Proof of a stronger version of the AJ Conjecture for torus knots

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For a knot $K$ in $S^3$, the $sl_2$–colored Jones function $J_K(n)$ is a sequence of Laurent polynomials in the variable $t$ that is known to satisfy non-trivial linear recurrence relations. The operator corresponding to the minimal linear recurrence relation is called the recurrence polynomial of $K$. The AJ Conjecture (see Garoufalidis [4]) states that when reducing $t = -1$, the recurrence polynomial is essentially equal to the $A$–polynomial of $K$. In this paper we consider a stronger version of the AJ Conjecture, proposed by Sikora [14], and confirm it for all torus knots.

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1 Introduction

1.1 The AJ Conjecture

For a knot $K$ in $S^3$, let $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$ be the colored Jones polynomial of $K$ colored by the $n$–dimensional simple $sl_2$–representation (Jones [8], and Reshetikhin and Turaev [13]), normalized so that for the unknot $U$,

$$J_U(n) = [n] := \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$ 

The color $n$ can be assumed to take negative integer values by setting $J_K(-n) = -J_K(n)$. In particular, $J_K(0) = 0$. It is known that $J_K(1) = 1$, and $J_K(2)$ is the ordinary Jones polynomial.

Define two operators $L, M$ acting on the set of discrete functions $f : \mathbb{Z} \to \mathcal{R} := \mathbb{C}[t^{\pm 1}]$ by

$$(L f)(n) := f(n + 1), \quad (M f)(n) := t^{2n} f(n).$$

It is easy to see that $LM = t^2 ML$. Besides, the inverse operators $L^{-1}, M^{-1}$ are well-defined. One can consider $L, M$ as elements of the quantum torus

$$\mathcal{T} := \mathcal{R}(L^{\pm 1}, M^{\pm 1})/(LM - t^2 ML),$$

which is not commutative, but almost commutative.
Let 
\[ A_K = \{ P \in T \mid PJ_K = 0 \}, \]
which is a left-ideal of \( T \), called the \textit{recurrence ideal} of \( K \). It was proved by Garoufalidis and Lê in [5] that for every knot \( K \), the recurrence ideal \( A_K \) is non-zero. An element in \( A_K \) is called a recurrence relation for the colored Jones polynomials of \( K \).

The ring \( T \) is not a principal left-ideal domain, ie, not every left-ideal of \( T \) is generated by one element. By adding all inverses of polynomials in \( t, M \) to \( T \), one gets a principal left-ideal domain \( \mathcal{T} \); cf [4]. The ring \( \mathcal{T} \) can be formally defined as follows. Let \( \mathcal{R}(M) \) be the fractional field of the polynomial ring \( \mathcal{R}[M] \). Let \( \mathcal{T} \) be the set of all Laurent polynomials in the variable \( L \) with coefficients in \( \mathcal{R}[M] \)

\[ \mathcal{T} = \left\{ \sum_{j \in \mathbb{Z}} f_j(M)L^j \mid f_j(M) \in \mathcal{R}(M), f_j = 0 \text{ almost everywhere} \right\}, \]

and define the product in \( \mathcal{T} \) by \( f(M)L^k \cdot g(M)L^l = f(M)g(t^{2k}M)L^{k+l} \).

The left-ideal extension \( \mathcal{A}_K := \mathcal{T}A_K \) of \( A_K \) in \( \mathcal{T} \) is then generated by a polynomial

\[ \alpha_K(t; M, L) = \sum_{j=0}^{d} \alpha_{K, j}(t, M)L^j, \]

where \( d \) is assumed to be minimal and all the coefficients \( \alpha_{K, j}(t, M) \in \mathbb{Z}[t^{\pm 1}, M] \) are assumed to be co-prime. That \( \alpha_K \) can be chosen to have integer coefficients follows from the fact that \( J_K(n) \in \mathbb{Z}[t^{\pm 1}] \). The polynomial \( \alpha_K \) is defined up to a polynomial in \( \mathbb{Z}[t^{\pm 1}, M] \). Moreover, one can choose \( \alpha_K \in A_K \), ie, it is a recurrence relation for the colored Jones polynomials. We will call \( \alpha_K \) the \textit{recurrence polynomial} of \( K \).

Let \( \varepsilon \) be the map reducing \( t = -1 \). Garoufalidis [4] formulated the following conjecture (see also Frohman, Gelca and Lofaro [3], and Gelca [6]).

\textbf{Conjecture 1 (AJ Conjecture)} \textit{For every knot \( K \), \( \varepsilon(\alpha_K) \) is equal to the \( A \)-polynomial, up to a polynomial depending on \( M \) only.}

The \( A \)-polynomial of a knot was introduced by Cooper, Culler, Gillet, Long and Shalen [1]; it describes the \( \text{SL}_2(\mathbb{C}) \)-character variety of the knot complement as viewed from the boundary torus. Here in the definition of the \( A \)-polynomial, we also allow the factor \( L - 1 \) coming from the abelian component of the character variety of the knot group. Hence the \( A \)-polynomial in this paper is equal to \( L - 1 \) times the \( A \)-polynomial defined in [1].
The AJ Conjecture was verified for the trefoil and figure 8 knots by Garoufalidis [4], and was partially checked for all torus knots by Hikami [7]. It was established for some classes of two–bridge knots and pretzel knots, including all twist knots and \((-2, 3, 6n \pm 1)\)–pretzel knots, by Lê and the author [9; 10]. Here we provide a full proof of the AJ Conjecture for all torus knots. Moreover, we show that a stronger version of the conjecture, due to Sikora, holds true for all torus knots.

