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Derived A_∞–algebras in an operadic context

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Derived A_∞–algebras were developed recently by Sagave. Their advantage over classical A_∞–algebras is that no projectivity assumptions are needed to study minimal models of differential graded algebras. We explain how derived A_∞–algebras can be viewed as algebras over an operad. More specifically, we describe how this operad arises as a resolution of the operad $d\mathcal{A}s$ encoding bidgas, ie bicomplexes with an associative multiplication. This generalises the established result describing the operad A_∞ as a resolution of the operad $\mathcal{A}s$ encoding associative algebras. We further show that Sagave's definition of morphisms agrees with the infinity-morphisms of dA_∞–algebras arising from operadic machinery. We also study the operadic homology of derived A_∞–algebras.

16E45, 18D50; 18G55, 18G10

Introduction

Mathematical areas in which A_∞–structures arise range from geometry, topology and representation theory to mathematical physics. One important application is to the study of differential graded algebras via A_∞–structures on their homology algebras. This is the theory of minimal models established by Kadeishvili [4] in the 1980s. However, the results concerning minimal models all have rather restrictive projectivity assumptions.

To bypass these projectivity assumptions, Sagave [9] recently developed the notion of derived A_∞–algebras. Compared to classical A_∞–algebras, derived A_∞–algebras are equipped with an additional grading. Using this definition one can define projective resolutions that are compatible with A_∞–structures. With these, Sagave established a notion of minimal models for differential graded algebras (dgas) whose homology is not necessarily projective.

Sagave's descriptions of derived A_∞–structures are largely formula-based. In this paper, we provide an alternative description of these structures using operads. It is not

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hard to write down an operad dA_∞ that encodes derived A_∞–structures, but we also explain the context into which this operad fits. The category we are going to work in is the category $\mathrm{BiCompl}_v$ of bicomplexes with no horizontal differential. We will start from an operad $d\mathcal{A}s$ in this category encoding bidgas, that is, monoids in bicomplexes (see Definition 1.3). Our main theorem shows that derived A_∞–algebras are algebras over the operad

$$dA_\infty = (d\mathcal{A}s)_\infty = \Omega((d\mathcal{A}s)^{¡}).$$

This means that the operad dA_∞ is a minimal model of a well-known structure.

We can summarise our main result and its relation to the classical case in the following table:

underlying category	operad $\mathcal{O}$	$\mathcal{O}$–algebra
differential graded $\boldsymbol{k}$–modules	$\mathcal{A}s$	dga
	A_∞	A_∞–algebra
$\mathrm{BiCompl}_v$	$d\mathcal{A}s$	bidga
	dA_∞	derived A_∞–algebra

We hope that this provides a useful way of thinking about derived A_∞–structures. It should allow many operadic techniques to be applied to their study and we give two examples. Firstly, we note a simple consequence of the homotopy transfer theorem. Secondly we develop operadic homology of derived A_∞–algebras and relate this to formality of dgas.

This paper is organised as follows. We start by recalling some previous results in Section 1. In the first part we summarise some definitions, conventions and results about derived A_∞–algebras. The second part is concerned with classical A_∞–algebras. We look at the operad $\mathcal{A}s$ encoding associative algebras and summarise how to obtain the operad A_∞ as a resolution of $\mathcal{A}s$.

In Section 2 we generalise this to the operad $d\mathcal{A}s$. More precisely, this operad lives in the category of bicomplexes with trivial horizontal differential. It encodes bidgas and can be described as the composition of the operad of dual numbers and $\mathcal{A}s$ using a distributive law. The main result of this section is computing its Koszul dual cooperad.

Section 3 contains our main result. We describe the operad dA_∞ encoding derived A_∞–algebras and show that it agrees with the cobar construction of the reduced Koszul dual cooperad of $d\mathcal{A}s$.

In Section 4 we consider ∞–morphisms and show that they coincide with the derived A_∞–morphisms defined by Sagave. We also give an immediate application of the operadic approach, by deducing the existence of a dA_∞–algebra structure on the vertical homology of a bidga over a field from the homotopy transfer theorem.

In Section 5, we study the operadic homology of derived A_∞–algebras. By comparing this to the previously defined Hochschild cohomology of the last two authors [8], we deduce a criterion for intrinsic formality of a dga.

We conclude with a short section outlining some areas for future investigation.

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1 A review of known results

Throughout this paper let $\boldsymbol{k}$ denote a commutative ring unless stated otherwise. All operads considered are nonsymmetric.

1.1 Derived A_∞–algebras

We are going to recall some basic definitions and results regarding derived A_∞–algebras. This is just a brief recollection; we refer to [9] and [8] for more details.

We start by considering $(\mathbb{N},\mathbb{Z})$–bigraded $\boldsymbol{k}$–modules

$$A = \bigoplus_{i\in\mathbb{N},j\in\mathbb{Z}} A_i^j.$$

The lower grading is called the *horizontal degree* and the upper grading the *vertical degree*. Note that the horizontal grading is homological whereas the vertical grading is cohomological. A morphism of bidegree (u,v) is then a morphism of bigraded modules that lowers the horizontal degree by u and raises the vertical degree by v. We are observing the *Koszul sign rule*, that is

$$(f\otimes g)(x\otimes y) = (-1)^{pi+qj} f(x)\otimes g(y)$$

if g has bidegree (p,q) and x has bidegree (i,j). Here we have adopted the grading conventions used in [8].

We can now say what a derived A_∞–algebra is.

Definition 1.1 [9] A *derived A_∞–structure* (or *dA_∞–structure* for short) on an $(\mathbb{N},\mathbb{Z})$–bigraded $\boldsymbol{k}$–module A consists of $\boldsymbol{k}$–linear maps

$$m_{ij}\colon A^{\otimes j} \longrightarrow A$$

of bidegree $(i, 2-(i+j))$ for each $i \geq 0,\ j \geq 1$, satisfying the equations

$$\sum_{\substack{u=i+p\\ v=j+q-1\\ j=1+r+t}} (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0 \tag{1}$$

for all $u \geq 0$ and $v \geq 1$. A *dA_∞–algebra* is a bigraded $\boldsymbol{k}$–module together with a dA_∞–structure.

Definition 1.2 [9] A map of dA_∞–algebras from (A, m^A) to (B, m^B) consists of a family of $\boldsymbol{k}$–module maps $f_{ij}\colon A^{\otimes j} \to B$ of bidegree $(i, 1-i-j)$ with $i \geq 0$, $j \geq 1$, satisfying

$$\sum_{\substack{u=i+p\\ v=j+q-1\\ j=1+r+t}} (-1)^{rq+t+pj} f_{ij}(1^{\otimes r} \otimes m^A_{pq} \otimes 1^{\otimes t}) = \sum_{\substack{u=i+p_1+\cdots+p_j,\\ v=q_1+\cdots+q_j}} (-1)^{\sigma} m^B_{ij}(f_{p_1q_1} \otimes \cdots \otimes f_{p_jq_j}), \tag{2}$$

with

$$\sigma = u + \sum_{k=1}^{j-1} (p_k + q_k)(j+k) + q_k\left(\sum_{s=k+1}^{j} p_s + q_s\right).$$

Sagave does not define composition of maps of dA_∞–algebras directly in terms of this definition. Instead this is done via a certain reformulation as maps on the reduced tensor algebra; see [9, Definition 4.5]. It follows that dA_∞–algebras form a category.

Examples of dA_∞–algebras include classical A_∞–algebras, which are derived A_∞–algebras concentrated in horizontal degree 0. Other examples are bicomplexes and bidgas, in the sense of the following definition.

Definition 1.3 A *bidga* is a derived A_∞–algebra with $m_{ij} = 0$ for $i + j \geq 3$. A *morphism of bidgas* is a morphism of derived A_∞–algebras f_{ij} with $f_{ij} = 0$ for $i + j \geq 2$.

Sagave notes that this is equivalent to saying that a bidga is a monoid in the category of bicomplexes.

For derived A_∞–algebras, the analogue of a quasi-isomorphism is called an E_2–equivalence. To explain this, we need to discuss twisted chain complexes. The terminology *multicomplex* is also used for a twisted chain complex.

Definition 1.4 A *twisted chain complex* C is an $(\mathbb{N},\mathbb{Z})$–bigraded $\boldsymbol{k}$–module with differentials $d_i^C\colon C\to C$ of bidegree $(i,1-i)$ for $i\geq 0$ satisfying

$$\sum_{i+p=u}(-1)^i d_i^C d_p^C=0$$

for $u\geq 0$. A *map of twisted chain complexes* $C\to D$ is a family of maps $f_i\colon C\to D$ of bidegree $(i,-i)$ satisfying

$$\sum_{i+p=u}(-1)^i f_i d_p^C=\sum_{i+p=u} d_i^D f_p.$$

The composition of maps $f\colon E\to F$ and $g\colon F\to G$ is defined by $(gf)_u=\sum_{i+p=u} g_i f_p$ and the resulting category is denoted tCh_k.

A derived A_∞–algebra has an underlying twisted chain complex, specified by the maps m_{i1} for $i\geq 0$.

If $f\colon C\to D$ is a map of twisted chain complexes, then f_0 is a d_0–chain map and $H_*^v(f_0)$ induces a d_1–chain map.

Definition 1.5 A map $f\colon C\to D$ of twisted chain complexes is an E_1*–equivalence* if $H_t^v(f_0)$ is an isomorphism for all $t\in\mathbb{Z}$ and an E_2*–equivalence* if $H_s^h(H_t^v(f_0))$ is an isomorphism for all $s\in\mathbb{N}$, $t\in\mathbb{Z}$.

The first main advantage of derived A_∞–structures over A_∞–structures is that one has a reasonable notion of a minimal model for differential graded algebras without any projectivity assumptions on the homology.

Theorem 1.6 [9] *Let A be a dga over $\boldsymbol{k}$. Then there is a degreewise $\boldsymbol{k}$–projective dA_∞–algebra E together with an E_2–equivalence $E\to A$ such that*

- *E is minimal (ie $m_{01}=0$),*
- *E is unique up to E_2–equivalence,*
- *together with the differential m_{11} and the multiplication m_{02}, E is a termwise $\boldsymbol{k}$–projective resolution of the graded algebra $H^*(A)$.*

The second and third authors then gave the analogue of Kadeishvili's formality criterion for dgas using Hochschild cohomology. They describe derived A_∞–structures in terms of a Lie algebra structure on morphisms of the underlying $\boldsymbol{k}$–module A. Then they use this Lie algebra structure to define Hochschild cohomology for a large class of derived A_∞–algebras and eventually reach the following result [8, Theorem 4.4]. Recall that a dga is called intrinsically formal if any other dga A' such that $H^*(A)\cong H^*(A')$ as associative algebras is quasi-isomorphic to A.

