On unstable modules over the Dickson algebras, the Singer functors $R_s$ and the functors $\text{Fix}_s$

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The category $D_s\mathcal{U}$ of unstable modules over the Steenrod algebra equipped with a compatible module structure over the Dickson algebra $D_s$ is studied at the prime 2, with applications to the Singer functor $R_s$, considered as a functor from unstable modules $\mathcal{U}$ to $D_s\mathcal{U}$. An explicit copresentation of $R_s M$ is given using Lannes’ $T$–functor when $M$ is a reduced unstable module; applying Lannes’ functor $\text{Fix}_s$, this is used to show that $R_s$ gives a fully-faithful embedding of $\mathcal{U}$ in $D_s\mathcal{U}$. In addition, the right adjoint $Z_s$ to $R_s$ is introduced and is related to the indecomposables functor and the functor $\text{Fix}_s$.

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1 Introduction

The Dickson algebras over the field with two elements, $\mathbb{F}$, play an important role in the theory of unstable algebras over the mod–2 Steenrod algebra; the Dickson algebra $D_s$ is the algebra of invariants $H^*V_s^{\text{Aut}(V_s)}$, where $H^*V_s$ denotes the group cohomology of the rank $s$ elementary abelian 2–group $V_s$. The category $D_s\mathcal{U}$ of $D_s$–modules in the category $\mathcal{U}$ of unstable modules arises naturally; for example, Singer introduced the functors $R_s$ in his work on the homology of the Steenrod algebra ([22] and related work), where $R_s$ can be considered as a functor from the category $\mathcal{U}$ to $D_s\mathcal{U}$. One of the aims of this paper is to study the functor $R_s$, considered both as a functor to $D_s\mathcal{U}$ and as a functor to $\mathcal{U}$, from the viewpoint of modern unstable module theory.

The functors $R_s$ can be applied in calculating the $E_2$–term of the Adams spectral sequence: Lannes and Zarati [11] related them to the derived functors of destabilization (the left adjoint to the inclusion of the category $\mathcal{U}$ in the category of graded modules over the Steenrod algebra $\mathcal{A}$) and these derived functors appear, via a Grothendieck spectral sequence, in the calculation of Ext groups in the category of $\mathcal{A}$–modules. The role of the Singer functors in calculating these derived functors on general modules over the Steenrod algebra has been clarified (at odd primes) by the author in [17], establishing the relationship between the approach of Lannes and Zarati [11] and that of Singer.
For these applications, it is important to understand the behaviour of the functor \( \text{Hom}_\mathcal{U}(R_s M, -) \), for \( M \in \text{Ob}\mathcal{U} \). Restricting to the full subcategory of \( \mathcal{U} \) with objects of the form \( H^* V \), this is equivalent to understanding \( R_s M \) in the category \( \mathcal{U}/\mathcal{Nil} \) of unstable modules localized away from the nilpotent unstable modules, using the work of Henn, Lannes and Schwartz [8].

The paper exploits the structure of the module categories \( D_s - \mathcal{U} \), their relation with the categories \( H^* V_s - \mathcal{U} \) of unstable modules over \( H^* V_s \) and the localized categories \( D_s - \mathcal{U}/\mathcal{Nil} \); important tools are the functor \( \text{Fix}_s: H^* V_s - \mathcal{U} \to \mathcal{U} \) (see Lannes [10]) and the study of \( \omega_s \)-torsion for unstable modules over \( H^* V_s \), which was initiated by Dwyer and Wilkerson [3; 4] and developed by Lannes and Zarati [12]. The latter leads to the notion of \( \omega_s \)-closure: an unstable \( D_s \)-module \( M \) is \( \omega_s \)-closed if it is \( \omega_s \)-torsion-free and is maximal with this property in the equivalence class up to \( \omega_s \)-torsion.

A key new ingredient in studying the functor \( R_s \) is an approximation \( \tilde{R}_s: \mathcal{U} \to D_s - \mathcal{U} \) which is defined as the equalizer of a diagram

\[
D_s \otimes M \xrightarrow{\sigma_M} H^* V_s \otimes TV_s M
\]

in the category \( D_s - \mathcal{U} \), where \( TV_s \) is Lannes’ \( T \)-functor.

**Theorem 1**  For \( s \in \mathbb{N} \), there is a natural monomorphism \( \gamma_s: R_s \to \tilde{R}_s \) of functors \( \mathcal{U} \to D_s - \mathcal{U} \) such that, for an unstable module \( M \),

1. \( \tilde{R}_s M \) is the \( \omega_s \)-closure of \( R_s M \);
2. the morphism \( \gamma_s: R_s M \to \tilde{R}_s M \) is an isomorphism if \( M \) is reduced.

This is derived from a model for the Singer functor modulo nilpotent unstable modules. The category \( \mathcal{U}/\mathcal{Nil} \) embeds in the category \( \mathcal{F} \) of functors from the category \( \mathcal{V}^f \) of finite-dimensional \( \mathbb{F} \)-vector spaces to \( \mathbb{F} \)-vector spaces and the nillocalization of \( D_s - \mathcal{U} \) embeds in a functor category \( \mathcal{F}^g(D_s) \), for which \( \mathcal{V}^f \) is replaced by a category with objects \((V, W)\), where \( W \leq V \) is a subspace of codimension at most \( s \) in \( V \in \text{Ob} \mathcal{V}^f \).

The functor \( R_s \) corresponds to the functor \( \kappa_s: \mathcal{F} \to \mathcal{F}^g(D_s) \) given on \( F \in \text{Ob} \mathcal{F} \) by \( \kappa_s F(V, W) := F(W) \). The above diagram corresponds to a copresentation of the functor \( \kappa_s \); throughout the paper, the comparison with the behaviour after nillocalization is a guiding principle.

**Theorem 1** provides a model for \( R_s M \) (considered either in \( \mathcal{U} \) or in \( D_s - \mathcal{U} \)) when \( M \) is a reduced unstable module. In this case, the calculation of \( TV R_s M \) is accessible.
by using standard techniques of unstable module theory; as such, it provides a way of approaching the calculation of the functor $\text{Hom}_\mathcal{U}(R_s M, -)$.

The functor $\tilde{R}_s$ leads to a proof of the following result, where the natural transformation $\text{Fix}_s(H^* V_s \otimes D_s, R_s(-)) \to 1_\mathcal{U}$ is constructed by adjunction from the canonical monomorphism $R_s M \leftarrow D_s \otimes M$.

**Theorem 2** For $s \in \mathbb{N}$, the natural transformation $\text{Fix}_s(H^* V_s \otimes D_s, R_s(-)) \to 1_\mathcal{U}$ of functors on $\mathcal{U}$ is an isomorphism.

This result is striking: $\text{Fix}_s(H^* V_s \otimes D_s, -)$ is equipped with a natural $\text{Aut} V_s$ action; when composed with $R_s$, the action is trivial. For instance, applying the functor $\text{Fix}_s(H^* V_s \otimes D_s, -)$ to the natural inclusion $R_s M \leftarrow M$ yields the natural inclusion $M \leftarrow T V_s M$, where $T V_s M$ is an $\text{Aut}(V_s)$–module by functoriality of $T$.

The functor $\text{Fix}_s(H^* V_s \otimes D_s, -)$: $D_s - \mathcal{U} \to \mathcal{U}$ can be identified in the nillocalized situation, where it corresponds to the composite functor $\Psi_s \text{Ind}_s: \mathcal{F}^0(D_s) \to \mathcal{F}$, which is given on an object $G \in \text{Ob} \mathcal{F}^0(D_s)$ by $\Psi_s \text{Ind}_s G(V) := G(V \oplus \mathbb{F}^s V)$. In this setting, Theorem 2 corresponds to the natural isomorphism $\Psi_s \text{Ind}_s \kappa_s \simeq 1_\mathcal{F}$ (see Lemma 7.1.6); the force of the theorem is that this lifts to unstable modules. Since the functor $\text{Fix}_s$ does not see $\omega_s$–torsion, in the proof of Theorem 2, $R_s$ can be replaced by the model $\tilde{R}_s$ of Theorem 1, which leads to the result.

As a consequence, one obtains the following.

**Corollary 3** The functor $R_s$ induces a fully-faithful embedding $R_s: \mathcal{U} \hookrightarrow D_s - \mathcal{U}$, for $s \in \mathbb{N}$.

The Singer functor $R_s: \mathcal{U} \to D_s - \mathcal{U}$ admits a right adjoint $\mathcal{Z}_s$. The functor $\mathcal{Z}_s$ leads to a stronger conclusion (Theorem 8.3.1); Corollary 3 corresponds to the fact that the adjunction unit $1_\mathcal{U} \to \mathcal{Z}_s R_s$ is an isomorphism.

The functor $\mathcal{Z}_s$ is of independent interest; there is a natural transformation $Q_s \to \mathcal{Z}_s$ of functors $D_s - \mathcal{U} \to \mathcal{U}$, where $Q_s$ is the indecomposables functor, and this is an isomorphism up to nilpotent unstable modules. After nillocalization, $Q_s$ corresponds to $\mathcal{R}_s^0: \mathcal{F}^0(D_s) \to \mathcal{F}$ given by $\mathcal{R}_s^0 G(V) = G(V, V)$; in particular the functor $Q_s$ becomes exact upon nillocalization.

The functor $\mathcal{Z}_s$ is also related to the functor $\text{Fix}_s$, via a natural transformation

$$\mathcal{Z}_s \to \text{Fix}_s(H^* V_s \otimes D_s, -).$$

In the case $s = 1$, this leads to a criterion (see Theorem 9.3.3) for an object of $D_1 - \mathcal{U}$ to be in the image of $R_1$.
Organization of the paper  Sections 2, 3 and 4 set the stage, providing background, introducing the categories of unstable modules over the Dickson algebras and the Singer functors respectively. Sections 5 and 6 introduce the tools of nillocalization as they apply to unstable modules over Dickson algebras and Section 7 gives the model for the Singer functors viewed through the filter of nillocalization.

The main results of the paper are proved in Sections 8 and 9.

2 Background

2.1 Unstable modules and unstable algebras

Throughout the paper, $\mathbb{F}$ is the field with two elements and $\mathcal{A}$ is the mod–2 Steenrod algebra (see Schwartz [19] for the basics of the theory of unstable modules over the Steenrod algebra). The category of graded $\mathcal{A}$–modules is denoted by $\mathcal{M}$ and the full subcategory of unstable modules $\mathcal{U}$; these are equipped with the usual tensor product. A commutative algebra $\mathcal{B}$ in $\mathcal{M}$ is unstable if the underlying module is unstable and satisfies $\text{Sq}^0 x = x^2$, where $\text{Sq}^0$ denotes the top Steenrod operation; the category of unstable algebras and algebra morphisms is denoted by $\mathcal{H}$. Observe that the degree zero part of an unstable algebra is a Boolean algebra. An unstable algebra is Noetherian if the underlying commutative algebra is finitely-generated.

The category of $\mathcal{B}$–modules in $\mathcal{M}$ is denoted by $\mathcal{B}_\mathcal{M}$ and, if $\mathcal{K}$ is an unstable algebra, the category of $\mathcal{K}$–modules in $\mathcal{U}$ is denoted by $\mathcal{K}_\mathcal{U}$. If $\mathcal{K} \to L$ is a morphism of unstable algebras, there is an adjunction

$$L \otimes_{\mathcal{K}} \to \mathcal{K}_\mathcal{U} \Rightarrow L \otimes \mathcal{U} : \text{Restrict}_{L}^{\mathcal{K}},$$

where $L \otimes_{\mathcal{K}} \to$ is the induction functor, left adjoint to the exact restriction functor.

The degree-doubling functor $\Phi: \mathcal{M} \to \mathcal{M}$ restricts to a functor $\Phi: \mathcal{U} \to \mathcal{U}$ and $\text{Sq}^0$ induces a natural transformation $\lambda: \Phi \to 1_{\mathcal{U}}$ (see [19, Section 1.7]). An unstable module $M$ is reduced if $\lambda_M$ is a monomorphism (equivalently, if $M$ does not contain a nilpotent submodule, where an unstable module $N$ is nilpotent if $\text{Sq}^0$ acts locally nilpotently); $M$ is nilclosed if $\text{Ext}^e_{\mathcal{U}}(N, M) = 0$, for every $e \in \{0, 1\}$ and nilpotent $N$. The full subcategory of nilpotent unstable modules $\mathcal{Nil} \subset \mathcal{U}$ is a localizing subcategory (see Gabriel [6] for generalities on localization of abelian categories).

An unstable algebra is reduced (resp. nilclosed) if the underlying unstable module has this property, hence an unstable algebra $K$ is reduced if and only if $K$ contains no nilpotent elements. The functor $\Phi$ commutes with tensor products, thus restricts to a functor $\Phi: \mathcal{H} \to \mathcal{H}$, and $\Phi$ induces an exact functor $\Phi: \mathcal{K}_\mathcal{U} \to \Phi \mathcal{K}_\mathcal{U}$. If $K$
is reduced, $\Phi K$ identifies via $\lambda_K$ as the unstable subalgebra of $K$ generated by the squares of elements of $K$.

The functor $\tilde{\Phi}: \mathcal{U} \to \mathcal{U}$ is the right adjoint to $\Phi$ (see [19, Examples 2.2.3]) and the adjunction unit $M \to \tilde{\Phi}\Phi M$ is a natural isomorphism. Proposition A.1.1 shows that $\tilde{\Phi}$ induces a right adjoint to $\Phi$: $K-\mathcal{U} \to \Phi K-\mathcal{U}$.

### 2.2 Lannes’ $T$–functor

For $V$ an elementary abelian 2–group ($V_s$ will be written to denote an elementary abelian 2–group of rank $s$), $H^*V$ denotes the group cohomology of $V$ with $\mathbb{F}$–coefficients, which is isomorphic to the symmetric algebra $S^*(V^*)$ on the dual of $V$; the underlying unstable module of $H^*V$ is injective in $\mathcal{U}$ (see [19, Chapter 3]).

Lannes’ $T$–functor $T_V: \mathcal{U} \to \mathcal{U}$ is the left adjoint to $H^*V \otimes -: \mathcal{U} \to \mathcal{U}$; it is exact and commutes with tensor products. Moreover, $T_V$ restricts to a functor $T_V: \mathcal{K} \to \mathcal{K}$ and, for an unstable algebra $K$, induces an exact functor $K-\mathcal{U} \to T_V K-\mathcal{U}$.

A morphism of unstable algebras $\varphi: K \to H^*V$ is adjoint to a morphism $T_V K \to \mathbb{F}$ of unstable algebras, which factors across a morphism of Boolean algebras $\tilde{\varphi}: T^0_V K \to \mathbb{F}$, where $T^0_V$ denotes the degree zero part of $T_V$; $\mathbb{F}$ is a flat $T^0_V K$–module with respect to this morphism, so that $\mathbb{F} \otimes T^0_V K -$ is exact on the category of $T^0_V K$–modules.

**Definition 2.2.1** For $\varphi: K \to H^*V$ a morphism of unstable algebras, let $T_{(V,\varphi)} K$ denote the unstable algebra $\mathbb{F} \otimes T^0_V K T_V K$, where $\mathbb{F}$ is a $T^0_V K$–algebra via $\tilde{\varphi}$, and let $T_{(V,\varphi)}: K-\mathcal{U} \to T_{(V,\varphi)} K-\mathcal{U}$ be the exact functor $\mathbb{F} \otimes T^0_V K T_V (-)$.