1.2 Main results

For a finitely generated group \(G\), let \(\chi(G)\) denote the \(\text{SL}_2(\mathbb{C})\)–character variety of \(G\); see eg Culler and Shalen [2], and Lubotzky and Magid [11]. For a manifold \(Y\) we use \(\chi(Y)\) also to denote \(\chi(\pi_1(Y))\). Suppose \(G = \mathbb{Z}^2\), the free abelian group with 2 generators. Every pair of generators \(\mu, \lambda\) will define an isomorphism between \(\chi(G)\) and \((\mathbb{C}^*)^2/\tau\), where \((\mathbb{C}^*)^2\) is the set of non-zero complex pairs \((M, L)\) and \(\tau\) is the involution \(\tau(M, L) := (M^{-1}, L^{-1})\), as follows: Every representation is conjugate to an upper diagonal one, with \(M\) and \(L\) being the upper left entries of \(\mu\) and \(\lambda\), respectively. The isomorphism does not change if one replaces \((\mu, \lambda)\) by \((\mu^{-1}, \lambda^{-1})\).

For an algebraic set \(V\) (over \(\mathbb{C}\)), let \(\mathbb{C}[V]\) denote the ring of regular functions on \(V\). For example, \(\mathbb{C}[(\mathbb{C}^*)^2/\tau] = t^\sigma\), the \(\sigma\)–invariant subspace of \(t := \mathbb{C}[M \pm 1, L \pm 1]\), where \(\sigma(M^k L^l) := M^{-k} L^{-l}\).

Let \(K\) be a knot in \(S^3\) and \(X = S^3 \setminus K\) its complement. The boundary of \(X\) is a torus whose fundamental group is free abelian of rank two. An orientation of \(K\) will define a unique pair of an oriented meridian \(\mu\) and an oriented longitude \(\lambda\) such that the linking number between the longitude and the knot is zero. The pair provides an identification of \(\chi(\partial X)\) and \((\mathbb{C}^*)^2/\tau\) that actually does not depend on the orientation of \(K\).

The inclusion \(\partial X \hookrightarrow X\) induces an algebra homomorphism
\[
\theta: \mathbb{C}[\chi(\partial X)] \cong t^\sigma \longrightarrow \mathbb{C}[\chi(X)].
\]
We will call the kernel \(p\) of \(\theta\) the \(A\)–ideal of \(K\); it is an ideal of \(t^\sigma\). The \(A\)–ideal was first introduced in [3]; it determines the \(A\)–polynomial of \(K\). In fact \(p = (A_K \cdot t)^\sigma\), the \(\sigma\)–invariant part of the ideal \(A_K \cdot t \subset t\) generated by the \(A\)–polynomial \(A_K\).

The involution \(\sigma\) acts on the quantum torus \(\mathcal{T}\) also by \(\sigma(M^k L^l) = M^{-k} L^{-l}\). Let \(A^\sigma_K\) be the \(\sigma\)–invariant part of the recurrence ideal \(A_K\); it is an ideal of \(\mathcal{T}^\sigma\). Sikora [14] proposed the following conjecture.

Conjecture 2 Suppose \(K\) is a knot. Then \(\sqrt{\varepsilon(A^\sigma_K)} = p\).
Here $\sqrt[\varepsilon]{(A_K^{\sigma})}$ denotes the radical of the ideal $\varepsilon(A_K^{\sigma})$ in the ring $t^\sigma = \varepsilon(T^\sigma)$.

It is easy to see that Conjecture 2 implies the AJ Conjecture. Conjecture 2 was verified for the unknot and the trefoil knot by Sikora [14]. In the present paper we confirm it for all torus knots.

**Theorem 1** Conjecture 2 holds true for all torus knots.

1.3 Plan of the paper

We provide a full proof of the AJ Conjecture for all torus knots in Section 2 and prove Theorem 1 in Section 2.

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2 Proof of the AJ Conjecture for torus knots

We will always assume that knots have framings 0.

Let $T(a, b)$ denote the $(a, b)$–torus knot. We consider the two cases, $a, b > 2$ and $a = 2$, separately. Lemmas 2.1 and 2.5 below were first proved in [7] using formulas for the colored Jones polynomials and the Alexander polynomial of torus knots given in Morton [12]. We present direct proofs here.

2.1 The case $a, b > 2$

**Lemma 2.1** One has

$$J_{T(a, b)}(n+2) = t^{-4ab(n+1)} J_{T(a, b)}(n) + t^{-2ab(n+1)} \frac{t^2 \lambda(a+b)(n+1) - t^{-2} \lambda(a-b)(n+1)}{t^2 - t^{-2}} ,$$

where $\lambda_k := t^{2k} + t^{-2k}$.