Theorem 1.7 [8] *Let A be a dga and E its minimal model with dA_∞–structure m. By $\widetilde{E}$, we denote the underlying bidga of E, ie $\widetilde{E} = E$ as $\boldsymbol{k}$–modules together with dA_∞–structure $\widetilde{m} = m_{11} + m_{02}$. If*

$$HH^{m,2-m}_{bidga}(\widetilde{E}, \widetilde{E}) = 0 \quad \text{for } m \geq 3,$$

then A is intrinsically formal.

1.2 The operad $\mathcal{A}s$

The goal of our paper is to describe derived A_∞–algebras as algebras over an operad, and to show that this operad is a minimal model of a certain Koszul operad. The operad in question is an operad called $d\mathcal{A}s$ (defined in Section 2), which is a generalisation of the operad $\mathcal{A}s$ that encodes associative algebras. So let us recall this strategy for $\mathcal{A}s$ itself. For this subsection only, let $\boldsymbol{k}$ be a field. We work in the category of (cohomologically) differential graded $\boldsymbol{k}$–vector spaces, denoted dg$\boldsymbol{k}$–vs.

We will use the notation $\mathcal{F}(M)$ for the free (nonsymmetric) operad generated by a collection $M = \{M(n)\}_{n\geq 1}$ of graded $\boldsymbol{k}$–vector spaces. It is weight graded by the number s of vertices in the planar tree representation of elements of $\mathcal{F}(M)$ and we denote by $\mathcal{F}_{(s)}(M)$ the corresponding graded $\boldsymbol{k}$–vector space. We denote by $\mathcal{P}(M, R)$ the operad defined by generators and relations, $\mathcal{F}(M)/(R)$. A *quadratic operad* is an operad such that $R \subset \mathcal{F}_{(2)}(M)$.

Definition 1.8 The operad $\mathcal{A}s$ in dg$\boldsymbol{k}$–vs is given by

$$\mathcal{A}s = \mathcal{P}(\boldsymbol{k}\mu, \boldsymbol{k}as),$$

where μ is a binary operation concentrated in degree zero and $as = \mu \circ_1 \mu - \mu \circ_2 \mu$. The differential is trivial.

It is easy to verify that an $\mathcal{A}s$–algebra structure on the differential graded $\boldsymbol{k}$–vector space A, ie a morphism of dg operads

$$\mathcal{A}s \xrightarrow{\Phi} \mathrm{End}_A,$$

endows A with the structure of an associative dga, with multiplication

$$\Phi(\mu)\colon A^{\otimes 2} \longrightarrow A.$$

Theorem 1.9 *The operad $\mathcal{A}s$ is a Koszul operad, ie the map of operads in* dg$\boldsymbol{k}$–vs

$$\Omega(\mathcal{A}s^{¡}) \longrightarrow \mathcal{A}s$$

is a quasi-isomorphism. Furthermore, an algebra over $\Omega(\mathcal{A}s^{¡})$ is precisely an A_∞–algebra.

Here, a quasi-isomorphism of operads is a quasi-isomorphism of dg-$\boldsymbol{k}$–vector spaces in each arity degree. We do not recall the definitions of the Koszul dual cooperad $(-)^{¡}$ or the cobar construction $\Omega(-)$ here. (This is going to be discussed in greater detail for our computations later). Let us just mention now that the cobar construction of a cooperad is a free graded operad endowed with a differential built from the cooperad structure, so we can think of the map above as a free resolution of the operad $\mathcal{A}s$. This result can be proved using beautiful geometric and combinatorial methods such as the Stasheff cell complex. Unfortunately, the derived case will not be as obviously geometric.

Our aim is to create an analogue of the above for the derived case. The first step is to consider working in a different category; instead of differential graded $\boldsymbol{k}$–vector spaces, we consider a category of graded chain complexes over a commutative ring $\boldsymbol{k}$.

The role of $\mathcal{A}s$ in this case is going to be played by an operad $d\mathcal{A}s$, which encodes bidgas rather than associative dgas.

The first goal is showing that $d\mathcal{A}s$ is a Koszul operad, ie that

$$(d\mathcal{A}s)_\infty := \Omega((d\mathcal{A}s)^{¡}) \longrightarrow d\mathcal{A}s$$

is a quasi-isomorphism of operads in an appropriate category. We are going to achieve this by "splitting" $d\mathcal{A}s$ into two parts, namely the operad of dual numbers and $\mathcal{A}s$ itself, via a distributive law.

Secondly, we are going to compute the generators and differential of $(d\mathcal{A}s)_\infty$ explicitly, so we can read off that $(d\mathcal{A}s)_\infty$–algebras give exactly derived A_∞–algebras in the sense of Sagave.

Our work will show that the operad controlling derived A_∞–algebras can be seen as a free resolution of the operad encoding bidgas, in the same sense that the classical A_∞–operad is a free resolution of the operad encoding associative dgas.

2 The operad $d\mathcal{A}s$

In the first part of this section, we recall some basic notions about the Koszul dual cooperad of a given operad and we compute the Koszul dual of $d\mathcal{A}s$. Further details can be found in Fresse [1], which covers Koszul duality for operads over a general commutative ground ring. We also refer to the book of Loday and Vallette [6].

We are first going to specify the category we work in. Again, let $\boldsymbol{k}$ be a commutative ring.

2.1 Vertical bicomplexes and operads in vertical bicomplexes

Definition 2.1 The category of *vertical bicomplexes* $\mathrm{BiCompl}_v$ consists of bigraded $\boldsymbol{k}$–modules as above together with a vertical differential

$$d_A\colon A_i^j \longrightarrow A_i^{j+1}$$

of bidegree $(0,1)$. The morphisms are those morphisms of bigraded modules commuting with the vertical differential. We denote by $\mathrm{Hom}(A,B)$ the set of morphisms (preserving the bigrading) from A to B.

If $c, d \in A$ have bidegree (c_1, c_2) and (d_1, d_2) respectively we denote by $|c||d|$ the integer $c_1 d_1 + c_2 d_2$.

We define a degree shift operation on $\mathrm{BiCompl}_v$ as follows. Let $A \in \mathrm{BiCompl}_v$. Then sA is defined as

$$(sA)_i^j = A_i^{j+1},$$

with

$$d_{sA}(sx) = -s(d_A x).$$

So if $c \in A$ is of bidegree (c_1, c_2), then $sc \in sA$ is of bidegree $(c_1, c_2 - 1)$.

This shift is compatible with the embedding of differential graded complexes into $\mathrm{BiCompl}_v$ given by $C_0^l = C^l$ and $C_k^l = 0$, if $k > 0$.

The tensor product of two vertical bicomplexes A and B is given by

$$(A \otimes B)_u^v = \bigoplus_{\substack{i+p=u,\\ j+q=v}} A_i^j \otimes B_p^q,$$

with $d_{A\otimes B} = d_A \otimes 1 + 1 \otimes d_B\colon (A\otimes B)_u^v \to (A \otimes B)_u^{v+1}$.

Note that $\mathrm{BiCompl}_v$ is isomorphic to the category of $\mathbb{N}$–graded chain complexes of $\boldsymbol{k}$–modules.

There are two other sorts of morphism that we will consider later and we introduce notation for these now. (Various alternative choices of notation are used in the literature.) Let A and B be two vertical bicomplexes. We write $\mathrm{Hom}_{\boldsymbol{k}}$ for morphisms of $\boldsymbol{k}$–modules. We will denote by $\mathrm{Mor}(A,B)$ the vertical bicomplex given by

$$\mathrm{Mor}(A,B)_u^v = \prod_{\alpha,\beta} \mathrm{Hom}_{\boldsymbol{k}}\left(A_\alpha^\beta, B_{\alpha-u}^{\beta+v}\right),$$

with vertical differential given by $\partial_{\mathrm{Mor}}(f) = d_B f - (-1)^j f d_A$ for f of bidegree (l, j).

We will denote by $\mathbf{Hom}(A,B)$ the (cohomologically) graded complex given by

$$\mathbf{Hom}(A,B)^k = \prod_{\alpha,\beta} \mathrm{Hom}_{\boldsymbol{k}}\big(A_\alpha^\beta, B_\alpha^{\beta+k}\big),$$

with the same differential as above. One has

$$\mathrm{Hom}(A,B) = \mathrm{Mor}(A,B)_0^0 \quad \text{and} \quad \mathbf{Hom}(A,B)^* = \mathrm{Mor}(A,B)_0^*.$$

Definition 2.2 A *collection* in $\mathrm{BiCompl}_v$ is a collection $A(n)_{n\geq 1}$ of vertical bicomplexes. We denote by $\mathcal{C}\mathrm{BiCompl}_v$ the category of collections of vertical bicomplexes. This category is endowed with a monoidal structure, the plethysm given by, for any two collections M and N,

$$(M \circ N)(n) = \bigoplus_{k,\, l_1+\cdots+l_k=n} M(k) \otimes N(l_1) \otimes \cdots \otimes N(l_k).$$

The unit for the plethysm is given by the collection

$$I(n) = \begin{cases} 0 & \text{if } n \neq 1, \\ \boldsymbol{k} \text{ concentrated in bidegree } (0,0) & \text{if } n = 1. \end{cases}$$

Given two collections A and B in $\mathrm{BiCompl}_v$, one can consider again the three collections:

- $\mathrm{Hom}(A,B)(n) := \{\mathrm{Hom}(A(n), B(n))\}_{n\geq 1}$ in the category of $\boldsymbol{k}$–modules.
- $\mathrm{Mor}(A,B)(n) := \{\mathrm{Mor}(A(n), B(n))\}_{n\geq 1}$ in the category of vertical bicomplexes.
- $\mathbf{Hom}(A,B)(n) := \{\mathbf{Hom}(A(n), B(n))\}_{n\geq 1}$ in the category of complexes.

Definition 2.3 A (nonsymmetric) *operad* in $\mathrm{BiCompl}_v$ is a monoid in the category $\mathcal{C}\mathrm{BiCompl}_v$. This is the usual definition of operads in the symmetric monoidal category $(\mathrm{BiCompl}_v, \otimes)$.

