R_s \cong \kappa_s l$ of functors from $\mathcal{U}$ to $\mathcal{F}^{\mathfrak{g}(D_s)}$.*

Proof The functors $\mathfrak{l}_{D_s} R_s$ and $\kappa_s l$ are exact and send nilpotent unstable modules to zero. Hence, by Lemma A.2.1, it suffices to prove that the two functors coincide naturally on the full subcategory of nilclosed unstable modules. On this subcategory, $\gamma_s\colon R_s \hookrightarrow \widetilde{R}_s$ is a natural isomorphism, by Theorem 8.1.5, hence it suffices to prove that there is a natural isomorphism $\mathfrak{l}_{D_s} \widetilde{R}_s \cong \kappa_s l$. This is by construction: applying the functor $\mathfrak{l}_{D_s}$ to the equalizer diagram defining $\widetilde{R}_s$ gives the copresentation of $\Omega^{\mathfrak{g}(D_s)}\kappa_s$ given in Proposition 7.2.3. □

8.2 The composite of R_s and $\mathrm{Fix}_s(H^*V_s \otimes_{D_s} -)$

Under the correspondence between D_s–$\mathcal{U}/\mathcal{N}il$ and the category $\mathcal{F}^{\mathfrak{g}(D_s)}$ given by Theorem 5.2.8, the Singer functor corresponds to the functor κ_s (by Corollary 8.1.6) and the functor $\Psi_s \mathrm{Ind}_s$ corresponds to the functor $\mathrm{Fix}_s(H^*V_s \otimes_{D_s} -)$ (by Proposition 6.3.2). Lemma 7.1.6 states that the composite $\Psi_s \mathrm{Ind}_s \kappa_s$ is isomorphic to the identity functor; the purpose of this section is to establish the corresponding result at the level of unstable modules.

Recall that $i\colon D_s \hookrightarrow H^*V_s$ denotes the canonical inclusion and that Proposition 2.3.2 implies that $T_{(V_s,i)}$ induces a functor D_s–$\mathcal{U} \to H^*V_s$–$\mathcal{U}$. The following result is the key input.

Lemma 8.2.1 *For $s \in \mathbb{N}$ and $M \in \mathrm{Ob}\, D_s$–$\mathcal{U}$,*

$$\mathrm{Fix}_s(H^*V_s \otimes_{D_s} M) \cong \mathbb{F} \otimes_{H^*V_s} T_{(V_s,i)} M.$$

In particular, there is an isomorphism of unstable algebras

$$\mathrm{Fix}_s(H^*V_s \otimes_{D_s} H^*V_s) \cong \mathbb{F}^{\mathrm{Aut} V_s}.$$

Proof By [10, 4.4.3], $\mathrm{Fix}_s X \cong \mathbb{F} \otimes_{T_{V_s} H^* V_s} T_{V_s} X$, for $X \in \mathrm{Ob}\, H^* V_s{-}\mathcal{U}$. The T–functor commutes with tensor products, so, taking $X = H^* V_s \otimes_{D_s} M$, there are natural isomorphisms

$$\begin{aligned}\mathrm{Fix}_s(H^* V_s \otimes_{D_s} M) &\cong \mathbb{F} \otimes_{T_{V_s} D_s} T_{V_s} M \\ &\cong \mathbb{F} \otimes_{T_{(V_s,i)} D_s} T_{(V_s,i)} M \cong \mathbb{F} \otimes_{H^* V_s} T_{(V_s,i)} M.\end{aligned}$$

For the case $M = H^* V_s$, one verifies that $T_{(V_s,i)} H^* V_s$ identifies with $\mathbb{F}^{\mathrm{Aut}(V_s)} \otimes H^* V_s$ as an $H^* V_s$–algebra, from which the result follows. □

Theorem 8.2.2 *For $s \in \mathbb{N}$, the natural transformation*

$$\mathrm{Fix}_s(H^* V_s \otimes_{D_s} R_s(-)) \to 1_{\mathcal{U}}$$

of functors on $\mathcal{U}$, which is adjoint to the canonical inclusion $R_s(-) \hookrightarrow D_s \otimes (-) \hookrightarrow H^* V_s \otimes (-)$, is an isomorphism.

Moreover, the natural monomorphism $R_s(-) \hookrightarrow D_s \otimes -$ induces the canonical inclusion

$$\mathrm{Fix}_s(H^* V_s \otimes_{D_s} R_s(-)) \cong 1_{\mathcal{U}} \hookrightarrow T_{V_s}(-) \cong \mathrm{Fix}_s(H^* V_s \otimes_{D_s} (D_s \otimes -)).$$

Proof The natural monomorphism $\gamma_s \colon R_s \hookrightarrow \widetilde{R}_s$ is an isomorphism up to ω_s–torsion, by Theorem 8.1.5, hence it suffices to prove the result with $\widetilde{R}_s$ in place of R_s, since Fix_s annihilates ω_s–torsion, by [12, Proposition 0.8]. The defining equalizer diagram for $\widetilde{R}_s M$ gives rise to an equalizer diagram in $H^* V_s{-}\mathcal{U}$:

$$H^* V_s \otimes_{D_s} \widetilde{R}_s M \longrightarrow H^* V_s \otimes M \rightrightarrows H^* V_s \otimes_{D_s} H^* V_s \otimes T_{V_s} M,$$

since $H^* V_s \otimes_{D_s} (-)$ is exact.

