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Some Ramsey-type results on intrinsic linking of n–complexes

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Define the *complete n–complex on N vertices*, K_N^n, to be the n–skeleton of an $(N-1)$–simplex. We show that embeddings of sufficiently large complete n–complexes in $\mathbb{R}^{2n+1}$ necessarily exhibit complicated linking behaviour, thereby extending known results on embeddings of large complete graphs in $\mathbb{R}^3$ (the case $n=1$) to higher dimensions. In particular, we prove the existence of links of the following types: r–component links, with the linking pattern of a chain, necklace or keyring; 2–component links with linking number at least λ in absolute value; and 2–component links with linking number a nonzero multiple of a given integer q. For fixed n the number of vertices required for each of our results grows at most polynomially with respect to the parameter r, λ or q.

57Q45; 57M15, 57Q35

1 Introduction

In the 1980s Sachs [15] and Conway and Gordon [1] proved that an embedding of the complete graph K_6 in $\mathbb{R}^3$ necessarily contains a pair of disjoint cycles that form a nonsplit link. This fact is expressed by saying that K_6 is *intrinsically linked*. Conway and Gordon also showed that every embedding of K_7 in $\mathbb{R}^3$ contains a cycle that forms a nontrivial knot, and we say that K_7 is *intrinsically knotted*.

Since these papers, the study of intrinsic knotting and linking has been pursued in several directions, and we refer the reader to Ramírez Alfonsín [14] for a survey of some known results. One such direction is to show that embeddings of larger complete graphs necessarily exhibit more complex knotting and linking behaviour. Restricting our attention to linking, Flapan, Pommersheim, Foisy and Naimi [4] and Fleming and Diesl [6] have shown that embeddings of sufficiently large complete graphs must contain nonsplit r–component links; Flapan [2] has shown that they must contain 2–component links with high linking number; and Fleming [5] has extended work by Fleming and Diesl [6] to show that, given an integer q, they must contain 2–component links with linking number a nonzero multiple of q.

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We will refer to results such as those described above as *Ramsey-type results* on intrinsic linking. Perhaps the strongest results in this direction are those of Negami [12] and Flapan, Mellor and Naimi [3]. Restricting attention to embeddings with a projection that is a "good drawing", Negami shows that, given a link L, for n, m sufficiently large every such embedding of the complete bipartite graph $K_{n.m}$ contains a link that is ambient isotopic to L. The restriction to embeddings with a projection that is a good drawing excludes local knots in the edges, which is necessary but not sufficient (Negami [13]) for the result to hold. With no restriction on the embedding, Flapan, Mellor and Naimi show that intrinsic knotting and linking are arbitrarily complex in the following sense: Given positive integers r and α, embeddings of sufficiently large complete graphs contain r–component links in which the second coefficient of the Conway polynomial of each component, and the linking number of each pair of components, is at least α in absolute value.

Extending the result of Sachs [15] and Conway and Gordon [1] in another direction, we may consider embeddings of n–complexes in $\mathbb{R}^d$. By a general position argument every n–complex embeds in $\mathbb{R}^{2n+1}$, and a pair of disjoint n–spheres in $\mathbb{R}^{2n+1}$ have a well-defined linking number (the homology class of one component in the n^{th} homology group of the complement of the second, which is isomorphic to $\mathbb{Z}$), so we take $d = 2n+1$. Define the *complete n–complex on N vertices*, K_N^n, to be the n–skeleton of an $(N-1)$–simplex. Then Lovász and Schrijver [9, Corollary 1.1], Taniyama [19], Melikhov [11, Example 4.7] and Melikhov [10, Example 4.9] show by various arguments that K_{2n+4}^n is intrinsically linked, in the sense that every embedding in $\mathbb{R}^{2n+1}$ contains a pair of disjoint n–spheres that have nonzero linking number. Since $K_N^1 \cong K_N$ this specialises to the K_6 result in the case $n = 1$. The $(n+1)$–fold join $(K_4^0)^{*(n+1)}$ has also been shown to be intrinsically linked in this sense, by M Skopenkov [18].

We may also consider links in which the components are of different dimensions. For $l < k$ Segal and Spież [16] construct a k–complex Q containing subcomplexes $\Sigma^k \cong S^k$ and $\Sigma^l \cong S^l$, such that Q embeds in $\mathbb{R}^{k+l+1}$, and moreover in every such embedding the images of Σ^k and Σ^l are homologically linked. See also Freedman, Krushkal and Teichner [7] for an example and application of such a complex in the case $(k, l) = (2, 1)$. We refer the reader to A Skopenkov [17] for a survey of these and other results on higher-dimensional intrinsic linking, and their implications for questions of embeddability of complexes in $\mathbb{R}^d$.

The purpose of this paper is to establish some Ramsey-type results for embeddings of complete n–complexes in $\mathbb{R}^{2n+1}$. Our results are already known for embeddings of complete graphs in $\mathbb{R}^3$, and our arguments will typically mimic the proof of the corresponding 1–dimensional result. However, in the case of Theorem 1.4 we will obtain a better bound for $n = 1$ than that previously known; and in addition, some constructions used in the arguments require modifications in higher dimensions. These modifications are needed for two main reasons: Firstly, $\partial D^n = S^{n-1}$ is disconnected for $n = 1$, but not for $n \geq 2$; and secondly, triangulations of D^n have simpler combinatorics for $n = 1$ than they do for $n \geq 2$.