For a vertical bicomplex A, the *endomorphism operad* End_A is the operad in vertical bicomplexes given by $\mathrm{End}_A(n) = \mathrm{Mor}(A^{\otimes n}, A)$, where the operad structure is given by the composition of morphisms, as usual.

2.2 The operad $d\mathcal{A}s$

We now describe the operad in $\mathrm{BiCompl}_v$ that encodes bidgas.

Definition 2.4 The operad $d\mathcal{A}s$ in $\mathrm{BiCompl}_v$ is defined as $\mathcal{P}(M_{d\mathcal{A}s}, R_{d\mathcal{A}s})$, where

$$M_{d\mathcal{A}s}(n) = \begin{cases} 0 & \text{if } n > 2, \\ \boldsymbol{k} m_{02} \text{ concentrated in bidegree } (0,0) & \text{if } n = 2, \\ \boldsymbol{k} m_{11} \text{ concentrated in bidegree } (1,0) & \text{if } n = 1, \end{cases}$$

and

$$R_{d\mathcal{A}s} = \boldsymbol{k}(m_{02} \circ_1 m_{02} - m_{02} \circ_2 m_{02}) \oplus \boldsymbol{k} m_{11}^2 \oplus \boldsymbol{k}(m_{11} \circ_1 m_{02} - m_{02} \circ_1 m_{11} - m_{02} \circ_2 m_{11}),$$

with trivial vertical differential.

This operad is clearly quadratic.

The following result is now essentially a matter of definitions, but we include the details for completeness.

Proposition 2.5 *The category of $d\mathcal{A}s$–algebras in* $\mathrm{BiCompl}_v$ *is isomorphic to the category of bidgas.*

Proof A $d\mathcal{A}s$–algebra structure on a vertical bicomplex A is given by a morphism of operads

$$\theta\colon d\mathcal{A}s \longrightarrow \mathrm{End}_A .$$

Since A is a vertical bicomplex, it is $(\mathbb{N}, \mathbb{Z})$–graded and comes with a vertical differential $d_A = d^v$ of bidegree $(0,1)$. From the images of the operad generators we have morphisms

$$m = \theta(m_{02})\colon A^{\otimes 2} \longrightarrow A \quad \text{and} \quad d^h = \theta(m_{11})\colon A \longrightarrow A,$$

of bidegree $(0,0)$ and $(1,0)$ respectively.

The operad relations tell us precisely that m is associative, that d^h is a differential and that d^h is a derivation with respect to m. The fact that θ is a morphism of operads in $\mathrm{BiCompl}_v$, and that the differential on each $d\mathcal{A}s(n)$ is trivial, gives us two further relations:

$$\partial_{\mathrm{Mor}}(m) = 0 \quad \text{and} \quad \partial_{\mathrm{Mor}}(d^h) = 0.$$

The first of these relations tells us that d^v is a derivation with respect to m and the second that $d^vd^h - d^hd^v = 0$. This gives A precisely the structure of a bidga (with exactly Sagave's sign conventions).

A morphism of $d\mathcal{A}s$–algebras $f\colon A \to B$ is a map of vertical bicomplexes which also commutes with m and d^h. This is precisely a morphism of bidgas. □

Let us describe the operad $d\mathcal{A}s$ in a little more detail. Let m_k denote any $(k-1)$–fold composite of m_{02}. (Because of the associativity relation, m_k does not depend on the choice of composition.) Due to the "Leibniz rule relation" every element of $d\mathcal{A}s$ in arity k can be written as a $\boldsymbol{k}$–linear combination of the elements

$$m_k(m_{11}^{\epsilon_1}, \dots, m_{11}^{\epsilon_k})$$

with $\epsilon_i \in \mathbb{Z}/2$. The partial composition $m_l(m_{11}^{\epsilon_1}, \dots, m_{11}^{\epsilon_l}) \circ_i m_k(m_{11}^{\delta_1}, \dots, m_{11}^{\delta_k})$ is given by

$$(-1)^\alpha \sum_{s=1}^{k} (-1)^\beta m_{k+l-1}\big(m_{11}^{\epsilon_1}, \dots, m_{11}^{\epsilon_{i-1}}, m_{11}^{\delta_1}, \dots, m_{11}^{\delta_s+1}, \dots, m_{11}^{\delta_k}, m_{11}^{\epsilon_{i+1}}, \dots, m_{11}^{\epsilon_l}\big)$$

if $\epsilon_i = 1$ and

$$(-1)^\alpha m_{k+l-1}\big(m_{11}^{\epsilon_1}, \dots, m_{11}^{\epsilon_{i-1}}, m_{11}^{\delta_1}, \dots, m_{11}^{\delta_s}, \dots, m_{11}^{\delta_k}, m_{11}^{\epsilon_{i+1}}, \dots, m_{11}^{\epsilon_l}\big)$$

if $\epsilon_i = 0$, where $\alpha = (\sum_{j=i+1}^{l} \epsilon_j)(\sum_{r=1}^{k} \delta_r)$ and $\beta = \sum_{r=1}^{s-1} \delta_r$.

We see that we have an isomorphism of bigraded $\boldsymbol{k}$–modules,

$$d\mathcal{A}s(n) \cong \boldsymbol{k}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2), \quad |x_i| = (1, 0),$$

determined by assigning the monomial $x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ to the element $m_n(m_{11}^{\epsilon_1}, \dots, m_{11}^{\epsilon_n})$.

Let $\mathcal{D}$ denote the operad of dual numbers in the category of vertical bicomplexes, namely

$$\mathcal{D} = \mathcal{P}(\boldsymbol{k}m_{11}, \boldsymbol{k}m_{11}^2)$$

with trivial differential.

We can now reformulate the above description of $d\mathcal{A}s$ in terms of plethysm and distributive laws; see [6, Section 8.6].

Lemma 2.6 *The map*

$$\varphi\colon \mathcal{D} \circ \mathcal{A}s \longrightarrow \mathcal{A}s \circ \mathcal{D}$$

determined by

$$\varphi\colon m_{11} \circ_1 m_{02} \longmapsto m_{02} \circ_1 m_{11} + m_{02} \circ_2 m_{11}$$

defines a distributive law, such that the induced operad structure on $\mathcal{A}s \circ \mathcal{D}$ coincides with the operad $d\mathcal{A}s$.

Proof We adopt the notation and terminology of [6, Section 8.6.2]. We define

$$\varphi\colon \boldsymbol{k}m_{11} \circ_{(1)} \boldsymbol{k}m_{02} \longrightarrow \boldsymbol{k}m_{02} \circ_{(1)} \boldsymbol{k}m_{11}$$

as above. This gives a rewriting rule for the quadratic operads $\mathcal{D}$ and $\mathcal{A}s$ and it is clear that $d\mathcal{A}s$ is isomorphic to $\mathcal{A}s \vee_\varphi \mathcal{D}$. From the description of the operad $d\mathcal{A}s$ above, we see that the induced map $\mathcal{A}s \circ \mathcal{D} \to \mathcal{A}s \vee_\varphi \mathcal{D} \cong d\mathcal{A}s$ is an isomorphism. So, by [6, Proposition 8.6.2], φ induces a distributive law and an isomorphism of operads $\mathcal{A}s \circ \mathcal{D} \to \mathcal{A}s \vee_\varphi \mathcal{D}$. □

For $\mathcal{P} = \mathcal{P}(M, R)$ a quadratic operad, the *Koszul dual cooperad* $\mathcal{P}^{¡}$ of $\mathcal{P}$ is given by

$$\mathcal{P}^{¡} = \mathcal{C}^c(sM, s^2 R).$$

Here $\mathcal{C}^c(E, R)$ denotes the cooperad cogenerated by E with corelations R. (For a description see [6, Section 7.2].)

There are two ways of describing the cooperad $(d\mathcal{A}s)^{¡}$, either by describing the distributive law

$$\mathcal{D}^{¡} \circ \mathcal{A}s^{¡} \longrightarrow \mathcal{A}s^{¡} \circ \mathcal{D}^{¡}$$

or by describing the elements of $\mathcal{C}^c(s(\boldsymbol{k}m_{11} \oplus \boldsymbol{k}m_{02}), s^2 R_{d\mathcal{A}s})$ in the cofree cooperad $\mathcal{F}^c(s(\boldsymbol{k}m_{11} \oplus \boldsymbol{k}m_{02}))$. The first description implies that for every n, $(d\mathcal{A}s)^{¡}(n)$ is a free $\boldsymbol{k}$–module.

Proposition 2.7 *The underlying collection of the cooperad $d\mathcal{A}s^{¡}$ is isomorphic to that of*

$$\mathcal{D}^{¡} \circ \mathcal{A}s^{¡} = \boldsymbol{k}[\mu_{11}] \circ \mathcal{A}s^{¡},$$

where μ_{11} has bidegree $(1, -1)$. Hence, as a $\boldsymbol{k}$–module, $(d\mathcal{A}s)^{¡}(n)$ is free with basis given by elements ν_{in} of bidegree $(i, 1-i-n)$. These elements are in 1–to–1 correspondence with the elements $(sm_{11})^i \circ \mu_n$ in $\mathcal{D}^{¡} \circ \mathcal{A}s^{¡}$.

Proof The first part of the claim follows from Lemma 2.6, since $d\mathcal{A}s \cong \mathcal{A}s \vee_\varphi \mathcal{D}$ and by [6, Proposition 8.6.6], there is an isomorphism of underlying collections $(\mathcal{A}s \vee_\varphi \mathcal{D})^{¡} \cong \mathcal{D}^{¡} \circ \mathcal{A}s^{¡}$.

