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The Lawrence–Sullivan construction is the right model for I^+

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We prove that the universal enveloping algebra of the Lawrence–Sullivan construction is a particular perturbation of the complete Baues–Lemaire cylinder of S^0. Together with other evidence we present, this exhibits the Lawrence–Sullivan construction as the right model for I^+. From this, we also deduce a generalized Euler formula on Bernoulli numbers.

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1 Introduction

R Lawrence and D Sullivan [8] constructed what they called a *free Lie model for the interval*. It is a complete free differential Lie algebra $\mathfrak{L} = (\widehat{\mathbb{L}}(a,b,z),\partial)$ in which a and b are *Maurer–Cartan* or *flat* elements and the differential on z is the only one which makes a and b gauge equivalent (see next section for a precise definition). In very general terms, it is based on the idea that 0–simplices (points) are represented by Maurer–Cartan elements and 1–simplices (paths) correspond to *gauge transformations* between the endpoints.

In this beautiful construction and the deep ideas behind it there is, however, a particular Maurer–Cartan element which has not received attention, namely $0 \in \mathfrak{L}_{-1}$. Taking that into account we find three Maurer–Cartan elements $0, a, b \in \mathfrak{L}_{-1}$, and thus three points in the space that it represents, together with a gauge transformation from a to b representing a path joining two of the three vertices. Hence, what $\mathfrak{L}$ really models is $I^+ = I \dot{\cup} \{*\}$, the disjoint union of an interval and an exterior point.

That the spatial realization of this complete Lie algebra is in fact of the homotopy type of two contractible components is a particular instance of the more general context of homotopy theory in the category of (unbounded) differential graded Lie algebras, or more generally, L_∞–algebras; see the authors' [5].

In this paper we give the Lawrence–Sullivan construction the consideration that it deserves by proving that it satisfies essential functorial properties which reflect the role

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of I^+ in the based homotopy category. We begin by observing (see Proposition 3.1) that the differential in $\mathfrak{L}$ is characterized by being the only one for which the following holds: two Maurer–Cartan elements $x, y \in L_{-1}$ of a given differential graded Lie algebra L, DGL henceforth, are gauge equivalent if and only if there exist a morphism of differential graded algebras $\mathfrak{L} \to L$ sending a to x and b to y. This readily implies, as we will remark, that the classical Quillen notion of homotopy of morphisms corresponds, under this scope, to the standard homotopy notion in the based category in which $S^0 \wedge I = I^+$ is the right cylinder. In other words, $\mathfrak{L}$ is a good cylinder of the model of S^0 given by $(\mathbb{L}(u), \partial)$, $\partial(u) = -\frac{1}{2}[u,u]$.

Our main result, Theorem 3.3, corroborates this assertion. We show that the universal enveloping algebra of $\mathfrak{L}$ is a complete tensor algebra whose differential is a perturbation of the classical Baues–Lemaire cylinder [1] of the tensor model of S^0 given by $T(u)$, with u of degree -1 and $d(u) = -u \otimes u$.

With all of the above a model of I is then $(\mathbb{L}(a,z), \partial)$, obtained as the quotient of $\mathfrak{L}$ by $(\mathbb{L}(b), \partial)$. Theorem 3.6 illustrates this situation.

We finish with an application of Theorem 3.3 in number theory by obtaining, in the same spirit as Parent and Tanré [13], a generalized *Euler formula* on Bernoulli numbers.