The functor $T_V$ is natural in $V$; in particular, there is a natural inclusion $1_\mathcal{U} \cong T_0 \hookrightarrow T_V$, for $V \in \text{Ob}\mathcal{V}^f$.

**Lemma 2.2.2** For $K \in \text{Ob}\mathcal{K}$, $M \in \text{Ob} K-\mathcal{U}$ and $\varphi: K \to H^*V$ a morphism of unstable algebras, there are morphisms of unstable algebras

$$K \to T_V K \to T_{(V,\varphi)} K \cong \mathbb{F} \otimes T^0_V K T_V K,$$

with respect to which the natural morphisms of unstable modules

$$M \to T_V M \to T_{(V,\varphi)} M \cong \mathbb{F} \otimes T^0_V K T_V M$$

are morphisms of $K-\mathcal{U}$.

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2.3 The Dickson algebras

The group Aut(V) of linear automorphisms acts on H*V on the right by morphisms of unstable algebras; the Dickson algebra DV ∈ ObH is the ring of invariants H*V Aut(V); DVs will be denoted by Ds. There is an isomorphism of graded algebras

Ds ≅ F[ωs,0, . . . , ωs,s−1],

where the generator ωs,i has degree 2s − 2i; see Wilkerson [25]. The top Dickson invariant ωs,0 will be written ωs and identifies with the product of the elements of (H1Vs)\{0}; there are related explicit descriptions of the other generators. The algebra H*Vs is free as a Ds–module, forgetting the action of the Steenrod algebra (see Neusel and Smith [16] for example), in particular is flat as a Ds–module.

**Lemma 2.3.1** [11, Définition-Proposition 4.4.7] Let α: U ↪ V be the inclusion of a subspace of codimension c. There is a canonical surjection of unstable algebras DV → Φc DU which fits into a commutative diagram

![Diagram](image)

In particular, for s ∈ N, the kernel of Ds → ΦDs−1 is the prime ideal ωsDs, which is invariant under the A–action.

The Boolean algebra T0W H*Vs identifies with FHom(W, Vs), and by [19, Proposition 3.9.8], the subalgebra T0Vs Ds is isomorphic to FHom(W, Vs)/Aut(Vs). The morphism of Boolean algebras Ti: T0Vs Ds ≅ FHom(Vs, Vs)/Aut(Vs) → F associated to the canonical inclusion i: Ds ↪ H*Vs is induced by evaluation on the element of Hom(Vs, Vs)/Aut(Vs) represented by the identity morphism of Vs.

**Proposition 2.3.2** For s ∈ N,

1. the unstable algebra T(Vs,i)Ds is isomorphic to H*Vs;
2. T(Vs,i) induces an exact functor T(Vs,i): Ds−0U → H*Vs−0U.

**Proof** The first statement is a case of the calculation of the T–functor on rings of invariants (cf [19, Proposition 3.9.8], Dwyer and Wilkerson [5, proof of 1.4]). The second is an immediate consequence, following from the definition of T(Vs,i). □
2.4 Unstable modules over \( H^*V_s \) and \( \text{Fix}_s \)

The module category \( D_s - \mathcal{U} \) is related to the category \( H^*V_s - \mathcal{U} \) via the adjunction

\[
H^*V_s \otimes_{D_s} : D_s - \mathcal{U} \cong H^*V_s - \mathcal{U} : \text{Restrict}^{H^*V_s}_{D_s}.
\]

The functor \( \text{Fix}_s : H^*V_s - \mathcal{U} \to \mathcal{U} \) is the left adjoint to the free \( H^*V_s \)-module functor \( H^*V_s \otimes - : \mathcal{U} \to H^*V_s - \mathcal{U} \); it commutes with tensor products and restricts to a functor \( \text{Fix}_s : H^*V_s \downarrow \mathcal{K} \to \mathcal{K} \) which is left adjoint to \( H^*V_s \otimes - : \mathcal{K} \to H^*V_s \downarrow \mathcal{K} \) (cf [12, Théorème 1.3.3]). See [10] and [12; 13] for further properties of the categories \( H^*V_s - \mathcal{U} \) and the functors \( \text{Fix}_s \).

Lemma 2.2.2 has the following analogue for the functor \( \text{Fix}_s \), using Lannes’ description of \( \text{Fix}_s \) in terms of \( T_Vs \).

Proposition 2.4.1 For \( s \in \mathbb{N} \), the natural transformation \( 1_{\mathcal{U}} \to T_Vs \) induces a natural transformation \( \text{Forget}_s \to \text{Fix}_s \), where \( \text{Forget}_s : H^*V_s - \mathcal{U} \to \mathcal{U} \) is the forgetful functor; this factors naturally

\[
\text{Forget}_s \quad \longrightarrow \quad \mathbb{F} \otimes H^*V_s (\quad) \quad \longrightarrow \quad \text{Fix}_s (\quad)
\]

across the \( H^*V_s \)-module indecomposables.

Proof By [10, 4.4.3], \( \text{Fix}_s M \cong \mathbb{F} \otimes_{T_Vs} H^*V_s T_Vs M \), where \( T_Vs H^*V_s \to \mathbb{F} \) is adjoint to the identity on \( H^*V_s \). The natural morphism \( M \to T_Vs M \) defines a morphism of \( H^*V_s \)-modules, as in Lemma 2.2.2. The \( H^*V_s \)-module structure on \( \text{Fix}_s M \) is induced by the morphism of unstable algebras \( H^*V_s \to \text{Fix}_s H^*V_s \cong \mathbb{F} \). The result follows. \( \square \)

3 Unstable modules over the Dickson algebras

3.1 Unstable modules over \( D_s \), \( \omega_s \)-torsion and \( \omega_s \)-closure

Recall from Lemma 2.3.1 that there is a canonical surjection of unstable algebras \( D_s \to \Phi D_{s-1} \); this induces the functors in the following standard result.

Proposition 3.1.1 For \( s \in \mathbb{N} \), there are adjunctions

\[
\Phi D_{s-1} - \mathcal{U} \downarrow \Phi D_{s-1} - \mathcal{U} \cong D_s - \mathcal{U},
\]

where \( (-)/\omega_s : M \mapsto M/\omega_s M \), for \( M \in \text{Ob} D_{s-1} - \mathcal{U} \) and \( \text{Ann}_{\omega_s} M \) is the submodule of elements \( x \) such that \( \omega_s x = 0 \).
Moreover, the adjunction $\text{Restr}_{\mathcal{M}} \dashv \text{Ann}_{\omega_s}$ identifies $\Phi D_{s-1} \otimes \mathcal{M}$ as the full subcategory of $D_{s-\mathcal{M}}$ of modules annihilated by $\omega_s$.

**Remark 3.1.2**

1. The functor $(-)/\omega_s$ identifies with $\Phi D_{s-1} \otimes D_s -$, the induction functor.

2. Localization away from the torsion associated to an invariant ideal of an unstable algebra has been considered by Henn [7, Section 3] and Meyer [15, Chapter 7]. Dwyer and Wilkerson [3; 4] and Lannes and Zarati [12] have considered localization away from $\omega_s$–torsion.

Localization inverting the top Dickson invariant is an important tool. For $s \in \mathbb{N}$, the localized algebras $D_s[\omega_s^{-1}]$ and $H^* V_s[\omega_s^{-1}]$ are commutative algebras in $\mathcal{M}$ (see Singer [21] and Wilkerson [24]); moreover, $D_s[\omega_s^{-1}] \otimes D_s -$ induces an exact functor

$$D_s[\omega_s^{-1}] \otimes D_s - : D_{s-\mathcal{M}} \to D_s[\omega_s^{-1}] - \mathcal{M},$$

which will be denoted $M \mapsto M[\omega_s^{-1}]$.

Recall that the inclusion $\mathcal{M} \hookrightarrow \mathcal{M}$ has a right adjoint $\text{Un}: \mathcal{M} \to \mathcal{M}$, which gives the largest unstable module of an $A$–module. If $X, Y$ are $A$–modules, there is a canonical monomorphism $(\text{Un}X) \otimes (\text{Un}Y) \to \text{Un}(X \otimes Y)$. It follows that, for $M \in \text{Ob} D_{s-\mathcal{M}}$, there is a natural morphism $M \to \text{Un}(M[\omega_s^{-1}])$ in $D_{s-\mathcal{M}}$.

**Definition 3.1.3** [7; 15] An unstable $D_s$–module $M \in \text{Ob} D_{s-\mathcal{M}}$ is $\omega_s$–closed if the map $M \to \text{Un}(M[\omega_s^{-1}])$ is an isomorphism. An unstable $H^* V_s$–module is $\omega_s$–closed if the underlying unstable $D_s$–module is $\omega_s$–closed.

**Proposition 3.1.4** For $s \in \mathbb{N}$, an unstable $D_s$–module $M$ is $\omega_s$–closed if the unstable $H^* V_s$–module $H^* V_s \otimes D_s M$ is $\omega_s$–closed.

In particular, for $N \in \text{Ob} \mathcal{M}$, $D_s \otimes N$ is $\omega_s$–closed.

**Proof** The unstable $D_s$–module $M$ embeds in $H^* V_s \otimes D_s M$ as the invariants (in $D_{s-\mathcal{M}}$) of the action of $\text{Aut}(V_s)$ induced by the action on the left hand factor. This implies that, if $H^* V_s \otimes D_s M$ is $\omega_s$–closed, then $M$ is $\omega_s$–closed, since the kernel of a morphism between $\omega_s$–closed objects is $\omega_s$–closed.

By [11, Proposition 2.5.2], the morphism

$$H^* V_s \otimes N \to \text{Un}(H^* V_s[\omega_s^{-1}] \otimes N)$$

is an isomorphism, for $N$ an unstable module, which shows that $H^* V_s \otimes N$ is $\omega_s$–closed. Since $H^* V_s \otimes N$ is isomorphic to $H^* V_s \otimes D_s (D_s \otimes N)$, it follows that $D_s \otimes N$ is $\omega_s$–closed.

$\square$
Remark 3.1.5 The converse is false in general. Consider $H^*V_\omega$ as a $D_s$–module (which is $\omega_s$–closed); for $s > 1$ an integer, $H^*V_\omega \otimes_{D_s} H^*V_\omega$ is not $\omega_s$–closed.

3.2 The indecomposables functor

The augmentation $D_s \to \mathbb{F}$ induces a functor $\text{triv}_s : \mathcal{U} \to D_s – \mathcal{U}$ which has left adjoint $Q_s : D_s – \mathcal{U} \to \mathcal{U}$ given by $M \mapsto M/\mathbb{F}D_s M \cong \mathbb{F} \otimes_{D_s} M$, $\mathbb{F}D_s$ denoting the augmentation ideal, and right adjoint which associates to an unstable $D_s$–module the largest unstable submodule with trivial $D_s$–action, thus identifying $\mathcal{U}$ as the full subcategory of $D_s – \mathcal{U}$ of objects with trivial $D_s$–module structure

$$\mathcal{U} \xleftarrow{\text{triv}_s} Q_s \xrightarrow{\text{triv}_s} D_s – \mathcal{U}. $$

4 Introducing the Singer functors

4.1 The Singer functor $R_s$

The definition and properties of the Singer functor $R_s : \mathcal{U} \to D_s – \mathcal{U}$ are reviewed in this section: for further details, the reader is referred to the original work of Singer [23; 20] (and related work) and Lannes and Zarati’s paper [11].

In the following, if $I$ is a sequence of nonnegative integers, $Sq^I$ denotes the Milnor basis element of the Steenrod algebra indexed by $I$. The linear map $St_s$ introduced below corresponds to the Steenrod total power.

Definition 4.1.1 For $s \geq 1$ an integer and $M$ an unstable module, let

1. $St_s : \Phi^s M \to D_s \otimes M$ be the linear map defined on a homogeneous element $\Phi^s x$ by

$$St_s(\Phi^s x) := \sum_{I=(i_1, \ldots, i_s)} \omega_{s,0}^{i_1} \omega_{s,1}^{i_2} \cdots \omega_{s,s-1}^{i_s} \otimes Sq^I(x)$$

where $|x| = \epsilon + i_1 + \cdots + i_s$;

2. $R_s M$ denote the sub $D_s$–module (ignoring the $A$–action) of $D_s \otimes M$ which is generated by the image of $St_s$.

Remark 4.1.2 (1) The linear maps $St_s$ can be constructed as iterates of the linear maps $St_1$; the above definition stresses the intimate relationship between the Dickson algebras and the dual Steenrod algebra.
Proposition 4.1.3 below contains the statement that \( R_s M \subset D_s \otimes M \) is stable under the action of the Steenrod algebra, which is not immediately obvious from the definition given.

By convention, \( R_0 \) is taken to be the identity functor on \( \mathcal{U} \) and \( R_{-1} \) to be the zero functor.

**Proposition 4.1.3** [11] For \( s \in \mathbb{N} \), \( R_s \) defines a functor \( R_s: \mathcal{U} \to D_s - \mathcal{U} \), equipped with a monomorphism \( R_s(-) \hookrightarrow D_s \otimes - \) in \( D_s - \mathcal{U} \).

1. For \( M \) an unstable module, the underlying graded \( D_s \)–module of \( R_s M \) is free on a vector space isomorphic to \( \Phi^s M \). Moreover, there is a natural isomorphism \( Q_s R_s M \cong \hat{F} \otimes D_s R_s M \cong \Phi^s M \) in \( \mathcal{U} \).
2. The functor \( R_s: \mathcal{U} \to D_s - \mathcal{U} \) is exact and commutes with limits and colimits.
3. For unstable modules \( M, N \), there is a natural isomorphism \( R_s(M \otimes N) \cong R_s M \otimes D_s R_s N \) in \( D_s - \mathcal{U} \).
4. There is a natural surjection \( \rho_s: R_s M \twoheadrightarrow \Phi R_{s-1} M \) in \( D_s - \mathcal{U} \) which makes the following diagram commute:

\[
\begin{array}{ccc}
R_s M & \to & D_s \otimes M \\
\downarrow \rho_s & & \downarrow \\
\Phi R_{s-1} M & \to & \Phi(D_{s-1} \otimes M) \cong \Phi D_{s-1} \otimes \Phi M \xrightarrow{1 \otimes \lambda_M} (\Phi D_{s-1}) \otimes M,
\end{array}
\]

where the terms of the bottom row are considered as \( D_s \)–modules via restriction of their natural \( \Phi D_{s-1} \)–module structures.

Moreover, there is a natural short exact sequence in \( D_s - \mathcal{U} \),

\[
0 \to \omega_s R_s M \to R_s M \to \Phi R_{s-1} M \to 0.
\]

5. If \( N \) is a nilpotent unstable module, then \( R_s N \) is nilpotent.
6. If \( M \) is a reduced (respectively nilclosed) unstable module, then \( R_s M \) is reduced (resp. nilclosed).

The following result will be strengthened in Section 8.3.

**Lemma 4.1.4** For \( s \in \mathbb{N} \) and unstable modules \( M, N \), the functor \( R_s \) induces a monomorphism \( \text{Hom}_{\mathcal{U}}(M, N) \hookrightarrow \text{Hom}_{D_s - \mathcal{U}}(R_s M, R_s N) \).