**Proof** By [12], we have

(1) $$J_{T(a, b)}(n) = t^{-ab(n^2-1)} \sum_{j=-(n-1)/2}^{(n-1)/2} t^{4bj(aj+1)}[2aj + 1] ,$$
where \([k] := (t^{2k} - t^{-2k})/(t^2 - t^{-2})\). Hence:

\[
J_{T(a,b)}(n+2) = t^{-ab(n+2)^2-1} \sum_{j=-(n+1)/2}^{(n+1)/2} t^{4bj(aj+1)}[2aj + 1] = t^{-ab(n+2)^2-1} \sum_{j=-(n-1)/2}^{(n-1)/2} t^{4bj(aj+1)}[2aj + 1] + t^{-ab(n+2)^2-1} \\
\times \left( t^{b(n+1)(a(n+1)^2)}[a(n + 1) + 1] - t^{b(n+1)(a(n+1)-2)}[a(n + 1) - 1] \right)
\]

\[
= t^{-4ab(n+1)} J_{T(a,b)}(n) + t^{-2ab(n+1)} \frac{t^{2\lambda(a+b)(n+1)} - t^{-2\lambda(a-b)(n+1)}}{t^2 - t^{-2}} \]

\(\Box\)

**Lemma 2.2** The colored Jones function of \(T(a, b)\) is annihilated by the operator \(F_{a,b} = c_3 L^3 + c_2 L^2 + c_1 L + c_0\) where:

\[
c_3 := t^2 \left( t^{2(a+b)} M^{a+b} + t^{-2(a+b)} M^{-(a+b)} \right) - t^{-2} \left( t^{2(a-b)} M^{a-b} + t^{-2(a-b)} M^{-(a-b)} \right)
\]

\[
c_2 := -t^{-2ab} \left( t^2 \left( t^{4(a+b)} M^{a+b} + t^{-4(a+b)} M^{-(a+b)} \right) + t^{-2} \left( t^{4(a-b)} M^{a-b} + t^{-4(a-b)} M^{-(a-b)} \right) \right)
\]

\[
c_1 := -t^{-8ab} M^{-2ab} c_3
\]

\[
c_0 := -t^{-4ab} M^{-2ab} c_2
\]

**Proof** It is easy to check that \(c_3 t^{-4ab(n+2)} + c_1 = c_2 t^{-4ab(n+1)} + c_0 = 0\) and

\[
c_3 \left( t^{2\lambda(a+b)(n+2)} - t^{-2\lambda(a-b)(n+2)} \right) + c_2 t^{2ab} \left( t^{2\lambda(a+b)(n+1)} - t^{-2\lambda(a-b)(n+1)} \right) = 0.
\]
Hence, by Lemma 2.1, \( F_{a,b} J_{T(a,b)}(n) \) is equal to:

\[
\begin{align*}
    c_3 J_{T(a,b)}(n + 3) + c_2 J_{T(a,b)}(n + 2) + c_1 J_{T(a,b)}(n + 1) + c_0 J_{T(a,b)}(n) \\
    &= c_3 \left( t^{-4ab(n+2)} J_{T(a,b)}(n + 1) + t^{-2ab(n+2)} \frac{t^2 \lambda(a+b)(n+2) - t^{-2} \lambda(a-b)(n+2)}{t^2 - t^{-2}} \right) \\
    &\quad + c_2 \left( t^{-4ab(n+1)} J_{T(a,b)}(n) + t^{-2ab(n+1)} \frac{t^2 \lambda(a+b)(n+1) - t^{-2} \lambda(a-b)(n+1)}{t^2 - t^{-2}} \right) \\
    &\quad + c_1 J_{T(a,b)}(n + 1) + c_0 J_{T(a,b)}(n) \\
    &= (c_3 t^{-4ab(n+2)} + c_1) J_{T(a,b)}(n + 1) + (c_2 t^{-4ab(n+1)} + c_0) J_{T(a,b)}(n) \\
    &\quad + t^{-2ab(n+1)} \left( c_3 \frac{t^2 \lambda(a+b)(n+2) - t^{-2} \lambda(a-b)(n+2)}{t^2 - t^{-2}} \\
    &\quad \quad + c_2 t^{2ab} \frac{t^2 \lambda(a+b)(n+1) - t^{-2} \lambda(a-b)(n+1)}{t^2 - t^{-2}} \right) \\
    &= 0
\end{align*}
\]

This proves Lemma 2.2. \( \square \)

Recall that \( \alpha_{T(a,b)} \) is the recurrence polynomial of \( T(a,b) \).

**Proposition 2.3** For \( a, b > 2 \), one has \( \alpha_{T(a,b)} = F_{a,b} \).

**Proof** By Lemma 2.2 it suffices to show that if an operator \( P = P_2 L^2 + P_1 L + P_0 \), where the \( P_j \) are polynomials in \( \mathbb{C}[t^\pm 1, M] \), annihilates the colored Jones polynomials of \( T(a,b) \) then \( P = 0 \).

Indeed, suppose \( P J_{T(a,b)}(n) = 0 \). Then, by Lemma 2.1:

\[
\begin{align*}
    0 &= P_2 J_{T(a,b)}(n + 2) + P_1 J_{T(a,b)}(n + 1) + P_0 J_{T(a,b)}(n) \\
    &= P_2 \left( t^{-4ab(n+1)} J_{T(a,b)}(n) + t^{-2ab(n+1)} \frac{t^2 \lambda(a+b)(n+1) - t^{-2} \lambda(a-b)(n+1)}{t^2 - t^{-2}} \right) \\
    &\quad + P_1 J_{T(a,b)}(n + 1) + P_0 J_{T(a,b)}(n) \\
    &= (t^{-4ab(n+1)} P_2 + P_0) J_{T(a,b)}(n) + P_1 J_{T(a,b)}(n + 1) \\
    &\quad + P_2 t^{-2ab(n+1)} \frac{t^2 \lambda(a+b)(n+1) - t^{-2} \lambda(a-b)(n+1)}{t^2 - t^{-2}}
\end{align*}
\]

Let \( P'_2 = t^{-4ab(n+1)} P_2 + P_0 \) and

\[
P'_0 = P_2 t^{-2ab(n+1)} \frac{t^2 \lambda(a+b)(n+1) - t^{-2} \lambda(a-b)(n+1)}{t^2 - t^{-2}}.
\]
Then,

\[ P'_2 J_{T(a,b)}(n) + P_1 J_{T(a,b)}(n + 1) + P'_0 = 0. \]

Note that \( P'_2 \) and \( P'_0 \) are polynomials in \( \mathbb{C}[t^{\pm 1}, M] \). We need the following lemma.