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2 Notation and tools

In this note, the coefficient field $\mathbb{K}$ is assumed to be of characteristic zero. Differential Lie algebras, DGL from now on, as well as any other graded object, are considered $\mathbb{Z}$–graded. Given a graded vector space V, $\mathbb{L}(V)$ denotes the *free lie algebra* generated by V. If in the tensor algebra $T(V) = \sum_{n\geq 0} T^n(V)$ we consider the Lie structure given by commutators, $\mathbb{L}(V)$ is the Lie subalgebra generated by V. In the same way, replacing $T(V)$ by the *complete tensor algebra* $\widehat{T}(V) = \prod_{n\geq 0} T^n(V)$, we obtain $\widehat{\mathbb{L}}(V)$, the *complete free Lie algebra* generated by V. A generic element of $\widehat{T}(V)$ will be written as a formal series $\sum_{n\geq 0} \phi_n$, $\phi_n \in T^n(V)$. Note that $T(V) \subset \widehat{T}(V)$ and $\mathbb{L}(V) \subset \widehat{\mathbb{L}}(V)$. The universal enveloping algebra $U\mathbb{L}(V)$ of $\mathbb{L}(V)$ extends to the complete free Lie algebra to produce a graded algebra $U\widehat{\mathbb{L}}(V)$ naturally isomorphic to $\widehat{T}(V)$.

A *Maurer–Cartan* or *flat* element of a given DGL is a degree -1 element a which satisfies $\partial(a) = -\frac{1}{2}[a,a]$. We denote by $MC(L)$ the set of all Maurer–Cartan elements of L.

Let L be either a complete free Lie algebra or any DGL in which the adjoint action of L_0 is locally nilpotent, ie for any $x \in L_0$ there is an integer i such that $\mathrm{ad}_x^i = 0$. The *gauge action* of L_0 on $MC(L)$ (see Lawrence and Sullivan [8] and Manetti [10]) is defined as follows: given $x \in L_0$ and $a \in MC(L)$,

$$x * a = e^{\mathrm{ad}_x}(a) - \frac{e^{\mathrm{ad}_x}(\partial x) - (\partial x)}{\mathrm{ad}_x(\partial x)} = e^{\mathrm{ad}_x}(a) - f_x(\partial x),$$

where $e^{\mathrm{ad}_x} = \sum_{n\geq 0} \frac{(\mathrm{ad}_x)^n}{n!}$ and, as an operator,

$$f_x = \frac{e^{\mathrm{ad}_x} - \mathrm{id}}{\mathrm{ad}_x}.$$

Explicitly,

$$x * a = \sum_{i\geq 0} \frac{\mathrm{ad}_x^i(a)}{i!} - \sum_{i\geq 0} \frac{\mathrm{ad}_x^i(\partial x)}{(i+1)!}.$$

This can also be introduced in this way: Consider $L \otimes \Lambda t = L[t]$, the DGL in which t has degree zero, the Lie bracket is given by the one in L and multiplication on $\mathbb{K}[t]$, and the differential arises from that on L and by setting $\partial t = 0$. Then $x * a = p(1)$, where $p(t) \in L[t]$ is the only formal power series with coefficients in L which is a solution of the differential equation

$$\begin{aligned} p'(t) &= \partial x - \mathrm{ad}_x\, p(t), \\ p(0) &= a. \end{aligned}$$

Indeed, if we write $p(t) = \sum_{i\geq 0} a_i t^i$, $a_i \in L$, the only solution for this equation is given recursively by

$$a_0 = a, \quad a_1 = \partial x - \mathrm{ad}_x(a), \quad a_{n+1} = -\frac{\mathrm{ad}_x(a_n)}{n+1}.$$

That is,

$$a_n = \frac{(-\mathrm{ad}_x)^{n-1}}{n!}(\partial x) + \frac{(-\mathrm{ad}_x)^n}{n!}(a),$$

and therefore $p(1) = e^{\mathrm{ad}_x}(a) - f_x(\partial x)$.

The gauge action determines an equivalence relation among flat elements which coincides with the usual homotopy relation on $MC(L)$ (see for instance [10, Theorem 5.5]), which we now recall. Two elements $u, v \in MC(L)$ are said to be homotopic, and we write $u \sim v$, if there is a Maurer–Cartan element $\phi \in L \otimes \Lambda(t, dt)$ such that $\varepsilon_0(\phi) = u$ and $\varepsilon_1(\phi) = v$. Here $\Lambda(t, dt)$ is the free commutative algebra generated by t and dt, of degrees 0 and 1 respectively, and $\varepsilon_0, \varepsilon_1\colon L \otimes \Lambda(t, dt) \to L$ are the DGL morphisms obtained by evaluating t at 0 and 1 respectively.