We note that for $n \geq 2$ an n–sphere does not knot in $\mathbb{R}^{2n+1}$ for reasons of codimension. Thus, we will not seek to establish any results on intrinsic knotting of complete n–complexes in $\mathbb{R}^{2n+1}$. A question of interest is to determine for which n there is an n–complex which embeds in $\mathbb{R}^{n+2}$, and which is intrinsically knotted in the sense that every such embedding contains a nontrivially knotted n–sphere.

1.1 Statement of results

In what follows, a *k–component link* means k disjoint n–spheres embedded in $\mathbb{R}^{2n+1}$. Given a 2–component link $L_1 \cup L_2$ we will write $\ell k(L_1, L_2)$ for their linking number, and $\ell k_2(L_1, L_2)$ for their linking number mod two. For $\{i, j\} = \{1, 2\}$ the integral linking number is given by the homology class $[L_i]$ in $H_n(\mathbb{R}^{2n+1} - L_j; \mathbb{Z}) \cong \mathbb{Z}$.

Our first result is similar to Flapan et al [4, Theorems 1 and 2], and shows that embeddings of sufficiently large complete n–complexes necessarily contain nonsplit r–component links. Moreover, the number of vertices required grows at most linearly with respect to each of r and n.

Theorem 1.1 *Let $r \in \mathbb{N}$, $r \geq 2$.*

(a) *For $N \geq (2n + 4)(r - 1)$ every embedding of K^n_N in $\mathbb{R}^{2n+1}$ contains an r–component link $L_1 \cup L_2 \cup \cdots \cup L_r$ such that*

$$\ell k_2(L_i, L_{i+1}) \neq 0 \tag{1}$$

for $i = 1, \ldots, r - 1$.

(b) *If $r \geq 3$ then for $N \geq (2n + 4)r$ every embedding of K^n_N in $\mathbb{R}^{2n+1}$ contains an r–component link $L_1 \cup L_2 \cup \cdots \cup L_r$ satisfying Equation (1) for $i = 1, \ldots, r$ (subscripts taken mod r).*

The link of Theorem 1.1(a) resembles a chain, and the link of Theorem 1.1(b) resembles a necklace, except that there is no requirement that nonadjacent components do not also link. Our next result generalises Fleming and Diesl [6, Lemma 2.2], and yields links that resemble a bunch of keys on a keyring. However, there is again no requirement that the "keys" do not also link each other, and following Flapan et al [3] we call such a link a *generalised keyring*. Generalised keyrings will play a crucial role in establishing our results for 2–component links, in Theorems 1.3–1.5.

Theorem 1.2 *For a natural number r define*

$$\kappa_n(r) = 4r^2(2n+4) + n + \left\lceil \frac{4r^2-2}{n} \right\rceil + 1.$$

Then every embedding of $K^n_{\kappa_n(r)}$ in $\mathbb{R}^{2n+1}$ contains an $(r+1)$–component link $R \cup L_1 \cup L_2 \cup \cdots \cup L_r$ such that

$$\ell k_2(R, L_i) = 1$$

for $i = 1, \ldots, r$.

Observe that $\kappa_n(r)$ grows quadratically in r and linearly in n. The existence of generalised keyrings in embeddings of K^n_N for N sufficiently large may be established by following Fleming and Diesl's argument, or that of Flapan et al [3, Lemma 1]; the Fleming–Diesl argument leads to a bound that grows exponentially with respect to r, and so we will follow the argument of Flapan et al, as this leads to the polynomial bound given above. For $n = 1$ the term $n + \lceil (4r^2-2)/n \rceil + 1$ of κ_n is not needed, so it suffices to take $\kappa_1(r) = 24r^2$. This bound follows from Flapan et al [3, Lemma 1], although they do not state the bound explicitly.

Our last three results concern linking number in 2–component links. The first extends Flapan [2, Theorem 2] to higher dimensions (although our proof will be based on a technique from Flapan et al [3, Lemma 2], as this leads to a better bound in higher dimensions):

Theorem 1.3 *Let $\lambda \in \mathbb{N}$ be given, and let*

$$N = \kappa_n(2\lambda - 1) + n + \left\lceil \frac{2\lambda - 1}{n} \right\rceil + 1.$$

Then every embedding of K^n_N in $\mathbb{R}^{2n+1}$ contains a two-component link $L \cup J$ such that, for some orientation of the components, $\ell k(L, J) \geq \lambda$.

Our last two results concern divisibility of the linking number. Fleming and Diesl [6] showed that for $q = 3$ or q a power of two, embeddings of sufficiently large complete graphs in $\mathbb{R}^3$ necessarily contain 2–component links with linking number a nonzero multiple of q, and Fleming [5] later extended this to all $q \in \mathbb{N}$. We now extend this further to embeddings of complete n–complexes in $\mathbb{R}^{2n+1}$, and by slightly modifying Fleming's argument, reduce the number of vertices required from exponentially many to only polynomially many. We state and prove two results in this direction: the first is for q arbitrary, and the second is for q prime, where a simpler argument leads to a bound with much slower growth.

Theorem 1.4 *Let q be a positive integer. Then for N sufficiently large every embedding of K_N^n in $\mathbb{R}^{2n+1}$ contains a two-component link $R \cup S$ such that $\ell k(R, S) = kq$ for some $k \neq 0$. The minimum number of vertices required is no greater than $c(n+1)\binom{2n+4}{n+1}q^{n+2}$ (c a constant), which for fixed n grows polynomially in q.*

When q is prime, a much simpler argument leads to a bound with growth $O(q^2)$ instead of $O(q^{n+2})$:

Theorem 1.5 *Suppose that the positive integer q is prime. Then the conclusion of Theorem 1.4 holds for*

$$N \geq \kappa_n(2q-1) + n + \left\lceil \frac{2q-3}{n} \right\rceil + 1.$$

Since the proof of Theorem 1.5 is simpler than that of Theorem 1.4 we will prove it first, in Section 4.2, and then prove Theorem 1.4 later in Section 6.