**Lemma 2.4** The lowest degree in \( t \) of \( J_{T(a,b)}(n) \) is

\[ l_n = -abn^2 + ab + \frac{1}{2}(1 - (-1)^{n-1})(a - 2)(b - 2). \]

**Proof** From (1), it follows easily that \( l_n = -abn^2 + ab \) if \( n \) is odd, and \( l_n = (-abn^2 + ab) + (ab - 2b - 2a + 4) \) if \( n \) is even. \( \square \)

Let us complete the proof of Proposition 2.3. Suppose \( P'_2, P_1 \neq 0 \). Let \( r_n \) and \( s_n \) be the lowest degrees (in \( t \)) of \( P'_2 \) and \( P_1 \) respectively. Note that, when \( n \) is large enough, \( r_n \) and \( s_n \) are polynomials in \( n \) of degrees at most 1. Equation (2) then implies that \( r_n + l_n = s_n + l_{n+1} \), ie:

\[ r_n - s_n = l_{n+1} - l_n = -ab(2n + 1) - (-1)^n(a - 2)(b - 2) \]

This cannot happen since the LHS is a polynomial in \( n \), when \( n \) is large enough, while the RHS is not (since \( (a - 2)(b - 2) > 0 \)). Hence \( P'_2 = P_1 = P'_0 = 0 \), which means \( P = 0 \). \( \square \)

It is easy to see that \( e(\alpha_{T(a,b)}) = M^{-2ab}(M^a - M^{-a})(M^b - M^{-b})A_{T(a,b)} \) where \( A_{T(a,b)} = (L - 1)(L^2 M^{2ab} - 1) \) is the \( A \)-polynomial of \( T(a,b) \) when \( a, b > 2 \). This means the AJ Conjecture holds true for \( T(a,b) \) when \( a, b > 2 \).

### 2.2 The case \( a = 2 \)

**Lemma 2.5** One has

\[ J_{T(2,b)}(n + 1) = -t^{-(4n+2)b} J_{T(2,b)}(n) + t^{-2nb}[2n + 1]. \]

**Proof** By (1), we have

\[ J_{T(2,b)}(n) = t^{-2b(n^2-1)} \sum_{j=-(n-1)/2}^{(n-1)/2} t^{4bj(2j+1)}[4j + 1]. \]

Hence

\[ J_{T(2,b)}(n + 1) = t^{-2b((n+1)^2-1)} \sum_{k=-n/2}^{n/2} t^{4bk(2k+1)}[4k + 1]. \]
Set $k = -(j + \frac{1}{2})$. Then:

$$J_{T(2,b)}(n + 1)$$

$$= t^{-2b((n+1)^2-1)} \sum_{j=(n-1)/2}^{-(n+1)/2} t^{4bj(2j+1)}[-(4j + 1)]$$

$$= t^{-2b((n+1)^2-1)} \left( \sum_{j=-(n-1)/2}^{(n-1)/2} t^{4bj(2j+1)}[4j + 1] + t^{2bn(n+1)}[2n + 1] \right)$$

$$= -t^{-(4n+2)b} J_{T(2,b)}(n) + t^{-2nb}[2n + 1]$$

This proves Lemma 2.5. \qed

**Lemma 2.6** The colored Jones function of $T(2, b)$ is annihilated by the operator $G_{2,b} = d_2 L^2 + d_1 L + d_0$ where

$$d_2 := t^2 M^2 - t^{-2} M^{-2},$$

$$d_1 := t^{-2b} (t^{-4b} M^{-2b} (t^2 M^2 - t^{-2} M^{-2}) - (t^6 M^2 - t^{-6} M^{-2})), $$

$$d_0 := -t^{-4b} M^{-2b} (t^6 M^2 - t^{-6} M^{-2}).$$

**Proof** From Lemma 2.5 we have

$$J_{T(2,b)}(n + 1) = -t^{-(4n+2)b} J_{T(2,b)}(n) + t^{-2nb}[2n + 1],$$

$$J_{T(2,b)}(n + 2) = t^{-8(n+1)b} J_{T(2,b)}(n) - t^{-6(n+1)b}[2n + 1] + t^{-2(n+1)b}[2n + 3].$$

It is easy to check that

$$t^{-8(n+1)b} d_2 - t^{-(4n+2)b} d_1 + d_0 = 0,$$

$$d_2(-t^{-6(n+1)b}[2n + 1] + t^{-2(n+1)b}[2n + 3]) + d_1 t^{-2nb}[2n + 1] = 0.$$

Hence:

$$G_{2,b} J_{T(2,b)}(n)$$

$$= d_2 J_{T(2,b)}(n + 2) + d_1 J_{T(2,b)}(n + 1) + d_0 J_{T(2,b)}(n)$$

$$= d_2 (t^{-8(n+1)b} J_{T(2,b)}(n) - t^{-6(n+1)b}[2n + 1] + t^{-2(n+1)b}[2n + 3])$$

$$+ d_1 (-t^{-(4n+2)b} J_{T(2,b)}(n) + t^{-2nb}[2n + 1]) + d_0 J_{T(2,b)}(n)$$

$$= (t^{-8(n+1)b} d_2 - t^{-(4n+2)b} d_1 + d_0) J_{T(2,b)}(n)$$

$$+ d_2(-t^{-6(n+1)b}[2n + 1] + t^{-2(n+1)b}[2n + 3]) + d_1 t^{-2nb}[2n + 1]$$

$$= 0$$
This proves Lemma 2.6.

\[ \square \]

**Proposition 2.7** One has \( \alpha_{T(2,b)} = G_{2,b} \).

**Proof** By Lemma 2.6, it suffices to show that if an operator \( P = P_1L + P_0 \), where the \( P_j \) are polynomials in \( \mathbb{C}[t^{\pm1}, M] \), annihilates the colored Jones polynomials of \( T(2,b) \) then \( P = 0 \).