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Proof  By Proposition 4.1.3, the composite $Q_s R_s$ is naturally isomorphic to the functor $\Phi^s$, hence gives rise to

$$\text{Hom}_\mathbb{U}(M, N) \xrightarrow{R_s} \text{Hom}_{D_s-\mathbb{U}}(R_s M, R_s N) \xrightarrow{Q_s} \text{Hom}_\mathbb{U}(\Phi^s M, \Phi^s N),$$

which identifies with the natural morphism corresponding to the functor $\Phi^s$. The functor $\Phi^s$ is fully faithful, hence the first morphism is injective, as required. \qed

Recall that an unstable module $N$ is locally finite if $A x \subset N$ is finite, for every element $x$ of $N$.

Proposition 4.1.5  For $s \in \mathbb{N}$ and $X$ a locally finite unstable module, the natural monomorphism $R_s X \hookrightarrow D_s \otimes X$ is the $\omega_s$–closure of $R_s X$ in $D_s \otimes X$.

Proof  The module $D_s \otimes X$ is $\omega_s$–closed in $D_s-\mathbb{U}$, by Proposition 3.1.4, hence it suffices to show that the cokernel of $R_s X \to D_s \otimes X$ is $\omega_s$–torsion. Since both functors commute with colimits, it suffices to consider the case where $X$ is a finite unstable module. This case is established by induction on the total dimension of $X$, using the cases $X = \Sigma^n F$, for $n \in \mathbb{N}$, for the inductive step. The monomorphism $R_s \Sigma^n F \hookrightarrow D_s \otimes \Sigma^n F$ identifies with the $n$–iterated suspension of the inclusion $\omega_s^n D_s \hookrightarrow D_s$, so the cokernel is $\omega_s$–torsion. \qed

Proposition 4.1.6  The functor $R_s : \mathbb{U} \to D_s-\mathbb{U}$ admits a left adjoint $\mathfrak{A}_s$ and a right adjoint $\mathfrak{Z}_s$,

$$D_s-\mathbb{U} \xleftarrow{\mathfrak{A}_s} \mathbb{U} \xrightarrow{R_s} D_s-\mathbb{U} \xrightarrow{\mathfrak{Z}_s} \mathbb{U}. $$

Moreover,

(1) the functor $\mathfrak{A}_s$ sends projective objects of $D_s-\mathbb{U}$ to projectives of $\mathbb{U}$;

(2) the functor $\mathfrak{Z}_s$ sends injective (respectively reduced) objects of $D_s-\mathbb{U}$ to injective (resp. reduced) objects of $\mathbb{U}$.

Proof  The result is a formal consequence of the properties of $R_s$. For example, for $M \in \text{Ob} D_s-\mathbb{U}$, $\mathfrak{Z}_s M$ is reduced if and only if $\text{Hom}_\mathbb{U}(N, \mathfrak{Z}_s M) \cong \text{Hom}_{D_s-\mathbb{U}}(R_s N, M)$ is trivial for every nilpotent unstable module $N$. The functor $R_s$ preserves nilpotent unstable modules; hence, if $M$ is reduced, then $\text{Hom}_{D_s-\mathbb{U}}(R_s N, M) = 0$ for nilpotent $N$. \qed
Remark 4.1.7  The category $D_s - \mathcal{U}$ has enough projectives and injectives [12; 15]. A family of projective generators of $D_s - \mathcal{U}$ is given by the family of unstable $D_s$–modules $D_s \otimes F(n)$, where $F(n)$ denotes the free unstable module on a generator of degree $n$. Similarly, there is a family of injective cogenerators given by the generalized Brown–Gitler modules $J_{D_s}(n)$, for $n \in \mathbb{N}$, where $J_{D_s}(n)$ corepresents the contravariant functor $M \mapsto (M^n)^*$, for $M \in \text{Ob } D_s - \mathcal{U}$.

The unstable modules $\mathfrak{A}_s(D_s \otimes F(n))$ and $\mathfrak{I}_s(J_{D_s}(n))$ are closely related and illuminate the relationship between the Dickson algebras and the Steenrod algebra.

Recall from Section 3 that $\text{triv}_s: \mathcal{U} \rightarrow D_s - \mathcal{U}$ gives an unstable module the trivial $D_s$–module structure and $\text{Restrict}_s: \Phi D_{s-1} - \mathcal{U} \rightarrow D_s - \mathcal{U}$ is induced by the canonical projection $D_s \rightarrow \Phi D_{s-1}$. The functor $\Phi: \Phi D_{s-1} - \mathcal{U} \rightarrow D_{s-1} - \mathcal{U}$ is provided by Proposition A.1.1.

Proposition 4.1.8  For $s \in \mathbb{N}$, there are natural isomorphisms

1. $\mathfrak{I}_s \circ \text{Restrict}_s \cong \mathfrak{I}_{s-1} \circ \Phi: \Phi D_{s-1} - \mathcal{U} \rightarrow \mathcal{U}$
2. $\mathfrak{I}_s \circ \text{triv}_s \cong \Phi^*: \mathcal{U} \rightarrow \mathcal{U}$.

Proof  For $M \in \text{Ob } \mathcal{U}$ and $N \in \text{Ob } \Phi D_{s-1} - \mathcal{U}$, there are natural isomorphisms

$$\text{Hom}_{\mathcal{U}}(M, \mathfrak{I}_s \text{Restrict}_s N) \cong \text{Hom}_{D_s - \mathcal{U}}(R_s M, \text{Restrict}_s N) \cong \text{Hom}_{\Phi D_{s-1} - \mathcal{U}}((R_s M)/\omega_s, N).$$

By Proposition 4.1.3, there is a natural isomorphism $(R_s M)/\omega_s \cong \Phi R_{s-1} M$, hence $\text{Hom}_{\Phi D_{s-1} - \mathcal{U}}((R_s M)/\omega_s, N) \cong \text{Hom}_{D_{s-1} - \mathcal{U}}(R_{s-1} M, \Phi N) \cong \text{Hom}_{\mathcal{U}}(M, \mathfrak{I}_{s-1} \Phi N)$, by adjunction. The first statement follows.

The second statement can either be proved directly by a similar argument, or deduced by induction from the first, since $\text{triv}_s$ is the composite of the functors $\text{Restrict}_i$ for $1 \leq i \leq s$. $\square$

Further results on the functors $\mathfrak{I}_s$ are given in Section 9, using deeper properties of the Singer functors $R_s$.

5  Functor categories and nillocalization

This section reviews the techniques of nillocalization, as they apply to the study of the category of unstable modules over an unstable algebra. This is based on the foundations of Henn, Lannes and Schwartz [8; 9], related to earlier work of Lam, Rector [18] and Adams and Wilkerson [1], and on subsequent work of Djament [2], Henn [7], Lannes and Zarati [12; 13], Mekkia [14] and Meyer [15].
5.1 Nillocalizations

The general theory of localization of abelian categories [6] provides an adjunction \( l : \mathcal{U} \xrightarrow{\simeq} \mathcal{U}/\text{Nil} : r \) and, moreover, the functor \( l \) is exact [19, Chapter 5]. The adjunction unit \( M \to rlM \) corresponds to nilclosure: \( M \) is reduced (respectively nilclosed) if and only if it is a monomorphism (resp. isomorphism).

**Notation 5.1.1** Write \( \mathcal{V}^f \) for the full subcategory of finite-dimensional spaces in \( \mathcal{V} \), the category of \( \mathbb{F} \)-vector spaces; the category of functors from \( \mathcal{V}^f \) to \( \mathcal{V} \) is denoted by \( \mathcal{F} \) and the full subcategory of locally finite (or analytic) functors, \( \mathcal{F}_\omega \) (see [8; 19]).

The category \( \mathcal{U}/\text{Nil} \) identifies with the full subcategory \( \mathcal{F}_\omega \subset \mathcal{F} \) of analytic functors via \( l : \mathcal{U} \to \mathcal{F}, lM(V) = T^0_V M, \) which has right adjoint equally denoted by \( r : \)

\[
l : \mathcal{U} \xrightarrow{\simeq} \mathcal{F} : r.
\]

**Example 5.1.2** For \( V \in \text{Ob} \mathcal{V}^f \), we have that the analytic functor \( lH^*V \) is the injective \( I_V(-) = F^{\text{Hom}(-,V)} \). The analogue of the functor \( T_V \) in the category \( \mathcal{F} \) is the shift functor \( \Delta_V : \mathcal{F} \to \mathcal{F}, \) defined by precomposition with \( - \oplus V : \mathcal{V}^f \to \mathcal{V}^f \).

The functors \( l \) and \( r \) both commute with tensor products, which is an important fact in considering module structures in the respective categories. Similarly, the functor \( l \) sends an unstable algebra to a functor with values in Boolean algebras. The category of Boolean algebras is equivalent to the opposite of the category of profinite sets, via the functor \( X \mapsto F^X \), where \( F^X \) denotes the space of continuous maps from the profinite set \( X \) to \( \mathbb{F} \).

**Notation 5.1.3** Let

1. \( \mathcal{PF} \) denote the category of presheaves of profinite sets on \( \mathcal{V}^f \), so that the continuous map functor induces \( F(-) : \mathcal{P} \text{F}^{\text{op}} \to \mathcal{F}; \)

2. \( g : \mathcal{H}^{\text{op}} \to \mathcal{PF} \) be the functor \( g(K) : V \mapsto \text{Hom}_{\mathcal{H}}(K, H^*V). \)

Lannes’ linearization principle fits into this framework via the isomorphism

\[
l(K)(V) = T^0_V K \cong F^{g(K)}(V).
\]

If \( K \) is a Noetherian unstable algebra, then \( g(K) \) takes values in finite sets (cf [8]).
Example 5.1.4  For \( s \in \mathbb{N} \),

1. \( \mathfrak{g}(H^*V_s)(W) = \text{Hom}_{\mathcal{V}f}(W, V_s) \);
2. \( \mathfrak{g}(D_s)(W) = \text{Hom}_{\mathcal{V}f}(W, V_s)/\text{Aut}(V_s) \), which is equivalent to the set of subspaces of \( W \) of codimension at most \( s \), regarded as a contravariant functor by pullback of subspaces.

The inclusion \( D_s \hookrightarrow H^*V_s \) induces the surjection to coinvariants \( \mathfrak{g}(H^*V_s) \twoheadrightarrow \mathfrak{g}(D_s) \).

5.2 Nillocalization of the category of modules over a Noetherian unstable algebra

Let \( K \) be a Noetherian unstable algebra; an object of \( K-\mathfrak{u} \) is said to be nilpotent if the underlying unstable module is nilpotent. There is an exact localization functor \( K-\mathfrak{u} \rightarrow K-\mathfrak{u}/\text{Nil} \). (This notation should not lead to confusion, since there is a forgetful functor to \( \mathfrak{u}/\text{Nil} \) and the category \( K-\mathfrak{u}/\text{Nil} \) only depends on \( K \) up to nillocalization.)

An element of \( K-\mathfrak{u} \) is nilclosed if and only if the underlying unstable module is nilclosed; in this case, the unstable \( K-\)module structure is the restriction of the induced unstable \( rl(K)-\)module structure (\( rl(K) \) has a canonical unstable algebra structure [8]).

An element \( \varphi \in \mathfrak{g}(K)(V) \) can be considered as a morphism of Boolean algebras \( T^0_VK \rightarrow \mathbb{F} \) and the functor \( T^0_{(V,\varphi)} \) is defined (cf Definition 2.2.1), which has degree zero part denoted \( T^0_{(V,\varphi)} \). The pair \( (V,\varphi) \) can be considered as an object of a comma category, which motivates the following.

Notation 5.2.1 For \( \mathfrak{X} \) a presheaf of finite sets on \( \mathcal{V}f \), denote by

1. \( \mathcal{V}f/\mathfrak{X} \) the comma category, with objects pairs \((V, x)\), where \( V \in \mathcal{V}f \) and \( x \in \mathfrak{X}(V) \), and a morphism \((V, x) \rightarrow (W, y)\) is a linear map \( f: V \rightarrow W \) such that \( \mathfrak{X}(f)y = x \);
2. \( \mathcal{F}^{\mathfrak{X}} \) the category of functors \( \text{Funct}(\mathcal{V}f/\mathfrak{X}, \mathcal{V}) \);
3. \( \mathcal{F}^{\mathfrak{X}}-\mathcal{F} \) the category of \( \mathcal{F}^{\mathfrak{X}}-\)modules in \( \mathcal{F} \).

Example 5.2.2  (1) The category \( \mathcal{V}f/_{\mathfrak{g}(H^*V_s)} \) is the over-category \( \mathcal{V}f/_{V_s} \).

(2) The category \( \mathcal{V}f/_{\mathfrak{g}(D_s)} \) has objects \((V, U)\), where \( U \leq V \) is a subspace of codimension at most \( s \); a morphism \((V, U) \rightarrow (V', U')\) is a linear map \( V \rightarrow V' \) sending \( U \) to \( U' \) and such that the induced map \( V/U \rightarrow V'/U' \) is a monomorphism.
The category $\mathcal{F}^\mathcal{X}$ is abelian equipped with a tensor product, this structure being inherited from $\mathcal{V}$. Moreover, Yoneda’s lemma shows that it has sufficiently many projectives and injectives.

**Example 5.2.3** Consider the case $\mathcal{X} = g(D_s)$, for $s \in \mathbb{N}$. If $(V, U)$ is an object of $\mathcal{V}^f_{/g(D_s)}$, then

1. $P_{(V, U)} \in \text{Ob} \mathcal{F}g(D_s)$ denotes the projective functor $\mathbb{F}[\text{Hom}((V, U), -)]$,  
2. $I_{(V, U)} \in \text{Ob} \mathcal{F}g(D_s)$ denotes the injective functor $\mathbb{F}\text{Hom}(-, (V, U))$, 

where Hom is taken in the category $\mathcal{V}^f_{/g(D_s)}$. This gives families of projective generators and injective cogenerators respectively, as $(V, U)$ runs over representatives of isomorphism classes of objects of $\mathcal{V}^f_{/g(D_s)}$.

The functor $I_{(V_s, 0)}$ plays an important role; it can be identified as follows. There is a natural isomorphism

$$\text{Hom}_{\mathcal{V}^f_{/g(D_s)}}((V, U), (V_s, 0)) \cong \text{Inj}(V/U, V_s),$$

where the right hand side is the set of injective linear maps. Hence we have that $I_{(V_s, 0)}(V, U) \cong \mathbb{F}\text{Inj}(V/U, V_s)$.

The functors $T^0_{(V, \varphi)}$ of Definition 2.2.1 are constructed using the splitting associated to the canonical idempotents of the finite-dimensional Boolean algebra $T^0_V K$, which gives an isomorphism for $M \in \text{Ob} K - \mathcal{U}$, namely

$$T^0_V M \cong \bigoplus_{\varphi \in \mathcal{U}(K)(V)} T^0_{(V, \varphi)} M.$$

This corresponds to a functor defined in the general framework which was introduced in Notation 5.2.1 (see [2, Chapitre 3]). Namely the forgetful functor $\mathcal{V}^f_{/\mathcal{X}} \rightarrow \mathcal{V}^f_{/\mathcal{X}}$ induces a functor $i^\mathcal{X}: \mathcal{F} \rightarrow \mathcal{F}^\mathcal{X}$ by precomposition, given explicitly by $i^\mathcal{X} F(V, x) = F(V)$. This admits a right adjoint $\hat{\Omega}^\mathcal{X}$ given by $\hat{\Omega}^\mathcal{X} G(V) = \bigoplus_{x \in \mathcal{X}(V)} G(V, x)$.