For $n = 1$, Theorem 1.4 may be proved using a total of

$$4q^2(6 + 15(q-1)) = 12q^2(5q-3)$$

vertices, in contrast to the exponentially many required by Fleming [5, Theorem 3.1]. This reduction to polynomial growth comes about for two reasons. The first is that we use Flapan et al's rather than Fleming and Diesl's construction of a generalised keyring, as this requires only polynomially many rather than exponentially many vertices. The second savings comes from modifying the method by which the keys of the keyring are combined, so that each key requires roughly $3q$ vertices rather than $O(q^{\log q})$. In fact it should be possible to reduce the number of vertices required further, by a factor of about $\frac{2}{3}$, because for $n = 1$ our method really only requires the keys to have about $2q$ vertices.

Clearly, the number of vertices required by Theorem 1.4 grows at most exponentially with respect to n, because $\binom{m}{k} \leq 2^m$. More precisely, Stirling's formula may be used to show that asymptotically we have

$$c(n+1)\binom{2n+4}{n+1}q^{n+2} \sim C\sqrt{n}4^n q^{n+2},$$

for some constant C.

1.2 Discussion

We briefly discuss the existence of more complex links in embeddings of large complete complexes in $\mathbb{R}^{2n+1}$.

1.2.1 More complex keyrings Each of Theorems 1.3–1.5 is proved by converting a suitable generalised keyring $R \cup L_1 \cup \cdots \cup L_m$ into a two component link $R \cup L'$, where L' is formed as a connected sum of some of the L_i (and perhaps an additional disjoint component S). Starting with a generalised keyring with mr keys, and working with them m at a time, we may therefore construct a link $R \cup L'_1 \cup \cdots \cup L'_r$ in which each linking number $\ell k(R, L'_i)$ satisfies the conclusion of the theorem. It follows for example that for $q \in \mathbb{N}$ and N sufficiently large, every embedding of K^n_N in $\mathbb{R}^{2n+1}$ contains a link $R \cup L'_1 \cup \cdots \cup L'_r$ in which each linking number $\ell k(R, L'_i)$ is a nonzero multiple of q.

1.2.2 More complex linking patterns Flapan et al [3, Theorem 1] show that intrinsic linking of graphs in $\mathbb{R}^3$ is arbitrarily complex in the following sense: Given natural numbers r and λ, for N sufficiently large every embedding of K_N in $\mathbb{R}^3$ contains an r–component link in which all pairwise linking numbers are at least λ in absolute value. We believe that, with minor adaptions to higher dimensions, their work shows that intrinsic linking of n–complexes in $\mathbb{R}^{2n+1}$ is arbitrarily complex in this sense also. The main adaption needed is to use our Lemma 2.5 in place of the 1–dimensional construction it replaces in higher-dimensional arguments. This adaption requires the addition of some extra vertices (to create the auxiliary sphere S_0 of the lemma), and is illustrated in the proofs of Lemma 3.2 and Theorem 1.3. These are based respectively on [3, Lemma 1] and a technique from the proof of [3, Lemma 2].

A step in their argument is to show that, for N sufficiently large, every embedding of K_N in $\mathbb{R}^3$ contains a link $X_1 \cup \cdots \cup X_m \cup Z_1 \cup \cdots \cup Z_m$ such that

$$\ell k_2(X_i, Z_j) = 1$$

for $1 \leq i, j \leq m$ (Flapan et al [3, Proposition 1]). We observe that this step certainly extends to embeddings of complete n–complexes in $\mathbb{R}^{2n+1}$, as their proof is a purely combinatorial argument that depends only on [3, Lemma 1] and the existence of generalised keyrings, which we extend here to higher dimensions as Lemma 3.2 and Theorem 1.2 respectively.

1.3 Organisation

The paper is organised as follows. We begin with some technical preliminaries in Section 2, and then prove Theorems 1.1 and 1.2 concerning many-component links in Section 3. In Section 4 we prove two of our results on linking numbers in 2–component links, Theorems 1.3 and 1.5.

We then construct some triangulations of an M–simplex in Section 5, as further technical preliminaries needed for our proof of our divisibility result Theorem 1.4.

This result is proved in Section 6. As a further application of the triangulations of Section 5 we conclude the paper in Section 7 with an alternate proof of Theorem 1.3, without the polynomial bound on the number of vertices required. This introduces an additional technique that may be used to prove Ramsey-type results on intrinsic linking of n–complexes.

2 Technical preliminaries I: Spheres and discs in K_N^n

In this section we construct some subcomplexes of K_N^n that are needed for our proofs. As an aid to understanding, in Section 2.1 we first illustrate the role the corresponding subcomplexes of K_N play in studying intrinsic linking of graphs in $\mathbb{R}^3$.

2.1 Tactics

A common technique of [2], [3], [4], [5] and [6] in proving Ramsey-type results for graphs is the use of connected sums and the additivity of linking number. These may be used to convert a link with several components to one with fewer components, but more complicated linking behaviour. We illustrate this technique by sketching the proofs for $n = 1$ of the four-to-three Lemmas 3.1 and 7.2. The $n = 1$ case of Lemma 7.2 corresponds to Flapan [2, Lemma 2], and Lemma 3.1 is a mod two version of this result that is similar to Flapan et al [4, Lemma 1].