Indeed, suppose \( PJ_{T(2,b)}(n) = 0 \). Then:

\[
0 = P_1J_{T(2,b)}(n + 1) + P_0J_{T(2,b)}(n) = P_1(-t^{-(4n+2)b}J_{T(2,b)}(n) + t^{-2nb}[2n + 1]) + P_0J_{T(2,b)}(n)
\]

\[
= (-t^{-(4n+2)b}P_1 + P_0)J_{T(2,b)}(n) + t^{-2nb}[2n + 1]P_1
\]

Let \( P'_1 = -t^{-(4n+2)b}P_1 + P_0 \) and \( P'_0 = t^{-2nb}[2n + 1]P_1 \). Then \( P'_1, P'_0 \) are polynomials in \( \mathbb{C}[t^{\pm1}, M] \) and \( P'_1J(n) + P'_0 = 0 \). This implies that \( P'_1 = P'_0 = 0 \) since the lowest degree in \( t \) of \( J_{T(2,b)}(n) \) is \(-2bn^2 + 2b\), which is quadratic in \( n \), by Lemma 2.4. Hence \( P = 0 \). \( \square \)

It is easy to see that \( \varepsilon(\alpha_{T(2,b)}) = M^{-2b}(M^2 - M^{-2})A_{T(2,b)} \) where \( A_{T(2,b)} = (L - 1)(LM^{2b} + 1) \) is the \( A \)-polynomial of \( T(2,b) \). This means the AJ Conjecture holds true for \( T(2,b) \).

3 Proof of Theorem 1

As in the previous section, we consider the two cases, \( a, b > 2 \) and \( a = 2 \), separately.

### 3.1 The case \( a, b > 2 \)

We claim that:

**Proposition 3.1** The colored Jones function of \( T(a,b) \) is annihilated by the operator \( PQ \) where:

\[
P := t^{-10ab}(L^3M^{2ab} + L^{-3}M^{-2ab}) - (t^{2(a-b)} + t^{2(b-a)})t^{-4ab}(L^2M^{2ab} + L^{-2}M^{-2ab}) + t^{2ab}(LM^{2ab} + L^{-1}M^{-2ab}) - (t^{2ab} + t^{-2ab})(L + L^{-1}) + (t^{2(a-b)} + t^{2(b-a)})(t^{4ab} + t^{-4ab})
\]

\[
Q := t^{-6ab}(L^3M^{2ab} + L^{-3}M^{-2ab}) - (t^{2(a+b)} + t^{-2(a+b)})q^{-ab}(L^2M^{2ab} + L^{-2}M^{-2ab}) + t^{-2ab}(LM^{2ab} + L^{-1}M^{-2ab}) - (t^{2ab} + t^{-2ab})(L + L^{-1}) + 2(t^{2(a+b)} + t^{-2(a+b)})
\]
Proof  We first prove the following two lemmas.

Lemma 3.2  One has

\[ QJ_{T(a,b)}(n) = t^{4ab-2}(\lambda_{a+b} - \lambda_{a-b}) \frac{t^{2ab}n^2 \lambda_{a-b}(n+1) - t^{-2ab}n \lambda_{a-b}(n-1)}{t^2 - t^{-2}}. \]

Proof  Let

\[ g(n) := t^{-2abn} \frac{t^2 \lambda_{a+b}n - t^{-2} \lambda_{a-b}n}{t^2 - t^{-2}}. \]

Then, by Lemma 2.1, \( J_{T(a,b)}(n+2) = t^{-4ab(n+1)} J_{T(a,b)}(n) + g(n+1) \). Hence:

\[
QJ_{T(a,b)}(n) \\
= t^{-6ab} \left( t^{4ab(n+3)} J_{T(a,b)}(n + 3) + t^{-4ab(n-3)} J_{T(a,b)}(n - 3) \right) \\
- \left( t^2(a+b) + t^{-2(a+b)} \right) t^{-4ab} \left( t^{4ab(n+2)} J_{T(a,b)}(n + 2) + t^{-4ab(n-2)} J_{T(a,b)}(n - 2) \right) \\
+ t^{-2ab} \left( t^{4ab(n+1)} J_{T(a,b)}(n + 1) + t^{-4ab(n-1)} J_{T(a,b)}(n - 1) \right) \\
- \left( t^2(a+b) + t^{-2(a+b)} \right) \left( J_{T(a,b)}(n + 1) + J_{T(a,b)}(n - 1) \right) \\
+ 2 \left( t^2(a+b) + t^{-2(a+b)} \right) J_{T(a,b)}(n)
\]