The functor Fix_s is exact, hence this gives the equalizer diagram in $\mathcal{U}$:

$$\mathrm{Fix}_s(H^* V_s \otimes_{D_s} \widetilde{R}_s M) \longrightarrow \mathrm{Fix}_s(H^* V_s \otimes M) \rightrightarrows \mathrm{Fix}_s(H^* V_s \otimes_{D_s} H^* V_s \otimes T_{V_s} M).$$

There are natural isomorphisms $\mathrm{Fix}_s(H^* V_s \otimes M) \cong T_{V_s} M$ and

$$\mathrm{Fix}_s(H^* V_s \otimes_{D_s} H^* V_s \otimes T_{V_s} M) \cong \mathbb{F}^{\mathrm{Aut}(V_s)} \otimes T_{V_s} T_{V_s} M,$$

obtained by viewing $H^* V_s \otimes_{D_s} H^* V_s \otimes T_{V_s} M$ as the tensor product over $H^* V_s$ of $H^* V_s \otimes_{D_s} H^* V_s$ and $H^* V_s \otimes T_{V_s} M$ and applying Lemma 8.2.1.

The equalizer diagram therefore identifies with

$$\mathrm{Fix}_s(H^* V_s \otimes_{D_s} \widetilde{R}_s M) \longrightarrow T_{V_s} M \underset{\widetilde{\tau}_M}{\overset{\widetilde{\sigma}_M}{\rightrightarrows}} \mathbb{F}^{\mathrm{Aut}(V_s)} \otimes T_{V_s} T_{V_s} M,$$

where $\widetilde{\sigma}_M := \mathrm{Fix}_s(H^* V_s \otimes_{D_s} \sigma_M)$ and $\widetilde{\tau}_M := \mathrm{Fix}_s(H^* V_s \otimes_{D_s} \tau_M)$ are identified below.

As in Remark 7.2.4, the result is a formal consequence of the following two points:

(1) the natural morphism $M \hookrightarrow T_{V_s} M$ equalizes the morphisms $\tilde{\sigma}_M$ and $\tilde{\tau}_M$;

(2) $M \hookrightarrow T_{V_s} M$ is the equalizer of the diagram of unstable modules
$$T_{V_s} M \rightrightarrows T_{V_s} T_{V_s} M,$$
which is obtained from $\tilde{\sigma}_M$, $\tilde{\tau}_M$ by composing with the surjection $\mathbb{F}^{\mathrm{Aut}(V_s)} \otimes T_{V_s} T_{V_s} M \twoheadrightarrow T_{V_s} T_{V_s} M$ induced by the augmentation $\mathbb{F}^{\mathrm{Aut}(V_s)} \twoheadrightarrow \mathbb{F}$.

The identification of the morphisms $\tilde{\sigma}_M$ and $\tilde{\tau}_M$ is a standard calculation with the T–functor; the precise form depends on the conventions used in the isomorphism $\mathrm{Fix}_s(H^* V_s \otimes_{D_s} H^* V_s) \cong \mathbb{F}^{\mathrm{Aut}(V_s)}$ of Lemma 8.2.1. The appropriate form can be deduced from the nillocalized case, as in Section 7.2, which leads to the following identifications.

For an automorphism $\alpha \in \mathrm{Aut}(V_s)$, the components
$$T_{V_s} M \underset{\tilde{\tau}_M^\alpha}{\overset{\tilde{\sigma}_M^\alpha}{\rightrightarrows}} T_{V_s} T_{V_s} M \cong T_{V_s \oplus V_s} M,$$
of $\tilde{\sigma}_M$ and $\tilde{\tau}_M$ indexed by α are induced by naturality of the T–functor by
$$V_s \underset{1_{V_s} \prod \alpha}{\overset{1_{V_s} \prod 0}{\rightrightarrows}} V_s \oplus V_s.$$

The two key points are established as in Remark 7.2.4: that M lies in the equalizer follows since $\tilde{\sigma}_M$, $\tilde{\tau}_M$ are derived from the naturality with respect to V of T_V; the second point follows by observing that the diagram
$$M \longrightarrow T_{V_s} M \rightrightarrows T_{V_s \oplus V_s} M$$
is a split equalizer in unstable modules, by applying Lemma 7.2.2, where the morphisms $T_{V_s} M \rightrightarrows T_{V_s \oplus V_s} M$ are induced respectively by $1_{V_s} \prod 0\colon V_s \to V_s \oplus V_s$ and the diagonal $\Delta\colon V_s \to V_s \oplus V_s$.