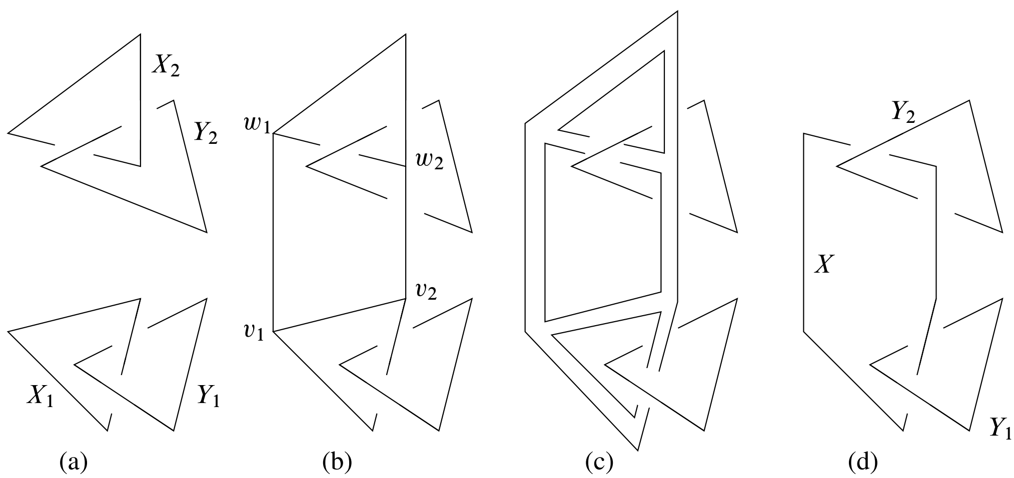

Figure 1: Illustrating the proof of Lemma 3.1 (the four-to-three lemma for mod two linking number) in the case $n = 1$.

Suppose that the 4–component link $Y_1 \cup X_1 \cup X_2 \cup Y_2$ in Figure 1(a) is part of an embedding of K_N in $\mathbb{R}^3$, and that we wish to replace the cycles X_1 and X_2 with a

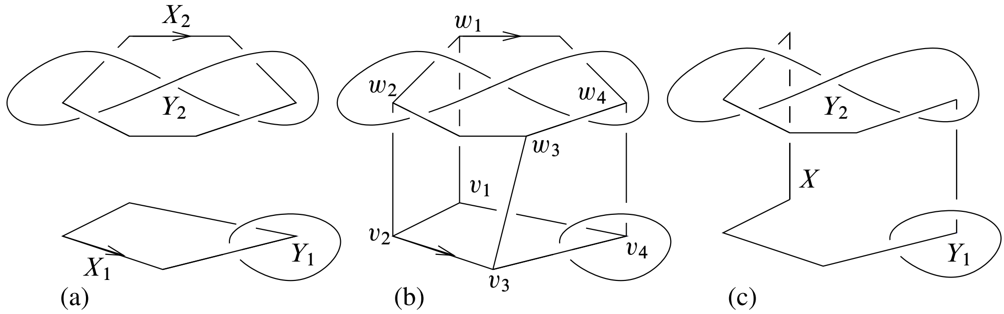

Figure 2: Illustrating the proof of Flapan's [2, Lemma 2], the $n = 1$ case of our Lemma 7.2 (the four-to-three lemma for integral linking number).

single cycle X linking both Y_1 and Y_2 mod two. We choose vertices v_1, v_2 on X_1 and w_1, w_2 on X_2, and consider the edges (v_i, w_i) as in Figure 1(b). Together with X_1 and X_2 these give us a collection of cycles (Figure 1(c)) whose linking numbers with each of Y_1 and Y_2 sum to zero mod two; and taking the connected sum of a suitably chosen subset as in Figure 1(d) we get the desired cycle X.

Working now with integer coefficients, consider the link $Y_1 \cup X_1 \cup X_2 \cup Y_2$ in Figure 2(a). Our goal here is to replace this with a three component link $L \cup Z \cup W$ such that $\ell k(L, Z)$ is nonzero, and $\ell k(L, W)$ is at least as large as $\ell k(X_2, Y_2)$ in absolute value. We again do this by constructing a series of cycles that sum to zero with X_1 and X_2, but now in order to ensure we can find one linking Y_2 with the correct sign it is necessary to have at least $q > |\ell k(X_2, Y_2)|$ such cycles. This is achieved by choosing vertices $v_1, \dots, v_q$ on X_1 and $w_1, \dots, w_q$ on X_2, such $v_1, \dots, v_q$ are encountered in increasing order following the orientation of X_1, and $w_1, \dots, w_q$ are encountered in decreasing order following the orientation of X_2. The needed cycles are formed by connecting X_1 and X_2 using the edges $(v_1, w_1), \dots, (v_q, w_q)$, as in Figure 2(b), and a suitable connected sum (Figure 2(c)) then gives us the desired 3–component link.

To prove analogous results in higher dimensions we will regard the intervals $[v_1, v_q]$ and $[w_1, w_q]$ as identically triangulated discs $D_1 \subseteq X_1$ and $D_2 \subseteq X_2$, and the correspondence $v_i \mapsto w_i$ as an orientation reversing simplicial isomorphism $\phi\colon D_1 \to D_2$ mapping one triangulation to the other. Given this data we then construct the collection of edges (v_i, w_i), which we regard as a complex $\mathcal{C}$ homeomorphic to $D_1^{(0)} \times I$ realising the restriction of ϕ to the zero skeleton of D_1. The pair of edges (v_i, w_i) and (v_{i+1}, w_{i+1}) may then be seen as a copy of $S^0 \times I$, which we cap with the intervals $[v_i, v_{i+1}], [w_i, w_{i+1}]$ to create a copy of S^1.