The cooperad structures of $\mathcal{D}^{¡}$ and $\mathcal{A}s^{¡}$ are well-known and can be shown by induction with the methods used in Theorem 2.8. In arity n, $\mathcal{A}s^{¡}(n)$ is a free $\boldsymbol{k}$–module on the generator μ_n. The element μ_n has bidegree $(0, 1-n)$. The cooperad $\mathcal{D}^{¡}$ is concentrated in arity 1. It is the free cooperad on the generator sm_{11}. This implies that $(d\mathcal{A}s)^{¡}(n)$ is free on the images ν_{in} in $(d\mathcal{A}s)^{¡}(n)$ of the generators

$$(sm_{11})^i \circ \mu_n \in (D^{¡} \circ \mathcal{A}s^{¡})(n).$$

We can read off a generator's bidegree as

$$|\nu_{in}| = i(|m_{11}| + |s|) + |\mu_n| = (i, 1-i-n). \qquad \square$$

Notation Let $\mathcal{C}$ be a cooperad and $c \in \mathcal{C}(n)$. We are going to describe the cocomposition

$$\Delta\colon \mathcal{C} \longrightarrow \mathcal{C} \circ \mathcal{C}.$$

We write

$$\Delta(c) = \sum_{j,|I|=n} c_j ; c_I.$$

Here, $I = (i_1, \dots, i_j)$ is a j–tuple with $|I| = i_1 + \dots + i_j$, and

$$c_I = c_{i_1} \otimes \dots \otimes c_{i_j} \in \mathcal{C}^{\otimes j}.$$

If $\mathcal{C} = \mathcal{F}^c(V)$ is a cofree cooperad cogenerated by a collection V, then it has a description in terms of trees whose vertices are labelled by elements of V; see [6, Section 5.8.6]. Moreover if $V(n)$ is a free $\boldsymbol{k}$–module for each n, then so is $\mathcal{C}(n)$, and a basis as a free $\boldsymbol{k}$–module is given by planar trees whose vertices are labelled by a basis of V. If the root of such a tree has arity k and is labelled by v we denote it by $v(t^1, \dots, t^k)$, where $t^1, \dots, t^k$ are elements of $\mathcal{C} = \mathcal{F}^c(V)$. Remembering that

$$\Delta(t^r) = \sum t^r_{j_r} ; t^r_{I_r}$$

one obtains the formula

$$\begin{aligned}(3)\quad \Delta(v(t^1, \dots, t^k)) = 1; v(t^1, \dots, t^k)& \\ + \sum (-1)^{\sum_{r=1}^{k-1} |t^r_{I_r}| \left(\sum_{s=r+1}^{k} |t^s_{j_s}| \right)} & v(t^1_{j_1}, \dots, t^k_{j_k}); t^1_{I_1} \otimes \dots \otimes t^k_{I_k}.\end{aligned}$$

We now compute the full structure of $(d\mathcal{A}s)^{¡}$. From Proposition 2.7 we already know the structure of its underlying bigraded $\boldsymbol{k}$–modules, and we can use (3) to write down the cocomposition of its basis elements.

We remark that we have chosen to work directly with the cooperad $(d\mathcal{A}s)^{¡}$, rather than with the operad $(d\mathcal{A}s)^{!}$. This is to avoid taking linear duals, which can be badly behaved over a general ground ring.

Theorem 2.8 *The cooperad $(d\mathcal{A}s)^{¡}$ is a subcooperad of $\mathcal{F}^c(sM_{d\mathcal{A}s})$ with trivial differential. Its underlying collection consists of free $\boldsymbol{k}$–modules with basis $\{\mu_{ij}, i\geq 0, j\geq 1\}$ such that μ_{01} is the identity of the cooperad, $\mu_{02}=sm_{02}$ and $\mu_{11}=sm_{11}\in\mathcal{F}^c(sM_{d\mathcal{A}s})$. The other μ_{ij} are defined inductively via*

$$\mu_{i1}=\mu_{11}(\mu_{i-1,1}) \qquad \text{for } i\geq 1,$$

$$\mu_{0n}=\sum_{p+q=n}(-1)^{p(q+1)}\mu_{02}(\mu_{0p},\mu_{0q}) \qquad \text{for } n\geq 2,$$

$$\mu_{ij}=\mu_{11}(\mu_{i-1,j})+\sum_{\substack{r+t=i\\ s+w=j}}(-1)^{|s\mu_{rs}||\mu_{tw}|+rw}\mu_{02}(\mu_{rs},\mu_{tw}) \qquad \text{for } i\geq 1,\ j\geq 2.$$

The element μ_{ij} has bidegree $(i,1-i-j)$. These elements satisfy

$$\Delta(\mu_{uv})=\sum_{\substack{i+p_1+\dots+p_j=u\\ q_1+\dots+q_j=v}}(-1)^{X((p_1,q_1),\dots,(p_j,q_j))}\mu_{ij};\mu_{p_1q_1}\otimes\dots\otimes\mu_{p_jq_j}, \tag{4}$$

where

$$\begin{aligned} X((p_1,q_1),\dots,(p_j,q_j)) &= \sum_{k=1}^{j-1}|s\mu_{p_kq_k}|\Big(\sum_{l=k+1}^{j}|\mu_{p_lq_l}|\Big)+\sum_{k=1}^{j-1}p_k\Big(\sum_{l=k+1}^{j}q_l\Big)\\ &=\sum_{k=1}^{j-1}\Big((p_k+q_k)(j+k)+q_k\sum_{l=k+1}^{j}(p_l+q_l)\Big). \end{aligned} \tag{5}$$

Proof Firstly we are going to show that those inductively defined elements form a subcooperad of $\mathcal{F}^c(sM_{d\mathcal{A}s})$. Then we will see that this subcooperad contains the quadratic relations $s^2R_{d\mathcal{A}s}$. Together with Proposition 2.7, this means that it must be $(d\mathcal{A}s)^{¡}$ itself.

For the first part we have to prove formula (4), which is done by induction on $u+v$.

One has

$$\Delta(\mu_{u1})=\sum_{i+p=u}\mu_{i1};\mu_{p1},$$

which is proved by induction from the definition

$$\mu_{u1}=\mu_{11}(\mu_{u-1,1}).$$

The case of $\Delta(\mu_{0v})$ is similar to the general case $\Delta(\mu_{uv})$, so we only prove formula (4) for $u\geq 1, v\geq 2$.

We would like to prove that

$$\Delta(\mu_{uv}) = \sum (-1)^{X(I)} \mu_{ij}; \mu_I,$$

where the sum is taken over $i, j, I = ((p_1, q_1), \dots, (p_j, q_j))$ such that $i + \sum_k p_k = u$, $\sum_k q_k = v$.

By formula (3) we have

$$\Delta(\mu_{uv}) = \Delta\Big(\mu_{11}(\mu_{u-1,v}) + \sum_{\substack{r+t=u\\ s+w=v}} (-1)^{|s\mu_{rs}||\mu_{tw}|+rw} \mu_{02}(\mu_{rs}, \mu_{tw})\Big). \tag{6}$$

We will evaluate the summands on the right hand side of the above formula separately using induction together with formula (3).

Assume that we have proved (4) for all μ_{kl} with $k + l < u + v$. This implies that

$$\Delta(\mu_{u-1,v}) = \sum (-1)^{X(I)} \mu_{i-1,j}; \mu_I.$$

Applying formula (3) allows us to relate this to $\Delta(\mu_{11}(\mu_{u-1,v}))$ with the result that

$$\Delta(\mu_{11}(\mu_{u-1,v})) = \mu_{01}; \mu_{11}(\mu_{u-1,v}) + \sum (-1)^0 (-1)^{X(I)} \mu_{11}(\mu_{i-1,j}); \mu_I.$$

Thus we have computed the first summand of (6). As for the second summand, the induction assumption gives us

$$\Delta(\mu_{rs}) = \sum (-1)^{X(I_1)} \mu_{\rho\tau}; \mu_{I_1} \quad \text{and} \quad \Delta(\mu_{tw}) = \sum (-1)^{X(I_2)} \mu_{\gamma\delta}; \mu_{I_2}$$

with $I_1 = ((p_1, q_1), \dots, (p_\tau, q_\tau))$ and $I_2 = ((p_{\tau+1}, q_{\tau+1}), \dots, (p_j, q_j))$. Putting this in (3) gives

$$\begin{aligned}\Delta(\mu_{02}(\mu_{rs}, \mu_{tw}))& \\ &= \sum (-1)^{\sum_{k=1}^{\tau} |\mu_{p_k q_k}||\mu_{\gamma\delta}|} (-1)^{X(I_1)+X(I_2)} \mu_{02}(\mu_{\rho\tau}, \mu_{\gamma\delta}); \mu_{I_1} \otimes \mu_{I_2} \\ &\qquad + \mu_{01}; \mu_{02}(\mu_{rs}, \mu_{tw}).\end{aligned}$$

We will feed these computations back into (6) and work out the signs to obtain the desired (4). Let $i \geq 1$ and $j \geq 2$. We are interested in computing the signs in front of elements of the type $\mu_{11}(\mu_{i-1,j}); \mu_I$ and of the type $\mu_{02}(\mu_{\rho\tau}, \mu_{\gamma\delta}); \mu_I$, where

$$\rho + \gamma = i, \quad \tau + \delta = j \quad \text{and} \quad I = ((p_1, q_1), \dots, (p_j, q_j)).$$

For the first type the sign is $(-1)^{X(I)}$. For the second type the sign is of the form $(-1)^{Y}$, where Y is computed mod 2:

$$\begin{aligned}
Y &= |s\mu_{rs}||\mu_{tw}| + rw + \sum_{k=1}^{\tau} |\mu_{p_k q_k}||\mu_{\gamma\delta}| + X(I_1) + X(I_2) \\
&= |s\mu_{rs}||\mu_{tw}| + rw + \sum_{k=1}^{\tau} |\mu_{p_k q_k}||\mu_{\gamma\delta}| \\
&\qquad + \sum_{k=1}^{\tau-1} |s\mu_{p_k q_k}| \Big(\sum_{l=k+1}^{\tau} |\mu_{p_l q_l}| \Big) + \sum_{k=1}^{\tau-1} p_k \Big(\sum_{l=k+1}^{\tau} q_l \Big) \\
&\qquad + \sum_{k=\tau+1}^{j-1} |s\mu_{p_k q_k}| \Big(\sum_{l=k+1}^{j} |\mu_{p_l q_l}| \Big) + \sum_{k=\tau+1}^{j-1} p_k \Big(\sum_{l=k+1}^{j} q_l \Big).
\end{aligned}$$

Let us now simplify the sign Y. Using the equalities

$$|\mu_{tw}| = |\mu_{\gamma\delta}| + \sum_{k=\tau+1}^{j} |\mu_{p_k q_k}|, \qquad \rho + \sum_{k=1}^{\tau} p_k = r,$$
$$|\mu_{rs}| = |\mu_{\rho\tau}| + \sum_{k=1}^{\tau} |\mu_{p_k q_k}|, \qquad \sum_{l=\tau+1}^{j} q_l = w,$$

one gets

$$\begin{aligned}
Y &= X(I) + |s\mu_{rs}||\mu_{tw}| + rw + \sum_{k=1}^{\tau} |s\mu_{p_k q_k}| \Big(\sum_{l=\tau+1}^{j} |\mu_{p_l q_l}| \Big) \\
&\qquad + \sum_{k=1}^{\tau} |\mu_{p_k q_k}|(|\mu_{\gamma\delta}|) + \Big(\sum_{l=1}^{\tau} p_k \Big) \Big(\sum_{l=\tau+1}^{j} q_l \Big) \\
&= X(I) + |s\mu_{\rho\tau}||\mu_{\gamma\delta}| + (|s\mu_{\rho\tau}| + \tau|s|) \Big(\sum_{k=\tau+1}^{j} |\mu_{p_k q_k}| \Big) + \rho w \\
&= X(I) + |s\mu_{\rho\tau}||\mu_{\gamma\delta}| + \rho(\delta - w) + \rho w \\
&= X(I) + |s\mu_{\rho\tau}||\mu_{\gamma\delta}| + \rho\delta.
\end{aligned}$$

Putting this together, we obtain a summand of the form

$$(-1)^{X(I)} (\mu_{11}(\mu_{i-1,j}); \mu_I + \sum_{\substack{\rho+\gamma=i \\ \tau+\delta=j}} (-1)^{|s\mu_{\rho\tau}||\mu_{\gamma\delta}|+\rho\delta} \mu_{02}(\mu_{\rho\tau}, \mu_{\gamma\delta}); \mu_I) = (-1)^{X(I)} \mu_{ij}; \mu_I,$$

for $i \geq 1$ and $j \geq 2$.