The *Lawrence–Sullivan construction* is the complete free DGL $(\widehat{\mathbb{L}}(a, b, z), \partial)$, denoted by $\mathfrak{L}$, in which a and b are flat elements and

$$\partial(z) = [z, b] + \sum_{i=0}^{\infty} \frac{B_i}{i!} \operatorname{ad}_z^i (b - a),$$

where B_i denotes the i–th Bernoulli number. Equivalently, as shown in [8],

$$\partial z = \operatorname{ad}_z(b) + f_z^{-1}(b - a),$$

where, again as an operator,

$$f_z^{-1} = \frac{\operatorname{ad}_z}{e^{\operatorname{ad}_z} - \operatorname{id}}.$$

Observe that f_z^{-1} is indeed the inverse operator of f_z. Another inductive description of the differential in this complete free Lie algebra was suggested in [8] and shown to be equivalent to the above in [13, Main Theorem].

Last result of next section assumes basic knowledge on L_∞–algebras. From the rational homotopy theory point of view on these objects, we refer, for instance, to [7].

3 The DGL cylinder of S^0 and a model of the interval

We begin by the following observation:

Proposition 3.1 *Two Maurer–Cartan elements $u, v \in L_{-1}$ are homotopic if and only if there is a DGL morphism $\Phi\colon \mathfrak{L} \to L$ such that $\Phi(a) = u$ and $\Phi(b) = v$.*

Proof Let $\Phi\colon \mathfrak{L} \to L$ such that $\Phi(a) = u$ and $\Phi(b) = v$, with $u, v \in MC(L)$, and consider $w = \Phi(z) \in L_0$. Then

$$\begin{aligned}
\partial w = \Phi(\partial z) &= \Phi\left(\operatorname{ad}_z(b) + \frac{\operatorname{ad}_z}{e^{\operatorname{ad}_z} - \operatorname{id}}(b - a)\right) \\
&= \operatorname{ad}_{\Phi(z)} \Phi(b) + \frac{\operatorname{ad}_{\Phi(z)}}{e^{\operatorname{ad}_{\Phi(z)}} - \operatorname{id}}\big(\Phi(b) - \Phi(a)\big) \\
&= \operatorname{ad}_w(v) + \frac{\operatorname{ad}_w}{e^{\operatorname{ad}_w} - \operatorname{id}}(v - u).
\end{aligned}$$

Then
$$\frac{e^{\mathrm{ad}_w}-\mathrm{id}}{\mathrm{ad}_w}(\partial w)=\frac{e^{\mathrm{ad}_w}-\mathrm{id}}{\mathrm{ad}_w}(\mathrm{ad}_w\, b)+v-u=e^{\mathrm{ad}_w}(v)-u,$$
and thus, $u=e^{\mathrm{ad}_w}(v)-f_w(\partial w)$. In other words, u and v are gauge equivalent, and thus, homotopy equivalent.

Reciprocally, if the elements $u,v\in MC(L)$ are homotopic, there is $w\in L_0$ such that $u=e^{\mathrm{ad}_w}(v)-f_w(\partial w)$. Define $\Phi(b)=v$, $\Phi(a)=u$ and $\Phi(z)=w$, then the above computation shows that this is a DGL morphism. □

