Algebraic & Geometric Topology 13 (2013) 905–925

Amenable category of three–manifolds

José Carlos Gómez-Larrañaga
Francisco González-Acuña
Wolfgang Heil

A closed topological n–manifold M^n is of ame–category $\leq k$ if it can be covered by k open subsets such that for each path-component W of the subsets the image of its fundamental group $\pi_1(W) \to \pi_1(M^n)$ is an amenable group. $\mathrm{cat}_{\mathrm{ame}}(M^n)$ is the smallest number k such that M^n admits such a covering. For $n = 3$, M^3 has ame–category ≤ 4. We characterize all closed 3–manifolds of ame–category 1, 2 and 3.

55M30, 57M27, 57N10; 57N16

1 Introduction

Categorical properties of a manifold M are those that deal with covers of M by open sets with certain properties. For example, the classical Lusternik–Schnirelmann category $\mathrm{cat}(M)$ of M is the smallest number k such that there is an open cover $W_1, \dots, W_k$ of M with each W_i contractible in M. An extensive survey for this category can be found in Cornea, Lupton, Oprea and Tanré [3]. M Clapp and D Puppe [2] proposed the following generalization $\mathrm{cat}(M)$: Let $\mathcal{K}$ be a non-empty class of spaces. A subset W of M is $\mathcal{K}$*–contractible* (in M) if the inclusion $\iota\colon W \to M$ factors homotopically through some $K \in \mathcal{K}$, ie, there exist maps $f\colon W \to K$, $\alpha\colon K \to M$, such that ι is homotopic to $\alpha \cdot f$. (W and K need not be connected.) The $\mathcal{K}$–category $\mathrm{cat}_{\mathcal{K}}(M)$ of M is the smallest number of open $\mathcal{K}$–contractible subsets of M that cover M. If no such finite cover exists, $\mathrm{cat}_{\mathcal{K}}(M)$ is infinite. When the family $\mathcal{K}$ contains just one space K, one writes $\mathrm{cat}_K(M)$ instead of $\mathrm{cat}_{\mathcal{K}}(M)$. In particular, if K is a single point, then $\mathrm{cat}_K(M) = \mathrm{cat}(M)$. For closed n–manifolds, $1 \leq \mathrm{cat}_{\mathcal{K}}(M) \leq \mathrm{cat}(M) \leq n+1$.

Note that for each path-component W' of a K–contractible subset W of M, the image of its fundamental group $\iota_*(\pi_1(W', *) \subset \pi_1(M))$ is a quotient of a subgroup of $\pi_1(K, f(*))$, for every basepoint $* \in W'$. This suggests considering coverings of M by open sets whose components satisfy certain group properties. For example, if $\pi_1(K', *)) = 1$ for each path-component K' of K, one may ask more generally: what is the smallest number k of open sets W_i that are needed to cover M and such that

Published: 30 March 2013 DOI: 10.2140/agt.2013.13.905

for each component of W_i the image of its fundamental group in $\pi_1(M)$) is trivial? This number is the π_1–category of M and has been calculated for closed 3–manifolds M^3 in [7] (Corollary 4.2).

When we considered the case of S^1–category [8; 9], ie, when $K = \{S^1\}$, J Porti pointed out a connection to the Gromov Vanishing Theorem [11], which states that if a closed orientable n–manifold M admits an open cover by n amenable sets, then the simplicial volume $|M|$ of M vanishes. Here a set $W \subset M$ is *amenable* if for each path-component W' of W the image of its fundamental group $\pi_1(W') \to \pi_1(M^n)$ is an amenable group. By Perelman's proof of the Geometrization Theorem for 3–manifolds, see eg Bessières, Besson, Maillot, Boileau and Porti [1], we know that a connected closed orientable 3–manifold M^3 is a connected sum of graph manifolds if and only if $|M^3| = 0$. Here a graph manifold is a union of Seifert fiber spaces along tori components in their boundaries. A good exposition of this is in chapter 13 of [1].

Motivated by the work of Gromov (see also Ivanov [13]) we define the ame–category $\mathrm{cat}_{\mathrm{ame}} M^n$ to be the smallest number of open amenable sets needed to cover M^n. For M^n compact one has $1 \le \mathrm{cat}_{\mathrm{ame}}(M^n) \le n+1$. By Gromov and Perelman, if M is a closed orientable 3–manifold with ame–category ≤ 3 then M is a connected sum of graph manifolds. We show the converse in Theorem 2. This answers a question of M Boileau that is to find a characterization of graph manifolds in terms of category concepts.

In this paper we study the ame–category of all compact 3–manifolds M. It turns out (Proposition 2) that the fundamental group of a compact 3–manifold is amenable if and only if it does not contain a free group of rank 2, which happens if and only if it is virtually solvable. These manifolds are classified in Proposition 3 (Section 4). As a preparation for this proposition we list in Section 3 the compact 3–manifolds with solvable fundamental group and with boundary containing projective planes. In Section 5 we classify the 3–manifolds with $\mathrm{cat}_{\mathrm{ame}} = 2$ (Theorem 1). It is perhaps interesting to note that the only closed prime 3–manifolds with ame–category 2 are non-orientable 3–manifolds that contain projective planes such the vertices of the P^2–graph of M are the manifolds in the examples of section 3. Finally, in Section 6, we classify all closed non-orientable 3–manifolds of ame–category ≤ 3 (Theorems 2 and 3).

2 Basic properties and $\mathrm{cat}_{\mathrm{ame}}$ for 2–manifolds

A group G is *solvable* if $G^n = 1$ for some n, where $G^0 = G$ and $G^{i+1} = [G^i, G^i]$, $(i = 0, \dots, n-1)$.

G is *virtually solvable* if it contains a solvable subgroup of finite index.

G is *amenable* if it a has finitely additive, left-invariant probability measure μ, ie, $\mu(gS) = \mu(S)$ for all subsets $S \subset G$, $g \in G$, $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint subsets $A, B \subset G$ and $\mu(G) = 1$.

Let us say that a group G is *hunfree* ("hereditarily unfree") if G does not contain the free group F_2 of rank 2 as a subgroup.

(Virtually) solvable groups are amenable and amenable groups are hunfree. Subgroups and quotient groups of solvable, resp. amenable, resp. hunfree groups are solvable, resp. amenable, resp. hunfree. Extensions of amenable groups by amenable groups are amenable; virtually amenable groups are amenable.

Definition 1 Let M be a manifold. A subset W of M is *amenable* (in M) if, for every basepoint $* \in W$, the image $\iota_*(\pi_1(W, *) \subset \pi_1(M, *)$ is an amenable group.

Note that a subset of an amenable set is amenable.

Definition 2 $\mathrm{cat}_{\mathrm{ame}}(M)$ is the smallest number of open amenable subsets of M that cover M.

For any compact n–manifold we have $1 \leq \mathrm{cat}_{\mathrm{ame}}(M) \leq n+1$.

For the case that $\mathrm{cat}_{\mathrm{ame}} M^n \leq 2$ we first observe that we may choose compact amenable submanifolds that intersect along their boundaries:

Lemma 1 *Let M be a closed n–manifold. Then $\mathrm{cat}_{\mathrm{ame}}(M) \leq 2$ if and only if there are compact amenable n–submanifolds W_i of M so that $M = W_1 \cup W_2$ and $W_1 \cap W_2 = \partial W_1 = \partial W_2$.*

Proof If $\mathrm{cat}_{\mathrm{ame}}(M) \leq 2$ there are open amenable subsets U_0 and U_1 of M whose union is M. By Lemma 1 of [8], there exist compact n–submanifolds W_0, W_1 such that $W_0 \cup W_1 = M^n$, $W_0 \cap W_1 = \partial W_0 = \partial W_1$ and $W_i \subset U_i$ $(i = 0, 1)$. Since subgroups of amenable groups are amenable, W_i is amenable. □

First we use this lemma to calculate the amenable category for compact 2–manifolds. Denote by $\chi(M)$ the Euler characteristic of M.