If $j = 1$, we are interested in computing the sign in front of the element of the type $\mu_{11}(\mu_{i-1,1}); \mu_{u-i,v}$ if $i \geq 1$ or in front of $\mu_{01}; \mu_{uv}$ if $i = 0$. In the first case one still gets $(-1)^{X(I)}$ with $I = (u-i, v)$ as well as in the second case.

If $i = 0$ and $j > 1$ we are interested in computing the sign in front of the elements of the type $\mu_{02}(\mu_{0\tau}, \mu_{0\delta}); \mu_I$ where $\tau + \delta = j$ which has already been computed and coincides with the desired sign. Consequently formula (4) is proved.

Hence the collection of μ_{ij} forms a subcooperad of the free cooperad $\mathcal{F}^c(sM_{d\mathcal{A}s})$. Furthermore it contains $s^2 R_{d\mathcal{A}s}$, since

$$\mu_{03} = sm_{02} \circ_1 sm_{02} - sm_{02} \circ_2 sm_{02},$$
$$\mu_{12} = sm_{11} \circ_1 sm_{02} - sm_{02} \circ_1 sm_{11} - sm_{02} \circ_2 sm_{11},$$
$$\mu_{21} = sm_{11} \circ_1 sm_{11}.$$

We also know that its $\boldsymbol{k}$–module structure coincides with the $\boldsymbol{k}$–module structure of $(d\mathcal{A}s)^{¡}$, since the $\boldsymbol{k}$–basis elements μ_{in} are in bijection with the ν_{in} of Proposition 2.7.

As a consequence, the cooperad described is the cooperad $(d\mathcal{A}s)^{¡}$. □

Corollary 2.9 *The infinitesimal cocomposition on $(d\mathcal{A}s)^{¡}$ is given by*

$$\Delta_{(1)}(\mu_{uv}) = \sum_{\substack{i+p=u \\ r+q+t=v \\ r+1+t=j}} (-1)^{r(1-p-q)+pt} \mu_{ij}; 1^{\otimes r} \otimes \mu_{pq} \otimes 1^{\otimes t}. \qquad \square$$

3 Derived A_∞–structures

In this section we will prove our main result, Theorem 3.2, describing derived A_∞–algebras as algebras over the operad $(d\mathcal{A}s)_\infty$. Again [1] is our main reference for the cobar construction of a cooperad over a general ground ring. We will also interpret our description in terms of coderivations and compare with Sagave's approach.

3.1 The operad dA_∞

We would now like to encode derived A_∞–algebras via an operad. Recall from Section 1 that a derived A_∞–structure on a bigraded module A consists of morphisms

$$m_{uv}\colon (A^{\otimes v})^*_* \longrightarrow A^{*+2-u-v}_{*-u}$$

such that for $u \geq 0, v \geq 1$,

$$\sum_{\substack{u=i+p\\ v=j+q-1\\ j=1+r+t}} (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0.$$

If one considers $-m_{01}$ as an internal differential of A the relation reads

$$(-m_{01})(m_{uv}) - (-1)^{u+v} \sum_{r+t+1=v} m_{uv}(1^{\otimes r} \otimes (-m_{01}) \otimes 1^{\otimes t})$$
$$= (-1)^u \sum_{\substack{u=i+p,\, v=j+q-1\\ j=1+r+t\\ (i,j)\neq(0,1),\, (p,q)\neq(0,1)}} (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}).$$

Definition 3.1 The operad dA_∞ in $\mathrm{BiCompl}_v$ is defined as the free operad

$$\mathcal{F}(\boldsymbol{k} m_{uv} : u \geq 0, v \geq 1, (u,v) \neq (0,1)),$$

together with the differential

$$\partial_\infty(m_{uv}) = (-1)^u \sum_{\substack{u=i+p,\, v=j+q-1\\ j=1+r+t\\ (i,j),(p,q)\neq(0,1)}} (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}).$$

Hence it is easily verified that an algebra over the operad dA_∞ in $\mathrm{BiCompl}_v$ is a derived A_∞–algebra in the above sense.

For a coaugmented cooperad $\mathcal{C}$, the *cobar construction* $\Omega(\mathcal{C})$ of $\mathcal{C}$ is the operad defined as $\mathcal{F}(s^{-1}\overline{\mathcal{C}})$, where $\overline{\mathcal{C}}$ is the cokernel of the coaugmentation, together with the differential $\partial_\Omega = d_1 + d_2$. Here, d_2 is induced by the infinitesimal cocomposition map $\Delta_{(1)}$ of $\mathcal{C}$ and d_1 is induced by the internal differential of $\mathcal{C}$ itself. Note that in our case $\mathcal{C} = d\mathcal{A}s$, this internal differential is trivial.

We can now state the main result of our paper.

Theorem 3.2 *The operads $(d\mathcal{A}s)_\infty = \Omega((d\mathcal{A}s)^{¡})$ and dA_∞ agree. Hence, a derived A_∞–algebra is a $(d\mathcal{A}s)_\infty$–algebra.*

Proof By definition, $\Omega((d\mathcal{A}s)^{¡})$ is the free operad on the shift of $\overline{(d\mathcal{A}s)^{¡}}$. Let us denote its generators by

$$\rho_{ij} = s^{-1}\mu_{ij} \quad \text{for } i \geq 0,\, j \geq 1, i+j \neq 1.$$

The elements μ_{ij} were described in Theorem 2.8. The element ρ_{ij} obviously has bidegree $(i, 2-i-j)$.

Recall that if $\mathcal{C}$ is a coaugmented cooperad then the differential on $\Omega(\mathcal{C})$ is obtained from $\Delta_{(1)}$ as follows. Assume

$$\Delta_{(1)}(c) = \sum c_i; 1^{\otimes r} \otimes c_j \otimes 1^{\otimes t},$$

then

$$\partial_\Omega(s^{-1}c) = \sum (-1)^{|s^{-1}||c_i|} s^{-1}c_i(1^{\otimes r} \otimes s^{-1}c_j \otimes 1^{\otimes t}).$$

From Corollary 2.9 one gets

$$\begin{aligned}
(7) \qquad \partial_\Omega(\rho_{uv}) &= -\sum_{\substack{u=i+p,\, v=j+q-1\\ j=1+r+t\\ (i,j),(p,q)\neq(0,1)}} (-1)^{r(1-p-q)+pt+i+j} \rho_{ij}(1^{\otimes r} \otimes \rho_{pq} \otimes 1^{\otimes t}) \\
&= (-1)^u \sum_{\substack{u=i+p,\, v=j+q-1\\ j=1+r+t\\ (i,j),(p,q)\neq(0,1)}} (-1)^{rq+pj+t} \rho_{ij}(1^{\otimes r} \otimes \rho_{pq} \otimes 1^{\otimes t}).
\end{aligned}$$

This is the Definition 3.1 of the operad dA_∞. □

Recall that a quadratic operad $\mathcal{P}$ is *Koszul* if the map of operads

$$\mathcal{P}_\infty := \Omega(\mathcal{P}^{¡}) \longrightarrow \mathcal{P}$$

is a quasi-isomorphism.

Proposition 3.3 *The operad $d\mathcal{A}s$ is Koszul. Thus, dA_∞ is a minimal model of $d\mathcal{A}s$.*

Proof We know that $d\mathcal{A}s = \mathcal{D} \circ \mathcal{A}s$ by Proposition 2.7. The operads $\mathcal{D}$ and $\mathcal{A}s$ are Koszul. Using [6, Theorem 8.6.5], $d\mathcal{A}s$ is Koszul. □

Remark If we do not put in the multiplication and consider just the operad $\mathcal{D}_\infty = \Omega\mathcal{D}^{¡}$ in $\mathrm{BiCompl}_v$, we obtain an operad whose algebras are precisely the twisted chain complexes. This can be seen either directly as a bigraded version of [6, Section 10.3.7] or by tracing just the $j = 1$ parts of the structure through our results.

3.2 Coderivations and Sagave's approach

We now relate derived A_∞–structures to coderivations. In the classical case, an A_∞–structure on the differential graded $\boldsymbol{k}$–module A is equivalent to a coderivation of degree $+1$ on the reduced tensor coalgebra

$$d\colon \overline{\mathcal{T}}^c(sA) \longrightarrow \overline{\mathcal{T}}^c(sA) \quad \text{such that } d^2 = 0.$$

Sagave generalised this viewpoint to derived A_∞–algebras in the following way [9, Section 4]. A derived A_∞–structure on the bigraded $\boldsymbol{k}$–module A is equivalent to a coderivation of degree $+1$,

$$\begin{array}{ccc} \overline{\mathcal{T}}^c(sA) & \xrightarrow{\quad d \quad} & \overline{\mathcal{T}}^c(sA) \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ \overline{\mathcal{T}}^c(sA)\otimes\overline{\mathcal{T}}^c(sA) & \xrightarrow{d\otimes 1+1\otimes d} & \overline{\mathcal{T}}^c(sA)\otimes\overline{\mathcal{T}}^c(sA), \end{array}$$

such that $(\mathcal{T}^c(SA), d)$ is a twisted chain complex, see Definition 1.4, [9, Lemma 4.1]. The definition of a differential of a twisted cochain complex differs from the condition $d^2 = 0$ by signs.