Triangulations of an interval have very simple combinatorics, and in Figure 2(b) it didn't matter that there was an additional vertex between w_2 and w_3. Thus, Flapan's argument only requires that each component has at least q vertices. In order to use similar techniques when $n \geq 2$ we will impose the more stringent requirement that our link components contain identically triangulated copies of D^n. Additional work will then be required to ensure that our links contain such discs.

2.2 Cylinders, spheres and discs in K_N^n

We now construct the needed subcomplexes of K_N^n.

Lemma 2.1 *Let (S_1, D_1) and (S_2, D_2) be disjoint subcomplexes of K_N^n each homeomorphic to (S^n, D^n). Suppose that there is a simplicial isomorphism*

$$\phi\colon D_1 \longrightarrow D_2.$$

Let $D_i^{(n-1)}$ be the $(n-1)$–skeleton of D_i. Then there is a subcomplex $\mathcal{C}$ of K_N^n and a homeomorphism

$$\Phi\colon D_1^{(n-1)} \times I \longrightarrow \mathcal{C}$$

such that

(1) *all vertices of $\mathcal{C}$ lie on $D_1 \cup D_2$;*

(2) *$\mathcal{C} \cap S_i = D_i^{(n-1)}$ for $i = 1, 2$;*

(3) *Φ restricts to the identity on $D_1^{(n-1)} \times \{0\}$; and*

(4) *$\Phi = \phi$ on $D_1^{(n-1)} \times \{1\}$.*

We note that the subcomplex $\mathcal{C}$ may be regarded as the mapping cylinder of the restriction of ϕ to the $(n-1)$–skeleton.

Proof To construct $\mathcal{C}$ we use the subdivision of $\Delta^m \times I$ into $(m+1)$–simplices used in the proof of the homotopy invariance of singular homology (see for example Hatcher [8, page 112]). Label the vertices of D_1 arbitrarily as $v_0, v_1, \dots, v_M$, and label the vertices of D_2 as $w_0, w_1, \dots, w_M$ so that $w_i = \phi(v_i)$. Now, for each m–simplex $\delta = [v_{i_0}, \dots, v_{i_m}]$ of $D_1^{(n-1)}$, with $i_0 < i_1 < \cdots < i_m$, we have

$$\delta \times I \cong C(\delta) = \bigcup_{j=0}^{m} [v_{i_0}, \dots, v_{i_j}, w_{i_j}, \dots, w_{i_m}].$$

Since $m \leq n-1$ each $(m+1)$–simplex involved in this union is a simplex of K_N^n, and we obtain a subcomplex of K_N^n homeomorphic to $\delta \times I$, meeting D_1 and D_2 in $\delta \times \{0\} = \delta$ and $\delta \times \{1\} = \phi(\delta)$ respectively. In addition, all vertices of $C(\delta)$ belong to $D_1 \cup D_2$.

Let δ_l denote the simplex $[v_{i_0}, \dots, \widehat{v}_{i_l}, \dots, v_{i_m}]$ belonging to $\partial\delta$, where the hat indicates that v_{i_l} is omitted. An m–simplex belonging to $C(\delta)$ is of one of several possible types:

(1) The simplex $[v_{i_0}, \dots, v_{i_m}] = \delta$ or $[w_{i_0}, \dots, w_{i_m}] = \phi(\delta)$.

(2) One of the simplices

$$[v_{i_0}, \dots, \widehat{v}_{i_l}, \dots, v_{i_j}, w_{i_j}, \dots, w_{i_m}],$$
$$[v_{i_0}, \dots, v_{i_j}, w_{i_j}, \dots, \widehat{w}_{i_l}, \dots, w_{i_m}],$$

with l fixed and $j \neq l$, which together make up $C(\delta_l)$.

(3) A simplex of the form $[v_{i_0}, \dots, v_{i_j}, w_{i_{j+1}}, \dots, w_{i_m}]$, which is interior to $\delta \times I$.

Inductively, this implies that if δ' is a simplex of δ, then $C(\delta')$ is a subcomplex of $C(\delta)$, and the diagram

$$\begin{tikzcd} \delta' \times I \arrow[r] \arrow[d, "\cong"'] & \delta \times I \arrow[d, "\cong"] \\ C(\delta') \arrow[r] & C(\delta) \end{tikzcd}$$

commutes. Moreover, our construction ensures that $C(\delta_1)$ and $C(\delta_2)$ are disjoint unless δ_1 and δ_2 intersect, in which case $C(\delta_1) \cap C(\delta_2) = C(\delta_1 \cap \delta_2)$. Thus, taking the union of $C(\delta)$ over all $(n-1)$–simplices of $D_1^{(n-1)}$ we obtain a subcomplex $\mathcal{C}$ of K_N^n homeomorphic to $D_1^{(n-1)} \times I$ meeting S_i in $D_i^{(n-1)}$ for each i, and the homeomorphism Φ may be constructed satisfying the given conditions. □

Corollary 2.2 *Let (S_1, D_1) and (S_2, D_2) be disjoint subcomplexes of K_N^n each homeomorphic to (S^n, D^n). Suppose that there is an orientation reversing simplicial isomorphism*

$$\phi\colon D_1 \longrightarrow D_2,$$

and let $\Delta_1, \dots, \Delta_k$ be the n–simplices of D_1. Then there exist subcomplexes $P_0, P_1, \dots, P_k$ of K_N^n such that

(1) *the vertices of $P_0, P_1, \dots, P_k$ all lie on $S_1 \cup S_2$;*

(2) *$P_i \cong S^n$ for each i;*

(3) *$P_0 \cap S_j = \overline{S_j \setminus D_j}$ for $i = 1, 2$;*

(4) *$P_i \cap S_1 = \Delta_i$, $P_i \cap S_2 = \phi(\Delta_i)$ for $i \geq 1$; and*

(5) *as an integral chain we have*

$$S_1 + S_2 + \sum_{i=0}^{k} P_i = 0.$$

Remark 2.3 Condition (1) implies that if A is a subcomplex of K_N^n disjoint from $S_1 \cup S_2$, then A is disjoint from P_i for all i.