\[
= t^{-6ab} \left( t^{4ab} \left( J_{T(a,b)}(n + 1) + J_{T(a,b)}(n - 1) \right) \right) \\
- \left( t^2(a+b) + t^{-2(a+b)} \right) t^{-4ab} \left( 2t^{4ab} J_{T(a,b)}(n) \right) \\
+ t^{-2ab} \left( t^{4ab} \left( J_{T(a,b)}(n - 1) + J_{T(a,b)}(n + 1) \right) + \left( t^{2ab(n+3)} - t^{-2ab(n-3)} \right) g(n) \right) \\
- \left( t^2(a+b) + t^{-2(a+b)} \right) \left( J_{T(a,b)}(n + 1) + J_{T(a,b)}(n - 1) \right) \\
+ 2 \left( t^2(a+b) + t^{-2(a+b)} \right) J_{T(a,b)}(n)
\]

\[
= t^{-6ab} \left( t^{2ab(n+5)} g(n + 2) - t^{-2ab(n-5)} g(n - 2) \right) \\
- \left( t^2(a+b) + t^{-2(a+b)} \right) t^{-4ab} \left( t^{2ab(n+4)} g(n + 1) - t^{-2ab(n-4)} g(n - 1) \right) \\
+ t^{-2ab} \left( t^{2ab(n+3)} - t^{-2ab(n-3)} \right) g(n)
\]

Using the definition of \( g(n) \), we get:

\[
QJ_{T(a,b)}(n) = t^{4ab} \left( t^{2ab} \frac{t^{2\lambda_{a+b}(n+2)} - t^{-2\lambda_{a-b}(n+2)}}{t^2 - t^{-2}} \right. \\
\left. - t^{-2ab} \frac{t^{2\lambda_{a+b}(n-2)} - t^{-2\lambda_{a-b}(n-2)}}{t^2 - t^{-2}} \right)
\]
\[-(t^2(a+b) + t^{-2}(a+b)) t^{4ab} \times \]
\[
\left( t^{2abn} \frac{t^2 \lambda(a+b)(n+1) - t^{-2} \lambda(a-b)(n+1)}{t^2 - t^{-2}} \right)
\left( t^{2abn} \frac{t^2 \lambda(a+b)(n-1) - t^{-2} \lambda(a-b)(n-1)}{t^2 - t^{-2}} \right)
\left( t^{4ab} \frac{t^2 \lambda(a-b)n - t^{-2} \lambda(a-b)n}{t^2 - t^{-2}} \right)
\]
\[+ t^{4ab} (t^{2abn} - t^{-2abn}) \frac{t^2 \lambda(a-b)(n+1) - t^{-2abn} \lambda(a-b)(n-1)}{t^2 - t^{-2}}.\]

Now applying the equality $\lambda_{k+l} + \lambda_{k-l} = \lambda_k \lambda_l$, we then obtain

\[ Q J_{T(a,b)}(n) = t^{4ab-2} (\lambda_{a+b} - \lambda_{a-b}) \frac{t^{2abn} \lambda(a-b)(n+1) - t^{-2abn} \lambda(a-b)(n-1)}{t^2 - t^{-2}}. \]

This proves Lemma 3.2.

Let $h(n) := t^{2abn} \lambda(a-b)(n+1) - t^{-2abn} \lambda(a-b)(n-1)$.

**Lemma 3.3** The function $h(n)$ is annihilated by the operator $P$, ie, $Ph(n) = 0$.

**Proof** Let $c = a - b$. Then:

\[ Ph(n) = t^{-10ab} \left( t^{4ab(n+3)} h(n+3) + t^{-4ab(n-3)} h(n-3) \right) \]
\[- \left( t^{2(a-b) + t^{-2}(b-a)} t^{4ab(n+2)} (n+2) + t^{-4ab(n-2)} h(n-2) \right) \]
\[+ t^{2ab} (t^{4ab(n+1)} h(n+1) + t^{-4ab(n-1)} h(n-1)) \]
\[- \left( t^{ab} + t^{-2ab} \right) \left( h(n+1) + h(n-1) \right) + \left( t^{2(a-b) + t^{-2}(b-a)} \right) t^{4ab} + t^{-4ab} h(n) \]
\[= \left( t^{2ab(3n+4)} \lambda_{c(n+4)} - t^{2ab(n-2)} \lambda_{c(n+2)} \right) \]
\[+ t^{2ab(n+2)} \lambda_{c(n-2)} - t^{-2ab(3n-4)} \lambda_{c(n-4)} \]
\[- \lambda_c \left( t^{2ab(3n+4)} \lambda_{c(n+3)} - t^{-2abn} \lambda_{c(n+1)} + t^{-2abn} \lambda_{c(n-1)} - t^{-2ab(3n-4)} \lambda_{c(n-3)} \right) \]
\[+ \left( t^{2ab(n+2)} \lambda_{c(n+2)} + t^{-2ab(n-2)} \lambda_{c(n-2)} \right) \]
\[- \left( t^{2ab} + t^{-2ab} \right) \left( t^{2ab(n+1)} \lambda_{c(n+1)} - t^{-2ab(n+1)} \lambda_{c(n+1)} \right) \]
\[+ t^{2ab(n-1)} \lambda_{c(n-1)} \lambda_{c(n-2)} \]
\[+ \lambda_c \left( t^{4ab} + t^{-4ab} \right) \left( t^{2abn} \lambda_{c(n+1)} - t^{-2abn} \lambda_{c(n-1)} \right). \]
Note that $\lambda_{k+l} + \lambda_{k-l} = \lambda_k \lambda_l$. Hence:

$$P h(n) = \left(-t^{2ab(n-2)}\lambda_{c(n+2)} + t^{-2ab(n+2)}\lambda_{c(n-2)}\right)$$

$$- \lambda_c \left(-t^{2abn}\lambda_{c(n+1)} + t^{-2abn}\lambda_{c(n-1)}\right)$$