Our approach varies from this. In the setting of associative algebras in dg-$\boldsymbol{k}$–modules, one has

$$\mathcal{A}s^{¡}(A) = \overline{\mathcal{T}}^c(sA).$$

However, $(d\mathcal{A}s)^{¡}(A)$ is *not* given by $\overline{\mathcal{T}}^c(sA)$ in the derived setting; we showed its structure in Theorem 2.8.

So in our setting, a derived A_∞–structure on the vertical bicomplex A is given by a coderivation of degree $+1$,

$$\begin{array}{ccc} (d\mathcal{A}s)^{¡}(A) & \xrightarrow{\quad d \quad} & (d\mathcal{A}s)^{¡}(A) \\ {\scriptstyle\Delta_{(1)}}\downarrow & & \downarrow{\scriptstyle\Delta_{(1)}} \\ \big((d\mathcal{A}s)^{¡}\circ_{(1)}(d\mathcal{A}s)^{¡}\big)(A) & \xrightarrow{d\circ_{(1)}1+1\circ_{(1)}d} & \big((d\mathcal{A}s)^{¡}\circ_{(1)}(d\mathcal{A}s)^{¡}\big)(A), \end{array}$$

such that $d^2 = 0$. Comparing those two equivalent conditions we see the following. Sagave's description has the advantage of a much easier coalgebra structure while the complexity of the derived A_∞–structure is encoded in the more complicated condition that a coderivation has to satisfy. In our description, a coderivation has to satisfy the relatively simple condition $d^2 = 0$ while the complexity lies in the more complicated coalgebra structure.

4 Infinity morphisms and an application

The main purpose of this section is to describe ∞–morphisms of $(d\mathcal{A}s)_\infty = dA_\infty$–algebras, and to prove that they coincide with the derived A_∞–morphisms defined by Sagave. At the end of the section, we give an application of the homotopy transfer theorem.

4.1 Infinity morphisms

Using the language of operads, the natural notion of morphism between two dA_∞–algebras A and B is a map $f\colon A \to B$ respecting the algebra structure. This is the notion of a strict morphism. However, in the context of $\mathcal{P}_\infty$–algebras where $\mathcal{P}$ is a Koszul operad, there is also a more general notion of ∞–morphism, which is more relevant to the homotopy theory of P_∞–algebras; see, for example, [6, Section 10.2]. In the case of A_∞–algebras, this gives rise to the usual notion of A_∞–morphism between two A_∞–algebras A and B and this can be formulated as a morphism of differential graded coalgebras between the bar constructions of A and B.

As seen at the end of the previous section, a dA_∞–structure m on the vertical bicomplex A is equivalent to a square-zero coderivation D_m of degree $+1$ on the $(d\mathcal{A}s)^{¡}$–coalgebra $(d\mathcal{A}s)^{¡}(A)$. This coalgebra corresponds to the bar construction for A_∞–algebras in our framework. This lends itself to the following definition.

Definition 4.1 Let (A, m) and (B, m') be dA_∞–algebras. An *∞–morphism of dA_∞–algebras* is a morphism

$$F\colon ((d\mathcal{A}s)^{¡}(A), D_m) \longrightarrow ((d\mathcal{A}s)^{¡}(B), D_{m'})$$

of $(d\mathcal{A}s)^{¡}$–coalgebras.

We will interpret this definition in terms of twisting morphisms, but first, we give a recollection of some facts based on the book of Loday and Vallette, adapted to the category of vertical bicomplexes. We will need these as a basis for our computation.

Definition 4.2 Let $(\mathcal{C}, d_{\mathcal{C}})$ be a cooperad and $(\mathcal{P}, d_{\mathcal{P}})$ an operad in vertical bicomplexes. Following the notation of Section 2.1, we consider the collection in complexes $\mathbf{Hom}(\mathcal{C}, \mathcal{P})$. It is a differential graded operad called the *convolution operad*.

There is an operation $\star$ on $\mathbf{Hom}(\mathcal{C}, \mathcal{P})$ defined by

$$f \star g\colon \mathcal{C} \xrightarrow{\Delta_{(1)}} \mathcal{C} \circ_{(1)} \mathcal{C} \xrightarrow{f \circ_{(1)} g} \mathcal{P} \circ_{(1)} \mathcal{P} \xrightarrow{\gamma_{(1)}} \mathcal{P},$$

where $\Delta_{(1)}$ and $\gamma_{(1)}$ are respectively the infinitesimal cocomposition and composition maps. As in [6, Section 6.4.2], this determines the structure of a differential graded pre-Lie algebra on $\prod_n \mathbf{Hom}(\mathcal{C},\mathcal{P})(n)$. The associated differential graded Lie algebra is called the *convolution Lie algebra*.

Definition 4.3 A *twisting morphism* is an element α of degree 1 in the complex $\mathbf{Hom}(\mathcal{C},\mathcal{P})$ satisfying the Maurer–Cartan equation

$$\partial(\alpha)+\alpha\star\alpha=0.$$

We denote the set of twisting morphisms by $\mathrm{Tw}(\mathcal{C},\mathcal{P})$.

By construction, the cobar construction Ω satisfies

$$\mathrm{Hom}_{\mathrm{BiCompl}_v\text{–}op}(\Omega(\mathcal{C}),\mathcal{P})\cong \mathrm{Tw}(\mathcal{C},\mathcal{P}),$$

where the left-hand side means morphisms of operads in vertical bicomplexes. This means that a dA_∞–structure m on the vertical bicomplex A, that is, a square-zero coderivation D_m of degree $+1$ on the $(d\mathcal{A}s)^{¡}$–coalgebra $(d\mathcal{A}s)^{¡}(A)$ as seen at the end of the previous section, is equivalent to a twisting morphism

$$\varphi_m\in \mathrm{Tw}((d\mathcal{A}s)^{¡},\mathrm{End}_A).$$

Let A and B be vertical bicomplexes, and let End^A_B, a collection in vertical bicomplexes, be given by

$$\mathrm{End}^A_B(n)=\mathrm{Mor}(A^{\otimes n},B).$$

The vertical differential is given by

$$\partial(f)=d_B f-(-1)^j\sum_{v=0}^{n-1} f(1^{\otimes v}\otimes d_A\otimes 1^{n-v-1})$$

for f in arity n and bidegree (i,j).

For $f\in\mathbf{Hom}((d\mathcal{A}s)^{¡},\mathrm{End}^A_B)$ and $\varphi\in\mathbf{Hom}((d\mathcal{A}s)^{¡},\mathrm{End}_A)$, the map $f\ast\varphi$ is given by the composite

$$f\ast\varphi\colon (d\mathcal{A}s)^{¡}\xrightarrow{\Delta_{(1)}}(d\mathcal{A}s)^{¡}\circ_{(1)}(d\mathcal{A}s)^{¡}\xrightarrow{f\circ_{(1)}\varphi}\mathrm{End}^A_B\circ_{(1)}\mathrm{End}_A\xrightarrow{\rho}\mathrm{End}^A_B,$$

where ρ is induced by the composition of maps. Similarly, for $\psi\in\mathbf{Hom}((d\mathcal{A}s)^{¡},\mathrm{End}_B)$ and f as above, $\psi\circledast f$ is given by

$$\psi\circledast f\colon (d\mathcal{A}s)^{¡}\xrightarrow{\Delta}(d\mathcal{A}s)^{¡}\circ(d\mathcal{A}s)^{¡}\xrightarrow{\psi\circ f}\mathrm{End}_B\circ\mathrm{End}^A_B\xrightarrow{\lambda}\mathrm{End}^A_B,$$

where λ is given by composition of maps.

Now let

$$\varphi_{m^A} \in \mathrm{Tw}((d\mathcal{A}s)^{¡}, \mathrm{End}_A) \quad \text{and} \quad \varphi_{m^B} \in \mathrm{Tw}((d\mathcal{A}s)^{¡}, \mathrm{End}_B)$$

be dA_∞–structures on the vertical bicomplexes A and B respectively. By [6, Theorem 10.2.3], an ∞–morphism

$$F\colon (d\mathcal{A}s)^{¡}(A) \longrightarrow (d\mathcal{A}s)^{¡}(B)$$

of dA_∞–algebras is equivalent to an element $f \in \mathbf{Hom}((d\mathcal{A}s)^{¡}, \mathrm{End}^A_B)$ of degree 0 such that

$$f * \varphi_{m^A} - \varphi_{m^B} \circledast f = \partial(f)$$

(note that the vertical bicomplex $(d\mathcal{A}s)^{¡}(n)$ has trivial differential). Taking this into account we arrive at the following.

Theorem 4.4 *An ∞–morphism $f\colon A \to B$ of dA_∞–algebras is a morphism of derived A_∞–algebras as defined by Sagave, that is, a collection of maps*

$$f_{uv}\colon A^{\otimes v} \longrightarrow B$$

of bidegree $(u, 1-u-v)$ satisfying Equation (2) of Definition 1.2.

Proof Assume that $f\colon (d\mathcal{A}s)^{¡} \to \mathrm{End}^A_B$ satisfies

$$f * \varphi_{m^A} - \varphi_{m^B} \circledast f = \partial(f).$$

We know the structure of $(d\mathcal{A}s)^{¡}$ from Theorem 2.8. The underlying $\boldsymbol{k}$–module of $(d\mathcal{A}s)^{¡}$ is free on generators μ_{uv} of bidegree $(u, 1-u-v)$. Write

$$f_{uv} := f(\mu_{uv})$$

and recall that $\varphi_{m^A}(\mu_{ij}) = m^A_{ij}$ and $\varphi_{m^B}(\mu_{ij}) = m^B_{ij}$.