Remark 3.2 We see that $\mathfrak{L}$ fits perfectly in the classical Quillen approach to rational homotopy [15] by which the homotopy category of reduced DGL is equivalent to the based homotopy category of simply connected rational spaces. In this category, two pointed maps $f,g\colon (X,x_0)\to(Y,y_0)$ are homotopic by means of a based homotopy $(X\times I,x_0\times I)\to(Y,y_0)$. But this corresponds, via the exponential, to a based map $S^0\wedge I=I^+\to\mathrm{map}^*(X,Y)$ sending the exterior point to the constant map. In the algebraic setting, let C and L be a differential graded coalgebra and a reduced DGL, models of X and Y respectively, with the appropriate connectivity restrictions, and let $\varphi,\psi\colon\mathcal{L}(C)\to L$ be models of f and g. Here $\mathcal{L}$ denotes the classical Quillen functor; see for instance [6, Section 22]. It is known [3, Theorem 10] that the convolution DGL $Hom(C_+,L)$ is a DGL model of the nonconnected pointed mapping space $\mathrm{map}^*(X,Y)$. Moreover, the restrictions $\varphi_{|C_+},\psi_{|C_+}\colon C_+\to L$ are Maurer–Cartan elements of this DGL. By the proposition above, these morphisms are homotopic if and only if there exists $\Phi\colon\mathfrak{L}\to Hom(C_+,L)$ such that $\Phi(a)=\varphi_{|C_+}$ and $\Phi(b)=\psi_{|C_+}$.

In [1, Section 1], H Baues and J M Lemaire defined a cylinder of a free differential graded algebra, denoted by DGA henceforth, $(T(V),d)$. It is the free tensor algebra $\mathrm{Cyl}\,T(V)=T(V\oplus V'\oplus sV)$ in which V' is a copy of V and sV denotes the suspension of V, $(sV)_p=V_{p-1}$. The differentials in V and V' are defined so that the inclusions $i_0,i_1\colon T(V)\hookrightarrow T(V\oplus V'\oplus sV)$, where $i_0(v)=v$, $i_1(v)=v'$, with $v\in V$, are DGA morphisms. Consider in $\mathrm{Cyl}\,T(V)$ the (i_0,i_1)–derivation S of degree -1 given by $Sv=Sv'=sv$ and $Ssv=0$. Then the differential in sV is defined by $d(sv)=v-v'-Sdv$, $v\in V$.

This is a right cylinder in the category of DGA, as both i_0,i_1 are injective and the projection $p\colon T(V\oplus V'\oplus sV)\to T(V)$, $p(v)=p(v')=v$, $p(sv)=0$, $v\in V$, is a quasi-isomorphism for which $pi_0=pi_1=1_{|T(V)}$.

In the case when $V=\langle u\rangle$, with u of degree -1 and $du=-u\otimes u$, we obtain $\mathrm{Cyl}\,T(u)=(T(u\oplus u'\oplus su),d)$ with differential $du=-u\otimes u$, $du'=-u'\otimes u'$ and $d(su)=u'-u+su\otimes u'-u\otimes su$. Note that $T(u)$ can be thought of as a model for

S^0 as $\operatorname{Cyl} T(u)$ can be used as a cylinder to describe the usual notion of homotopy in the category of DGA [9]. In $\widehat{T}(u \oplus u' \oplus su)$ we define D as the only derivation which extends d in u and u' and satisfies

$$(1) \quad D(su) = su \otimes u' - u' \otimes su + \sum_{n \geq 0} \sum_{p+q=n} (-1)^q \frac{B_n}{p!q!} su^{\otimes p} \otimes (u' - u) \otimes su^{\otimes q},$$

where each B_n denotes the n–th Bernoulli number. Denote by $\widehat{\operatorname{Cyl}} T(u)$ the pair $(\widehat{T}(u \oplus u' \oplus su), D)$. We now show that D is indeed a differential for which:

Theorem 3.3 *$U\mathfrak{L}$ is isomorphic as a DGA to $\widehat{\operatorname{Cyl}} T(u)$.*

Proof We show that the natural isomorphism of graded algebras $U\mathfrak{L} \cong \widehat{\operatorname{Cyl}} T(u)$ commutes with differentials on generators. This also guarantees that D is indeed a differential. Write $Du = du = -u \otimes u = -\frac{1}{2} u \otimes u - \frac{1}{2} u \otimes u$, which arises from $-\frac{1}{2}[a,a] = \partial a$ via the injective map $\mathfrak{L} \hookrightarrow U\mathfrak{L} \cong \widehat{\operatorname{Cyl}} T(u)$ provided by the Poincaré–Birkhoff–Witt theorem. The same applies to Du' and ∂b.