The final statement has been established in the course of the proof. □

8.3 The Singer functor is a fully-faithful embedding

Proposition 7.1.4 shows that the unit $1_{\mathscr{F}} \to \mathcal{R}_s^0 \kappa_s$ of the adjunction $\kappa_s \dashv \mathcal{R}_s^0$ is a natural isomorphism. This section shows that Theorem 8.2.2 implies the analogous statement for the adjunction $R_s \dashv \mathfrak{Z}_s$; in particular, the functor $R_s\colon \mathcal{U} \to D_s$–$\mathcal{U}$ is rigid, considered as a functor to unstable D_s–modules.

Theorem 8.3.1 *For $s \in \mathbb{N}$, the adjunction unit $1_{\mathcal{U}} \to \mathfrak{Z}_s R_s$ is a natural isomorphism and the natural inclusions $R_s \hookrightarrow D_s \otimes (-) \hookrightarrow H^*(V_s) \otimes (-)$ in $D_s{-}\mathcal{U}$ induce isomorphisms*

$$1_{\mathcal{U}} \cong \mathfrak{Z}_s R_s \cong \mathfrak{Z}_s(D_s \otimes -) \cong \mathfrak{Z}_s(H^* V_s \otimes -).$$

In particular, R_s induces a fully-faithful embedding $R_s\colon \mathcal{U} \hookrightarrow D_s{-}\mathcal{U}$.

Proof For the first statement, it suffices to prove that, for $M, N \in \mathrm{Ob}\,\mathcal{U}$, the functor R_s induces a natural isomorphism:

$$\mathrm{Hom}_{\mathcal{U}}(M, N) \longrightarrow \mathrm{Hom}_{D_s\text{-}\mathcal{U}}(R_s M, R_s N).$$

This is a monomorphism by Lemma 4.1.4; composing with the natural inclusion $R_s N \hookrightarrow D_s \otimes N \hookrightarrow H^* V_s \otimes N$, there is a natural monomorphism

$$\begin{aligned}\mathrm{Hom}_{D_s\text{-}\mathcal{U}}(R_s M, R_s N) \hookrightarrow{}& \mathrm{Hom}_{D_s\text{-}\mathcal{U}}(R_s M, H^* V_s \otimes N)\\ &\cong \mathrm{Hom}_{H^* V_s\text{-}\mathcal{U}}(H^* V_s \otimes_{D_s} R_s M, H^* V_s \otimes N).\end{aligned}$$

By adjunction,

$$\mathrm{Hom}_{H^* V_s\text{-}\mathcal{U}}(H^* V_s \otimes_{D_s} R_s M, H^* V_s \otimes N) \cong \mathrm{Hom}_{\mathcal{U}}(\mathrm{Fix}_s(H^* V_s \otimes_{D_s} R_s M), N),$$

and, by Theorem 8.2.2, $\mathrm{Fix}_s(H^* V_s \otimes_{D_s} R_s M) \cong M$. Thus, there are natural monomorphisms

$$\begin{aligned}\mathrm{Hom}_{D_s\text{-}\mathcal{U}}(R_s M, R_s N) \hookrightarrow \mathrm{Hom}_{D_s\text{-}\mathcal{U}}(R_s M, D_s \otimes N) \hookrightarrow{}& \mathrm{Hom}_{D_s\text{-}\mathcal{U}}(R_s M, H^* V_s \otimes N)\\ &\cong \mathrm{Hom}_{\mathcal{U}}(M, N).\end{aligned}$$

The composite with the natural inclusion $\mathrm{Hom}_{\mathcal{U}}(M, N) \hookrightarrow \mathrm{Hom}_{D_s\text{-}\mathcal{U}}(R_s M, R_s N)$ is the identity, which establishes the natural isomorphisms.

The property of the adjunction unit is a formal consequence. □

9 The functors $\mathfrak{Z}_s$, Q_s and Fix_s

The purpose of this section is to provide a better understanding of the right adjoint $\mathfrak{Z}_s$ to the Singer functor R_s, in particular its relationship with the indecomposables functor Q_s and with the functor $\mathrm{Fix}_s(H^* V_s \otimes_{D_s} -)$.

9.1 The Singer functor R_s and the indecomposables Q_s

In [3, Section 3] and [4, Section 3], Dwyer and Wilkerson studied a linear operation constructed from the Steenrod total power St_1. This is related to the natural transformation defined below (defined for arbitrary s), where $\mathrm{forget}_s\colon D_s\text{–}\mathcal{U} \to \mathcal{U}$ is the forgetful functor.

Definition 9.1.1 For $s \in \mathbb{N}$, let $\varepsilon_s\colon R_s\,\mathrm{forget}_s \to 1_{D_s\text{–}\mathcal{U}}$ be the natural transformation defined on $M \in \mathrm{Ob}\,D_s\text{–}\mathcal{U}$ as the composite

$$R_s M \hookrightarrow D_s \otimes M \to M$$

of the canonical inclusion followed by the product.