Using the formulas given by Theorem 2.8, Corollary 2.9 and because φ_{m^A} is of bidegree $(0,1)$ we obtain

$$\begin{aligned}(f * \varphi_{m^A})(\mu_{uv}) &= \sum_{\substack{u=i+p\\ v=j+q-1\\ j=r+t+1}} (-1)^{r(1-p-q)+pt+1+i+j} f_{ij}(1^{\otimes r} \otimes m^A_{pq} \otimes 1^{\otimes t})\\ &= \sum_{\substack{u=i+p\\ v=j+q-1\\ j=r+t+1}} (-1)^{rq+pj+t+u} f_{ij}(1^{\otimes r} \otimes m^A_{pq} \otimes 1^{\otimes t})\end{aligned}$$

and

$$(\varphi_{m^B} \circledast f)(\mu_{uv}) = \sum (-1)^X m^B_{ij}(f_{p_1q_1} \otimes \cdots \otimes f_{p_jq_j}),$$

where

$$X = X((p_1,q_1),\dots,(p_j,q_j)) = \sum_{k=1}^{j-1}\Big((p_k+q_k)(j+k) + q_k \sum_{l=k+1}^{j}(p_l+q_l)\Big).$$

Also,

$$\partial_{End}(f)(\mu_{uv}) = d_B f_{uv} - (-1)^{1+u+v}\sum_{l=0}^{v-1} f_{uv}(1^{\otimes l}\otimes d_A \otimes 1^{v-l-1}).$$

With $d_A = m_{01}^A$ and $d_B = m_{01}^B$, this equals

$$\partial_{End}(f)(\mu_{uv}) = m_{01}^B(f_{uv}) - (-1)^{1+u+v}\sum_{l=0}^{v-1} f_{uv}(1^{\otimes l}\otimes m_{01}^A \otimes 1^{v-l-1}).$$

Putting this together, we arrive at

$$(-1)^u \sum_{\substack{u=i+p\\ v=j+q-1\\ j=r+t+1}} (-1)^{rq+t+pj} f_{ij}(1^{\otimes r}\otimes m_{pq}^A \otimes 1^{\otimes t}) = \sum (-1)^u(-1)^\sigma m_{ij}^B(f_{p_1q_1}\otimes\cdots\otimes f_{p_jq_j}),$$

which is exactly formula (2) of Sagave's definition. □

4.2 The homotopy transfer theorem for $d\mathcal{A}s$

As an immediate application of our operadic description, we can apply the homotopy transfer theorem; see [6, Section 10.3]. To do so, we will need to now work over a ground field. Although this takes us out of the context which motivated the introduction of derived A_∞–algebras, it nonetheless gives us a new family of examples.

Let $\mathcal{P}$ be a Koszul operad, W a $\mathcal{P}_\infty$–algebra and V a homotopy retract of W. Recall that a $\mathcal{P}_\infty$–structure on W is equivalent to an element $\varphi \in \mathrm{Tw}(\mathcal{P}^{¡}, \mathrm{End}_W)$. The homotopy transfer theorem [6, Theorem 10.3.3] says that the homotopy retract V can be given a $\mathcal{P}_\infty$–structure by the twisting morphism given by the following composite

$$\mathcal{P}^{¡} \xrightarrow{\Delta} \mathcal{F}^c(\bar{\mathcal{P}}^{¡}) \xrightarrow{\mathcal{F}^c(s\varphi)} \mathcal{F}^c(s\,\mathrm{End}_W) \xrightarrow{\Psi} \mathrm{End}_V\,.$$

(The map Δ is the coproduct map defined in [6, Section 5.8.8].) Moreover there is a standard way to interpret this formula in terms of the combinatorics of trees.

We adopt the usual notation for this setting: we have the inclusion $i\colon V\to W$ and projection $p\colon W\to V$ such that pi is the identity on V, and a homotopy $h\colon W\to W$ between ip and the identity on W, $1_W - ip = d_W h + h d_W$.

As a special case, we consider $\mathcal{P} = d\mathcal{A}s$ and we let $V = A$ be a bidga over a field. The vertical homology $W = H^v(A)$ of A is a homotopy retract and we therefore obtain a derived A_∞–algebra structure on this. Write $d_h = m_{11}$ for the horizontal differential and $m = m_{02}$ for the multiplication. Making the transferred structure explicit for this special case yields the following.

Proposition 4.5 *There is a derived A_∞–algebra structure on the vertical homology $H^v(A)$ of a bidga A over a field, which can be described as follows. We obtain m_{ij} as a (suitably signed) sum over the maps corresponding to planar trees with j leaves, where each vertex has been assigned a weight of either 2 or 3, and the number of vertices of weight 2 is i. The procedure for assigning a map to such a tree is as follows. We adorn the trees with the map i on the leaves, the map p at the root and the map h on internal edges. On vertices, we put the multiplication m at every vertex of weight 3 and the horizontal differential d_h at every vertex of weight 2.* □

This construction specialises to the A_∞–case which involves binary trees with no vertices of degree 2. That is, we recover the expected A_∞–algebra structure on the part concentrated in degrees $(0, j)$; see [6, Theorem 10.3.4].

The signs can be calculated recursively from the explicit signs appearing in the formula (4) for Δ.

5 Operadic and Hochschild cohomology

In this section, we compute the tangent complex of a derived A_∞–algebra A, define the Hochschild cohomology of A and make the link with the formality theorem of [8]. Hochschild cohomology has previously only been defined, in [8], for a special class of derived A_∞–algebras, the "orthogonal" ones.

Given a vertical bicomplex A, the trigraded $\boldsymbol{k}$–module $C^{*,*}_*(A, A)$ is defined by

$$C^{n,i}_k(A, A) = \mathrm{Mor}(A^{\otimes n}, A)^i_k.$$

We will describe a graded Lie structure on $CH^{*+1}(A, A)$, where the grading is the total grading

$$CH^N(A, A) = \prod_{n\geq 1} \prod_{k,\, j \mid k+j+n=N} C^{n,j}_k(A, A),$$

that is, an element in $C^{n,j}_k(A, A)$ has *total degree* $j + k + n$.

5.1 Lie structures

Let us make explicit Definition 4.2 for the differential graded pre-Lie structure on $\prod_n \mathbf{Hom}((d\mathcal{A}s)^{¡}, \mathrm{End}_A)(n)$. From Corollary 2.9, knowing the infinitesimal cocomposition on $(d\mathcal{A}s)^{¡}$, the $\star$ operation on $\mathbf{Hom}((d\mathcal{A}s)^{¡}, \mathrm{End}_A)$ is given by

$$(8)\quad (f \star g)(\mu_{uv}) = \sum_{\substack{u=i+p\\ v=r+q+t\\ j=r+t+1}} (-1)^{r(1+p+q)+pt+|g||\mu_{ij}|} f(\mu_{ij})(1^{\otimes r} \otimes g(\mu_{pq}) \otimes 1^{\otimes t}),$$

where $|g|$ denotes the vertical grading.

For every N, there is a bijection

$$\Phi = \prod_n \Phi_n \colon \prod_n \mathbf{Hom}((d\mathcal{A}s)^{¡}, \mathrm{End}_A)(n)^N \longrightarrow \prod_n \prod_u C_u^{n,N+1-n-u}(A, A),$$

where $\Phi_n \colon \mathbf{Hom}((d\mathcal{A}s)^{¡}, \mathrm{End}_A)(n)^N \to \prod_u C_u^{n,N+1-n-u}(A, A)$ is given by evaluation:

$$\Phi_n(f_n) = \prod_u f_n(\mu_{un}).$$

The unique preimage of a family $(G_n)_n$, where $G_n = (G_u^{n,N+1-n-u})_u$, is given by the family $g = (g_n)_n = (\Phi_n^{-1}(G_n))_n$ in degree N defined via

$$g_n(\mu_{un}) = G_u^{n,N+1-n-u}.$$

We can transport the pre-Lie structure on $\prod_n \mathbf{Hom}((d\mathcal{A}s)^{¡}, \mathrm{End}_A)(n)$ to $CH^{*+1}(A,A)$ as follows: let $F = (F_n)_{n\geq 1}$ be of total degree $N+1$ and let $G = (G_m)_{m\geq 1}$ be of total degree $M+1$. There are unique families $f = (f_n)_n, g = (g_m)_m$ of degree N and M respectively such that $F = \Phi(f)$ and $G = \Phi(g)$. Then

$$F \star G := \Phi(f \star g).$$

Note that the total degree of $F \star G$ is $N+M+1$. Hence the pre-Lie product decreases the total degree by one. That is, this pre-Lie product endows $CH^{*+1}(A,A)$ with the structure of a graded pre-Lie algebra.

Naturally, this gives rise to a graded Lie algebra structure on $CH^{*+1}(A,A)$ via

$$[F, G] = F \star G - (-1)^{(N+1)(M+1)} G \star F.$$

Let us compare the pre-Lie structure above with the pre-Lie structure on $C_*^{*,*}(A,A)$ built in [8]. Let $\mathfrak{f} \in C_k^{n,i}(A,A)$ and $\mathfrak{g} \in C_l^{m,j}(A,A)$. Then

$$\mathfrak{f} = f_n(\mu_{kn}) \quad \text{with } |f_n| = n+i+k-1$$

and

$$\mathfrak{g} = g_m(\mu_{lm}) \quad \text{with } |g_m| = m + j + l - 1.$$

Putting this into formula (8) yields

$$\mathfrak{f} \star \mathfrak{g} = \sum_{r=0}^{n-1} (-1)^{(n+1)(m+1)+r(m+1)+j(n+1)+k(m+j+l+1)} \mathfrak{f}(1^{\otimes r} \otimes \mathfrak{g} \otimes 1^{\otimes n-r-1})$$
$$\in C_{k+l}^{n+m-1,i+j}.$$

Hence we can see that the sign in this formula differs from the sign in the other pre-Lie algebra structure $\mathfrak{f} \circ_{RW} \mathfrak{g}$ given in [8, Definition 2.11] by the sign $(-1)^{k(m+j+l+1)}$.

We can read off the following.

Lemma 5.1 *Let $m \in CH^2(A,A)$. Then m defines a dA_∞–structure on A if and only if $m \star m = 0$.* □

5.2 Hochschild cohomology

We now use this new Lie structure to define another notion of Hochschild cohomology of derived A_∞–algebras. This definition differs from that constructed in [8] by the different signs in the Lie structure, as explained above. It has the advantage that it applies to all dA_∞–algebras rather than just the "orthogonal" ones.

Definition 5.2 Let (A, m) be a dA_∞–algebra. Then the *Hochschild cohomology of* A is defined as

$$\mathrm{HH}^*(A,A) := \mathrm{H}^*(CH(A,A), [m, -]).$$

The morphism

$$[m, -]\colon CH^*(A,A) \longrightarrow CH^*(A,A)$$

is indeed a differential. Since m has total degree 2 and $[-,-]$ has total degree -1, it raises degree by 1. By [5, Lemma 1.10] (with respect to the pre-Lie product $\circ$), one has $[m,[m,-]] = [m \star m, -]$, and the right-hand side vanishes because of Lemma 5.1.