Proposition 1 *Let M^2 be a compact 2–manifold. Then*

$$\mathrm{cat}_{\mathrm{ame}} M^2 = \begin{cases} 1 & \text{if } \chi(M^2) \geq 0, \\ 2 & \text{if } \partial M \neq \varnothing \text{ and } \chi(M^2) < 0, \\ 3 & \text{otherwise.} \end{cases}$$

Proof If M^2 is not a disk, an annulus, a Möbius band, S^2, P^2, a torus, or a Klein bottle, then $\pi_1(M^2)$ is not hunfree, so the manifolds with $\chi(M^2) \geq 0$ are the only 2–manifolds with amenable fundamental group.

If $\partial M^2 \neq \varnothing$ then M^2 can be decomposed into two disks and therefore $\mathrm{cat}_{\mathrm{ame}} M^2 \leq 2$.

We show that if M^2 is closed with $\mathrm{cat}_{\mathrm{ame}} M^2 \leq 2$, then $\chi(M^2) \geq 0$. We write $M = W_1 \cup W_2$ and $W_1 \cap W_2 = \partial W_1 = \partial W_2$ as in Lemma 1 and assume that the number c of components of $\partial W_1 = \partial W_2 = W_1 \cap W_2$ is minimal. If $c = 0$ then $W_2 = \varnothing$, say, and M is amenable (case 1). So assume $c \neq 0$. If a simple closed curve of $W_1 \cap W_2$ is null-homotopic in M^2, it bounds a disk in M^2 and we let D be an innermost such disk (ie, $\mathrm{int}(D) \cap W_1 \cap W_2 = \varnothing$). Then D is equal to a component of W_2, say, and we obtain a new decomposition $M = W_1' \cup W_2'$, $W_1' \cap W_2' = \partial W_1' = \partial W_2'$, where $W_1' = W_1 \cup D$ and $W_2' = W_2 - D$ are amenable and the number of components of $W_1' \cap W_2'$ is less than c, a contradiction. So each component of W_i is π_1–injective, ie, its fundamental group is amenable, and it must be an annulus or Möbius band. It follows that $\chi(M^2) \geq 0$. □

For 3–manifolds we observe that the amenable category of a connected sum is bounded by the highest amenable category of the factors:

Lemma 2 *Let $M = M_1 \# M_2$ be a connected sum of 3–manifolds.*

If $\mathrm{cat}_{\mathrm{ame}}(M_i) \leq k_i$ for $i = 1, 2$ and $k_i \geq 2$, then $\mathrm{cat}_{\mathrm{ame}}(M) \leq \max\{k_1, k_2\}$.

Proof There are 3–balls $B_i \subset M_i$ so that $M = (M_1 - \mathrm{int}\, B_1) \cup (M_2 - \mathrm{int}\, B_2)$ and $(M_1 - \mathrm{int}\, B_1) \cap (M_2 - \mathrm{int}\, B_2) = \partial B_1 = \partial B_2$. Deleting a ball from an open amenable contractible subset does not change amenability, so we may assume $M_i = W_{i1} \cup \cdots \cup W_{ik_i}$ is an amenable cover such that $B_i \subset W_{i1}$, $B_i \cap W_{ij} = \varnothing$ for $j \neq 1$ and $B_1 \cap \overline{W}_{12} = \varnothing$. Note that $W_{ij} - \mathrm{int}\, B_i$ is amenable in $M_i - \mathrm{int}\, B_i$ and therefore in M. Let N be an open product neighborhood of ∂B_1 in $W_{11} - \mathrm{int}\, B_1$ with $N \cap W_{12} = \varnothing$. Assume that $k_1 \leq k_2$. Then $M = W_1 \cup \cdots \cup W_{k_2}$, where $W_1 = (W_{11} - B_1) \cup W_{22}$, $W_2 = W_{12} \cup (W_{21} - B_2) \cup N$, $W_j = W_{1j} \cup W_{2j}$ for $3 \leq j \leq k_1$ and $W_j = W_{2j}$ for $k_1 < j \leq k_2$ are amenable in M. □

Corollary 1 *Let M be a closed 3–manifold with prime decomposition*

$$M = M_1 \# M_2 \# \cdots \# M_m.$$

Then for $k \geq 2$, $\mathrm{cat}_{\mathrm{ame}}(M) \leq k$ if and only if $\mathrm{cat}_{\mathrm{ame}}(M_i) \leq k$, for $i = 1, \ldots, m$.

Proof If $\mathrm{cat}_{\mathrm{ame}}(M_i) \leq 2$ for each i, then by Lemma 2, $\mathrm{cat}_{\mathrm{ame}}(M) \leq 2$.

Conversely, suppose $\{W_1, \dots, W_k\}$ is an amenable open cover of

$$M = (M_1 - \mathrm{int}\, B_1) \cup \cdots \cup (M_m - \mathrm{int}\, B_m).$$

Then $W_j \cap M_i$ is an amenable subset of M, and since $\pi_1(M_i - \mathrm{int}\, B_i, *) \to \pi_1(M, *)$ is injective, it follows that $W_j \cap M_i$ is amenable in M_i (for $1 \leq i \leq m$, $1 \leq j \leq k$). Hence $(W_1 \cap M_i) \cup \cdots \cup (W_k \cap M_i)$ is an amenable cover of M_i. □

We use the following terminology: A *closed* manifold is a compact manifold without boundary. We also assume that a closed manifold is connected unless stated otherwise.

The manifold that is obtained from a manifold M by filling in all boundary spheres with 3–balls is denoted by $\widehat{M}$.

$T \mathbin{\tilde{\times}} I$, $K \mathbin{\tilde{\times}} I$, $S^1 \mathbin{\tilde{\times}} D^2$, $S^1 \mathbin{\tilde{\times}} S^2$ denote, respectively, an I–bundle over the torus, an I–bundle over the Klein bottle, a D^2–bundle over S^1, an S^2–bundle over S^1. The bundles may be trivial (ie, product bundles) or non-trivial.

By a *Seifert manifold* we mean a compact 3–manifold (orientable or not, closed or with boundary) that is decomposed into disjoint simple closed curves, the *fibers*, such that each fiber has a neighborhood that forms a fibered solid torus in the sense of Seifert [20]. The fibering of such a solid torus is obtained from the mapping torus of a rotation of a disk by an angle of $2\pi\beta/\alpha$, for some coprime $\alpha > 0$, β. If $\alpha > 1$, the middle fiber is an *exceptional* fiber of multiplicity α. The *orbifold Euler characteristic* of a Seifert manifold is $\chi(S) - \sum_{i=1}^{k}(1 - 1/\alpha_i)$, where $\chi(S)$ is the usual Euler characteristic of the orbit surface S and the α_i are the multiplicities of the exceptional fibers.

A *graph manifold* is a union of Seifert manifolds along tori or Klein bottle components in their boundaries.