Proposition 9.1.2 *The natural transformation* $\mathrm{forget}_s \twoheadrightarrow Q_s$ *of functors from* $D_s\text{–}\mathcal{U}$ *to* $\mathcal{U}$ *induces a factorization*

$$\begin{tikzcd} R_s\,\mathrm{forget}_s \arrow[r, two heads] \arrow[rr, bend right, "\varepsilon_s"'] & R_s Q_s \arrow[r, "\overline{\varepsilon_s}"] & 1_{D_s\text{–}\mathcal{U}} \end{tikzcd}$$

of endofunctors of $D_s\text{–}\mathcal{U}$.

Proof The proof proceeds by reduction to the behaviour on D_s. For $M \in \mathrm{Ob}\,D_s\text{–}\mathcal{U}$, there is an exact sequence of unstable modules

$$\overline{D_s} \otimes M \to M \to Q_s M \to 0,$$

where the first morphism is induced by multiplication. The functor R_s is exact, hence, by naturality of ε_s, it suffices to prove that the composite morphism

$$R_s(\overline{D_s} \otimes M) \xrightarrow{\varepsilon_s} \overline{D_s} \otimes M \to M$$

is zero.

There is a natural isomorphism $R_s(\overline{D_s} \otimes M) \cong R_s(\overline{D_s}) \otimes_{D_s} R_s M$, by the monoidal property of R_s (see Proposition 4.1.3) and, with respect to this, the above composite is induced by the tensor product over D_s of $\varepsilon_s\colon R_s\overline{D_s} \to \overline{D_s}$ and $\varepsilon_s\colon R_s M \to M$. Therefore, to prove the result, it is sufficient to show that the morphism $\varepsilon_s\colon R_s\overline{D_s} \to \overline{D_s}$ is trivial.

This can be proved directly, generalizing [3, Lemma 3.3(ii)], by reducing to the case $s = 1$, using the fact [12] that St_s is the s–fold iterate of St_1.

An alternative method is to use passage to nillocalization. Since R_s preserves reduced objects and $\overline{D_s}$ is reduced, it is sufficient to prove that the induced morphism

$\mathfrak{l}_{D_s} R_s(\overline{D_s}) \to \mathfrak{l}_{D_s} \overline{D_s}$ is trivial. The natural transformation ε_s corresponds to the natural transformation

$$\kappa_s \Omega^{\mathfrak{g}(D_s)} \to 1_{\mathscr{F}^{\mathfrak{g}(D_s)}}$$

of endofunctors of $\mathscr{F}^{\mathfrak{g}(D_s)}$ given by the composite

$$\kappa_s \Omega^{\mathfrak{g}(D_s)} \hookrightarrow \iota^{\mathfrak{g}(D_s)} \Omega^{\mathfrak{g}(D_s)} \to 1_{\mathscr{F}^{\mathfrak{g}(D_s)}}$$

induced by the inclusion $\kappa_s \hookrightarrow \iota^{\mathfrak{g}(D_s)}$ and the counit of the $\iota^{\mathfrak{g}(D_s)} \dashv \Omega^{\mathfrak{g}(D_s)}$ adjunction. For $G \in \mathrm{Ob}\mathscr{F}^{\mathfrak{g}(D_s)}$ and $(V, W) \in \mathrm{Ob}\mathscr{V}^f{}_{/\mathfrak{g}(D_s)}$, by using the explicit form of the $\iota^{\mathfrak{g}(D_s)} \dashv \Omega^{\mathfrak{g}(D_s)}$ adjunction counit, this identifies as the surjection

$$\bigoplus_{\operatorname{codim} U \leq s} G(W, U) \twoheadrightarrow G(V, W)$$

given by projection onto the summand indexed by $U = W$ followed by the morphism $G(W, W) \to G(V, W)$ induced by $(W, W) \to (V, W)$ in $\mathscr{V}^f{}_{/\mathfrak{g}(D_s)}$.

Now, $\mathfrak{l}_{D_s} D_s$ is the constant functor $\mathbb{F} \in \mathscr{F}^{\mathfrak{g}(D_s)}$ and the augmentation ideal gives the subfunctor $\mathfrak{l}_{D_s} \overline{D_s}$:

$$(V, W) \mapsto \begin{cases} \mathbb{F} & V \neq W, \\ 0 & V = W. \end{cases}$$

The result follows. □

The following corollary is formal.

Corollary 9.1.3 *For $s \in \mathbb{N}$, the natural transformation $\zeta_s \colon Q_s \to \mathfrak{Z}_s$ adjoint to $\overline{\varepsilon_s} \colon R_s Q_s \to 1_{D_s\text{-}\mathscr{U}}$, fits into a commutative diagram:*

$$\begin{tikzcd} \mathrm{forget}_s \arrow[r, two heads] \arrow[rr, bend right, "\widetilde{\varepsilon_s}"'] & Q_s \arrow[r, "\zeta_s"] & \mathfrak{Z}_s, \end{tikzcd}$$

where $\widetilde{\varepsilon_s} \colon \mathrm{forget}_s \to \mathfrak{Z}_s$ is adjoint to $\varepsilon_s \colon R_s \,\mathrm{forget}_s \to 1_{D_s\text{-}\mathscr{U}}$.