The importance of this adjunction is through the following.

**Proposition 5.2.4** [2, Proposition 3.3.10] For $\mathcal{X}$ a presheaf of finite sets on $\mathcal{V}^f$, the adjunction $i^\mathcal{X} \dashv \hat{\Omega}^\mathcal{X}$ induces an equivalence between $\mathcal{F}^\mathcal{X}$ and the category $\mathcal{F}^\mathcal{X} - \mathcal{F}$ of $\mathcal{F}^\mathcal{X} - \mathcal{F}$–modules in $\mathcal{F}$.

There is a relative version of the above construction [2, Définition et Proposition 3.3.4]. For $\alpha: \mathcal{X} \rightarrow \mathcal{Y}$ a morphism of presheaves of finite sets, $\mathcal{V}^f_{/\alpha}$ induces a functor
\(\alpha^!: \mathcal{F}^\Omega \to \mathcal{F}^X\) by precomposition, which admits a right adjoint \(\alpha^!\): \(\mathcal{F}^X \to \mathcal{F}^\Omega\) given on \(F \in \text{Ob} \mathcal{F}^X\) by

\[(\alpha^! F)(V, y) = \bigoplus_{x \in \alpha^{-1}_V y} F(V, x).\]

Consult [2, Chapitre 3] for the general properties.

**Remark 5.2.5** For \(X\) a presheaf of finite sets, \(i^X = (X \to *)^!\) and \(\Omega^X = (X \to *)_!\), where * is the terminal presheaf.

**Example 5.2.6** For \(m: K \to L\) a morphism of Noetherian unstable algebras, we have that \(g(m): g(L) \to g(K)\) induces an adjunction

\[g(m)^!: \mathcal{F}g(K) \rightarrow\leftarrow \mathcal{F}g(L) : g(m)_!.\]

This is related to the induction-restriction adjunction \(L \otimes_K - \cong L \otimes_L : \text{Restrict}_K^L\) in Theorem 5.2.8.

**Definition 5.2.7** A functor \(G\) of \(\mathcal{F}^X\) is analytic if \(\Omega^X G \in \text{Ob} \mathcal{F}\) is analytic; the full subcategory of analytic functors in \(\mathcal{F}^X\) is denoted \(\mathcal{F}^X_\omega\).

**Theorem 5.2.8** For \(K\) a Noetherian unstable algebra, the adjunction \(l: \mathcal{U} \cong \mathcal{F} : r\) induces an adjunction

\[l_K: K-\mathcal{U} \cong \mathcal{F}g(K) : r_K,\]

where \((l_K M)(V, \varphi) = T^0_{(V, \varphi)} M\) and the underlying functor \(r_K: \mathcal{F}g(K) \to \mathcal{U}\) is the composite \(r\Omega g(K)\).

The functor \(l_K\) is exact and commutes with tensor products. Moreover, \(l_K\) induces an equivalence of categories

\[K-\mathcal{U}/Nil \cong \mathcal{F}g(K)\]

For \(m: K \to L\) a morphism of Noetherian unstable algebras,

1. \(l_L (L \otimes_K -): K-\mathcal{U} \to \mathcal{F}g(L)\) is naturally equivalent to \(g(m)^! l_K\);
2. \(\text{Restrict}_K^L r_L: \mathcal{F}g(L) \to K-\mathcal{U}\) is naturally equivalent to \(r_K g(m)_!\).

**Proof** The functor \(l: \mathcal{U} \to \mathcal{F}\) commutes with tensor products, hence induces a functor \(l: K-\mathcal{U} \to \mathcal{F}g(K)-\mathcal{F}\). The category \(\mathcal{F}g(K)-\mathcal{F}\) is equivalent to \(\mathcal{F}g(K)\) by Proposition 5.2.4, and this yields the functor \(l_K\). Likewise, the composite \(r\Omega g(K)\) induces a functor to \(rl(K)-\mathcal{U}\); restriction along the adjunction unit \(K \to rl(K)\), which is a morphism of unstable algebras, gives \(r_K\). That these functors are adjoint is formal and the basic properties follow from the general theory of nillocalization [8].
Consider the morphism \( m: K \to L \). Statement (1) can be verified directly by using the explicit form of \( l_K \) and \( l_L \), as follows. Consider \( M \in \text{Ob} \mathcal{K} \) and an element \( \psi \in \mathfrak{g}(L)(V) \); there are natural isomorphisms

\[
T_0^0(\psi)(L \otimes_K M) \cong \mathbb{F} \otimes_{T_0^0 L} (T_0^0 L \otimes_{T_0^0 K} T_0^0 M) \cong \mathbb{F} \otimes_{T_0^0 K} T_0^0 M,
\]

where the latter tensor product is formed with respect to \( \mathfrak{g}(m) \psi \in \mathfrak{g}(K)(V) \). This establishes the first identification. Statement (2) follows by adjunction from (1).

**Example 5.2.9** For \( s \in \mathbb{N} \),

1. \( D_s - \mathcal{U}/\mathcal{N}il \) is equivalent to the category \( \mathcal{F}_\mathfrak{g}(D_s) \), embedded as a full subcategory of \( \mathcal{F}_\mathfrak{g}(D) \);

2. \( H^* V_s - \mathcal{U}/\mathcal{N}il \) is equivalent to \( \mathcal{F}_\mathfrak{g}(H^* V) \).

### 6 Nillocalization of unstable modules over the Dickson algebras

The results of Section 5.2 are applied to the categories \( D_s - \mathcal{U} \) to obtain the analogues of the structures considered in Section 3. The reader is referred to [2] for further results; in particular, the adjunctions considered here fit into recollement diagrams of abelian categories.

Throughout the section, the identification of \( \mathcal{V}^f/\mathfrak{g}(D_s) \) given in Example 5.2.2 is used without further comment.

#### 6.1 Restriction

For \( 0 < s \in \mathbb{Z} \), the surjection \( D_s \to \Phi D_{s-1} \) of Lemma 2.3.1 induces an inclusion \( \mathfrak{g}(D_{s-1}) \cong \mathfrak{g}(\Phi D_{s-1}) \hookrightarrow \mathfrak{g}(D_s) \). As in Section 3.1 in the setting of modules over the Dickson algebras, there are associated adjunctions.

**Proposition 6.1.1** For \( 1 \leq s \in \mathbb{Z} \), there is an adjunction

\[
\mathcal{R}_s: \mathcal{F}_\mathfrak{g}(D_s) \rightleftarrows \mathcal{F}_\mathfrak{g}(D_{s-1}) : \mathcal{P}_s,
\]

in which \( \mathcal{R}_s = \mathfrak{g}( D_s \to \Phi D_{s-1} ) \) is restriction and \( \mathcal{P}_s = \mathfrak{g}(D_s 
rightarrow \Phi D_{s-1}) ! \) is extension by zero.

Moreover, there are natural equivalences of exact functors

1. \( \mathcal{R}_s \downharpoonleft D_s \cong \downharpoonleft D_{s-1} \left( ( - ) / \omega_s \right): D_s - \mathcal{U} \to \mathcal{F}_\mathfrak{g}(D_{s-1}) ; \)

2. \( \mathcal{P}_s \downharpoonleft D_{s-1} \cong \downharpoonleft D_s \circ \text{Restrict}_s: \Phi D_{s-1} - \mathcal{U} \to \mathcal{F}_\mathfrak{g}(D_s) . \)
The identification of the functors is straightforward (cf the general results of [2, Appendix C.6]); the final statement follows from \textbf{Theorem 5.2.8}. □

6.2 Full restriction

The augmentation $D_s \rightarrow \mathbb{F}$ gives rise to an adjunction, as in \textbf{Proposition 6.1.1}. In the following statement, $F \in \text{Ob} \mathbb{F}$ and $G \in \text{Ob} \mathbb{F}_g(D_s)$.

\textbf{Proposition 6.2.1} For $s \in \mathbb{N}$, there is an adjunction

$$\mathcal{R}_s^0, \mathbb{F}_g(D_s) \rightleftarrows \mathcal{P}_s^0,$$

where $\mathcal{R}_s^0 = g(D_s \rightarrow \mathbb{F})^!$ and $\mathcal{P}_s^0 = g(D_s \rightarrow \mathbb{F})_!$.

Explicitly, $\mathcal{R}_s^0: \mathbb{F}_g(D_s) \rightarrow \mathbb{F}$ is the restriction functor defined by $\mathcal{R}_s^0 G(V) := G(V, V)$ and $\mathcal{P}_s^0: \mathbb{F} \rightarrow \mathbb{F}_g(D_s)$ is extension by zero $\mathcal{P}_s^0 F(V, U) = 0$ unless $V = U$, when $\mathcal{P}_s^0 F(V, V) = F(V)$. In particular, the composite $\mathcal{R}_s^0 \mathcal{P}_s^0$ is naturally equivalent to $1_\mathbb{F}$.

Moreover, there are natural equivalences of exact functors:

(1) $\mathcal{R}_s^0 l \cong l Q_s: D_s \rightarrow \mathbb{F}$;

(2) $\mathcal{P}_s^0 l \cong l D_s \circ \text{triv}_s: \mathbb{F} \rightarrow \mathbb{F}_g(D_s)$.

6.3 Induction and restriction

The canonical inclusion $D_s \hookrightarrow H^*V_s$ induces the functor $\mathcal{V}_\mathbb{F}(V) \rightarrow \mathcal{V}_\mathbb{F}(D_s)$, which sends an object $f: V \rightarrow V_s$ of $\mathcal{V}_\mathbb{F}(V)/V_s$ to $(V, \ker f)$. As above, one has the following.

\textbf{Proposition 6.3.1} For $s \in \mathbb{N}$, there is an adjunction

$$\text{Ind}_s: \mathbb{F}_g(D_s) \rightleftarrows \mathbb{F}_g(H^*V_s): \text{Res}_s,$$

where $\text{Ind}_s = g(D_s \rightarrow H^*V_s)^!$ and $\text{Res}_s = g(D_s \rightarrow H^*V_s)_!$. The functors $\text{Ind}_s$ and $\text{Res}_s$ are exact and $\text{Ind}_s$ commutes with tensor products.

The induction functor is given explicitly by $(\text{Ind}_s G)(V \rightarrow V_s) = G(V, \ker f)$, for $G \in \text{Ob} \mathbb{F}_g(D_s)$.

Recall from \textbf{Section 5.2} that there is an adjunction

$$l^g(D_s): \mathbb{F} \rightleftarrows \mathbb{F}_g(D_s): \Omega^g(D_s).$$

In the following, $F \in \text{Ob} \mathbb{F}$, $G \in \text{Ob} \mathbb{F}_g(D_s)$ and $H \in \text{Ob} \mathbb{F}_g(H^*V_s)$. Moreover, $I(V_s, 0)$ denotes the injective cogenerator of $\mathbb{F}_g(D_s)$ introduced in \textbf{Example 5.2.3}. 

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Proposition 6.3.2  For \( s \in \mathbb{N} \),

1. there are adjunctions

\[
\Psi_s : H(V) = H(V \oplus V_s \to V_s) \quad \text{and the functor } \Psi_s \text{ is exact and commutes with tensor products;}
\]

2. \( \text{Ind}_s \, \mathfrak{g}^{(D_s)} \) is naturally isomorphic to \( \mathfrak{g}(H^*V_s) \) and \( \Omega \mathfrak{g}^{(D_s)} \text{Res}_s \) is naturally isomorphic to \( \Omega \mathfrak{g}(H^*V_s) \);

3. the map \( \Psi_s \text{Ind}_s : \mathfrak{g}^{(D_s)} \to \mathfrak{F} \) is determined by \( \Psi_s \text{Ind}_s G(V) = G(V \oplus V_s, V) \); it is exact, commutes with tensor products and is left adjoint to the functor \( \text{Res}_s \mathfrak{g}(H^*V_s) : \mathfrak{F} \to \mathfrak{g}^{(D_s)} \);

4. \( \text{Res}_s \mathfrak{g}(H^*V_s) : \mathfrak{F} \to \mathfrak{g}^{(D_s)} \) is exact and is naturally equivalent to the functor

\[
F \mapsto \mathfrak{g}(D_s) F \otimes I(V_s,0).
\]

Moreover, there are natural isomorphisms

\[
/\text{Fix}_s \cong \Psi_s \mathfrak{I}_{H^*V_s} : H^*V_s \to \mathfrak{F},
\]

\[
/\text{Fix}_s (H^*V_s \otimes D_s -) \cong \Psi_s \text{Ind}_s \mathfrak{I}_{D_s} : D_s - \to \mathfrak{F}.
\]

**Proof**  The first statement follows from the natural isomorphism

\[
\text{Hom}_{\mathfrak{F} / V_s} ((A \to V_s), (V \oplus V_s \to V_s)) \cong \text{Hom}_{\mathfrak{F}} (A, V).
\]

The remaining numbered statements are straightforward and follow from [2, Définition et Proposition 3.3.4].

The functor \( /\text{Fix}_s \) is left adjoint to the functor

\[
F \mapsto H^*V_s \otimes rF \cong r(I_{V_s} \otimes F),
\]

for \( F \in \text{Ob} \mathfrak{F} \). The latter is isomorphic to \( \tau_{H^*V_s} \mathfrak{g}(H^*V_s) F \), since the underlying functor of \( \tau_{H^*V_s} \mathfrak{g}(H^*V_s) \) is \( r \Omega \mathfrak{g}(H^*V_s) \) and \( \Omega \mathfrak{g}(H^*V_s) F \cong I_{V_s} \otimes F \) (cf [2, Définition et Proposition 3.3.4]). Since \( \mathfrak{I}_{H^*V_s} \) is left adjoint to \( \tau_{H^*V_s} \) and \( \Psi_s \) is left adjoint to \( \mathfrak{g}(H^*V_s) \), it follows by unicity of adjoints that \( /\text{Fix}_s \) is equivalent to \( \Psi_s \mathfrak{I}_{H^*V_s} \).

The final identification follows from the natural isomorphism given by Theorem 5.2.8,

\[
/\mathfrak{I}_{H^*V_s} (H^*V_s \otimes D_s -) \cong \text{Ind}_s \mathfrak{I}_{D_s}.
\]

\( \Box \)
Example 6.3.3 Proposition 6.3.2(4) contains the isomorphism
\[ \text{Res}_{s}^{(H^{*}V_{s})} \cong I_{(V_{s},0)}. \]

The importance of \( I_{(V_{s},0)} \) is shown by the isomorphism \( \tau_{D_{s}} I_{(V_{s},0)} \cong \text{Restrict}^{H^{*}V_{s}}_{D_{s}} H^{*}V_{s} \) in \( D_{s} - \emptyset \), which follows from Theorem 5.2.8.

7 The Singer functors up to nilpotent unstable modules

This section introduces the functors \( \kappa_{s} : \mathcal{F} \rightarrow \mathcal{F}^{\emptyset}(D_{s}) \) which model the Singer functors up to nilpotent unstable modules. The proof that these correspond to the functors \( R_{s} \) is postponed until Section 8.

7.1 Avatars of the Singer functors

Some of the material of this section is available in [2] under a dual formulation using comodules over Boolean coalgebras.