3 Six 3–manifolds with solvable fundamental groups

We describe some well-known 3–manifolds containing projective planes. In the examples below, M is a compact orientable 3–manifold that admits an orientation-reversing involution $\tau\colon M \to M$ with zero-dimensional fixed point set and $m > 0$ fixed points. Choose invariant 3–ball neighborhoods $C_1, \dots, C_m$ of the fixed points and let $M_* = \overline{M - (C_1 \cup \cdots \cup C_m)}/\tau$ be the orbit manifold. The boundary of M_* contains m projective planes and $\pi_1(M_*)$ is a semi-direct product of Z_2 with $\pi_1(M)$.

Example 1 The *geminus* $M = D^2 \times S^1$. There is only one (up to conjugates) orientation-reversing involution τ with non-empty zero-dimensional fixed point set (see Luft and Sjerve [16, Corollary 3.4]). $\tau(x,z) = (-x, \overline{z})$ (where $x \in D^2$), $m = 2$. The *geminus* is $M_* = (P^2 \times I) \#_b (P^2 \times I)$, the disk sum of two copies of $(P^2 \times I)$. The boundary of the geminus consists of 2 projective planes and a Klein bottle.

Example 2 The *quadripus* $M = S^1 \times S^1 \times I$. There is only one orientation-reversing involution τ with non-empty zero-dimensional fixed point set (see [16], Kim and Sanderson [14]). $\tau(z_1, z_2, t) = (\overline{z_1}, \overline{z_2}, 1-t)$, $m = 4$. The orbit manifold M_* is the *quadripus*; its boundary consists of 4 projective planes and one incompressible torus.

Example 3 The *dipus* $M = (K \tilde{\times} I)_o = S^1 \times S^1 \times [0,1]/(z_1, z_2, 1) = (-z_1, \overline{z_2}, 1)$, the orientable twisted I–bundle over the Klein bottle K with boundary the torus $T = S^1 \times S^1 \times \{0\}$. There is only one orientation-reversing involution τ with non-empty zero-dimensional fixed point set on M, given by $\tau[z_1, z_2, t] = [-\overline{z_1}, -z_2, t]$ (see [16, Corollary 4.8]), $m = 2$. The orbit manifold M_* is the *dipus*; its boundary consists of 2 projective planes and an incompressible Klein bottle.

The dipus D is also obtained from the geminus $P = (P^2 \times I) \#_b (P^2 \times I)$ and the solid Klein bottle $m_0 \times I$ (where m_0 is the Möbius band) by gluing a nonseparating annulus A_1 in the Klein bottle boundary of P to the incompressible annulus $A_2 = \partial m_0 \times I$ [16, page 333].

Example 4 The *octopod* and *tetrapod* M is an orientable torus bundle over S^1. There are only two torus bundles over S^1 that admit orientation-reversing involutions τ with non-empty zero-dimensional fixed point set, and each admits only one such involution [14], [16].

(i) The *octopod* $M = S^1 \times S^1 \times S^1$, $\tau(z_1, z_2, z_3) = (\overline{z_1}, \overline{z_2}, \overline{z_3})$, $m = 8$. The orbit manifold M_* is the *octopod*; its boundary consists of 8 projective planes.

Any self-homeomorphism of the torus boundary T_0 of the quadripus Q (Example 2) extends to a homeomorphism of Q. The octopod may also be viewed as $Q \cup_{T_0} Q$, the union of two copies of Q along the torus boundary.

(ii) The *tetrapod* $M = S^1 \times S^1 \times [-1,1]/(z_1, z_2, 1) \sim (\overline{z_1}, \overline{z_2}, -1)$, $\tau[z_1, z_2, t] = [-\overline{z_1}, \overline{z_2}, -t]$ (see [14, page 106]), $m = 4$. The orbit manifold M_* is the *tetrapod*; its boundary consists of 4 projective planes.

M can also be described as the double of $(K \tilde{\times} I)_o$: The torus in

$$S^1 \times S^1 \times I/(z_1, z_2, 0) \sim (\overline{z_1}, \overline{z_2}, 1)$$

that is the union of the two annuli $\{\pm i\} \times S^1 \times [0,1]$ cuts M into two copies of $(K \tilde{\times} I)_o$. With this description the tetrapod is the union of two copies of the dipus along the Klein bottle boundary.

The tetrapod may also be viewed as $Q \cup_{T_0} T \tilde{\times} I$ (where $T \tilde{\times} I$ is the non-orientable twisted I–bundle) and as $Q \cup_{T_0} K \tilde{\times} I$.

Example 5 The *bipod* $M = ((K \tilde{\times} I)_o) \cup_\varphi ((K \tilde{\times} I)_o)'$, the twisted double of $(K \tilde{\times} I)_o$, where

$$\varphi\colon S^1 \times S^1 \times \{0\} \to (S^1 \times S^1 \times \{0\})'$$

is $\varphi(z_1, z_2, 0) = (z_2, z_1, 0)'$. This "Hantzsche–Wendt manifold" is the only twisted double of $(K \tilde{\times} I)_o$, besides the double in (4)(ii), that admits an orientation-reversing involution τ with non-empty zero-dimensional fixed point set [16, Corollary 6.6], given by $\tau[z_1, z_2, t] = [-\bar{z}_1, -z_2, t]$, $\tau[z_1, z_2, t]' = [-z_1, -\bar{z}_2, t]'$, $m = 2$. The orbit manifold M_* is the *bipod*; its boundary consists of 2 projective planes.

The bipod B may also be viewed as $D \cup (K \tilde{\times} I)$, where $K \tilde{\times} I$ is the non-orientable I–bundle over the Klein bottle K and D is the dipus from Example 3, with $D \cap (K \tilde{\times} I) = \partial_K D = \partial(K \tilde{\times} I)$.

Two projective planes P_1, P_2 in a closed prime 3–manifold M are *pseudo-parallel* if they cobound a submanifold homotopy equivalent to $P^2 \times I \subset M$. By Perelman, pseudo-parallel is the same as *parallel*, ie, P_1, P_2 cobound a submanifold homeomorphic to $P^2 \times I \subset M$.

Remark 1 Let M_* be a geminus, quadripus, dipus, bipod, tetrapod, or octopod. If P_0 is a projective plane in $\mathrm{int}(M_*)$, then P_0 is parallel to a boundary component of M_*.

To see this, let $p\colon \overline{M - (C_1 \cup \cdots \cup C_m)} \to M_*$ be the 2–sheeted covering and let $S_0 = p^{-1}(P_0)$. Since M is irreducible, the 2–sphere S_0 bounds a punctured ball B_0 in $\overline{M - (C_1 \cup \cdots \cup C_m)}$ (where ∂B_0 consists of S_0 and some of the 2–spheres ∂C_i). Then $p\colon B_0 \to p(B_0)$ is a 2–sheeted covering. Hence $\pi_1(p(B_0)) = \mathbb{Z}_2$ and by Epstein [4] and Perelman's proof of the Poincaré Conjecture, $p(B_0)$ is homeomorphic to $P^2 \times I \subset M_*$, where $P^2 \times 0 = P_0$ and $P^2 \times 1 = p(\partial C_i)$, for some i.

4 Hunfree 3–manifolds

In this section we obtain a complete list of all compact 3–manifolds whose fundamental groups do not contain F_2 and show that these are precisely the compact 3–manifolds whose fundamental groups are virtually solvable.