The following natural transformation is used in Theorem 9.2.2.

Lemma 9.1.4 *For $s \in \mathbb{N}$, the functor Q_s applied to the adjunction counit $R_s \mathfrak{Z}_s \to 1_{D_s\text{-}\mathscr{U}}$ induces a natural transformation $\xi_s \colon \Phi^s \mathfrak{Z}_s \to Q_s$.*

Proof This is an immediate consequence of the natural isomorphism $Q_s R_s \cong \Phi^s$ of Proposition 4.1.3. □

9.2 The relationship between Q_s and $\mathfrak{Z}_s$

In order to understand the relationship between Q_s and $\mathfrak{Z}_s$, further information on the behaviour of $\mathfrak{Z}_s$ is required. Recall that the category $D_s\text{–}\mathcal{U}$ has enough injectives [12; 7; 15].

Lemma 9.2.1 *If $I \in \mathrm{Ob}\, D_s\text{–}\mathcal{U}$ is injective, then the natural morphism $\zeta_s\colon Q_s I \twoheadrightarrow \mathfrak{Z}_s I$ is surjective.*

Proof It suffices to show that the morphism $\mathrm{forget}_s\, I \to \mathfrak{Z}_s I$ of Corollary 9.1.3 is surjective. For $k \in \mathbb{N}$, there is a canonical embedding $R_s F(k) \hookrightarrow D_s \otimes F(k)$ in $D_s\text{–}\mathcal{U}$, and hence, by injectivity of I, a surjection

$$\mathrm{Hom}_{D_s\text{–}\mathcal{U}}(D_s \otimes F(k), I) \twoheadrightarrow \mathrm{Hom}_{D_s\text{–}\mathcal{U}}(R_s F(k), I).$$

This corresponds to the degree k part of the morphism $\mathrm{forget}_s\, I \to \mathfrak{Z}_s I$, which is therefore surjective. □

Recall that $\lambda\colon \Phi \to 1_{\mathcal{U}}$ is the natural transformation induced by Sq_0, which induces $\lambda^s\colon \Phi^s \to 1_{\mathcal{U}}$ by iteration, for $s \in \mathbb{N}$. The natural transformation ξ_s was introduced in Lemma 9.1.4.

Theorem 9.2.2 *For $s \in \mathbb{N}$,*

(1) *the composite $\Phi^s Q_s \xrightarrow{\Phi^s \zeta_s} \Phi^s \mathfrak{Z}_s \xrightarrow{\xi_s} Q_s$ is the natural transformation $\lambda^s_{Q_s}$;*

(2) *the composite $\Phi^s \mathfrak{Z}_s \xrightarrow{\xi_s} Q_s \xrightarrow{\zeta_s} \mathfrak{Z}_s$ is the natural transformation $\lambda^s_{\mathfrak{Z}_s}$.*

In particular, $\zeta_s\colon Q_s \to \mathfrak{Z}_s$ and $\xi_s\colon \Phi^s \mathfrak{Z}_s \to Q_s$ are isomorphisms up to nilpotent unstable modules and the functor $\mathfrak{Z}_s$ sends nilpotents to nilpotents.

Proof The first natural transformation is given by applying the functor Q_s to the composite $R_s Q_s \to R_s \mathfrak{Z}_s \to 1_{D_s\text{–}\mathcal{U}}$. The identification follows from the fact that, modulo decomposables, St_s identifies with the linear map $(\mathrm{Sq}_0)^s$.

The functor $\mathfrak{Z}_s$ is left exact and $D_s\text{–}\mathcal{U}$ has enough injectives, hence it suffices to show that the natural transformation identifies with $\lambda^s_{\mathfrak{Z}_s}$ when evaluated on any injective object I.

Consider the composite

$$\begin{tikzcd} \Phi^s Q_s I \arrow[r, two heads, "\Phi^s \zeta_s"'] \arrow[rr, bend left, "\lambda^s_{Q_s}"] & \Phi^s \mathfrak{Z}_s I \arrow[r, "\xi_s"'] \arrow[rr, bend right, dotted, "\lambda^s_{\mathfrak{Z}_s}"'] & Q_s I \arrow[r, "\zeta_s"] & \mathfrak{Z}_s I. \end{tikzcd}$$

Here the surjection $\Phi^s \zeta_s$ is given by Lemma 9.2.1, since the functor Φ^s is exact, and the identification of $\lambda^s_{Q_s}$ follows from the first part of the theorem.

To prove the result, by surjectivity of $\Phi^s \zeta_s$, it suffices to show that the composites $\zeta_s \xi_s (\Phi^s \zeta_s)$ and $\lambda^s_{\mathfrak{Z}_s}(\Phi^s \zeta_s)$ coincide evaluated on I. This follows from the naturality of λ^s, which gives the commutative diagram of natural transformations

$$\begin{tikzcd}
\Phi^s Q_s \arrow[r, "\Phi^s \zeta_s"] \arrow[d, "\lambda^s_{Q_s}"'] & \Phi^s \mathfrak{Z}_s \arrow[d, "\lambda^s_{\mathfrak{Z}_s}"] \\
Q_s \arrow[r, "\zeta_s"'] & \mathfrak{Z}_s .
\end{tikzcd}$$

The final statements are immediate consequences of these identifications, since the functor Q_s sends objects with nilpotent underlying unstable module to nilpotent unstable modules. $\square$

In the following, $\mathcal{R}^0_s$ is the functor of Proposition 6.2.1.