Proof We obtain the required spheres P_i using the subcomplex $\mathcal{C}$ and homeomorphism $\Phi\colon D_1^{(n-1)} \times I \to \mathcal{C}$ constructed in Lemma 2.1 above. For each $i = 1, \dots, k$ let

$$P_i = \Delta_i \cup \phi(\Delta_i) \cup \Phi(\partial \Delta_i \times I),$$

and let

$$P_0 = \overline{S_1 \setminus D_1} \cup \overline{S_2 \setminus D_2} \cup \Phi(\partial D_1 \times I).$$

Then Lemma 2.1 ensures that each P_i is a subcomplex of K_N^n satisfying conditions (1)–(4) above.

To obtain (5) we must orient each sphere P_i. For $i \geq 1$ we orient P_i so that Δ_i receives the opposite orientation from P_i as it does from S_1, and we orient P_0 analogously using the disc $\overline{S_1 \setminus D_1}$. This ensures that $\phi_\sharp \Delta_i$ receives opposite orientations from S_2 and P_i also, since ϕ is orientation reversing on Δ_i with respect to both S_2 and P_i (on $P_i \cong S^n$ it is induced by reflection in an equatorial S^{n-1}). Similar considerations apply to P_0, as ϕ extends to a (not necessarily simplicial) orientation reversing homeomorphism $(S_1, \overline{S_1 \setminus D_1}) \to (S_2, \overline{S_2 \setminus D_2})$.

It remains to consider the subcomplexes $C(\delta)$, for δ an $(n-1)$–simplex of D_1. Each such simplex belongs to two n–simplices of S_1, and receives opposite orientations from each (since $\partial S_1 = 0$); consequently, each subcomplex $C(\delta)$ belongs to two spheres P_i and P_j, and is also oppositely oriented by each. This completes the proof. $\square$

Remark 2.4 The n–spheres P_i of Corollary 2.2 may be expressed explicitly as chains as follows. We assume throughout that all simplices of D_1 are written with the labels on their vertices in increasing order.

For each m–simplex $\delta = [v_{i_0}, \dots, v_{i_m}]$ of $D_1^{(n-1)}$ define

$$\mathcal{P}(\delta) = \sum_{j=0}^{m} (-1)^j [v_{i_0}, \dots, v_{i_j}, w_{i_j}, \dots, w_{i_m}].$$

Let $\varepsilon_i \in \{\pm 1\}$ be the coefficient of Δ_i in the chain S_1, and set

$$P_i = -\varepsilon_i(\Delta_i + \mathcal{P}(\partial \Delta_i) - \phi_\sharp(\Delta_i))$$

for $i \geq 1$, and

$$P_0 = (D_1 - S_1) + (D_2 - S_2) + \mathcal{P}\partial D_1.$$

We verify below that $\partial P_i = 0$, and that $S_1 + S_2 + \sum_i P_i = 0$.

Suitably adapted, the calculation in Hatcher [8, page 112] shows that

$$\partial \mathcal{P} = \phi_\sharp - \mathrm{id}_\sharp - \mathcal{P}\partial,$$

so for $i \geq 1$ we have

$$\begin{aligned} -\varepsilon_i \partial P_i &= \partial \Delta_i + \partial \mathcal{P} \partial \Delta_i - \partial \phi_\sharp \Delta_i \\ &= \partial \Delta_i + \phi_\sharp \partial \Delta_i - \mathrm{id}_\sharp \partial \Delta_i - \mathcal{P}\partial^2 \Delta_i - \phi_\sharp \partial \Delta_i = 0. \end{aligned}$$

Similarly

$$\begin{aligned} \partial P_0 &= \partial(D_1 - S_1) + \partial(D_2 - S_2) + \partial \mathcal{P} \partial D_1 \\ &= \partial D_1 + \partial D_2 + \phi_\sharp \partial D_1 - \mathrm{id}_\sharp \partial D_1 - \mathcal{P}\partial^2 D_1 \\ &= \partial D_1 + \partial D_2 - \partial D_2 - \partial D_1 & (\phi_\sharp \partial D_1 = \partial \phi_\sharp D_1 = -\partial D_2) \\ &= 0, \end{aligned}$$

as required. Summing, we have $D_1 = \sum_{i=1}^k \varepsilon_i \Delta_i$, so

$$\begin{aligned} \sum_{i=1}^k P_k = -\sum_{i=1}^k \varepsilon_i \Delta_i - \mathcal{P}\partial \sum_{i=1}^k \varepsilon_i \Delta_i + \phi_\sharp \sum_{i=1}^k \varepsilon_i \Delta_i &= -D_1 - \mathcal{P}\partial D_1 + \phi_\sharp D_1 \\ &= -D_1 - \mathcal{P}\partial D_1 - D_2 \\ &= -P_0 - S_1 - S_2, \end{aligned}$$

and it follows that $S_1 + S_2 + \sum_{i=0}^k P_i = 0$.