Virtually solvable groups are amenable and amenable groups are hunfree. First we show that these three classes of groups agree for compact 3–manifold groups, by showing:

Proposition 2 *If the fundamental group of a compact 3–manifold N is hunfree, then it is virtually solvable. In fact, if N is not covered by the dodecahedral manifold and $\pi_1(N)$ is hunfree, then $\pi_1(N)$ is solvable.*

Proof The manifold that is obtained from a manifold W by filling in all boundary spheres with 3–balls is denoted by $\widehat{W}$ or $W^\wedge$. For a compact 3–manifold N we denote by $\widetilde{N}$ its minimal orientable cover (ie, if N is orientable, $\widetilde{N} = N$; if N is non-orientable, then $\widetilde{N}$ is the 2–sheeted orientable cover of N).

We start by applying Theorem 2.9 in Evans and Jaco's paper [5]:

Theorem [5] Let N be a compact 3–manifold. If $\widetilde{N}^\wedge$ is a closed 3–manifold with $\pi_2(\widetilde{N}^\wedge) = 0$, assume that $\widetilde{N}^\wedge$ is virtually Haken. Then if $\pi_1(N)$ is hunfree, $\pi_1(N)$ is polycyclic.

By Theorem 5.2 of Evans and Moser [6], $\pi_1(N)$ is polycyclic if and only if $\pi_1(N)$ is solvable.

So now consider the remaining case where $\widetilde{N}^\wedge$ is a closed 3–manifold with $\pi_2(\widetilde{N}^\wedge)=0$, but $\widetilde{N}^\wedge$ is not virtually Haken, hence orientable. Perelman's Geometrization Theorem implies that if $\widetilde{N}^\wedge$ is not virtually Haken with $\pi_2(\widetilde{N}^\wedge) = 0$, then $\widetilde{N}^\wedge$ is hyperbolic or spherical (see eg [1, Theorem 1.1.6]). By the Tits alternative for finitely generated linear groups [22], closed hyperbolic 3–manifold groups are not hunfree. Now if $\pi_1(N)$ is hunfree, then so is $\pi_1(\widetilde{N}^\wedge)$, and it follows that $\widetilde{N}^\wedge$ is spherical and $\pi_1(N)$ is finite, hence virtually solvable.

In fact it follows from Theorem 3.1 of [6] that in the latter case $\pi_1(N)$ is solvable with the exception of those for which N is covered by the dodecahedral manifold, which are the groups $SL(2,5) \times Z_m$, with $\gcd(m, 30) = 1$. □

The next proposition lists all compact 3–manifolds with hunfree (or amenable or virtually solvable) fundamental groups. The manifold $(K \tilde{\times} I)_0$ that appears in case (3) is the unique orientable non-trivial I–bundle over K. A *twisted double* of $(K \tilde{\times} I)_0$ is a closed 3–manifold obtained by gluing two copies of $(K \tilde{\times} I)_0$ along their boundary components.

Proposition 3 *Let W be a compact connected 3–manifold. Then $\pi_1(W)$ does not contain F_2 if and only if $\widehat{W}$ is one of the following manifolds:*

(1) *A closed Seifert manifold with non-negative orbifold Euler characteristic*

(2) *A torus bundle over* S^1

(3) *A twisted double of* $(K \tilde{\times} I)_0$

(4) $T \tilde{\times} I$, $K \tilde{\times} I$, $S^1 \tilde{\times} D^2$

(5) *The quadripus, dipus, or geminus*

(6) *The octopod, tetrapod, bipod,* $P^2 \times I$, $(P^2 \times I) \# P^3$, *or* $(P^2 \times I) \# (P^2 \times I)$

Proof Let $M = \widehat{W}$. Note that $\pi_1(W) = \pi_1(M)$ and suppose $\pi_1(M)$ does not contain F_2.

If M is covered by the dodecahedral manifold then M belongs to case (1) of the proposition. Therefore by Proposition 2 we may assume that $\pi_1(M)$ is solvable.

Case 1 $\pi_2(M) = 0$

If M is sufficiently large, then by Theorems 4.2 and 4.5 of [6], M is as in cases (1), (2), (3), (4) of the proposition.

If M is not sufficiently large (and therefore orientable), Perelman's Geometrization Theorem implies that M is hyperbolic or Seifert and, as before, the Tits alternative shows that M is Seifert. By Theorem 6.4 of Evans and Moser [6], M is as in case (1) of the proposition. (Note that Klein bottle bundles over S^1 and twisted doubles of non-orientable I–bundles over the Klein bottle belong to case (1) of the proposition).

Case 2 $\pi_2(M) \neq 0$

If there is an essential 2–sphere $S \subset M$ then $\pi_1(M) = \mathbb{Z} * \pi_1(M_1)$ (if S is non-separating) or $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$ is a non-trivial free product (if S is separating). Since $\pi_1(M)$ does not contain F_2, we obtain in the first case that $\pi_1(M) = \mathbb{Z}$ and in the second case that $\pi_1(M) = \mathbb{Z}_2 * \mathbb{Z}_2$. In the first case $M = S^1 \tilde{\times} S^2$ (case (1) of the proposition). In the second case, by Kneser's Conjecture (proved by Stallings [21]) $M \approx M_1 \# M_2$, where $\pi_1(M_i) \cong \mathbb{Z}_2$, and it follows that $M \approx (P^2 \times I) \# (P^2 \times I)$ or $(P^2 \times I) \# P^3$ (case (6) of the proposition), or $P^3 \# P^3$ (case (1) of the proposition).

Thus assume there is no essential 2–sphere $S \subset M$, ie, M is prime. By the Projective Plane Theorem of Epstein, there is a 2–sided $P^2 \subset M$.

If P^2 does not separate M, then the orientable double cover $\widetilde{M} = S^1 \tilde{\times} S^2 \# M_1$. Since $\pi_1(\widetilde{M})$ does not contain F_2, M_1 is a punctured S^3 and $\pi_1(M)$ is an extension of $\mathbb{Z}$ by $\mathbb{Z}_2$. Hence $\pi_1(M) \cong \mathbb{Z} \times \mathbb{Z}_2$ or $\mathbb{Z}_2 * \mathbb{Z}_2$. In the first case $M = P^2 \times S^1$ (case (1) of the proposition) and the second case does not occur by Kneser's Conjecture and since M is prime.

If $P^2 \subset M$ separates M but is not boundary parallel, then

$$\pi_1(M) = \pi_1(M_1) *_{\mathbb{Z}_2} \pi_1(M_2),$$

a free product with amalgamation over $\mathbb{Z}_2$. Now $\pi(M_i)$ is not finite, otherwise $M_i = P^2 \times I$ and P^2 would be boundary parallel. Then $\pi_1 \widetilde{M} = \pi_1(\widetilde{M}_1) * \pi_1(\widetilde{M}_2)$ would be a non-trivial free product. Since $\pi_1(\widetilde{M})$ is hunfree, this case can not happen.

So assume that all 2–sided P^2's in M are boundary parallel. It follows that all essential 2–spheres in $\widetilde{M}$ are boundary parallel and $\pi_2(\widetilde{M}\hat{\ }) = 0$. Hence $\widetilde{M}\hat{\ }$ is as in case 1, ie, as in cases (1)–(4) of the proposition. We also assume that $\pi_1(M)$ is infinite, since otherwise $M = P^2 \times I$ (case (5) of the proposition).

Extending the covering translation $\tau\colon \widetilde{M} \to \widetilde{M}$ to an orientation-reversing involution $\tau\colon \widetilde{M}\hat{\ } \to \widetilde{M}\hat{\ }$ (with isolated fixed points corresponding to the lifts of the projective planes) we obtain M from $\widetilde{M}\hat{\ }/\tau$ by removing neighbourhoods of the fixed points. We now consider all possible orientation-reversing involutions τ of $\widetilde{M}\hat{\ }$ with non-empty finite fixed point set in cases (1)–(4) of the proposition.