Corollary 9.2.3 *For $s \in \mathbb{N}$,*

(1) *there are natural isomorphisms*
  (a) $\mathfrak{Z}_s \mathfrak{r}_{D_s} \cong r\mathcal{R}^0_s \colon \mathscr{F}^{\mathfrak{g}(D_s)} \to \mathscr{U}$,
  (b) $l\mathfrak{Z}_s \cong \mathcal{R}^0_s \mathfrak{l}_{D_s} \colon D_s{-}\mathscr{U} \to \mathscr{F}$,

  and, in particular, the functor $l\mathfrak{Z}_s$ is exact;

(2) *for $N \in \operatorname{Ob} D_s{-}\mathscr{U}$ which is reduced, $\mathfrak{Z}_s N = 0$ if and only if $Q_s N$ is nilpotent.*

Proof The first isomorphism is a formal consequence of Corollary 8.1.6. Namely, $\mathfrak{Z}_s \mathfrak{r}_{D_s}$ is right adjoint to $\mathfrak{l}_{D_s} R_s$, which is naturally equivalent to $\kappa_s l$, by Corollary 8.1.6. The latter is left adjoint to $r\mathcal{R}^0_s$, by Proposition 7.1.4.

Precomposing with $\mathfrak{l}_{D_s}$ and postcomposing with l gives a natural isomorphism $l\mathfrak{Z}\mathfrak{r}_{D_s}\mathfrak{l}_{D_s} \cong \mathcal{R}^0_s \mathfrak{l}_{D_s}$. The natural transformation $1_{D_s - \mathscr{U}} \to \mathfrak{r}_{D_s}\mathfrak{l}_{D_s}$ is an isomorphism modulo nilpotent objects, hence, by Theorem 9.2.2, the induced natural transformation $l\mathfrak{Z}_s \longrightarrow l\mathfrak{Z}\mathfrak{r}_{D_s}\mathfrak{l}_{D_s}$ is an isomorphism; this gives the second natural isomorphism.

Suppose now that $N \in \operatorname{Ob} D_s{-}\mathscr{U}$ is a reduced object; hence, by Proposition 4.1.6, $\mathfrak{Z}_s N$ is reduced. Thus, $\mathfrak{Z}_s N = 0$ if and only if $l\mathfrak{Z}_s N = 0$, which is equivalent to $lQ_s N = 0$, by Theorem 9.2.2. $\square$

Corollary 9.2.4 *If $M \in \operatorname{Ob} D_s{-}\mathscr{U}$ is a reduced object, then $\mathfrak{Z}_s(\omega_s M) = 0$.*

Proof The D_s–module $\omega_s M$ is a submodule of M, hence is reduced, so it suffices to show that $Q_s(\omega_s M)$ is nilpotent. The short exact sequence

$$0 \to \omega_s M \to M \to M/\omega_s M \to 0$$

in D_s–$\mathcal{U}$ induces an exact sequence

$$Q_s(\omega_s M) \to Q_s M \to Q_s(M/\omega_s M) \to 0,$$

which is short exact modulo nilpotent unstable modules, since by Proposition 6.2.1 lQ_s is exact. The surjection $Q_s M \twoheadrightarrow Q_s(M/\omega_s M)$ is an isomorphism, so the result follows. □

Example 9.2.5 For $s \in \mathbb{N}$, $N \in \mathrm{Ob}\,\mathcal{U}$ and $M \in \mathrm{Ob}\, D_s$–$\mathcal{U}$ which is reduced,

$$\mathrm{Hom}_{D_s\text{–}\mathcal{U}}(R_s N, \omega_s M) = 0.$$

For example, take $M = D_s$.

9.3 The relationship between $\mathfrak{Z}_s$ and $\mathrm{Fix}_s(H^* V_s \otimes_{D_s} -)$

It is interesting to have a criterion for the counit $R_s \mathfrak{Z}_s M \to M$ (for $M \in \mathrm{Ob}\, D_s$–$\mathcal{U}$) to be an isomorphism. By Theorem 8.3.1, if $M \cong R_s N$ for some $N \in \mathrm{Ob}\,\mathcal{U}$, then $N \cong \mathfrak{Z}_s M$.

Proposition 9.3.1 *For $s \in \mathbb{N}$, there is a natural transformation*

$$\mathfrak{Z}_s \to \mathrm{Fix}_s(H^* V_s \otimes_{D_s} (-))$$

of functors from D_s–$\mathcal{U}$ to $\mathcal{U}$.