On the other hand, if we denote by cls the class of the corresponding element of $U\mathfrak{L} = \widehat{T}(\mathfrak{L})/\sim$, we have:

$$\begin{aligned}
\operatorname{cls}\{Dsu\} &= \operatorname{cls}\Big\{ su \otimes u' - u' \otimes su + \sum_{n \geq 0} \sum_{p+q=n} (-1)^q \frac{B_n}{p!q!} su^{\otimes p} \otimes (u' - u) \otimes su^{\otimes q} \Big\} \\
&= \operatorname{cls}\Big\{ \operatorname{ad}_{su} u' + \sum_{n \geq 0} \sum_{p+q=n} (-1)^q \frac{B_n}{n!} \binom{n}{q} su^{\otimes p} \otimes (u' - u) \otimes su^{\otimes q} \Big\} \\
&= \operatorname{cls}\Big\{ \operatorname{ad}_{su} u' + \sum_{n=0}^{\infty} \frac{B_n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} su^{\otimes n-k} \otimes (u' - u) \otimes su^{\otimes k} \Big\} \\
&= \operatorname{cls}\Big\{ \operatorname{ad}_{su} u' + \sum_{n=0}^{\infty} \frac{B_n}{n!} [\underbrace{su, [su, \dots, [su}_{n}, u' - u]] \dots] \Big\} \\
&= \operatorname{cls}\Big\{ \operatorname{ad}_{su} u' + \sum_{n=0}^{\infty} \frac{B_n}{n!} \operatorname{ad}_{su}^n (u' - u) \Big\}
\end{aligned}$$

Again, by the Poincaré–Birkhoff–Witt theorem, this class arises from ∂z via the injection $\mathfrak{L} \hookrightarrow U\mathfrak{L} \cong \widehat{\operatorname{Cyl}} T(u)$ and the theorem is proved. □

Recall that, in a complete tensor algebra $\widehat{T}(V)$, the *Baker–Campbell–Hausdorff formula* of $x, y \in V$ reads

$$\mathrm{BCH}(y,x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\Bigg(\sum_{\substack{p,q=0\\ p+q>0}}^{\infty}\frac{y^p x^q}{p!q!}\Bigg)^{k}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum\frac{y^{p_1}x^{q_1}\cdots y^{p_k}x^{q_k}}{p_1!q_1!\cdots p_k!q_k!},$$

where in the last term, the internal sum is taken over all possible collections of integral nonnegative numbers $(p_1,\dots,p_k,q_1,\dots,q_k)$ such that $p_1+q_1>0,\dots,p_k+q_k>0$. Here, and in what follows, for simplicity in the notation, we have omitted $\otimes$ in all tensor products. As in the classical case, this formula is obtained as

$$\mathrm{BCH}(y,x)=\log(e^y e^x),$$

where

$$e^x=\sum_{n\geq 0}\frac{x^n}{n!}$$

and, for a given $\varphi\in\widehat{T}^{+}(V)=\prod_{i\geq 1}T^i(V)$,

$$\log(1+\varphi)=\sum_{n\geq 1}(-1)^{n-1}\frac{\varphi^n}{n}.$$

We denote by $\mathrm{BCH}_y(y,x)$ the part of $\mathrm{BCH}(y,x)$ obtained by considering only those summands which are linear on y. Then it is known (see for instance [14, Lecture 6] for the ungraded case) that in $U\widehat{\mathbb{L}}(V)\cong\widehat{T}(V)$,

$$\mathrm{cls}\{\mathrm{BCH}_y(y,x)\}=\mathrm{cls}\Bigg\{\sum_{i\geq 0}\frac{B_i}{i!}\,\mathrm{ad}_x^i\,y\Bigg\}.$$

Hence, from Theorem 3.3 we immediately obtain:

Corollary 3.4 *In the notation of Theorem 3.3,*

$$\mathrm{BCH}_{u'-u}(u'-u,su)=\sum_{p+q=n}(-1)^q\frac{B_{p+q}}{p!q!}su^{\otimes p}\otimes(u'-u)\otimes su^{\otimes q}$$

and the formula for Dsu in (1) can be rewritten as

$$Dsu=su\otimes u'-u'\otimes su+\mathrm{BCH}_{u'-u}(u'-u,su).\qquad\square$$

Remark 3.5 (1) Note that the projection $p\colon\widehat{\mathrm{Cyl}}(u)\to(T(u),d)$, $p(u)=p(u')=u$, $p(su)=0$, is again a quasi-isomorphism, as it is so on the indecomposables. Thus, $\widehat{\mathrm{Cyl}}(u)$ is again a cylinder for $(T(u),d)$.

(2) Consider the DGL $(\mathbb{L}(u),\partial)$, $\partial u=-\frac{1}{2}[u,u]$, and observe that the universal enveloping algebra functor on the inclusions $(\mathbb{L}(u),\partial)\hookrightarrow\mathfrak{L}$ mapping u to a and b respectively, provides, by Theorem 3.3, the natural inclusions $i_0,i_1\colon (T(u),d)\hookrightarrow \widehat{\mathrm{Cyl}}\,T(u)$. That is, the Lawrence–Sullivan construction is a good DGL cylinder of $(\mathbb{L}(u),\partial)$. Now observe that this is a DGL model of S^0. Indeed, its commutative cochain graded algebra [6, Section 23], $C^*(\mathbb{L}(u),\partial)$, can be easily computed to yield $(\Lambda(x,y),d)$ with x and y of degree 0 and -1 respectively, $dx=0$ and $dy=\frac{1}{2}(x^2-x)$. This is a model of S^0 under any possible interpretation. On the one hand, it is an easy exercise to show that the geometric simplicial realization of this algebra has the homotopy type of S^0, each of its points being given by the augmentations sending x to 0 and 1 respectively. On the other hand, consider the DGA model of S^0 given by $\mathbb{Q}\alpha\oplus\mathbb{Q}\beta$ with α and β of degree 0, and products $\alpha^2=\alpha$, $\beta^2=\beta$ and $\alpha\beta=0$. Note that the identity in this algebra is $\alpha+\beta$. Hence, replacing $\alpha+\beta$ by 1 and β by x, this DGA is isomorphic to $\mathbb{Q}\oplus\mathbb{Q}x$ with $x^2=x$, which is quasi-isomorphic to $(\Lambda(x,y),d)$ with $dy=\frac{1}{2}(x^2-x)$.

In other words, $\mathfrak{L}$ geometrically describes the cylinder of S^0 in the pointed category, namely, I^+.

(3) In the same way, a good candidate for a complete model of the interval I would be then a model of the cofiber of the based map $S^0\to I\dot{\cup}*$, which sends the non base point of S^0 to any of the endpoints of the interval. In DGL this would be modeled by the inclusion $(\mathbb{L}(b),\partial)\hookrightarrow\mathfrak{L}$ whose cofiber is

$$\mathfrak{L}_I=(\widehat{\mathbb{L}}(a,z),\partial),\quad \text{in which}\quad \partial(a)=-\tfrac{1}{2}[a,a],\quad \partial(z)=-\sum_{i\geq 0}\frac{B_i}{i!}\,\mathrm{ad}_z^i(a).$$

Again, the fact that the realization of $\mathfrak{L}_I$ is a contractible space is a particular case of a much broader setting [5]. Here we corroborate this fact with a variant of [4, Theorem 4.1] that we now present.