(1) If $\widetilde{M}\hat{\ }$ is a closed Seifert manifold, then by Theorems 8.2 and 8.5 of Neumann and Raymond [19] $\widetilde{M}\hat{\ }$ fibers over S^1. Since $\widetilde{M}\hat{\ }$ is orientable, $\pi_1(\widetilde{M}\hat{\ })$ contains no F_2, and $\pi_2(\widetilde{M}\hat{\ }) = 0$, the fiber is a torus. So $\widetilde{M}\hat{\ }$ is as in the next case:

(2) If $\widetilde{M}\hat{\ }$ is an orientable torus bundle over S^1, then by Example 4, M is the octopod or the tetrapod.

(3) If $\widetilde{M}\hat{\ }$ is a twisted double of $(K \tilde{\times} I)_o$, then by Example 5, M is the bipod.

(4) If $\widetilde{M}\hat{\ } = T \times I$, then by Example 2, M is the quadripus.

If $\widetilde{M}\hat{\ } = (K \tilde{\times} I)_o$, then by Example 3, M is the dipus.

If $\widetilde{M}\hat{\ } = S^1 \tilde{\times} D^2$, then by Example 1, $M = (P^2 \times I) \#_b (P^2 \times I)$.

Conversely, all fundamental groups of the manifolds in the proposition are virtually solvable, hence amenable and hunfree:

The groups of the manifolds in cases (1)–(4) are solvable with the exception of those covered by the dodecahedral manifold, which are the finite groups $SL(2,5) \times Z_m$, with $\gcd(m, 30) = 1$ [6, Theorem 3.1]. All the remaining fundamental groups are solvable: the groups $Z_2 * Z_2$, Z_2 in cases (5) and (6) are solvable, the fundamental groups of the quadripus and dipus are extensions of the solvable fundamental groups of the torus and Klein bottle by $\mathbb{Z}_2$, the fundamental groups of the octopod and tetrapod are extensions of the solvable fundamental groups of torus bundles over S^1 by $\mathbb{Z}_2$ and the fundamental group of the bipod are extensions of the solvable fundamental group of a twisted double of $(K \tilde{\times} I)_0$ by $\mathbb{Z}_2$. □

Corollary 2 *Let M be a closed 3–manifold. Then $\mathrm{cat}_{\mathrm{ame}}(M) = 1$ if and only if M is one of the following:*

(1) *A closed Seifert manifold with non-negative orbifold Euler characteristic*

(2) *A torus bundle over S^1*

(3) *A twisted double of $(K \tilde{\times} I)_0$*

For future use we need the following two corollaries.

Corollary 3 *Let W be a compact connected 3–manifold such that $\pi_1(W)$ does not contain F_2. If G is a torus or a Klein bottle in ∂W with inclusion $\iota\colon G \to W$, then $|\pi_1(W) : \iota_*(\pi_1(G))| \leq 2$.*

Proof W is as in cases (4) or (5) of Proposition 3. If $\widehat{W} = G \times I$, $(P^2 \times I) \#_b (P^2 \times I)$ or $S^1 \tilde{\times} D^2$, then $|\pi_1(W) : i_*(\pi_1(G))| = 1$. If $\widehat{W}$ is a nontrivial I–bundle then it is a mapping cylinder of a 2–fold covering and so $|\pi_1(W) : \iota_*(\pi_1(G))| = 2$.

If $\widehat{W}$ is the quadripus or dipus, let $\rho\colon \widetilde{W} \to \widehat{W}$ be the orientable 2–fold covering. For the quadripus, $\widetilde{W}$ is a punctured $T \times I$. So for a torus component T of $\partial \widetilde{W}$ the inclusion induced homomorphism $j_*\colon \pi_1(T) \to \pi_1(\widetilde{W})$ is an isomorphism and ρ maps T homeomorphically onto G. Hence $|\pi_1(W) : \iota_* \pi_1(G)| = |\pi_1(\widehat{W}) : \rho_* j_* \pi_1(T)| = 2$.

If $\widehat{W}$ is the dipus, G is a Klein bottle and in the commutative diagram of injections

$$\begin{tikzcd}
\pi_1(\rho^{-1}(G)) \arrow[r] \arrow[d] & \pi_1(\widetilde{W}) \arrow[d] \\
\pi_1(G) \arrow[r] & \pi_1(\widehat{W})
\end{tikzcd}$$

with vertical monomorphisms induced by ρ and horizontal monomorphisms induced by inclusions, the upper and vertical monomorphism have images of index 2. Therefore image(ι_*) has index 2. □

Corollary 4 *Suppose $W = W_1 \cup W_2$ such that $W_1 \cap W_2 = \partial W_1 \cap \partial W_2$ is a torus or a Klein bottle. If, for $i = 1, 2$, W_i is a compact connected 3–manifold and $\pi_1(W_i)$ does not contain F_2, then $\pi_1(W)$ is solvable.*

Proof Let $G = W_1 \cap W_2$. Then $\pi_1(W) = \mathrm{I}(W_1) *_{\mathrm{I}(G)} \mathrm{I}(W_2)$ is the free product of $\mathrm{I}(W_1)$ and $\mathrm{I}(W_2)$ amalgamated along $\mathrm{I}(G)$, where $\mathrm{I}(X)$ denotes the image of $\pi_1(X)$ in $\pi_1(W)$ under the inclusion-induced homomorphism $\pi_1(X) \to \pi_1(W)$.

By Corollary 3 the index of $\mathrm{I}(G)$ in $\pi_1(W_i)$ is ≤ 2, hence $\mathrm{I}(G)$ is normal in $\mathrm{I}(W)$ and $\pi_1(W)/\mathrm{I}(G)$ is $Z_2 * Z_2$, Z_2, or 1. Since $\mathrm{I}(G)$ is solvable, it follows that $\pi_1(W)$ is solvable. □

5 $\mathrm{cat}_{\mathrm{ame}}(M^3) \leq 2$

In this section we classify the closed 3–manifolds of amenable category 2. The main result is the following:

Theorem 1 *A closed 3–manifold has amenable category ≤ 2 if and only if there is a disjoint collection of embedded 2–spheres and projective planes that splits M into submanifolds with amenable groups. Moreover, after filling all the boundary 2–spheres with balls, each component is closed, a bipod, a tetrapod or an octopod.*

Recall that two projective planes P_1, P_2 in a 3–manifold M are *parallel* if they cobound a submanifold homeomorphic to $P^2 \times I \subset M$. By [18] there is a (possibly empty) maximal disjoint collection $\mathcal{P}$ of 2–sided projective planes in M, unique up to isotopy, such that no two projective planes of $\mathcal{P}$ are parallel in M and every projective plane in $\overline{M - N(\mathcal{P})}$ is parallel (in M) to a component of $\mathcal{P}$. We call such a system a *complete* system $\mathcal{P}$ of projective planes of M. By Remark 1 of Section 3, if M' is a bipod, a tetrapod, or an octopod, such a complete system is formed by the boundary components of M'.

Corollary 5 *A closed prime 3–manifold M has amenable category 2 if and only if M is non-orientable and every component of the exterior of the complete system $\mathcal{P}$ of projective planes in M is a bipod, tetrapod, or octopod.*

In particular, there are no closed Seifert fiber spaces or graph-manifolds of amenable category 2. This is because a Seifert fiber space M that contains a 2–sided projective plane is homeomorphic to $P^2 \times S^1$, which is of amenable category 1.