2.3 connected sums of several spheres

Our next technical lemma takes several spheres $S_1, \dots, S_k$ and an additional sphere S_0, and constructs a sphere $\mathcal{S}$ meeting each of $S_1, \dots, S_k$ in a single n–simplex. The case $n = 1$, $k = 3$ is illustrated in Figure 3. This lemma is an adaption to higher dimensions of a construction used by Flapan et al [3] in the case $n = 1$. In that case the additional sphere S_0 is not needed, as it is only necessary to choose edges joining S_i to S_{i+1} and S_k to S_1. This depends on the fact that the cylinder $S^0 \times I$ is disconnected, and our additional sphere S_0 is necessary for $n \geq 2$, when $S^{n-1} \times I$ is connected.

Lemma 2.5 *Let $S_0, S_1, \dots, S_k$ be disjoint subcomplexes of K_N^n each homeomorphic to S^n, and suppose that S_0 has at least k n–simplices. Then there is a subcomplex $\mathcal{S}$ of K_N^n such that:*

(1) *The vertices of* $\mathcal{S}$ *all lie on* $S_0 \cup \cdots \cup S_k$.

(2) $\mathcal{S}$ *is homeomorphic to* S^n.

(3) *For* $i = 1, \ldots, k$ *there is an* n*–simplex* δ_i *of* S_i *such that* $\mathcal{S} \cap S_i = \delta_i$.

Moreover, if each sphere S_i *is oriented, then* $\mathcal{S}$ *may be chosen and oriented such that* δ_i *receives opposite orientations from* $\mathcal{S}$ *and from* S_i.

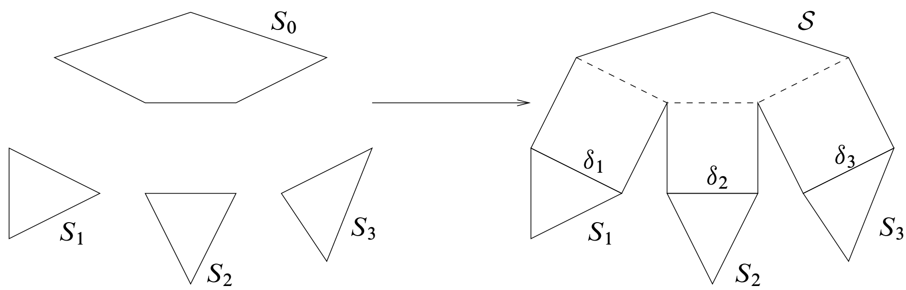

Figure 3: Illustrating Lemma 2.5 in the case $n = 1$, $k = 3$. The sphere S_0 is used to construct a sphere $\mathcal{S}$ meeting S_i in a single n–simplex δ_i for $i = 1, 2, 3$.

Proof We will assume that the S_i are oriented. Choose an n–simplex δ_i belonging to S_i for each $i \geq 1$, distinct n–simplices δ'_i belonging to S_0 for $i = 1, \ldots, k$, and orientation reversing simplicial isomorphisms $\phi_i \colon \delta_i \to \delta'_i$. Applying Corollary 2.2 to the pairs (S_i, δ_i) and (S_0, δ'_i) we obtain a sphere Q_i with all its vertices on $S_i \cup S_0$, and such that Q_i meets S_i in δ_i and S_0 in δ'_i. Note that this implies $Q_i \cap Q_j = \delta'_i \cap \delta'_j$.

We set $T_0 = S_0$, and for $i = 1, \ldots, k$ we inductively define T_i to be the complex obtained from T_{i-1} and Q_i by omitting the interior of the disc δ'_i. Then at each stage T_i is an n–sphere, because it is the result of gluing two discs along their common boundary $\partial\delta'_i$, and setting $\mathcal{S} = T_k$ we obtain the desired subcomplex. □

To conclude this section we establish a bound on the number of vertices required to construct an n–sphere with a specified number of n–simplices.

Lemma 2.6 *Given* $\ell \in \mathbb{N}$ *there is a triangulation of* S^n *with* $n + \ell + 1$ *vertices and* $\ell n + 2$ n*–simplices.*

Proof We construct the triangulation from a suitable triangulation of D^{n+1} with ℓ $(n+1)$–simplices. For $i = 1, \ldots, \ell$ let Δ_i be an $(n+1)$–simplex, and choose distinct n–simplices δ_i, σ_i belonging to Δ_i. Choose a simplicial isomorphism $\phi_i \colon \delta_i \to \sigma_{i+1}$

for each $i = 1, \dots, \ell - 1$, and let D be the $(n+1)$–disc that results from gluing the Δ_i according to the ϕ_i. We claim that $S = \partial D$ is the required triangulated n–sphere.

The union $\Delta_1 \cup \cdots \cup \Delta_\ell$ has a total of $\ell(n+2)$ n–simplices, of which $2(\ell - 1)$ are identified in pairs to form D. The n–simplices involved in the identifications lie in the interior of D, and the rest on the boundary, so S has $\ell(n+2) - 2(\ell - 1) = \ell n + 2$ n–simplices, as claimed. Similarly, each gluing identifies $2(n+1)$ vertices in pairs, leaving a total of $\ell(n+2) - (n+1)(\ell - 1) = \ell + n + 1$; alternately, we may carry the gluings out sequentially, and we see that we start with $n+2$ vertices, and each gluing adds just one, for a total of $(n+2) + (\ell - 1) = n + \ell + 1$.