Let L be any DGL and consider, for each $\lambda\in\mathbb{K}$, the DGL morphism

$$\varepsilon_\lambda\colon L\otimes\Lambda(t,dt)\longrightarrow L,\quad \varepsilon(p(t)+q(t)dt)=p(\lambda),$$

given by evaluating at λ. Here $p,q\in L\otimes\Lambda t=L[t]$ are considered, as before, as polynomials in t with coefficients in L. If we restrict to polynomials with no constant terms and evaluate at 1, the resulting DGL morphism,

$$\varepsilon_1\colon L\otimes\Lambda^+(t,dt)\longrightarrow L,$$

is a DGL model in the classical Quillen sense of the evaluation fibration

$$ev\colon \mathrm{map}^*(I,Y)\longrightarrow Y,\quad ev(\gamma)=\gamma(1),$$

whenever L is a reduced DGL model of Y [3]. Recall that map^* denotes the pointed mapping space. We will not assume here any bounding condition on L but will keep in mind this geometrical interpretation.

On the other hand, consider the chain map

$$\varepsilon_a\colon s^{-1}Der(\mathfrak{L}_I, L) \longrightarrow L, \quad \varepsilon_a(\theta) = \theta(a),$$

in which $s^{-1}Der(\mathfrak{L}_I, L)$ are the *desuspended* derivations with the usual differential. This chain complex admits a DGL structure only *up to homotopy*, ie an L_∞–structure in which the higher brackets $\{\ell_k\}_{k\geq 2}$ are given as follows (see [2; 4]):

$$s\ell_k(s^{-1}\theta_1,\dots,s^{-1}\theta_k)(z) = (-1)^{\epsilon}\frac{B_{k-1}}{(k-1)!}\sum_{\sigma\in S_k}\epsilon'[\theta_{\sigma(1)}z,[\theta_{\sigma(2)}z,\dots,[\theta_{\sigma(k-1)}z,\theta_{\sigma(k)}a]]\cdots],$$

where $\epsilon = k+1+\sum_{j=1}^{k}(k+1-j)|\theta_j|$ and ϵ' is given by the Koszul convention. Then we have:

Theorem 3.6 *The map*

$$\varphi\colon s^{-1}Der(\mathfrak{L}_I, L) \longrightarrow L\otimes\Lambda^+(t,dt), \quad \varphi(s^{-1}\theta) = \theta(a)\otimes t + \theta(z)\otimes dt,$$

is a quasi-isomorphism of L_∞–algebras which makes the following diagram commute:

$$\begin{tikzcd} s^{-1}Der(\mathfrak{L}_I, L) \arrow[rr, "\varphi", "\simeq"'] \arrow[dr, "\varepsilon_a"'] & & L\otimes\Lambda^+(t,dt) \arrow[dl, "\varepsilon_1"] \\ & L & \end{tikzcd}$$

Proof Observe that φ is precisely the quasi-isomorphism of L_∞–algebras denoted by Q in the proof of [4, Theorem 4.1]. We have just extend it to unbounded derivations and replaced $s^{-1}Der(\mathcal{L}(\Lambda(t,dt))^\sharp, L)$ by the isomorphic DGL $L\otimes\Lambda(t,dt)$. With this in mind, the diagram above trivially commutes. □

4 A generalized Euler identity

Recall that the Bernoulli numbers can be recursively defined in several ways starting with $B_0 = 1$. One is via the identity

$$-nB_n = \sum_{k=1}^{n}\binom{n}{k}B_kB_{n-k} + nB_{n-1},$$

which becomes the *Euler formula*

$$-\frac{(n+1)B_n}{n!}=\sum_{k=2}^{n-2}\frac{B_k}{k!}\frac{B_{n-k}}{(n-k)!}$$

when n is an even integer greater than 2. Here we prove an extended version which may be compared, for instance, to the Miki [12] or Matiyasevich [11] identities:

Theorem 4.1 *For any even $n\geq 2$ and $0\leq m\leq n-1$,*

$$-\frac{B_n}{n!}\binom{n+1}{n-m}=\sum_{i=2}^{m}\frac{B_i}{i!}\frac{B_{n-i}}{(n-i)!}\binom{n-i}{n-m-1}-\sum_{j=2}^{n-m-1}\frac{B_j}{j!}\frac{B_{n-j}}{(n-j)!}\binom{n-j}{m}.$$

Here, whenever m or $n-m-1$ are smaller than 2, the corresponding summand in the formula above is considered to be zero. Note also that for $m=n-1$ we recover Euler's formula. In [13, Proposition 8] a different and interesting formula involving Bernoulli numbers is deduced from the inductive definition of the differential in the Lawrence–Sullivan construction.