Moreover, if $M \in \mathrm{Ob}\, D_s$–$\mathcal{U}$ such that the counit $R_s \mathfrak{Z}_s M \to M$ is an isomorphism, then

$$\mathfrak{Z}_s M \to \mathrm{Fix}_s(H^* V_s \otimes_{D_s} M)$$

is an isomorphism.

Proof The natural transformation is given by applying the functor $\mathrm{Fix}_s(H^* V_s \otimes_{D_s} (-))$ to the counit $R_s \mathfrak{Z}_s \to 1_{D_s\text{–}\mathcal{U}}$, using the isomorphism of Theorem 8.2.2. If the counit is an isomorphism, then so is the induced natural morphism. □

Remark 9.3.2 (1) Composition with the natural transformation $\mathrm{forget}_s \to \mathfrak{Z}_s$ of Corollary 9.1.3 induces a natural morphism

$$\mathrm{forget}_s \to \mathrm{Fix}_s(H^* V_s \otimes_{D_s} (-)).$$

This is induced by the natural transformation of Proposition 2.4.1.

(2) The natural transformation $\mathfrak{Z}_s \to \mathrm{Fix}_s(H^* V_s \otimes_{D_s} (-))$ corresponds to the natural transformation $\mathcal{R}_s^0 \to \Psi_s \mathrm{Ind}_s$ of functors from $\mathscr{F}^{\mathfrak{g}(D_s)}$ to $\mathscr{F}$ which is given on $G \in \mathrm{Ob}\,\mathscr{F}^{\mathfrak{g}(D_s)}$ by $\mathcal{R}_s^0 G(V) = G(V, V) \to \Psi_s \mathrm{Ind}_s G(V) = G(V \oplus V_s, V)$, induced by $(V, V) \to (V \oplus V_s, V)$ (cf Propositions 6.3.2 and 6.2.1).

For $s > 1$, it is straightforward to see that $\mathfrak{Z}_s M \to \mathrm{Fix}_s(H^* V_s \otimes_{D_s} M)$ being an isomorphism does not imply in general that the counit is an isomorphism. However, in the case $s = 1$, one has the following.

Theorem 9.3.3 *For $M \in \mathrm{Ob}\, D_1{-}\mathcal{U}$ such that the underlying unstable module is reduced, the following conditions are equivalent:*

(1) *the counit $R_1 \mathfrak{Z}_1 M \to M$ is an isomorphism;*

(2) *the natural morphism $\mathfrak{Z}_1 M \to \mathrm{Fix}_1 M$ is an isomorphism.*

Proof Proposition 9.3.1 gives (1) $\Rightarrow$ (2).

For the converse, consider the exact sequence in $D_1{-}\mathcal{U}$

$$0 \to \mathrm{Ker} \to R_1 \mathfrak{Z}_1 M \to M \to \mathrm{Coker} \to 0.$$

The hypothesis (2) implies that $\mathrm{Fix}_1(R_1 \mathfrak{Z}_1 M \to M)$ is an isomorphism. Thus, by [12, Proposition 0.8], both Ker and Coker are ω_1–torsion. However, by construction, Ker is a subobject of $R_1 \mathfrak{Z}_1 M$, which is ω_1–torsion free, hence $\mathrm{Ker} = 0$ and $R_1 \mathfrak{Z}_1 M \hookrightarrow M$ is a monomorphism with ω_1–torsion cokernel.

By hypothesis, M is reduced, hence $\mathfrak{Z}_1 M \cong \mathrm{Fix}_1 M$ is reduced. Theorem 8.1.5 implies that $R_1 \mathfrak{Z}_1 M$ is ω_1–closed. Hence, to complete the proof, it suffices to show that M is ω_1–torsion free. Consider the submodule $A := \mathrm{Ann}_{\omega_1} M \subset M$, so that A is in the image of $\mathrm{triv}_1\colon \mathcal{U} \to D_1{-}\mathcal{U}$. The module M is ω_1–torsion free if and only if $A = 0$.

Applying the functor $\mathfrak{Z}_1$ yields a monomorphism

$$\widetilde{\Phi} A \cong \mathfrak{Z}_1 A \hookrightarrow \mathfrak{Z}_1 M \cong \mathrm{Fix}_1 M,$$

where the first isomorphism is given by Proposition 4.1.8. By naturality of $\mathfrak{Z}_1 \to \mathrm{Fix}_1$, this factors across $\mathrm{Fix}_1 A$, which is trivial (since A is ω_1–torsion). Thus $\widetilde{\Phi} A = 0$. However, A is a reduced unstable module, since it is a submodule of M, hence A must be zero, as required. □

Appendix A General results

A.1 The right adjoint to Φ on categories of modules

Proposition A.1.1 *For K an unstable algebra, $\widetilde{\Phi}$ induces a functor $\widetilde{\Phi}\colon \Phi K\text{–}\mathcal{U} \to K\text{–}\mathcal{U}$ which is right adjoint to $\Phi\colon K\text{–}\mathcal{U} \to \Phi K\text{–}\mathcal{U}$.*