$$+ \left(-t^{2ab(n+2)}\lambda_{cn} + t^{-2ab(n-2)}\lambda_{cn}\right)$$

$$- (t^{2ab} + t^{-2ab})\left(t^{2ab(n+1)}\lambda_{c(n+2)} - t^{-2ab(n+1)}\lambda_{cn}\right)$$

$$+ t^{2ab(n-1)}\lambda_{cn} - t^{-2ab(n-1)}\lambda_{c(n-2)}$$

$$+ \lambda_c \left(t^{4ab} + t^{-4ab}\right)\left(t^{2abn}\lambda_{c(n+1)} - t^{-2abn}\lambda_{c(n-1)}\right)$$

$$= -(t^{4ab} + t^{-4ab} + 1)t^{2abn}\lambda_{c(n+2)} + (t^{4ab} + t^{-4ab} + 1)t^{-2abn}\lambda_{c(n-2)}$$

$$- (t^{4ab} + t^{-4ab} + 1)(t^{2abn} - t^{-2abn})\lambda_{cn}$$

$$+ \lambda_c \left(t^{4ab} + t^{-4ab} + 1\right)\left(t^{2abn}\lambda_{c(n+1)} - t^{-2abn}\lambda_{c(n-1)}\right)$$

$$= -(t^{4ab} + t^{-4ab} + 1)t^{2abn}\left(\lambda_{c(n+2)} + \lambda_{cn} - \lambda_c \lambda_{c(n+1)}\right)$$

$$+ (t^{4ab} + t^{-4ab} + 1)t^{-2abn}\left(\lambda_{c(n-2)} + \lambda_{cn} - \lambda_c \lambda_{c(n-1)}\right)$$

$$= 0$$

This proves Lemma 3.3. \[\square\]

Proposition 3.1 follows directly from Lemmas 3.2 and 3.3. \[\square\]

### 3.2 The case $a = 2$

We claim that:

Proposition 3.4 The colored Jones function of $T(2, b)$ is annihilated by the operator

$$R = t^{-4b}(L^2 M^{2b} + L^{-2} M^{-2b}) + (t^{2b} + t^{-2b})(L + L^{-1})$$

$$- (t^4 + t^{-4})t^{-2b}(LM^{2b} + L^{-1} M^{-2b}) + (M^{2b} + M^{-2b}) - 2(t^4 + t^{-4}).$$

**Proof** From Lemma 2.5 we have

$$J_{T(2, b)}(n + 1) = -t^{-4(n+2)}b \ J_{T(2, b)}(n) + t^{-2nb}[2n + 1],$$

$$J_{T(2, b)}(n + 2) = t^{-8(n+1)b} \ J_{T(2, b)}(n) - t^{-6(n+1)b}[2n + 1] + t^{-2(n+1)b}[2n + 3],$$

$$J_{T(2, b)}(n - 1) = -t^{4(n-2)b} \ J_{T(2, b)}(n) + t^{2nb}[2n - 1],$$

$$J_{T(2, b)}(n - 2) = t^{8(n-1)b} \ J_{T(2, b)}(n) - t^{6(n-1)b}[2n - 1] + t^{-2(n-1)b}[2n - 3].$$
Hence
\[
RJ_{T(2,b)}(n) = t^{-4b} (t^{4(n+2)b} J_{T(2,b)}(n + 2) + t^{-4(n-2)b} J_{T(2,b)}(n - 2)) \\
+ (t^{2b} + t^{-2b}) (J_{T(2,b)}(n + 1) + J_{T(2,b)}(n - 1)) \\
- (t^4 + t^{-4}) t^{-2b} (t^{4(n+1)b} J_{T(2,b)}(n + 1) + t^{-4(n-1)b} J_{T(2,b)}(n - 1)) \\
+ ((t^{4nb} + t^{-4nb}) - 2(t + t^{-4})) J_{T(2,b)}(n) \\
= t^{-4b} (-t^{-2(n-1)b}[2n + 1] + t^{2(n+3)b}[2n + 3] \\
- i^{2(n+1)b}[2n - 1] + t^{-2(n-3)b}[2n - 3]) \\
+ (t^{2b} + t^{-2b}) (t^{-2nb}[2n + 1] + t^{2nb}[2n - 1]) \\
- (t^4 + t^{-4}) t^{-2b} (t^{2n+4b}[2n + 1] + t^{-2n+4b}[2n - 1]) \\
- (t + t^{-4}) t^{2b} (t^{2nb}[2n + 1] + t^{-2nb}[2n - 1]) \\
= t^{2b} t^{2nb} ([2n + 3] + [2n - 1] - (t^4 + t^{-4})[2n + 1]) \\
+ t^{2b} t^{-2nb} ([2n - 3] + [2n + 1] - (t^4 + t^{-4})[2n - 1]) \\
= 0,
\]
since \([k + l] + [k - l] = (t^{2l} + t^{-2l})[k].\]

\[\square\]

### 3.3 Proof of Theorem 1

We first note that the \(A\)-ideal \(\mathfrak{p}\), the kernel of \(\theta\): \(t^\sigma \mapsto \mathbb{C}[\chi(X)]\), is radical, ie, \(\sqrt{\mathfrak{p}} = \mathfrak{p}\), since the character ring \(\mathbb{C}[\chi(X)]\) is reduced, ie, has nil-radical 0, by definition.