For the proof of Theorem 1 note that the "if" part is clear; $M = W_1 \cup W_2$, where W_1 is a regular neighborhood of the collection of the 2–spheres and projective planes, and $W_2 = \overline{M - W_1}$.

For the "only if" part suppose M is a closed 3–manifold with $\mathrm{cat}_{\mathrm{ame}}(M) \leq 2$. We first observe that there is a (not necessarily connected) 2–sided surface F in M such that F and $\overline{M - N(F)}$ are amenable.

To see this, apply Lemma 1 to write $M = W_1 \cup W_2$, where W_i is a compact amenable 3–submanifold and so that $W_1 \cap W_2 = \partial W_1 = \partial W_2$. In particular, $F := W_1 \cap W_2$ is a 2–sided closed (pl)–surface in M and a regular neighborhood $N(F)$ in M is homeomorphic to a product $F \times I$. Note that W_i and F need not be connected. For each component F' of F, $\mathrm{im}(\pi_1(F') \to \pi_1(M))$ is contained in $\mathrm{im}(\pi_1(W_i') \to \pi_1(M))$, where W_i' is a component of W_i, and it follows that F is amenable. Furthermore the *exterior* $\overline{M - N(F)}$ of F is amenable.

Our goal is to show that we can find such an F such that every component of F is a 2–sphere or projective plane.

Lemma 3 *Let F be a 2–sided surface in the closed 3–manifold M such that F and $\overline{M - N(F)}$ are amenable. If F is compressible and if F_1 is the surface obtained from F after surgery on a compressing disk, then F_1 and $\overline{M - N(F_1)}$ are amenable.*

Proof Suppose D is a compressing disk for a component F' of F. Let $D \times I$ be a regular neighborhood such that $(D \times I) \cap F = \partial D \times I$ and $\partial D \times 0$ is an essential curve in F'. For the component F_1' of $F_1 = (F - \partial D \times I) \cup (D \times \partial I)$ that contains $D \times \{0\}$ or $D \times \{1\}$, $\mathrm{im}(\pi_1(F_1') \to \pi_1(M))$ is a subgroup of $\mathrm{im}(\pi_1(F') \to \pi_1(M))$. Since F' is amenable, so is F_1.

If a component M_1' of $\overline{M - N(F_1)}$ is different from a component of $\overline{M - N(F)}$, then either M_1' is contained in a component M' of $\overline{M - N(F)}$, or M_1' can be written as $M_1' = M' \cup (D' \times I_1)$, where I_1 is a subinterval of I and D is a subdisk of D' such that $(D' \times I_1) \cap M' = \partial D' \times I_1$. In the first case M_1' is amenable as a subset of the amenable set M'. In the second case $\pi_1(M_1')$ and $\pi_1(M')$ have the same image in $\pi_1(M)$, and since M' is amenable, so is M_1'. □

Define the *complexity* $c(F)$ of a closed connected 2–manifold F to be $c(F) = 1$, if F is the sphere, and otherwise, $c(F) = (2g - 1)\omega$ where g is the (orientable or non-orientable) genus of F and ω is the first infinite ordinal. If F is a closed non-connected 2–manifold with components $F_1, F_2, \ldots, F_n$, define $c(F) = c(F_1) + c(F_2) + \cdots + c(F_n)$.

Note the following:

If F is a surface with minimal complexity such that F and $\overline{M - N(F)}$ are amenable, if F_0 is a component of F and $F_1 = F - F_0$, then the component of $\overline{M - F_1}$ containing F_0 is not amenable.

Lemma 4 *Suppose M is a closed 3–manifold with* $\mathrm{cat}_{\mathrm{ame}}(M) \leq 2$. *If F is of minimal complexity such that F and $\overline{M - N(F)}$ are amenable, then every component of F is a 2–sphere or projective plane.*

Proof First we show that F is incompressible.

If not, let F_1 be obtained from F by surgery on a compressing disk. By Lemma 3, F_1 and $\overline{M - F_1}$ are amenable. However, $c(F_1) < c(F)$, a contradiction to the minimality of $c(F)$. Hence $F \to M$ is π_1–injective. (Here one says that $Y \to X$ is π_1–injective, if $\pi_1(Y,*) \to \pi_1(X,*)$ is injective for each basepoint $* \in F$).

In particular, since amenable groups are hunfree, all the components of F have non-negative Euler characteristic.

Let $F \times [0,1]$ be a tubular neighborhood of F and let $E = \overline{M - F \times [0,1]}$ be the exterior of F. Then the inclusions $\partial E \to E$ and $\partial E \to F \times I$ are π_1–injective and so $E \to M$ is π_1–injective (see for example [10, Lemma 2.2]). Since E is amenable, its components have amenable fundamental groups and so are as in Proposition 3.

Now suppose a component F_0 of F is a torus or Klein bottle. Let $F_0 \times [0,1]$ be the component of $F \times [0,1]$ containing F_0. If a component C of E contains $\partial(F_0 \times [0,1])$ then $\widehat{C}$ is homeomorphic to $F_0 \times I$ (only (4) of Proposition 3 applies) so $C \cup (F_0 \times [0,1])$ is a punctured F_0–bundle over S^1 and its fundamental group is amenable; hence $F - F_0$ and its complement in M are amenable and $c(F - F_0) < c(F)$, contradicting the minimality of $c(F)$.

If no component of E contains $\partial(F_0 \times [0,1])$, let C_1, C_2 be the two components of E intersecting $F_0 \times [0,1]$. Note that C_i is not a trivial I–bundle because of the minimal complexity condition. We claim that $C = C_1 \cup F_0 \times [0,1] \cup C_2$ is amenable, which again leads to a contradiction, since then $F - F_0$ and its complement are amenable with $c(F - F_0) < c(F)$.

$\widehat{C} = \widehat{C}_1 \cup_{F_0} \widehat{C}_2$ is a union along an incompressible torus or Klein bottle F_0, where $\widehat{C}_i$ is as in cases (4) or (5) of Proposition 3. If $\widehat{C}_i = T \mathbin{\tilde{\times}} I$ or $K \mathbin{\tilde{\times}} I$, then $\pi_1(C)$ is solvable. In the remaining cases, (where one or both of $\widehat{C}_i$ is a quadripus Q or dipus D), $\widehat{C} = T \mathbin{\tilde{\times}} I \cup_{F_0} Q = K \mathbin{\tilde{\times}} I \cup_{F_0} Q$ is the tetrapus, $\widehat{C} = Q \cup_{F_0} Q$ is the octopus, $\widehat{C} = K \mathbin{\tilde{\times}} I \cup_{F_0} \cup D$ is the bipod and $\widehat{C} = D \cup_{F_0} D$ is the tetrapod. All their groups are amenable (in fact solvable). □

We now complete the proof of the "only if" part of Theorem 1.

Proof Choose F of minimal complexity such that F and $\overline{M - N(F)}$ are amenable. By the preceding lemma, ∂C consists of 2–spheres and projective planes, for every component C of $\overline{M - N(F)}$, and C is as in Proposition 3.