Proof The functor Φ commutes with tensor products, hence the adjunction counit $\Phi\widetilde{\Phi} \to 1_{\mathcal{U}}$ induces a natural morphism

$$\widetilde{\Phi}M \otimes \widetilde{\Phi}N \to \widetilde{\Phi}(M \otimes N),$$

for $M, N \in \operatorname{Ob}\mathcal{U}$. Thus, if $M \in \operatorname{Ob}\Phi K\text{–}\mathcal{U}$, $\widetilde{\Phi}M$ is an object of $K\text{–}\mathcal{U}$ with respect to the structure morphism:

$$K \otimes \widetilde{\Phi}M \cong \widetilde{\Phi}\Phi K \otimes \widetilde{\Phi}M \to \widetilde{\Phi}(\Phi K \otimes M) \to \widetilde{\Phi}M,$$

where the last morphism is induced by the structure morphism of M. (By construction, this is a morphism of $\mathcal{U}$; the associativity and unit axioms are straightforward verifications.)

By definition, for $N \in \operatorname{Ob} K\text{–}\mathcal{U}$, $\operatorname{Hom}_{K\text{–}\mathcal{U}}(N, \widetilde{\Phi}M)$ is the equalizer of

$$\operatorname{Hom}_{\mathcal{U}}(N, \widetilde{\Phi}M) \rightrightarrows \operatorname{Hom}_{\mathcal{U}}(K \otimes N, \widetilde{\Phi}M).$$

By adjunction, this is equivalent to the diagram

$$\operatorname{Hom}_{\mathcal{U}}(\Phi N, M) \rightrightarrows \operatorname{Hom}_{\mathcal{U}}(\Phi K \otimes \Phi N, M).$$

A simple verification shows that this corresponds to the equalizer diagram defining $\operatorname{Hom}_{\Phi K\text{–}\mathcal{U}}(\Phi N, M)$, which completes the proof. □

A.2 Formal results for endofunctors of $\mathcal{U}$

The following results explain how to study exact endofunctors of the category $\mathcal{U}$ via passage to $\mathcal{U}/\mathcal{N}il$.

Lemma A.2.1 *Let $\Theta\colon \mathcal{U} \to \mathcal{U}$ be an exact functor which preserves the subcategory $\mathcal{N}il$, then*

1. *$\mathcal{U} \xrightarrow{\Theta} \mathcal{U} \xrightarrow{l} \mathcal{U}/\mathcal{N}il$ induces an exact functor $\overline{\Theta}\colon \mathcal{U}/\mathcal{N}il \to \mathcal{U}/\mathcal{N}il$ such that $\overline{\Theta}l \cong l\Theta$;*
2. *if, moreover, Θ preserves the class of nilclosed unstable modules, then there is a natural isomorphism $\Theta rl \cong r\overline{\Theta}l$. In particular, $\overline{\Theta}$ determines the restriction of Θ to the full subcategory of nilclosed unstable modules.*

Recall that $\mathcal{U}/\mathcal{N}il$ is equivalent to the category of analytic functors, $\mathcal{F}_\omega$. The category $\mathcal{F}_\omega$ is locally Noetherian [19, Proposition 5.3.3], hence any coproduct of injective cogenerators of $\mathcal{F}_\omega$ of the form I_V is injective in $\mathcal{F}_\omega$ and any analytic functor admits an injective resolution in which each term is of the form $\bigoplus_\alpha I_{V_\alpha}$. This implies the following result.

Lemma A.2.2 *Let $\Theta_1, \Theta_2\colon \mathcal{F}_\omega \to \mathcal{F}_\omega$ be two exact functors which commute with arbitrary coproducts. If the restrictions Θ_1, Θ_2 to the full subcategory with objects $\{I_{V_s} | s \in \mathbb{N}\}$ are naturally isomorphic, then Θ_1 and Θ_2 are naturally isomorphic.*

A.3 Preservation of reduced unstable modules

Lemma A.3.1 *Let $\gamma\colon G_1 \hookrightarrow G_2$ be a natural monomorphism of endofunctors of $\mathcal{U}$ such that*

(1) *G_1 is exact and G_2 is left exact;*

(2) *γ_M is an isomorphism if M is a nilclosed unstable module.*

Then $\gamma_N\colon G_1 N \to G_2 N$ is an isomorphism if N is a reduced unstable module.

Proof Consider a reduced unstable module N and the associated short exact sequence of unstable modules $0 \to N \to rlN \to (rlN)/N \to 0$. The natural monomorphism γ induces a commutative diagram in $\mathcal{U}$

$$\begin{tikzcd} 0 \arrow[r] & G_1 N \arrow[r] \arrow[d, hook, "\gamma_N"'] & G_1(rlN) \arrow[r] \arrow[d, "\cong"] & G_1((rlN)/N) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & G_2 N \arrow[r] & G_2(rlN) \arrow[r] & G_2((rlN)/N) & \end{tikzcd}$$

in which the rows are exact and the middle vertical morphism is an isomorphism, since rlN is nilclosed. The result follows from the five-lemma. □

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