Definition 7.1.1 For \( s \in \mathbb{N} \), let \( \kappa_{s} : \mathcal{F} \rightarrow \mathcal{F}^{\emptyset}(D_{s}) \) denote the functor defined on \( F \in \text{Ob} \mathcal{F} \) by
\[ (\kappa_{s} F)(V, W) := F(W). \]

Notation 7.1.2 For \( s \in \mathbb{N} \), \( V \in \text{Ob} \mathcal{F}^{f} \), let \( \text{Stab}(V, V \oplus V_{s}) \subset \text{Aut}(V \oplus V_{s}) \) denote the pointwise stabilizer of \( V \).

The following is clear.

Lemma 7.1.3 Let \( V, W \in \text{Ob} \mathcal{F}^{f} \) and \( f : V \rightarrow W \) be a linear morphism. Then

1. \( \text{Stab}(V, V \oplus V_{s}) \) is isomorphic to the semidirect product \( \text{Hom}(V_{s}, V) \rtimes \text{Aut}(V_{s}) \), where \( \text{Aut}(V_{s}) \) acts on the right on \( \text{Hom}(V_{s}, V) \) by precomposition;
2. the action of \( \text{Aut}(V \oplus V_{s}) \) on \( V \oplus V_{s} \) induces an action of \( \text{Stab}(V, V \oplus V_{s}) \) on \( (V \oplus V_{s}, V) \in \text{Ob} \mathcal{F}^{f}_{\emptyset}(D_{s}) \);
3. \( f : V \rightarrow W \) induces a group morphism \( \text{Stab}(V, V \oplus V_{s}) \rightarrow \text{Stab}(W, W \oplus V_{s}) \) which, with respect to the semidirect product decomposition, is induced by \( \text{Hom}(V_{s}, f) : \text{Hom}(V_{s}, V) \rightarrow \text{Hom}(V_{s}, W) \).

In the following statement, \( G \in \text{Ob} \mathcal{F}^{\emptyset}(D_{s}) \), \( V \in \text{Ob} \mathcal{F}^{f} \) and \( (V, U) \in \text{Ob} \mathcal{F}^{f}_{\emptyset}(D_{s}) \).
Proposition 7.1.4  Let $s$ be a natural number.

(1) The functor $\kappa_s: \mathcal{F} \to \mathcal{F}^{\bullet(D_s)}$ is exact and commutes with tensor products, limits and colimits.

(2) There is a natural monomorphism $\kappa_s \hookrightarrow I^{\mathcal{F}}(D_s)$ of functors from $\mathcal{F}$ to $\mathcal{F}^{\bullet(D_s)}$.

(3) The functor $\kappa_s$ is left adjoint to $R^0_s$ and admits a left adjoint $\alpha_s: \mathcal{F}^{\bullet(D_s)} \to \mathcal{F}$ given by $\alpha_s G(V) \cong G(V \oplus V_s, V)/\text{Stab}(V, V \oplus V_s)$.

(4) The adjunction counit $\alpha_s \kappa_s \to 1_\mathcal{F}$ is an isomorphism.

(5) The adjunction unit $1_\mathcal{F} \to R^0_s \kappa_s$ is an isomorphism.

Proof  The first statement is clear; for the second, the inclusion $U \subseteq V$ induces a natural morphism $F(U) \to F(V)$.

The fact that $R^0_s$ is right adjoint to $\kappa_s$ follows from the isomorphism

$$\text{Hom}_{\mathcal{F}^{\bullet(D_s)}}((V, V), (A, B)) \cong \text{Hom}_{\mathcal{F}}(V, B),$$

for $V \in \text{Ob} \mathcal{V}^g$ and $(A, B) \in \text{Ob} \mathcal{V}^g/_{\mathcal{V}^{\bullet(D_s)}}$.

The left adjoint $\alpha_s$ exists for formal reasons and $\alpha_s$ is a right exact functor which preserves projective objects. Lemma 7.1.3 implies that the association given by $V \mapsto G(V \oplus V_s, V)/\text{Stab}(V, V \oplus V_s)$ defines a right exact functor. Hence, since $\mathcal{F}^{\bullet(D_s)}$ has enough projectives (see Example 5.2.3), it suffices to check that this coincides with $\alpha_s$ on the full subcategory of projective objects in $\mathcal{F}^{\bullet(D_s)}$. Let $(A, B)$ and $V$ be as above, then there is a natural isomorphism

$$\text{Hom}_{\mathcal{V}^f}(A, B) \cong \text{Hom}_{\mathcal{V}/_{\mathcal{V}^{\bullet(D_s)}}}((A, B), (V \oplus V_s, V))/\text{Stab}(V, V \oplus V_s).$$

It follows that there is a natural isomorphism

$$\alpha_s P_{(A, B)}(V) \cong P_{(A, B)}(V \oplus V_s, V)/\text{Stab}(V, V \oplus V_s),$$

as required. The identification of the adjunction morphisms is clear.

Recall from Example 5.1.2 that, for $U \in \text{Ob} \mathcal{V}^f$, $I_U$ is the injective functor $\mathcal{F}^{\text{Hom}_{\mathcal{V}^f}(-, U)}$ of $\mathcal{F}$, which is contravariantly functorial in $U$.

The composite functor $\Omega^{\mathcal{V}^f(D_s)} \kappa_s: \mathcal{F} \to \mathcal{F}$ is of particular interest; the following result is required to relate the functor $\kappa_s$ to the Singer functor $R_s$.

Proposition 7.1.5  For $U \in \text{Ob} \mathcal{V}^f$, there is a natural isomorphism

$$\Omega^{\mathcal{V}^f(D_s)} \kappa_s I_U \cong I_{\text{Stab}(U, U \oplus V_s)}^{U \oplus V_s}.$$
The functor $\Omega^{g(D_s)}\kappa_s$ is right adjoint to the functor $a_s I^{g(D_s)}$, which identifies with the composite functor $F \mapsto (\Delta V_s F)/\text{Stab}(-, - \oplus V_s)$, by Proposition 7.1.4. The functor $\Delta V_s$ is left adjoint to the functor $- \otimes I_{V_s}$ (see Example 5.1.2), hence it follows that there is a natural isomorphism

$$\text{Hom}_F((\Delta V_s F)/\text{Stab}(-, - \oplus V_s), I_U) \cong \text{Hom}_F(F, (I_U \otimes I_{V_s})^{\text{Stab}(U, U \oplus V_s)})$$

and the group action is induced by the natural right action on $I_U \otimes I_{V_s} \cong I_{U \oplus V_s}$. The result follows.

We record the following, which is clear.

**Lemma 7.1.6** The composite functor $\Psi_s I_s \kappa_s$, considered as a functor $F \rightarrow F$, is naturally equivalent to the identity functor.

### 7.2 Copresentations of $\kappa_s$

In order to understand the underlying object of $\Omega^{g(D_s)}\kappa_s$, an alternative description is used; this is obtained by giving a copresentation of $\kappa_s$ via an equalizer diagram.

Recall that a diagram

$$
\begin{array}{ccc}
X & \overset{g}{\longrightarrow} & Y \\
\downarrow d & & \downarrow f \\
Z & \overset{h}{\longrightarrow} & V
\end{array}
$$

is a split equalizer if there exist morphisms $g: Y \rightarrow X$ and $h: Z \rightarrow Y$ such that $gd = 1_X$, $hf = 1_Y$ and $dg = he: Y \rightarrow Y$. A split equalizer is, in particular, an equalizer diagram.

**Notation 7.2.1** For $s \in \mathbb{N}$, let $\delta_s: \Psi_f^{g(D_s)} \Psi^f$ denote the functor $(V, U) \mapsto V \oplus V/U$.

**Lemma 7.2.2** For $(V, U) \in \text{Ob}^{g(D_s)} \Psi^f$, there is a natural equalizer diagram in $\Psi^f$

$$
\begin{array}{ccc}
U & \overset{1 \coprod 0}{\longrightarrow} & V \\
\downarrow q & & \downarrow q \\
V \oplus V/U & = \delta_s(V, U)
\end{array}
$$

where $q: V \rightarrow V/U$ is the quotient morphism. If $s: V/U \rightarrow V$ is a section of $q$, then the equalizer is split by the morphisms $1 \coprod -s: V \oplus V/U \rightarrow V$ and the induced projection $V \rightarrow U$.

In the following, the notation introduced in Example 5.2.3 for the injective cogenerators of $\Psi^{g(D_s)}$ is used.
Proposition 7.2.3  For $s \in \mathbb{N}$,

1. there is a monomorphism
   $$(\delta_s)^!(-) \hookrightarrow I(V_s, 0) \otimes \ell^g(D_s) \Delta V_s(-)$$
   of functors from $\mathcal{F}$ to $\mathcal{F}_g(D_s)$;

2. the equalizer of Lemma 7.2.2 induces an equalizer diagram of functors from $\mathcal{F}$ to $\mathcal{F}_g(D_s)$
   \[\kappa_s \longrightarrow \ell^g(D_s) \overrightarrow{\longrightarrow} (\delta_s)^!\]
   and hence an equalizer diagram
   \[\begin{array}{ccc}
   \kappa_s & \longrightarrow & \ell^g(D_s) \\
   & \overrightarrow{\longrightarrow} & (\delta_s)^!
   \end{array}\]

3. applying the functor $\Omega^g(D_s)$ to the equalizer (1) gives the equalizer diagram of functors from $\mathcal{F}$ to $\mathcal{F}_g(D_s) - \mathcal{F}$
   \[\Omega^g(D_s) \kappa_s \longrightarrow \mathcal{F}_g(D_s) \otimes (-) \overrightarrow{\longrightarrow} \sigma \tau I(V_s) \otimes \Delta V_s(-),\]
   where $\sigma, \tau$ are induced by the natural morphisms of $\mathcal{F}$
   \[\begin{array}{ccc}
   & \sigma'_{F} & \\
   F & \overleftarrow{\otimes} & I(V_s) \otimes \Delta V_s F,
   \end{array}\]
   where $\sigma'_{F}$ is the tensor product of the unit $\mathbb{F} \rightarrow I(V_s)$ with the natural inclusion $F \cong \Delta_0 F \hookrightarrow \Delta V_s F$ and $\tau'_{F}$ is the adjunction unit for $\Delta V_s \dashv (I(V_s) \otimes -)$.

Proof  The first statement can be established by an adjunction argument or be seen as follows. Recall from Example 5.2.3 there is an identification $I(V_s, 0)(V, U) \cong \mathcal{F}^{\text{Inj}}(V/U, V_s)$. Consider $F \in \text{Ob} \mathcal{F}$; the natural monomorphism
   $$(\delta_s)^!F(V, U) \rightarrow I(V_s, 0)(V, U) \otimes \ell^g(D_s) \Delta V_s F(V, U) \cong \mathcal{F}^{\text{Inj}}(V/U, V_s) \otimes F(V \oplus V_s)$$
   has component indexed by a monomorphism $V/U \leftrightarrow V_s$ given by the induced morphism $F(V \oplus V/U) \rightarrow F(V \oplus V_s)$.

The first diagram of the second statement is obtained by precomposition with the natural diagram of Lemma 7.2.2. Since limits are computed in $\mathcal{F}_g(D_s)$ pointwise, it suffices to show that, for $F \in \text{Ob} \mathcal{F}$ and $(V, U) \in \text{Ob} \mathcal{V}_g(D_s)$, the diagram in $\mathcal{V}$
   \[\begin{array}{ccc}
   F(U) & \rightarrow & F(V) \\
   & \overrightarrow{\longrightarrow} & F(V \oplus V/U)
   \end{array}\]
is a split equalizer; this follows from Lemma 7.2.2, since split equalizers are preserved by functors. Composing with the monomorphism of the first statement gives the second equalizer diagram.

The third statement follows by applying the exact functor $\Omega^{\varrho(D_s)}$ to the previous equalizer diagram, using [2, Définition et Proposition 3.3.4] to identify the functors. Namely, for $F \in \text{Ob} \mathcal{F}$, there is a natural isomorphism $\Omega^{\varrho(D_s)} \varrho(D_s) F \cong \mathbb{F}^{\varrho(D_s)} \otimes F$ in $\mathbb{F}^{\varrho(D_s)} \mathcal{F}$ and there are natural isomorphisms $\Omega^{\varrho(D_s)}(I_{V_s,0}) \otimes \varrho(D_s) \Delta V_s F \cong (\Omega^{\varrho(D_s)} I_{V_s,0} \otimes \varrho(D_s) \Delta V_s F) \cong (I_{V_s,0} \otimes \Delta V_s F) \cong I_{V_s} \otimes \Delta V_s F$, where the second isomorphism follows from $\Omega^{\varrho(D_s)} I_{V_s,0} \cong I_{V_s}$, which is a formal consequence of the adjunction $\varrho(D_s) \dashv \Omega^{\varrho(D_s)}$.

The identification of the natural transformations $\sigma, \tau$ follows by unravelling the definitions. \hfill \Box

**Remark 7.2.4** Lemma 7.1.6 shows that $\Psi_s \text{Ind}_s \kappa_s$ is naturally equivalent to the identity functor. It is instructive to see how this can be recovered from the copresentation of $\kappa_s$ given in Proposition 7.2.3; this is a guiding principle in the proof of Theorem 8.2.2.

The functor $\Psi_s \text{Ind}_s$ is exact and an explicit description is given in Proposition 6.3.2. Applying $\Psi_s \text{Ind}_s$ to the parallel arrows of (1), evaluated on $F \in \text{Ob} \mathcal{F}$, gives the diagram

$$ F(- \oplus \mathbb{F}_1^s) \xrightarrow{\mathbb{F} \text{Inj}(\mathbb{F}_1^s, \mathbb{F}_2^s)} F(- \oplus \mathbb{F}_1^s \oplus \mathbb{F}_2^s), $$

where the suffixes are used to distinguish the direct factors. Fixing an element $\alpha \in \text{Inj}(\mathbb{F}_1^s, \mathbb{F}_2^s) \cong \text{Aut}(\mathbb{F}^s)$, the associated components evaluated on $V \in \text{Ob} \mathcal{V}^f$ are

$$ (3) \quad F(V \oplus \mathbb{F}_1^s) \xrightarrow{F(1_V \| 1_{\mathbb{F}^s} \| 0)} F(V \oplus \mathbb{F}_1^s \oplus \mathbb{F}_2^s). $$

It is clear that

1. the natural inclusion $F(V) \hookrightarrow F(V \oplus \mathbb{F}_1^s)$ equalizes the parallel arrows,
2. for $\alpha = 1_{\mathbb{F}^s}$, the equalizer of (3) is $F(V)$,

where the second point is seen by applying Lemma 7.2.2 to $(V \oplus \mathbb{F}_1^s \oplus \mathbb{F}_2^s, V \oplus \mathbb{F}_1^s)$.

It follows formally that there is a natural isomorphism $\Psi_s \text{Ind}_s \kappa_s F(V) \cong F(V)$, as expected.
7.3 Compatibility

The functors $\kappa_s$ introduced above are related under restriction and induction, via the adjunctions $R_s: \mathcal{F}(D_s) \rightleftarrows \mathcal{F}(D_{s-1}) : \mathcal{P}_s$ of Proposition 6.1.1.

**Proposition 7.3.1**  Let $s$ be a positive integer.

1. There is a natural isomorphism $R_s t \theta(D_s) \to t \theta(D_{s-1})$ and the adjoint $t \theta(D_s) \to \mathcal{P}_s t \theta(D_{s-1})$ is a surjection with kernel $R_s \mathcal{P}_s t \theta(D_s)$.