If $\widehat{C} = P^2 \times I$, $(P^2 \times I) \# (P^2 \times I)$ or $(P^2 \times I) \# P^3$, let P be a P^2–component of F parallel to a boundary component of C, and (if $C \neq P^2 \times I$) let S be a 2–sphere in C splitting it into two punctured copies of $P^2 \times I$ (resp. into a punctured $P^2 \times I$ and a punctured P^3) (that is, the 2–sphere used for the connected sum #). Then $F_1 = (F - P) \cup S$ and $M - F_1$ are amenable and $c(F_1) < c(F)$, a contradiction.

Hence $\widehat{C}$ is as in cases (1) or (2) of Proposition 3 or is a bipod, tetrapod or octopod. □

6 $\mathrm{cat}_{\mathrm{ame}}(M^3) \leq 3$

In this section we classify the closed 3–manifolds M with $\mathrm{cat}_{\mathrm{ame}}(M) \leq 3$. The main result is the following:

A closed 3–manifold has amenable category ≤ 3 if and only if its minimal orientable 2–fold cover is a connected sum of graph manifolds.

This follows from Theorems 2 and 3. In the orientable case, Theorem 2 follows from Proposition 4, which establishes the converse of the statement that a closed orientable 3–manifold that can be covered by 3 open amenable sets has trivial simplicial volume by Gromov's Vanishing Theorem, and therefore is a connected sum of graph manifolds by Perelman's Geometrization Theorem. Theorem 3 gives a more detailed description of the non-orientable 3–manifolds with amenable category ≤ 3.

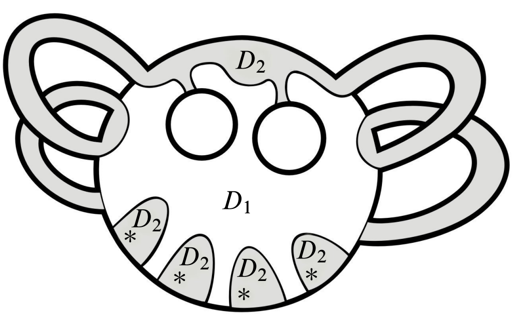
Lemma 5 (a) *If M is a Seifert fiber space with non-empty boundary, then*

$$\mathrm{cat}_{\mathrm{ame}}(M) \leq 2.$$

(b) *If M is a closed Seifert fiber space, then* $\mathrm{cat}_{\mathrm{ame}}(M) \leq 3$.

Proof (a) Let $p\colon M \to S$ be the projection to the orbit surface. Starting with a decomposition of S into two disks and choosing proper disjoint disk neighborhoods of the exceptional points, we obtain a decomposition of S into a disk D_1 and a disjoint collection D_2 of $r+1$ disks, where r is the number of exceptional points, such that $D_1 \cap D_2 = \partial D_1 \cap \partial D_2$. (The figure below illustrates the case when S is of genus 2 with 3 boundary components and 4 exceptional points). Then $W_1 = p^{-1}(D_1)$ and the components of $W_2 = p^{-1}(D_2)$ are solid tori and $W = W_1 \cup W_2$, where each component of W_i has cyclic fundamental group.

(b) The proof is as in (a) by starting with a decomposition of S into three disks. □



Proposition 4 *If M is a graph manifold, then* $\mathrm{cat}_{\mathrm{ame}}(M) \leq 3$.

Proof By Lemma 5 we may assume that M is not a Seifert fiber space. So there is a non-empty collection of tori and Klein bottles that splits M into Seifert fiber spaces. Let W_3 be a regular neighborhood (in M) of this collection and let $\overline{M - W_3} = M_1 \cup M_2 \cup \cdots \cup M_n$, where each M_i is a Seifert fiber space with non-empty boundary. By Lemma 5(a), $M_i = W_{i1} \cup W_{i2}$, for some amenable W_{i1} and W_{i2}. Now let $W_1 = \bigcup_i W_{i1}$, $W_2 = \bigcup_i W_{i2}$. Then $M = W_1 \cup W_2 \cup W_3$ is a cover by 3 amenable 3–submanifolds. □

Theorem 2 *Let M be a closed orientable 3–manifold M. Then* $\mathrm{cat}_{\mathrm{ame}} M \leq 3$ *if and only if M is a connected sum of graph manifolds.*

Proof As pointed out in the introduction, it follows from Gromov [11] and Perelman (see [1]) that $\mathrm{cat}_{\mathrm{ame}} M \geq 4$, if some factor M_i of the prime decomposition of M is not a graph manifold. The converse follows from Proposition 4 and Corollary 1. □

We now consider the case that M is non-orientable.

Lemma 6 *Let $p\colon \widetilde{M} \to M$ be any covering map. Then* $\mathrm{cat}_{\mathrm{ame}}(\widetilde{M}) \leq \mathrm{cat}_{\mathrm{ame}}(M)$.

Proof It suffices to show that if W is amenable in M, then $\widetilde{W} = p^{-1}(W)$ is amenable in $\widetilde{M}$. Assume W, $\widetilde{W}$ are connected (otherwise we look at components). Let $\iota\colon W \to M$ and $\tilde{\iota}\colon \widetilde{W} \to \widetilde{M}$ be the inclusions and let $p'\colon \widetilde{W} \to W$ be the restriction of p to $\widetilde{W}$. Then $p_*\tilde{\iota}_*(\pi_1(\widetilde{W})) = \iota_* p'_*(\pi_1(\widetilde{W}))$ is a subgroup of $\iota_*(\pi_1(W))$, which is amenable. Now $p_*\colon \tilde{\iota}_*(\pi_1(\widetilde{W})) \to p_*\tilde{\iota}_*(\pi_1(\widetilde{W}))$ is an isomorphism, hence $\tilde{\iota}_*(\pi_1(\widetilde{W}))$ is amenable. □

In particular, if M is a closed 3–manifold with $\mathrm{cat}_{\mathrm{ame}}(M) \leq 3$ and $p\colon \widetilde{M} \to M$ is its orientable 2–fold cover it follows from Theorem 2 that $\widetilde{M}$ is a graph manifold. By Meeks and Scott [17] there exists a torus decomposition of $\widetilde{M}$ that is equivariant under the covering translation, except in the special case when $\widetilde{M}$ is a torus bundle over S^1 with hyperbolic monodromy.

Lemma 7 *Suppose $\widehat{C}$ is a Haken graph manifold but not a torus bundle over S^1 with hyperbolic monodromy and suppose that $h\colon C \to C$ is a fixed-point free orientation-reversing involution. Then there is a disjoint (possibly empty) collection F of tori and Klein bottles in C such that every component of the orbit manifold $\overline{C - N(F)}/h$ is a punctured S^1–bundle or geminus.*

Proof Let $p\colon C \to C/h$ be the natural 2–fold covering and extend h to an involution $\widehat{h}\colon \widehat{C} \to \widehat{C}$, where possible fixed points of $\widehat{h}$ are the centers of ball components of $\widehat{C} - C$. Since $\widehat{C}$ is Haken but not a torus bundle with hyperbolic monodromy, there is by Meeks and Scott [17] an $\widehat{h}$–invariant disjoint collection T' of tori in $\widehat{C}$, such that the components of

$$\overline{\widehat{C} - N(T')}$$

are Seifert fibered. If a component of T' intersects $\mathrm{Fix}(\widehat{h})$ replace it in T' by the two boundary components of an $\widehat{h}$–invariant product neighborhood of this component.

The new collection T is an h–invariant union of tori in $\operatorname{int} C$.

Let V_o denote the union of all components C_i of $\overline{C - N(T)}$ for which $h(C_i) \cap C_i = \varnothing$ and let V denote the union of those components C_j for which $h(C_j) = C_j$.