**Lemma 3.5** Suppose \(\delta(t, M, L) \in A_K\). Then there are polynomials \(g(t, M) \in \mathbb{C}[t^\pm 1, M]\) and \(\gamma(t, M, L) \in T\) such that

\[\delta(t, M, L) = \frac{1}{g(t, M)} \gamma(t, M, L) \alpha_K(t, M, L).\]

Moreover, \(g(t, M)\) and \(\gamma(t, M, L)\) can be chosen so that \(\varepsilon(g) \neq 0\).

**Proof** By definition \(\alpha_K\) is a generator of \(\widetilde{A}_K\), the extension of \(A_K\) in the principal left-ideal domain \(\widetilde{T}\). Since \(\delta \in A_K\), it is divisible by \(\alpha_K\) in \(\widetilde{T}\). Hence (3) follows.

We can assume that \(t + 1\) does not divide both \(g(t, M)\) and \(\gamma(t, M, L)\) simultaneously. If \(\varepsilon(g) = 0\) then \(g\) is divisible by \(t + 1\), and hence \(\gamma\) is not. But then from the equality \(g\delta = \gamma \alpha_K\), it follows that \(\alpha_K\) is divisible by \(t + 1\), which is impossible, since all the coefficients of powers of \(L\) in \(\alpha_K\) are supposed to be co-prime. \[\square\]
Showing \( \sqrt{\varepsilon(A_K^\sigma)} \subset p \) For torus knots, by Section 1, we have \( \varepsilon(\alpha_K) = f(M)A_K \), where \( f(M) \in \mathbb{C}[M^{\pm 1}] \). For every \( \delta \in A_K \), by Lemma 3.5, there exist \( g(t, M) \in \mathbb{C}[t^{\pm 1}, M] \) and \( \gamma \in T \) such that \( \delta = \frac{1}{g(t, M)} \gamma \alpha_K \) and \( \varepsilon(g) \neq 0 \). It implies that

\[
(4) \quad \varepsilon(\gamma) = \frac{1}{\varepsilon(g(M))} \varepsilon(\gamma) \varepsilon(\alpha_K) = \frac{1}{\varepsilon(g(M))} \varepsilon(\gamma) f(M) A_K.
\]

The \( A \)-polynomial of a torus knot does not contain any non-trivial factor depending on \( M \) only. Since \( \varepsilon(\gamma) \in \mathbb{C}[L^{\pm 1}, M^{\pm 1}] \), equation (4) implies that

\[
h := \frac{1}{\varepsilon(g(M))} \varepsilon(\gamma) f(M)
\]

is an element of \( t \). Hence \( \varepsilon(\gamma) \in A_K \cdot t \), the ideal of \( t \) generated by \( A_K \). It follows that \( \varepsilon(A_K) \subset A_K \cdot t \) and thus \( \varepsilon(A_K^\sigma) \subset (A_K \cdot t)^\sigma = p \). Hence \( \sqrt{\varepsilon(A_K^\sigma)} \subset \sqrt{p} = p \).

**Showing \( p \subset \sqrt{\varepsilon(A_K^\sigma)} \)** For \( a, b > 2 \), by Proposition 3.1 the colored Jones function of \( T(a, b) \) is annihilated by the operator \( PQ \). Note that

\[
\varepsilon(PQ) = (L + L^{-1} - 2)(L^2 M^{2ab} + L^{-2} M^{-2ab} - 2)
= L^{-2}(L^{-1} M^{-ab}(L - 1)(L^2 M^{2ab} - 1))^4.
\]

If \( u \in p \) then \( u = v A_{T(a, b)}' \), where

\[ A_{T(a, b)}' := L^{-1} M^{-ab}(L - 1)(L^2 M^{2ab} - 1) = L^{-1} M^{-ab} A_{T(a, b)} \]

and \( v \in \mathbb{C}[M^{\pm 1}, L^{\pm 1}] \). It is easy to see that \( \sigma(v) = L v \), since \( \sigma(u) = u \) and \( \sigma(A_{T(a, b)}') = L^{-1} A_{T(a, b)}' \). This implies that \( \sigma(v^2 L) = \sigma(v)^2 L^{-1} = v^2 L \). We then have

\[ u^4 = v^4 A_{T(a, b)}'^4 = \varepsilon(v^4 L^2 PQ) \in \varepsilon(A_K^\sigma), \]

hence \( u \in \sqrt{\varepsilon(A_K^\sigma)} \).

For \( a = 2 \), by Proposition 3.4 the colored Jones function of \( T(2, b) \) is annihilated by the operator \( R \). Note that \( \sigma(R) = R \) and

\[
\varepsilon(R) = (L + L^{-1} - 2)(LM^{2b} + L^{-1} M^{-2b} + 2) = (L^{-1} M^{-b}(L - 1)(LM^{2b} + 1))^2.
\]

If \( u \in p \) then \( u = v A_{T(2, b)}' \), where

\[ A_{T(2, b)}' := L^{-1} M^{-b}(L - 1)(LM^{2b} + 1) = L^{-1} M^{-b} A_{T(2, b)} \]

and \( v \in \mathbb{C}[M^{\pm 1}, L^{\pm 1}] \). It is easy to see that \( \sigma(v) = -v \) and hence \( \sigma(v^2) = \sigma(v) \sigma(v) = v^2 \). We then have

\[ u^2 = v^2 A_{T(2, b)}'^2 = \varepsilon(v^2 R) \in \varepsilon(A_K^\sigma), \]
hence $u \in \sqrt{\varepsilon(A^n_K)}$.

In both cases $p \subset \sqrt{\varepsilon(A^n_K)}$. Hence $\sqrt{\varepsilon(A^n_K)} = p$ for all torus knots.

References


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