2. There is a natural isomorphism $R_s \kappa_s \to \kappa_{s-1}$ and the adjoint $\kappa_s \to \mathcal{P}_s \kappa_{s-1}$ is a surjection with kernel $R_s \mathcal{P}_s \kappa_s$.

3. There is a commutative diagram of natural transformations

$$
\begin{array}{ccc}
R_s \mathcal{P}_s \kappa_s & \xrightarrow{c} & \kappa_s \\
\downarrow & & \downarrow \\
R_s \mathcal{P}_s t \theta(D_s) & \xrightarrow{c} & \mathcal{P}_s t \theta(D_{s-1})
\end{array}
$$

in which the rows are short exact sequences.

**Proof**  Straightforward. \(\Box\)

**Remark 7.3.2**  For $F \in \text{Ob} \mathcal{F}$, the short exact sequence $R_s \mathcal{P}_s \kappa_s F \to \kappa_s F \to \mathcal{P}_s \kappa_{s-1} F$ is the analogue of the short exact sequence $\omega_s R_s M \to R_s M \to \Phi R_{s-1} M$ from Proposition 4.1.3.

8 Deeper properties of the Singer functors

This section introduces an approximation $\tilde{R}_s$ to the functor $R_s$, by lifting the copresentation of $\kappa_s$ of Section 7.2 to the category $D_s-\mathcal{U}$. The functors $\tilde{R}_s$ and $R_s$ are shown to coincide on reduced unstable modules; in general, $\tilde{R}_s$ is the $\omega_s$–closure of $R_s$.

In Section 8.2, the composite of the functor $\text{Fix}_s$ with $H^* V_s \otimes_{D_s} R_s(-)$ is shown to be naturally equivalent to the identity. This is used to deduce that the functor $R_s$ defines a fully-faithful embedding of $\mathcal{U}$ in $D_s-\mathcal{U}$.
8.1 Lifting the functor $\Omega^{g(D_s)_{K_s}}$ to $\mathcal{U}$

The parallel arrows of diagram (2) of Proposition 7.2.3 lift to a natural diagram in $D_s\mathcal{U}$:

$$D_s \otimes M \xrightarrow{\sigma_M} H^*V_s \otimes TV_s M,$$

for $M \in \text{Ob} \mathcal{U}$, where

1. $\sigma_M$ is the tensor product of the inclusions $M \cong T_0 M \hookrightarrow TV_s M$ and $D_s \hookrightarrow H^*V_s$;
2. $\tau_M$ is the morphism of $D_s$–modules induced by the adjunction unit (in $\mathcal{U}$) $M \rightarrow H^*V_s \otimes TV_s M$.

Remark 8.1.1 The context should ensure that there is no ambiguity with the notation used in Section 7.2.

Definition 8.1.2 For $s \in \mathbb{N}$, let $\tilde{R}_s : \mathcal{U} \rightarrow D_s\mathcal{U}$ be the functor determined on an unstable module $M$ by

$$\tilde{R}_s M := \ker \left\{ D_s \otimes M \xrightarrow{\sigma_M} H^*V_s \otimes TV_s M \right\}.$$

Recall that an unstable module $N$ is locally finite if and only if the natural monomorphism $N \hookrightarrow TN$ is an isomorphism [19, Theorem 6.2.1].

Proposition 8.1.3  For $s \in \mathbb{N}$,

1. there is a natural monomorphism $\tilde{R}_s \hookrightarrow D_s \otimes -$;
2. $\tilde{R}_s : \mathcal{U} \rightarrow D_s\mathcal{U}$ is left exact and commutes with coproducts;
3. $\tilde{R}_s$, considered as a functor with values in $\mathcal{U}$, preserves the class of reduced (respectively nilclosed) unstable modules;
4. $\tilde{R}_s$ takes values in the class of $\omega_s$–closed objects of $D_s\mathcal{U}$;
5. for $M, X$ unstable modules, with $X$ locally finite, there is a natural isomorphism $\tilde{R}_s (M \otimes X) \cong (\tilde{R}_s M) \otimes X$; in particular, $\tilde{R}_s$ commutes with suspension.

Proof The first three statements are straightforward. The fact that $\tilde{R}_s$ takes values in the category of $\omega_s$–closed unstable $D_s$–modules follows from Proposition 3.1.4.

For the final statement, since $X$ is locally finite, the natural inclusion $X \hookrightarrow TV_s X$ is an isomorphism, hence there is a natural isomorphism $TV_s (M \otimes X) \cong (TV_s M) \otimes X$. It is straightforward to check that, via this isomorphism, there are identifications $\sigma_M \otimes X = \sigma_M \otimes 1_X$ and $\tau_M \otimes X = \tau_M \otimes 1_X$, which implies the result. \[\Box\]
**Proposition 8.1.4** For $s \in \mathbb{N}$, the restrictions of the functors $R_s$ and $\tilde{R}_s$ to the full subcategory of nilclosed unstable modules are naturally isomorphic.

**Proof** By Proposition 4.1.3, $R_s$ commutes with coproducts, preserves the class of nilpotent unstable modules and sends nilclosed unstable modules to nilclosed unstable modules; Proposition 8.1.3 establishes the analogous properties for $\tilde{R}_s$. It follows from Lemma A.2.2 that it is sufficient to show that the two functors coincide on the full subcategory of $\mathcal{U}$ with objects $\{H^*V_s|s \in \mathbb{N}\}$. Since there are natural monomorphisms $R_s \hookrightarrow D_s \otimes -$ and $\tilde{R}_s \hookrightarrow D_s \otimes -$, by composing with the natural monomorphism $i \otimes -: D_s \otimes - \hookrightarrow H^*V_s \otimes -$, it is sufficient to show that the images of $R_s H^*V$ and $\tilde{R}_s H^*V$ in $H^*V_s \otimes H^*V$ coincide, for every $V \in \text{Ob} \mathcal{U}$.

Lannes and Zarati prove that $R_s H^*V$ is isomorphic to $H^*(V \oplus V_s)^{\text{Stab}(V,V \oplus V_s)}$ in [11, Section 5.4.7.5]; Proposition 7.1.5 implies that this is isomorphic to $\tilde{R}_s H^*V$. The result follows.

**Theorem 8.1.5** For $s \in \mathbb{N}$, there is a natural monomorphism $\gamma_s: R_s \hookrightarrow \tilde{R}_s$ of functors $\mathcal{U} \rightarrow D_s-\mathcal{U}$ such that

1. $\gamma_s$ identifies $\tilde{R}_s$ as the $\omega_s$–closure of $R_s$;
2. the morphism $\gamma_s: R_s M \rightarrow \tilde{R}_s M$ is an isomorphism if $M$ is reduced.

**Proof** The construction of the natural monomorphism $\gamma_s$ generalizes the argument employed in the proof of Lemma A.2.2. Recall (cf [19, Section 3.11]) that the set of objects $H^*V_m \otimes J(n)$ (where $J(n)$ denotes the $n$–th Brown–Gitler module), indexed over nonnegative integers $m,n$, forms a set of injective cogenerators of $\mathcal{U}$ and that, since $\mathcal{U}$ is locally Noetherian, any unstable module $M$ admits a copresentation of the form $0 \rightarrow M \rightarrow I^0 \rightarrow I^1$, where each $I^j$ is a coproduct of objects of this form. Hence, writing $W$ for $V_m$, it suffices to show that there is a factorization

$$R_s(H^*W \otimes J(n)) \leftarrow \tilde{R}_s(H^*W \otimes J(n)) \rightarrow D_s \otimes H^*W \otimes J(n).$$

Now we have that $R_s(H^*W \otimes J(n)) \cong R_s H^*W \otimes D_s R_s J(n)$, $D_s \otimes H^*W \otimes J(n) \cong (D_s \otimes H^*W) \otimes D_s (D_s \otimes J(n))$ and the vertical inclusion is the tensor product over $D_s$ of $R_s H^*W \hookrightarrow D_s \otimes H^*W$ and $R_s J(n) \hookrightarrow D_s \otimes J(n)$; the latter is the $\omega_s$–closure of $R_s J(n)$, by Proposition 4.1.5.
Similarly, by Proposition 8.1.3, the horizontal inclusion identifies with the tensor product over $D_s$ of $\tilde{R}_s H^* W \hookrightarrow D_s \otimes H^* W$ and the isomorphism $\tilde{R}_s J(n) \rightarrow D_s \otimes J(n)$. Since the images of $R_s H^* W$ and $\tilde{R}_s H^* W$ in $D_s \otimes H^* W$ coincide, by Proposition 8.1.3, this provides the required factorization.

The cokernel of $R_s (H^* W \otimes J(n)) \hookrightarrow \tilde{R}_s (H^* W \otimes J(n))$ is $\omega_s$–torsion, by the above discussion. It follows that the cokernel of $R_s M \rightarrow \tilde{R}_s M$ is $\omega_s$–torsion, for any unstable module $M$; since, $\tilde{R}_s M$ is $\omega_s$–closed (by Proposition 8.1.3), this exhibits $\tilde{R}_s M$ as the $\omega_s$–closure of $R_s M$.

To prove the final statement, one can forget the $D_s$–module structure. The result follows from Lemma A.3.1, since $\gamma_s$ is an isomorphism on nilclosed unstable modules, by Proposition 8.1.4.

**Corollary 8.1.6** For $s \in \mathbb{N}$, there is a natural isomorphism $\mathfrak{l}_{D_s} R_s \cong \kappa_s \mathfrak{l}$ of functors from $\mathfrak{u}$ to $\mathfrak{F}^\theta(D_s)$.

**Proof** The functors $\mathfrak{l}_{D_s} R_s$ and $\kappa_s \mathfrak{l}$ are exact and send nilpotent unstable modules to zero. Hence, by Lemma A.2.1, it suffices to prove that the two functors coincide naturally on the full subcategory of nilclosed unstable modules. On this subcategory, $\gamma_s : R_s \hookrightarrow \tilde{R}_s$ is a natural isomorphism, by Theorem 8.1.5, hence it suffices to prove that there is a natural isomorphism $\mathfrak{l}_{D_s} \tilde{R}_s \cong \kappa_s \mathfrak{l}$. This is by construction: applying the functor $\mathfrak{l}_{D_s}$ to the equalizer diagram defining $\tilde{R}_s$ gives the copresentation of $\Omega^\theta(D_s) \kappa_s$ given in Proposition 7.2.3.

**8.2 The composite of $R_s$ and Fix$_s (H^* V_s \otimes_{D_s} –)$**

Under the correspondence between $D_s – \mathfrak{u}/\mathfrak{N}il$ and the category $\mathfrak{F}^\theta(D_s)$ given by Theorem 5.2.8, the Singer functor corresponds to the functor $\kappa_s$ (by Corollary 8.1.6) and the functor $\Psi_s \text{Ind}_s$ corresponds to the functor $\text{Fix}_s (H^* V_s \otimes_{D_s} –)$ (by Proposition 6.3.2). Lemma 7.1.6 states that the composite $\Psi_s \text{Ind}_s \kappa_s$ is isomorphic to the identity functor; the purpose of this section is to establish the corresponding result at the level of unstable modules.

Recall that $i : D_s \hookrightarrow H^* V_s$ denotes the canonical inclusion and that Proposition 2.3.2 implies that $T(V_s, i)$ induces a functor $D_s – \mathfrak{u} \rightarrow H^* V_s – \mathfrak{u}$. The following result is the key input.

**Lemma 8.2.1** For $s \in \mathbb{N}$ and $M \in \text{Ob} D_s – \mathfrak{u}$,

$$\text{Fix}_s (H^* V_s \otimes_{D_s} M) \cong \mathbb{F} \otimes_{H^* V_s} T(V_s, i) M.$$  

In particular, there is an isomorphism of unstable algebras

$$\text{Fix}_s (H^* V_s \otimes_{D_s} H^* V_s) \cong \mathbb{F}^{\text{Aut} V_s}.$$  

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Proof. By [10, 4.4.3], $\text{Fix}_s X \cong \mathbb{F} \otimes_{T_{V_s}} H^*V_s \quad T_{V_s} X$, for $X \in \text{Ob} \ H^*V_s - \mathcal{U}$. The $T$–functor commutes with tensor products, so, taking $X = H^*V_s \otimes D_s \ M$, there are natural isomorphisms

$$\text{Fix}_s(H^*V_s \otimes D_s \ M) \cong \mathbb{F} \otimes_{T_{V_s}} D_s \quad T_{V_s} \ M \cong \mathbb{F} \otimes_{T_{(V_s,i)}} D_s \quad T_{(V_s,i)} \ M \cong \mathbb{F} \otimes H^*V_s \quad T_{(V_s,i)} \ M.$$

For the case $M = H^*V_s$, one verifies that $T_{(V_s,i)} H^*V_s$ identifies with $\mathbb{F}^{\text{Aut}(V_s)} \otimes H^*V_s$ as an $H^*V_s$–algebra, from which the result follows.

\[\square\]

**Theorem 8.2.2** For $s \in \mathbb{N}$, the natural transformation

$$\text{Fix}_s(H^*V_s \otimes D_s \quad R_s(-)) \to 1_{\mathcal{U}}$$

of functors on $\mathcal{U}$, which is adjoint to the canonical inclusion $R_s(-) \hookrightarrow D_s \otimes (-) \hookrightarrow H^*V_s \otimes (-)$, is an isomorphism.

Moreover, the natural monomorphism $R_s(-) \hookrightarrow D_s \otimes -$ induces the canonical inclusion

$$\text{Fix}_s(H^*V_s \otimes D_s \quad R_s(-)) \cong 1_{\mathcal{U}} \hookrightarrow T_{V_s}(-) \cong \text{Fix}_s(H^*V_s \otimes D_s \quad (D_s \otimes -)).$$

Proof. The natural monomorphism $\gamma_s: R_s \hookrightarrow \tilde{R}_s$ is an isomorphism up to $\omega_s$–torsion, by Theorem 8.1.5, hence it suffices to prove the result with $\tilde{R}_s$ in place of $R_s$, since $\text{Fix}_s$ annihilates $\omega_s$–torsion, by [12, Proposition 0.8]. The defining equalizer diagram for $\tilde{R}_s M$ gives rise to an equalizer diagram in $H^*V_s - \mathcal{U}$:

$$H^*V_s \otimes D_s \quad \tilde{R}_s \ M \longrightarrow H^*V_s \otimes M \longrightarrow H^*V_s \otimes D_s \quad H^*V_s \otimes T_{V_s} \ M,$$

since $H^*V_s \otimes D_s \ (-)$ is exact.

The functor $\text{Fix}_s$ is exact, hence this gives the equalizer diagram in $\mathcal{U}$:

$$\text{Fix}_s(H^*V_s \otimes D_s \quad \tilde{R}_s \ M) \longrightarrow \text{Fix}_s(H^*V_s \otimes M) \longrightarrow \text{Fix}_s(H^*V_s \otimes D_s \quad H^*V_s \otimes T_{V_s} \ M).$$

There are natural isomorphisms $\text{Fix}_s(H^*V_s \otimes M) \cong T_{V_s} \ M$ and

$$\text{Fix}_s(H^*V_s \otimes D_s \quad H^*V_s \otimes T_{V_s} \ M) \cong \mathbb{F}^{\text{Aut}(V_s)} \otimes T_{V_s} T_{V_s} \ M,$$

obtained by viewing $H^*V_s \otimes D_s \quad H^*V_s \otimes T_{V_s} \ M$ as the tensor product over $H^*V_s$ of $H^*V_s \otimes D_s \quad H^*V_s$ and $H^*V_s \otimes T_{V_s} \ M$ and applying Lemma 8.2.1.