In the case of (A, m) being an associative algebra, this definition recovers the classical definition of Hochschild cohomology of associative algebras.

Remark Because of the bijection Φ the complex computing the Hochschild cohomology of A coincides with the operadic cohomology. Recall that given a $\mathcal{P}$–algebra A, its operadic cohomology with coefficients in itself is $H^*(\mathbf{Hom}(\mathcal{P}^{¡}(A), A), \partial_\pi)$ where π depends on the twisting cochain defining the structure on A.

As an example, when A is a bidga with $m = m_{11} + m_{02}$, ie if A is a bidga with trivial horizontal differential, the external grading is preserved by both bracketing with m_{11} and m_{02}. Hence we can, as in [8, Section 3.1], consider bigraded Hochschild cohomology

$$\mathrm{HH}^{s,r}(A,A) = \mathrm{H}^s\Big(\prod_n C^{n,r}_{*-n}(A,A), [m,-]\Big).$$

We denote this special case by $\mathrm{HH}^{*,*}_{bidga}(A,A)$. It corresponds to the operadic cohomology with respect to the operad $d\mathcal{A}s$.

When $\mathcal{P}$ is a Koszul operad, given a $\mathcal{P}_\infty$–algebra, one can still define its operadic cohomology as the homology of the complex

$$(\mathbf{Hom}(\mathcal{P}^{¡}(A), A), \partial_\pi), \tag{9}$$

where π represents the twisting cochain associated to the $\mathcal{P}_\infty$–structure on A.

If A is a derived A_∞–algebra, the complex (9) is exactly the complex of Definition 5.2. That is, operadic cohomology for derived A_∞–algebras is Hochschild cohomology as defined at the beginning of the subsection.

Note however, that in order to identify this cohomology theory with the André–Quillen cohomology of derived A_∞–algebras as in [6, Proposition 12.4.7] one needs to assume that A is bounded below for the vertical grading and is free as a $\boldsymbol{k}$–module.

This more compact definition of Hochschild cohomology has some structural advantage over HH^*_{RW}, the Hochschild cohomology defined in [8]. In particular, we see that the Lie bracket $[-,-]$ on $CH^*(A,A)$ induces a Lie bracket on

$$\mathrm{HH}^*(A,A) = \mathrm{H}^*\big(CH^*(A,A), D = [m,-]\big).$$

This is the case because D is an inner derivation with respect to $[-,-]$ due to the graded Jacobi identity. Hence, the bracket of two cycles is again a cycle, and the bracket of a boundary and a cycle is a boundary.

Proposition 5.3 *The (shifted) Hochschild cohomology $\mathrm{HH}^{*+1}(A,A)$ of a dA_∞–algebra A has the structure of a graded Lie algebra.* □

5.3 Uniqueness and formality

Definition 5.4 Let A be a bidga with $m_{01} = 0, \partial = m_{11}, \mu = m_{02}$. Then

$$a = \sum_{i,j} a_{ij}, \quad a_{ij} \in C_i^{j,2-i-j}(A,A),\ i+j \geq 3,$$

is a *twisting cochain* if $\partial + \mu + a$ is a derived A_∞–structure.

One can read off the following result immediately.

Lemma 5.5 *The element a is a twisting cochain if and only if*

$$-D(a) = a \star a$$

for $D = [\partial + \mu, -]$. □

The above is the *Maurer–Cartan formula*.

A key step in the obstruction theory leading to uniqueness of dA_∞–structures is perturbing an existing twisting cochain by an element b of total degree 1. Roughly speaking, this new perturbed dA_∞–structure satisfies the following: it equals the existing dA_∞–structure below a certain bidegree, is modified using b in this bidegree and E_2–equivalent to the "old" dA_∞–structure. This has been shown in detail in [8, Lemma 3.6], but we verify briefly that this also works with our new Lie bracket.

Lemma 5.6 *Let A be a bidga with multiplication μ, horizontal differential ∂ and trivial vertical differential. Let a be a twisting cochain. Let either of the following conditions* (A) *or* (B) *hold.*

(A) $b \in C_k^{n-1,2-(n+k)}(A,A)$ *for some k, n such that $k+n \geq 3$, satisfying* $[\partial, b] = 0$.

(B) $b \in C_{k-1}^{n,2-(n+k)}(A,A)$ *for some k, n such that $k+n \geq 3$, satisfying* $[\mu, b] = 0$.

Then there is a twisting cochain $\bar{a}$ satisfying:

- *The dA_∞–structures $\partial + \mu + a$ and $\overline{m} = \partial + \mu + \bar{a}$ are E_2–equivalent.*
- $\bar{a}_{uv} = a_{uv}$ *for $u < k$ or $v < n-1$ or $(u,v) = (k, n-1)$ in case* (A) *and for $u < k-1$ or $v < n$ or $(u,v) = (k-1, n)$ in case* (B).
- $\bar{a}_{kn} = a_{kn} - [\mu, b]$ *in case* (A).
- $\bar{a}_{kn} = a_{kn} - [\partial, b]$ *in case* (B).

Proof A quick check of the signs in both Lie brackets shows that

$$[\partial, b]_{RW} = [\partial, b] \quad \text{and} \quad [\mu, b]_{RW} = [\mu, b].$$

Hence this is identical to [8, Lemma 3.6], where the $\bar{a}_{uv}$ are constructed inductively. □

We can now proceed to our uniqueness theorem, which has been shown in the context of $[-,-]_{RW}$ and $\mathrm{HH}_{RW}^{*,*}$ in [8, Theorem 3.7].

Theorem 5.7 *Let A be a bidga with multiplication μ, horizontal differential ∂ and trivial vertical differential. If*

$$\mathrm{HH}^{r,2-r}_{bidga}(A,A)=0 \quad \text{for } r\geq 3,$$

then every dA_∞–structure on A with $m_{01}=0$, $m_{11}=\partial$ and $m_{02}=\mu$ is E_2–equivalent to the trivial one.

Proof Let m be a dA_∞–structure on A as given in the statement. We want to show that it is equivalent to the dA_∞–structure $\partial+\mu$. We can write $m=\partial+\mu+a$ with a a twisting cochain.

We look at a_{kn}, $k+n=t\geq 3$. We show that m is equivalent to a dA_∞–structure $\overline{m}=\partial+\mu+\bar{a}$ with $\bar{a}_{kn}=0$ for fixed t by induction on k.

To start this induction we assume that

$$a_{ij}=0 \quad \text{for} \quad i+j<t \quad \text{and for} \quad i+j=t, \quad \text{if} \quad i<k.$$

The new equivalent dA_∞–structure $\overline{m}$ will also satisfy

$$\bar{a}_{ij}=a_{ij}=0 \quad \text{for} \quad i+j<t \quad \text{and for} \quad i+j=t, \quad \text{if} \quad i<k$$

as well as further

$$\bar{a}_{kn}=0.$$

So to construct $\overline{m}$, we "kill" a_{kn} but leave the trivial lower degree a_{ij} invariant.

Since a is a twisting cochain, it satisfies the Maurer–Cartan formula

$$-D(a)=a\star a.$$

However, an argument similar to [8, Theorem 3.7] shows that this implies $D(a_{kn})=0$ for degree reasons. Hence a_{kn} is a cycle and gives us a class

$$[a_{kn}]\in \mathrm{HH}^{k+n,2-k-n}_{bidga}(A,A)$$

in the Hochschild cohomology of A. This cohomology group has been assumed to be zero, hence a_{kn} must be a boundary too. Thus, there is a b of total degree 1 with $D(b)=a_{kn}$. For degree reasons, this b has to be of the form

$$b=b_0+b_1, \quad b_0\in C^{n,2-n-k}_{k-1}(A,A), \quad b_1\in C^{n-1,2-n-k}_{k}(A,A),$$

with

$$[\mu,b_0]=0 \quad \text{and} \quad [\partial,b_1]=0,$$

meaning that

$$D(b)=D(b_0+b_1)=[\mu,b_1]+[\partial,b_0].$$

Then, just as in the proof of [8, Theorem 3.7], applying Lemma 5.6 to b_1 yields a dA_∞–structure $\overline{m} = \partial + \mu + \overline{a}$ with

$$\bar{a}_{kn} = a_{kn} - [\mu, b_1] - [\partial, b_0] = a_{kn} - D(b) = 0. \qquad \square$$

It was shown in [8, Section 4] that $\mathrm{HH}^{*,*}_{RW}(A,A)$ is invariant under E_2–equivalences. Since this argument is independent of choice of signs in the Lie bracket, it also holds for our $\mathrm{HH}^{*,*}_{bidga}(A,A)$. Hence we can now give a criterion for intrinsic formality of a dga. (Recall that a dga A is intrinsically formal if for any other dga B with $\mathrm{H}^*(A) \cong \mathrm{H}^*(B)$ as associative algebras, A and B are quasi-isomorphic.)

Corollary 5.8 *Let A be a dga and E its minimal model with dA_∞–structure m. By $\widetilde{E}$, we denote the underlying bidga of E, ie $\widetilde{E} = E$ as k–modules together with dA_∞–structure $\widetilde{m} = m_{11} + m_{02}$. If*

$$\mathrm{HH}^{m,2-m}_{bidga}(\widetilde{E}, \widetilde{E}) = 0 \quad \text{for } m \geq 3,$$

then A is intrinsically formal. $\square$

6 Directions for further work

In this paper we have given an operadic perspective on derived A_∞–structures, allowing us to view derived A_∞–algebras as algebras over an operad. By results of Fresse [2], Harper [3] and Muro [7], it follows from our description that there is a model category structure on derived A_∞–algebras such that the weak equivalences are the E_1–equivalences (see Definition 1.5). However, we do not expect this model structure to be homotopically meaningful. Indeed, in order to view Sagave's minimal models as some kind of cofibrant replacement, one would need a model structure in which the weak equivalences are the E_2–equivalences. Producing such a model structure will involve a change of underlying category, probably to the category of twisted chain complexes. One would then need a suitable model structure on this underlying category and also to develop the appropriate notion of cobar construction. The apparent complication in carrying out such a programme explains our choice to work with vertical bicomplexes in this paper. We expect to return to this in future work.

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