The components of $p(V_o)$ are punctured orientable Seifert fiber spaces and we let E_o be the collection of torus boundaries of fibered solid torus neighborhoods of the exceptional fibers of $p(\widehat{V}_o)$.

Every component $p(C_j)$ of $p(V)$ is non-orientable and $\pi_1(C_j)$ contains a non-trivial cyclic normal subgroup. By Theorem 1 of Heil and Whitten [12] there is a collection B of 2–sided Klein bottles in $\widehat{p(V)}$ such that $\widehat{p(V)} = W_s \cup W_t$, $W_s \cap W_t = B$, the components of W_t are gemini, and each component of W_s is a Seifert bundle, ie, it admits a decomposition into disjoint circle-fibers each having a regular neighborhood that is either a fibered solid torus or a fibered solid Klein bottle.

Let E_s be the collection of torus boundaries of fibered solid torus neighborhoods of the exceptional fibers of W_s.

Let K_s be the union of all fibers in W_s that have solid Klein bottle neighborhoods. K_s is a union of tori and Klein bottles.

Now $F\colon = T \cup B \cup E_o \cup E_s \cup K_s$ satisfies the conclusion of the lemma. □

We now consider the case when $\widetilde{M}$ is a torus bundle over S^1 with hyperbolic monodromy, more generally:

Lemma 8 *Suppose $\widehat{C}$ is an orientable torus bundle over S^1 whose monodromy is not the identity or the inversion $(z_1, z_2) \to (\overline{z_1}, \overline{z_2})$. Let $h\colon C \to C$ be an orientation-reversing PL involution. Then C/h is a punctured torus bundle.*

Proof Extend h to an involution $\widehat{h}\colon \widehat{C} \to \widehat{C}$ by coning. By Corollary 1 of [14], the only orientable torus bundle over S^1 admitting orientation-reversing PL involutions with non-empty fixed point sets are the two that are excluded in the lemma. Hence $\widehat{h}$ has no fixed points and it follows that no (sphere) boundary component of C is h–invariant. Furthermore, by Theorem B of [14], $\widehat{C}/\widehat{h}$ is a torus bundle. Hence C/h is a punctured torus bundle. □

We close with the following theorem, that together with Corollary 2 and Theorem 1 provides a complete classification of closed 3–manifolds of amenable category 3.

Theorem 3 *Let M be a closed nonorientable 3–manifold and let $\widetilde{M}$ be its orientable 2–fold cover. The following are equivalent:*

(i) $\mathrm{cat}_{\mathrm{ame}}(M) \le 3$

(ii) $\mathrm{cat}_{\mathrm{ame}}(\widetilde{M}) \le 3$

(iii) *$\widetilde{M}$ is a connected sum of graph manifolds.*

(iv) *M contains a disjoint collection F of 2–spheres, projective planes, tori, and Klein bottles such that every component of $\overline{M - N(F)}$ is a punctured S^1–bundle or geminus.*

Proof (i) $\Rightarrow$ (ii) by Lemma 6.

(ii) $\Leftrightarrow$ (iii) by Theorem 2 and Corollary 2.

(iii) $\Rightarrow$ (iv): Let $p\colon \widetilde{M} \to M$ be the 2–fold covering and $h\colon \widetilde{M} \to \widetilde{M}$ the covering involution. By Kim and Tollefson [15] there is a collection $\widetilde{S}_0$ of disjoint h–invariant 2–spheres in $\widetilde{M}$ with an h–invariant neighborhood $N(\widetilde{S}_0)$. Every component C of

$$\widetilde{E} := \overline{\widetilde{M} - N(\widetilde{S}_0)}$$

is a punctured graph manifold and either $h(C) \cap C = \varnothing$ or C is h–invariant. Let $S_0 = p(\widetilde{S}_0)$, a disjoint union of 2–spheres and projective planes.

Let $\widetilde{V}_1$ be the union of the components C of $\widetilde{E}$ for which $h(C) \cap C = \varnothing$. Then $p(\widetilde{V}_1)$ is a disjoint union of punctured graph manifolds and there is a disjoint union T_1 of tori in the interior of $p(\widetilde{V}_1)$ such that the components of

$$V_1 := \overline{p(\widetilde{V}_1 - N(T_1))}$$

are punctured Seifert manifolds. Let E_1 be the union of the torus boundaries of fibered solid tori of the exceptional fibers of $\widehat{V}_1$ (which we may assume are contained in $\operatorname{int} V_1$).

Let $\widetilde{V}_2$ be the union of those components of $\widetilde{E}$ that are h–invariant torus bundles over S^1 with hyperbolic monodromy. By Lemma 8, $p(\widetilde{V}_2)$ is a disjoint union of punctured torus bundles and therefore there is a union T_2 of tori in int $p(\widetilde{V}_2)$ such that the components of

$$\overline{p(\widetilde{V}_2) - N(T_2)}$$

are punctured $T^2 \times I$'s.

Finally let $\widetilde{V}_3$ be the union of the components of E that are h–invariant but not torus bundles with hyperbolic monodromy. If C is such a component, then $\widehat{C}$ is Haken because either it has an incompressible torus or it is an irreducible closed Seifert manifold admitting an orientation-reversing involution and, by Neumann and Raymond [19], fibers over S^1. Hence by Lemma 7 there is a disjoint collection T_3 of tori and Klein bottles in $\operatorname{int}(p(\widetilde{V}_3))$ such that every component of

$$\overline{p(\widetilde{V}_3) - N(T_3)}$$

is a punctured S^1–bundle or geminus.

Now take $F := S_0 \cup T_1 \cup E_1 \cup T_2 \cup T_3$, and the conclusion follows.

(iv) $\Rightarrow$ (i): Let V_T (respectively V_S) be the union of the components of $\overline{M - N(F)}$ that are (respectively are not) gemini. There is an S^1–fibration $p\colon \widehat{V}_S \to B$ where B is a compact 2–manifold.

For every component of B with empty boundary take an annulus embedded in it and let A be the union of these annuli. We may assume that $p^{-1}(A) \subset \operatorname{int} V_S$. Let $W_1 = N(F) \cup p^{-1}(A)$.

Now, since every component of $\overline{B-A}$ has nonempty boundary we obtain a decomposition $\overline{B-A} = D \cup D'$ where D and D' are disjoint unions of disks and $D \cap D' = \partial D \cap \partial D'$. We may assume that

$$p(\overline{\widehat{V}_S - V_S}) \subset \operatorname{int} D'.$$

Let $W_2 = p^{-1}(D) \cup V_T$ and $W_3 = p^{-1}(D') \cap V_S$.

Then $M = W_1 \cup W_2 \cup W_3$. The components of W_1 are tubular neighborhoods of 2–spheres, projective planes, tori or Klein bottles. The components of W_2 are solid tori or gemini and the components of W_3 are punctured solid tori. All these components have amenable fundamental groups and it follows that $\operatorname{cat}_{ame(M)} \leq 3$. $\square$

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JCG, FG: *Centro de Investigación en Matemáticas, A.P. 402, 36000 Guanajuato, Mexico*

FG: *Instituto de Matemáticas, UNAM Morelos Campus, 62210 Cuernavaca,, Mexico*

WH: *Department of Mathematics, Florida State University, Tallahassee, FL 32306-4510, USA*

jcarlos@cimat.mx, fico@matem.unam.mx, heil@math.fsu.edu

http://www.math.fsu.edu/People/faculty.php?id=16

Received: 31 August 2011 Revised: 23 October 2012