The equalizer diagram therefore identifies with

$$\text{Fix}_s(H^*V_s \otimes D_s \quad \tilde{R}_s \ M) \longrightarrow T_{V_s} \ M \xrightarrow{\tilde{\sigma}_M} \mathbb{F}^{\text{Aut}(V_s)} \otimes T_{V_s} T_{V_s} \ M,$$

where $\tilde{\sigma}_M : = \text{Fix}_s(H^*V_s \otimes D_s \quad \sigma_M)$ and $\tilde{\tau}_M : = \text{Fix}_s(H^*V_s \otimes D_s \quad \tau_M)$ are identified below.
As in Remark 7.2.4, the result is a formal consequence of the following two points:

1. The natural morphism \( M \hookrightarrow T_{V_s} M \) equalizes the morphisms \( \tilde{\sigma}_M \) and \( \tilde{\tau}_M \);
2. \( M \hookrightarrow T_{V_s} M \) is the equalizer of the diagram of unstable modules

\[
T_{V_s} M \rightrightarrows T_{V_s} T_{V_s} M,
\]

which is obtained from \( \tilde{\sigma}_M \), \( \tilde{\tau}_M \) by composing with the surjection \( \mathbb{F}^{\text{Aut}(V_s)} \otimes T_{V_s} T_{V_s} M \to T_{V_s} T_{V_s} M \) induced by the augmentation \( \mathbb{F}^{\text{Aut}(V_s)} \to \mathbb{F} \).

The identification of the morphisms \( \tilde{\sigma}_M \) and \( \tilde{\tau}_M \) is a standard calculation with the \( T \)–functor; the precise form depends on the conventions used in the isomorphism \( \text{Fix}_s(H^*V_s \otimes D_s H^*V_s) \cong \mathbb{F}^{\text{Aut}(V_s)} \) of Lemma 8.2.1. The appropriate form can be deduced from the nillocalized case, as in Section 7.2, which leads to the following identifications.

For an automorphism \( \alpha \in \text{Aut}(V_s) \), the components

\[
T_{V_s} M \xrightarrow{\tilde{\sigma}_M^\alpha} T_{V_s} T_{V_s} M \cong T_{V_s} \oplus V_s M,
\]

of \( \tilde{\sigma}_M \) and \( \tilde{\tau}_M \) indexed by \( \alpha \) are induced by naturality of the \( T \)–functor by

\[
V_s \xrightarrow{1_V} \xrightarrow{\alpha} V_s \oplus V_s.
\]

The two key points are established as in Remark 7.2.4: that \( M \) lies in the equalizer follows since \( \tilde{\sigma}_M \), \( \tilde{\tau}_M \) are derived from the naturality with respect to \( V \) of \( T \); the second point follows by observing that the diagram

\[
M \longrightarrow T_{V_s} M \xrightarrow{\|} T_{V_s} \oplus V_s M
\]

is a split equalizer in unstable modules, by applying Lemma 7.2.2, where the morphisms \( T_{V_s} M \rightrightarrows T_{V_s} \oplus V_s M \) are induced respectively by \( 1_V \| 0 \): \( V_s \to V_s \oplus V_s \) and the diagonal \( \Delta : V_s \to V_s \oplus V_s \).

The final statement has been established in the course of the proof. \( \Box \)

### 8.3 The Singer functor is a fully-faithful embedding

Proposition 7.1.4 shows that the unit \( 1_{\mathbb{F}} \to \mathcal{R}_s^0 \kappa_s \) of the adjunction \( \kappa_s \dashv \mathcal{R}_s^0 \) is a natural isomorphism. This section shows that Theorem 8.2.2 implies the analogous statement for the adjunction \( R_s \dashv \mathfrak{U} \); in particular, the functor \( R_s : \mathfrak{U} \to D_s - \mathfrak{U} \) is rigid, considered as a functor to unstable \( D_s \)–modules.

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Theorem 8.3.1 For $s \in \mathbb{N}$, the adjunction unit $1_{\mathcal{U}} \to \mathcal{Z}_s R_s$ is a natural isomorphism and the natural inclusions $R_s \hookrightarrow D_s \otimes (-) \hookrightarrow H^*(V_s) \otimes (-)$ in $D_s - \mathcal{U}$ induce isomorphisms

$$1_{\mathcal{U}} \cong \mathcal{Z}_s R_s \cong \mathcal{Z}_s (D_s \otimes -) \cong \mathcal{Z}_s (H^* V_s \otimes -).$$

In particular, $R_s$ induces a fully-faithful embedding $R_s : \mathcal{U} \hookrightarrow D_s - \mathcal{U}$.

Proof For the first statement, it suffices to prove that, for $M, N \in \text{Ob} \mathcal{U}$, the functor $R_s$ induces a natural isomorphism:

$$\text{Hom}_{\mathcal{U}}(M, N) \to \text{Hom}_{D_s - \mathcal{U}}(R_s M, R_s N).$$

This is a monomorphism by Lemma 4.1.4; composing with the natural inclusion $R_s N \hookrightarrow D_s \otimes N \hookrightarrow H^* V_s \otimes N$, there is a natural monomorphism

$$\text{Hom}_{D_s - \mathcal{U}}(R_s M, R_s N) \hookrightarrow \text{Hom}_{D_s - \mathcal{U}}(R_s M, H^* V_s \otimes N) \cong \text{Hom}_{H^* V_s - \mathcal{U}}(H^* V_s \otimes D_s R_s M, H^* V_s \otimes N).$$

By adjunction,

$$\text{Hom}_{H^* V_s - \mathcal{U}}(H^* V_s \otimes D_s R_s M, H^* V_s \otimes N) \cong \text{Hom}_{\mathcal{U}}(\text{Fix}_s(H^* V_s \otimes D_s R_s M), N),$$

and, by Theorem 8.2.2, $\text{Fix}_s(H^* V_s \otimes D_s R_s M) \cong M$. Thus, there are natural monomorphisms

$$\text{Hom}_{D_s - \mathcal{U}}(R_s M, R_s N) \hookrightarrow \text{Hom}_{D_s - \mathcal{U}}(R_s M, D_s \otimes N) \hookrightarrow \text{Hom}_{D_s - \mathcal{U}}(R_s M, H^* V_s \otimes N) \cong \text{Hom}_{\mathcal{U}}(M, N).$$

The composite with the natural inclusion $\text{Hom}_{\mathcal{U}}(M, N) \to \text{Hom}_{D_s - \mathcal{U}}(R_s M, R_s N)$ is the identity, which establishes the natural isomorphisms.

The property of the adjunction unit is a formal consequence. \qed

9 The functors $\mathcal{Z}_s$, $Q_s$ and $\text{Fix}_s$

The purpose of this section is to provide a better understanding of the right adjoint $\mathcal{Z}_s$ to the Singer functor $R_s$, in particular its relationship with the indecomposables functor $Q_s$ and with the functor $\text{Fix}_s(H^* V_s \otimes D_s -)$. 
9.1 The Singer functor \( R_s \) and the indecomposables \( Q_s \)

In [3, Section 3] and [4, Section 3], Dwyer and Wilkerson studied a linear operation constructed from the Steenrod total power \( \text{St}_1 \). This is related to the natural transformation defined below (defined for arbitrary \( s \)), where \( \text{forget}_s: D_s \to \mathcal{U} \) is the forgetful functor.

**Definition 9.1.1** For \( s \in \mathbb{N} \), let \( \varepsilon_s: R_s \text{forget}_s \to 1_{D_s \to \mathcal{U}} \) be the natural transformation defined on \( M \in \text{Ob} D_s \to \mathcal{U} \) as the composite

\[
R_s M \hookrightarrow D_s \otimes M \rightarrow M
\]

of the canonical inclusion followed by the product.

**Proposition 9.1.2** The natural transformation \( \text{forget}_s \to Q_s \) of functors from \( D_s \to \mathcal{U} \) to \( \mathcal{U} \) induces a factorization

\[
R_s \text{forget}_s \to R_s Q_s \to 1_{D_s \to \mathcal{U}}
\]

of endofunctors of \( D_s \to \mathcal{U} \).

**Proof** The proof proceeds by reduction to the behaviour on \( D_s \). For \( M \in \text{Ob} D_s \to \mathcal{U} \), there is an exact sequence of unstable modules

\[
\overline{D}_s \otimes M \to M \to Q_s M \to 0,
\]

where the first morphism is induced by multiplication. The functor \( R_s \) is exact, hence, by naturality of \( \varepsilon_s \), it suffices to prove that the composite morphism

\[
R_s(\overline{D}_s \otimes M) \xrightarrow{\varepsilon_s} \overline{D}_s \otimes M \to M
\]

is zero.

There is a natural isomorphism \( R_s(\overline{D}_s \otimes M) \cong R_s(\overline{D}_s) \otimes_{D_s} R_s M \), by the monoidal property of \( R_s \) (see Proposition 4.1.3) and, with respect to this, the above composite is induced by the tensor product over \( D_s \) of \( \varepsilon_s: R_s \overline{D}_s \to \overline{D}_s \) and \( \varepsilon_s: R_s M \to M \). Therefore, to prove the result, it is sufficient to show that the morphism \( \varepsilon_s: R_s \overline{D}_s \to \overline{D}_s \) is trivial.

This can be proved directly, generalizing [3, Lemma 3.3(ii)], by reducing to the case \( s = 1 \), using the fact [12] that \( \text{St}_s \) is the \( s \)-fold iterate of \( \text{St}_1 \).

An alternative method is to use passage to nillocalization. Since \( R_s \) preserves reduced objects and \( \overline{D}_s \) is reduced, it is sufficient to prove that the induced morphism
\( l_{D^s}R_s(D_s) \rightarrow l_{D^s}D_s \) is trivial. The natural transformation \( \varepsilon_s \) corresponds to the natural transformation

\[
\kappa_s \omega_{\mathfrak{g}(D^s)} \rightarrow 1_{\mathfrak{g}(D_s)}
\]

of endofunctors of \( \mathfrak{g}(D^s) \) given by the composite

\[
\kappa_s \omega_{\mathfrak{g}(D^s)} \hookrightarrow \iota_{\mathfrak{g}(D^s)} \omega_{\mathfrak{g}(D^s)} \rightarrow 1_{\mathfrak{g}(D^s)}
\]

induced by the inclusion \( \kappa_s \hookrightarrow \iota_{\mathfrak{g}(D^s)} \) and the counit of the \( \iota_{\mathfrak{g}(D^s)} \dashv \omega_{\mathfrak{g}(D^s)} \) adjunction. For \( G \in \text{Ob} \mathfrak{g}(D^s) \) and \( (V, W) \in \text{Ob} \mathcal{V} /_{\mathfrak{g}(D^s)} \), by using the explicit form of the \( \iota_{\mathfrak{g}(D^s)} \dashv \omega_{\mathfrak{g}(D^s)} \) adjunction counit, this identifies as the surjection

\[
\bigoplus_{\text{codim} U \leq s} G(W, U) \twoheadrightarrow G(V, W)
\]

given by projection onto the summand indexed by \( U = W \) followed by the morphism \( G(W, W) \rightarrow G(V, W) \) induced by \( (W, W) \rightarrow (V, W) \) in \( \mathcal{V} /_{\mathfrak{g}(D^s)} \).

Now, \( l_{D^s}D_s \) is the constant functor \( \mathbb{F} \in \mathfrak{g}(D^s) \) and the augmentation ideal gives the subfunctor \( l_{D^s}D_s \):

\[
(V, W) \mapsto \begin{cases} 
\mathbb{F} & V \neq W, \\
0 & V = W.
\end{cases}
\]

The result follows. \( \square \)

The following corollary is formal.

**Corollary 9.1.3** For \( s \in \mathbb{N} \), the natural transformation \( \zeta_s : Q_s \rightarrow \mathfrak{3}_s \) adjoint to \( \varepsilon_s : R_s Q_s \rightarrow 1_{D^s-\mathfrak{g}} \), fits into a commutative diagram:

\[
\text{forget}_s \xrightarrow{Q_s} \mathfrak{3}_s \xrightarrow{\xi_s} \tilde{3}_s,
\]

where \( \tilde{3}_s \) is adjoint to \( \varepsilon_s : R_s \text{forget}_s \rightarrow 1_{D^s-\mathfrak{g}} \).

The following natural transformation is used in **Theorem 9.2.2**.

**Lemma 9.1.4** For \( s \in \mathbb{N} \), the functor \( Q_s \) applied to the adjunction counit \( R_s \mathfrak{3}_s \rightarrow 1_{D^s-\mathfrak{g}} \) induces a natural transformation \( \xi_s : \Phi^s \mathfrak{3}_s \rightarrow Q_s \).

**Proof** This is an immediate consequence of the natural isomorphism \( Q_s R_s \cong \Phi^s \) of **Proposition 4.1.3**. \( \square \)
9.2 The relationship between $Q_s$ and $Z_s$

In order to understand the relationship between $Q_s$ and $Z_s$, further information on the behaviour of $Z_s$ is required. Recall that the category $D_s\text{-}\mathfrak{u}$ has enough injectives [12; 7; 15].

**Lemma 9.2.1** If $I \in \text{Ob } D_s\text{-}\mathfrak{u}$ is injective, then the natural morphism $\zeta_s: Q_s I \to Z_s I$ is surjective.

**Proof** It suffices to show that the morphism $\text{forget}_s I \to Z_s I$ of Corollary 9.1.3 is surjective. For $k \in \mathbb{N}$, there is a canonical embedding $R_s F(k) \hookrightarrow D_s \otimes F(k)$ in $D_s\text{-}\mathfrak{u}$, and hence, by injectivity of $I$, a surjection

$$\text{Hom}_{D_s\text{-}\mathfrak{u}}(D_s \otimes F(k), I) \twoheadrightarrow \text{Hom}_{D_s\text{-}\mathfrak{u}}(R_s F(k), I).$$

This corresponds to the degree $k$ part of the morphism $\text{forget}_s I \to Z_s I$, which is therefore surjective. \hfill \Box

Recall that $\lambda: \Phi \to 1_\mathfrak{u}$ is the natural transformation induced by $\text{Sq}_0$, which induces $\lambda^s: \Phi^s \to 1_\mathfrak{u}$ by iteration, for $s \in \mathbb{N}$. The natural transformation $\xi_s$ was introduced in Lemma 9.1.4.

**Theorem 9.2.2** For $s \in \mathbb{N}$,

1. the composite $\Phi^s Q_s \xrightarrow{\Phi^s \xi_s} \Phi^s Z_s \xrightarrow{\xi_s} Q_s$ is the natural transformation $\lambda^s_{Q_s}$;
2. the composite $\Phi^s Z_s \xrightarrow{\xi_s} Q_s \xrightarrow{\xi_s} Z_s$ is the natural transformation $\lambda^s_{Z_s}$.

In particular, $\zeta_s: Q_s \to Z_s$ and $\xi_s: \Phi^s Z_s \to Q_s$ are isomorphisms up to nilpotent unstable modules and the functor $Z_s$ sends nilpotents to nilpotents.