Proof In $\widehat{\mathrm{Cyl}}\,T(u)$, from now on and to avoid excessive notation, we often omit the symbol $\otimes$. For the same purpose we define $y=u'-u$ and $x=su$.

Using the formula for Dx in Corollary 3.4, a short computation shows that $D^2x=0$ translates to

$$D^2x=D\,\mathrm{BCH}_y(y,x)+\mathrm{BCH}_y(y,x)\otimes u'+u'\otimes \mathrm{BCH}_y(y,x), \tag{2}$$

in which

$$\mathrm{BCH}_y(y,x)=\sum_{n=0}^{\infty}\sum_{p+q=n}c_{(p,q)}x^pyx^q,$$

where

$$c_{(p,q)}=(-1)^q\frac{B_{p+q}}{(p+q)!}\binom{p+q}{q}. \tag{3}$$

As $Dx^m=x^mu'-u'x^m+\sum_{i=0}^{m-1}x^i\,\mathrm{BCH}_y(y,x)x^{m-1-i}$, a straightforward computations shows that

$$D(x^pyx^q)=-u'x^pyx^q-x^pyx^qu'+\Gamma,$$

with

$$\Gamma = x^p y^2 x^q + \sum_{i=0}^{p-1} x^i \,\mathrm{BCH}_y(y,x) x^{p-i-1} y x^q - \sum_{i=0}^{q-1} x^p y x^i \,\mathrm{BCH}_y(y,x) x^{q-i-1}.$$

Hence

$$\begin{aligned} D\,\mathrm{BCH}_y(y,x) &= \sum_{n=0}^{\infty} \sum_{p+q=n} c_{(p,q)} D(x^p y x^q) \\ &= -u' \otimes \sum_{n=0}^{\infty} \sum_{p+q=n} c_{(p,q)} x^p y x^q - \sum_{n=0}^{\infty} \sum_{p+q=n} c_{(p,q)} x^p y x^q \otimes u' \\ &\qquad + \sum_{n=0}^{\infty} \sum_{p+q=n} c_{(p,q)} \Gamma \\ &= -\mathrm{BCH}_y(y,x) \otimes u' - u' \otimes \mathrm{BCH}_y(y,x) + \sum_{n=0}^{\infty} \sum_{p+q=n} c_{(p,q)} \Gamma. \end{aligned}$$

Therefore, in view of (2),

$$\sum_{n=0}^{\infty} \sum_{p+q=n} c_{(p,q)} \Gamma = 0.$$

In particular, for each p and q, the coefficient of $x^p y^2 x^q$ in this formula is zero. Another straightforward computation shows that this coefficient is

$$c_{(p,q)} + \sum_{i=0}^{p} c_{(p+1-i,q)} c_{(i,0)} - \sum_{j=0}^{q} c_{(p,q+1-j)} c_{(0,j)} = 0. \tag{4}$$

Observe that, in this expression, the term for $i = j = 1$ cancels with $c_{(p,q)}$, since $c_{(1,0)} = -c_{(0,1)} = -\frac{1}{2}$. Also, the term for $i = j = 0$ is $c_{(p+1,q)} - c_{(p,q+1)}$, as $c_{(0,0)} = 1$. Thus (4) reads

$$c_{(p,q+1)} - c_{(p+1,q)} = \sum_{i=2}^{p} c_{(p+1-i,q)} c_{(i,0)} - \sum_{j=2}^{q} c_{(p,q+1-j)} c_{(0,j)}.$$

Finally, replacing each of these numbers by its value from (3), the theorem follows. □

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