**Proof** The first natural transformation is given by applying the functor $Q_s$ to the composite $R_s Q_s \to R_s Z_s \to 1_{D_s\text{-}\mathfrak{u}}$. The identification follows from the fact that, modulo decomposables, $S_{t_s}$ identifies with the linear map $(\text{Sq}_0)^s$.

The functor $Z_s$ is left exact and $D_s\text{-}\mathfrak{u}$ has enough injectives, hence it suffices to show that the natural transformation identifies with $\lambda^s_{Z_s}$ when evaluated on any injective object $I$.

Consider the composite

$$\Phi^s Q_s I \xrightarrow{\Phi^s \xi_s} \Phi^s Z_s I \xrightarrow{\xi_s} Q_s I \xrightarrow{\xi_s} Z_s I.$$
Here the surjection \( \Phi^s \xi_s \) is given by Lemma 9.2.1, since the functor \( \Phi^s \) is exact, and the identification of \( \lambda^s_{Q_s} \) follows from the first part of the theorem.

To prove the result, by surjectivity of \( \Phi^s \xi_s \), it suffices to show that the composites \( \xi_s(\Phi^s \xi_s) \) and \( \lambda^s_{3s}(\Phi^s \xi_s) \) coincide evaluated on \( I \). This follows from the naturality of \( \lambda^s \), which gives the commutative diagram of natural transformations

\[
\begin{array}{ccc}
\Phi^s Q_s & \xrightarrow{\Phi^s \xi_s} & \Phi^s 3_s \\
\downarrow{\lambda^s_{Q_s}} & & \downarrow{\lambda^s_{3s}} \\
Q_s & \xrightarrow{\xi_s} & 3_s.
\end{array}
\]

The final statements are immediate consequences of these identifications, since the functor \( Q_s \) sends objects with nilpotent underlying unstable module to nilpotent unstable modules. \( \square \)

In the following, \( R^0_s \) is the functor of Proposition 6.2.1.

**Corollary 9.2.3** For \( s \in \mathbb{N} \),

1. there are natural isomorphisms
   
   (a) \( 3_s r D_s \cong r R^0_s : \mathcal{F}^0(D_s) \to \mathcal{U} \),
   
   (b) \( l3_s \cong R^0_s l D_s : D_s \to \mathcal{U} \to \mathcal{F} \),

   and, in particular, the functor \( l3_s \) is exact;

2. for \( N \in \text{Ob} D_s - \mathcal{U} \) which is reduced, \( 3_s N = 0 \) if and only if \( Q_s N \) is nilpotent.

**Proof** The first isomorphism is a formal consequence of Corollary 8.1.6. Namely, \( 3_s r D_s \) is right adjoint to \( l D_s R_s \), which is naturally equivalent to \( \kappa_s l \), by Corollary 8.1.6. The latter is left adjoint to \( r R^0_s \), by Proposition 7.1.4.

Precomposing with \( l D_s \) and postcomposing with \( l \) gives a natural isomorphism \( l3_s r D_s l D_s \cong R^0_s l D_s \). The natural transformation \( 1_{D_s - \mathcal{U}} \to r D_s l D_s \) is an isomorphism modulo nilpotent objects, hence, by Theorem 9.2.2, the induced natural transformation \( l3_s \to l3_s r D_s l D_s \) is an isomorphism; this gives the second natural isomorphism.

Suppose now that \( N \in \text{Ob} D_s - \mathcal{U} \) is a reduced object; hence, by Proposition 4.1.6, \( 3_s N \) is reduced. Thus, \( 3_s N = 0 \) if and only if \( l3_s N = 0 \), which is equivalent to \( lQ_s N = 0 \), by Theorem 9.2.2. \( \square \)

**Corollary 9.2.4** If \( M \in \text{Ob} D_s - \mathcal{U} \) is a reduced object, then \( 3_s (\omega_s M) = 0 \).
Proof The $D_s$–module $\omega_s M$ is a submodule of $M$, hence is reduced, so it suffices to show that $Q_s(\omega_s M)$ is nilpotent. The short exact sequence

$$0 \to \omega_s M \to M \to M/\omega_s M \to 0$$

in $D_s-\mathcal{U}$ induces an exact sequence

$$Q_s(\omega_s M) \to Q_s M \to Q_s(M/\omega_s M) \to 0,$$

which is short exact modulo nilpotent unstable modules, since by Proposition 6.2.1 $lQ_s$ is exact. The surjection $Q_s M \to Q_s(M/\omega_s M)$ is an isomorphism, so the result follows.

Example 9.2.5 For $s \in \mathbb{N}$, $N \in \text{Ob} \mathcal{U}$ and $M \in \text{Ob} D_s-\mathcal{U}$ which is reduced,

$$\text{Hom}_{D_s-\mathcal{U}}(R_s N, \omega_s M) = 0.$$

For example, take $M = D_s$.

9.3 The relationship between $\mathcal{Z}_s$ and $\text{Fix}_s(H^*V_s \otimes D_s -)$

It is interesting to have a criterion for the counit $R_s \mathcal{Z}_s M \to M$ (for $M \in \text{Ob} D_s-\mathcal{U}$) to be an isomorphism. By Theorem 8.3.1, if $M \cong R_s N$ for some $N \in \text{Ob} \mathcal{U}$, then $N \cong \mathcal{Z}_s M$.

Proposition 9.3.1 For $s \in \mathbb{N}$, there is a natural transformation

$$\mathcal{Z}_s \to \text{Fix}_s(H^*V_s \otimes D_s (-))$$

of functors from $D_s-\mathcal{U}$ to $\mathcal{U}$.

Moreover, if $M \in \text{Ob} D_s-\mathcal{U}$ such that the counit $R_s \mathcal{Z}_s M \to M$ is an isomorphism, then

$$\mathcal{Z}_s M \to \text{Fix}_s(H^*V_s \otimes D_s M)$$

is an isomorphism.

Proof The natural transformation is given by applying the functor $\text{Fix}_s(H^*V_s \otimes D_s (-))$ to the counit $R_s \mathcal{Z}_s \to 1_{D_s-\mathcal{U}}$, using the isomorphism of Theorem 8.2.2. If the counit is an isomorphism, then so is the induced natural morphism.

Remark 9.3.2 (1) Composition with the natural transformation $\text{forget}_s \to \mathcal{Z}_s$ of Corollary 9.1.3 induces a natural morphism

$$\text{forget}_s \to \text{Fix}_s(H^*V_s \otimes D_s (-)).$$

This is induced by the natural transformation of Proposition 2.4.1.
The natural transformation \( \mathcal{Z}_s \to \text{Fix}_s(H^*V_s \otimes_{D_s} (-)) \) corresponds to the natural transformation \( \mathcal{R}_s^0 \to \Psi_s \text{Ind}_s \) of functors from \( \mathcal{F}_0(D_s) \) to \( \mathcal{F} \) which is given on \( G \in \text{Ob} \mathcal{F}_0(D_s) \) by \( \mathcal{R}_s^0 G(V) = G(V, V) \to \Psi_s \text{Ind}_s G(V) = G(V \oplus V_s, V) \), induced by \( (V, V) \to (V \oplus V_s, V) \) (cf Propositions 6.3.2 and 6.2.1).

For \( s > 1 \), it is straightforward to see that \( \mathcal{Z}_s M \to \text{Fix}_s(H^*V_s \otimes_{D_s} M) \) being an isomorphism does not imply in general that the counit is an isomorphism. However, in the case \( s = 1 \), one has the following.

**Theorem 9.3.3** For \( M \in \text{Ob} D_1-\mathcal{U} \) such that the underlying unstable module is reduced, the following conditions are equivalent:

1. the counit \( R_1 \mathcal{Z}_1 M \to M \) is an isomorphism;
2. the natural morphism \( \mathcal{Z}_1 M \to \text{Fix}_1 M \) is an isomorphism.

**Proof** Proposition 9.3.1 gives \( (1) \Rightarrow (2) \).

For the converse, consider the exact sequence in \( D_1-\mathcal{U} \)

\[
0 \to \text{Ker} \to R_1 \mathcal{Z}_1 M \to M \to \text{Coker} \to 0.
\]

The hypothesis (2) implies that \( \text{Fix}_1(R_1 \mathcal{Z}_1 M \to M) \) is an isomorphism. Thus, by [12, Proposition 0.8], both \( \text{Ker} \) and \( \text{Coker} \) are \( \omega_1 \)-torsion. However, by construction, \( \text{Ker} \) is a subobject of \( R_1 \mathcal{Z}_1 M \), which is \( \omega_1 \)-torsion free, hence \( \text{Ker} = 0 \) and \( R_1 \mathcal{Z}_1 M \to M \) is a monomorphism with \( \omega_1 \)-torsion cokernel.

By hypothesis, \( M \) is reduced, hence \( \mathcal{Z}_1 M \cong \text{Fix}_1 M \) is reduced. **Theorem 8.1.5** implies that \( R_1 \mathcal{Z}_1 M \) is \( \omega_1 \)-closed. Hence, to complete the proof, it suffices to show that \( M \) is \( \omega_1 \)-torsion free. Consider the submodule \( A := \text{Ann}_{\omega_1} M \subset M \), so that \( A \) is in the image of \( \text{triv}_1: \mathcal{U} \to D_1-\mathcal{U} \). The module \( M \) is \( \omega_1 \)-torsion free if and only if \( A = 0 \).

Applying the functor \( \mathcal{Z}_1 \) yields a monomorphism

\[
\tilde{\Phi} A \cong \mathcal{Z}_1 A \hookrightarrow \mathcal{Z}_1 M \cong \text{Fix}_1 M,
\]

where the first isomorphism is given by **Proposition 4.1.8**. By naturality of \( \mathcal{Z}_1 \to \text{Fix}_1 \), this factors across \( \text{Fix}_1 A \), which is trivial (since \( A \) is \( \omega_1 \)-torsion). Thus \( \tilde{\Phi} A = 0 \). However, \( A \) is a reduced unstable module, since it is a submodule of \( M \), hence \( A \) must be zero, as required. \( \square \)
Appendix A  General results

A.1 The right adjoint to \( \Phi \) on categories of modules

**Proposition A.1.1**  For \( K \) an unstable algebra, \( \tilde{\Phi} \) induces a functor \( \tilde{\Phi} : \Phi K-\mathcal{U} \to K-\mathcal{U} \) which is right adjoint to \( \Phi : K-\mathcal{U} \to \Phi K-\mathcal{U} \).

**Proof**  The functor \( \Phi \) commutes with tensor products, hence the adjunction counit \( \Phi \tilde{\Phi} \to 1_{\mathcal{U}} \) induces a natural morphism

\[ \tilde{\Phi} M \otimes \tilde{\Phi} N \to \tilde{\Phi}(M \otimes N), \]

for \( M, N \in \text{Ob}\,\mathcal{U} \). Thus, if \( M \in \text{Ob}\,\Phi K-\mathcal{U} \), \( \tilde{\Phi} M \) is an object of \( K-\mathcal{U} \) with respect to the structure morphism:

\[ K \otimes \tilde{\Phi} M \cong \tilde{\Phi} \Phi K \otimes \tilde{\Phi} M \to \tilde{\Phi}(\Phi K \otimes M) \to \tilde{\Phi} M, \]

where the last morphism is induced by the structure morphism of \( M \). (By construction, this is a morphism of \( \mathcal{U} \); the associativity and unit axioms are straightforward verifications.)

By definition, for \( N \in \text{Ob}\,K-\mathcal{U} \), \( \text{Hom}_{K-\mathcal{U}}(N, \tilde{\Phi} M) \) is the equalizer of \( \text{Hom}_{\mathcal{U}}(N, \tilde{\Phi} M) \Rightarrow \text{Hom}_{\mathcal{U}}(K \otimes N, \tilde{\Phi} M) \).

By adjunction, this is equivalent to the diagram

\[ \text{Hom}_{\mathcal{U}}(\Phi N, M) \Rightarrow \text{Hom}_{\mathcal{U}}(\Phi K \otimes \Phi N, M). \]

A simple verification shows that this corresponds to the equalizer diagram defining \( \text{Hom}_{\Phi K-\mathcal{U}}(\Phi N, M) \), which completes the proof. \( \square \)

A.2 Formal results for endofunctors of \( \mathcal{U} \)

The following results explain how to study exact endofunctors of the category \( \mathcal{U} \) via passage to \( \mathcal{U}/\text{Nil} \).

**Lemma A.2.1**  Let \( \Theta : \mathcal{U} \to \mathcal{U} \) be an exact functor which preserves the subcategory \( \text{Nil} \), then

1. \( \mathcal{U} \xrightarrow{\Theta} \mathcal{U} \xrightarrow{l} \mathcal{U}/\text{Nil} \) induces an exact functor \( \tilde{\Theta} : \mathcal{U}/\text{Nil} \to \mathcal{U}/\text{Nil} \) such that \( \tilde{\Theta} l \cong l\Theta \);

2. if, moreover, \( \Theta \) preserves the class of nilclosed unstable modules, then there is a natural isomorphism \( \Theta rl \cong r\tilde{\Theta} l \). In particular, \( \tilde{\Theta} \) determines the restriction of \( \Theta \) to the full subcategory of nilclosed unstable modules.
Recall that $\mathcal{U}/\mathcal{Nil}$ is equivalent to the category of analytic functors, $\mathcal{F}_\omega$. The category $\mathcal{F}_\omega$ is locally Noetherian [19, Proposition 5.3.3], hence any coproduct of injective cogenerators of $\mathcal{F}_\omega$ of the form $I_V$ is injective in $\mathcal{F}_\omega$ and any analytic functor admits an injective resolution in which each term is of the form $\bigoplus_\alpha I_{V_\alpha}$. This implies the following result.

**Lemma A.2.2** Let $\Theta_1, \Theta_2 : \mathcal{F}_\omega \to \mathcal{F}_\omega$ be two exact functors which commute with arbitrary coproducts. If the restrictions $\Theta_1, \Theta_2$ to the full subcategory with objects $\{I_V | s \in \mathbb{N}\}$ are naturally isomorphic, then $\Theta_1$ and $\Theta_2$ are naturally isomorphic.

**A.3 Preservation of reduced unstable modules**

**Lemma A.3.1** Let $\gamma : G_1 \hookrightarrow G_2$ be a natural monomorphism of endofunctors of $\mathcal{U}$ such that

1. $G_1$ is exact and $G_2$ is left exact;
2. $\gamma_M$ is an isomorphism if $M$ is a nilclosed unstable module.

Then $\gamma_N : G_1 N \to G_2 N$ is an isomorphism if $N$ is a reduced unstable module.

**Proof** Consider a reduced unstable module $N$ and the associated short exact sequence of unstable modules $0 \to N \to r/N \to (r/N)/N \to 0$. The natural monomorphism $\gamma$ induces a commutative diagram in $\mathcal{U}$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G_1 N & \longrightarrow & G_1(r/N) & \longrightarrow & G_1((r/N)/N) & \longrightarrow & 0 \\
\gamma_N & & \downarrow \cong & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_2 N & \longrightarrow & G_2(r/N) & \longrightarrow & G_2((r/N)/N) & & 
\end{array}
$$

in which the rows are exact and the middle vertical morphism is an isomorphism, since $r/N$ is nilclosed. The result follows from the five-lemma.

**References**


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