To complete the proof we show that the vertices of D all lie on S. For $n = 1$ a circle with $\ell + 2$ edges necessarily has $\ell + 2$ vertices, by Euler characteristic; while for $n \geq 2$ each vertex of Δ_i belongs to at least three n–simplices, and so to at least one n–simplex belonging to ∂D after the identifications. □

Corollary 2.7 *If $k \in \mathbb{N}$ and $N \geq n + \lceil k/n \rceil + 1$ then K_N^n contains a subcomplex $S \cong S^n$ with at least $k + 2$ n–simplices.*

Proof Set $\ell = \lceil k/n \rceil$. Then $\ell \in \mathbb{N}$ and $\ell \geq k/n$, so the construction of Lemma 2.6 yields an n–sphere S in K_N^n with at least $k+2$ n–simplices. □

3 Many-component links

We now prove Theorems 1.1 and 1.2, thereby showing that embeddings of sufficiently large complete complexes necessarily contain nonsplit links with many components.

3.1 Necklaces and chains

In this section we establish Theorem 1.1. The key step is the following lemma, which plays the role of Flapan et al [4, Lemma 1].

Lemma 3.1 (The four-to-three lemma for mod two linking number) *Let $Y_1 \cup X_1 \cup X_2 \cup Y_2$ be a 4–component link contained in some embedding of K_N^n in $\mathbb{R}^{2n+1}$, satisfying*

$$\ell k_2(X_1, Y_1) = \ell k_2(X_2, Y_2) = 1.$$

Then there is an n–sphere X in K_N^n, all of whose vertices lie on $X_1 \cup X_2$, such that

$$\ell k_2(Y_1, X) = \ell k_2(X, Y_2) = 1.$$

Proof If $\ell k_2(X_1, Y_2) = 1$ then we may simply let $X = X_1$, and if $\ell k_2(X_2, Y_1) = 1$ then we may simply let $X = X_2$. So suppose that

$$\ell k_2(X_1, Y_2) = \ell k_2(X_2, Y_1) = 0.$$

Choose n–simplices δ_1, δ_2 belonging to X_1, X_2 respectively, and apply Corollary 2.2 to the pairs (X_1, δ_1), (X_2, δ_2) to obtain spheres P_0, P_1 satisfying

$$X_1 + X_2 + P_0 + P_1 = 0.$$

In the homology groups $H_n(\mathbb{R}^{2n+1} - Y_i; \mathbb{Z}/2\mathbb{Z})$ we have

$$[X_1] + [X_2] + [P_0] + [P_1] = 0,$$

and since $[X_1] + [X_2] = 1$ in each group we have also $[P_0] + [P_1] = 1$ in each group. Hence, for each i, precisely one of $[P_0]$, $[P_1]$ must equal 1 in $H_n(\mathbb{R}^{2n+1} - Y_i; \mathbb{Z}/2\mathbb{Z})$.

If $[P_1]$ takes the same value in both groups then we are done by setting $X = P_0$ if $[P_1] = 0$ in both groups, and $X = P_1$ if $[P_1] = 1$. Otherwise, without loss of generality suppose that $[P_1]$ is zero in $H_n(\mathbb{R}^{2n+1} - Y_1; \mathbb{Z}/2\mathbb{Z})$ and nonzero in $H_n(\mathbb{R}^{2n+1} - Y_2; \mathbb{Z}/2\mathbb{Z})$, and let X be the n–sphere obtained from X_1 and P_1 by omitting the interior of the simplex δ_1. Then

$$[X] = [X_1] + [P_1] = \begin{cases} [X_1] = 1 & \text{in } H_n(\mathbb{R}^{2n+1} - Y_1; \mathbb{Z}/2\mathbb{Z}), \\ [P_1] = 1 & \text{in } H_n(\mathbb{R}^{2n+1} - Y_2; \mathbb{Z}/2\mathbb{Z}), \end{cases}$$

and the result follows. □

We now prove Theorem 1.1, using the above lemma.

Proof of Theorem 1.1 The proof of part (a) is by induction on r, with the base case $r = 2$ given by Taniyama [19], and the inductive step following from Lemma 3.1. Given an embedding of $K^n_{(2n+4)r}$ in $\mathbb{R}^{2n+1}$, choose disjoint copies of $K^n_{(2n+4)(r-1)}$ and K^n_{2n+4} contained in the embedding. By the inductive hypothesis the $K^n_{(2n+4)(r-1)}$ contains an r–component link $L_1 \cup L_2 \cup \cdots \cup L_r$ satisfying Equation (1) for $i = 1, \ldots, r-1$, and the K^n_{2n+4} contains a two component link $J \cup K$ such that $\ell k_2(J, K) = 1$. Applying Lemma 3.1 to the (ordered) link $L_{r-1} \cup L_r \cup J \cup K$ we obtain an n–sphere X with all its vertices on $L_r \cup J$ such that

$$\ell k_2(L_{r-1}, X) = \ell k_2(X, K) = 1.$$

The link $L_1 \cup \cdots \cup L_{r-1} \cup X \cup K$ is then the desired r–component link.

To prove (b) we apply Lemma 3.1 to suitably chosen components of an $(r+1)$–component link as given by part (a). Given an embedding of $K^n_{(2n+4)r}$ in $\mathbb{R}^{2n+1}$, there is an $(r+1)$–component link $L_1 \cup L_2 \cup \cdots \cup L_r \cup L_{r+1}$ satisfying Equation (1) for $i = 1, \ldots, r$. We apply Lemma 3.1 to the (ordered) link $L_r \cup L_{r+1} \cup L_1 \cup L_2$ to obtain an n–sphere X, with all its vertices on $L_{r+1} \cup L_1$, and satisfying

$$\ell k_2(L_r, X) = \ell k_2(X, L_2) = 1.$$

The link $L_2 \cup \cdots \cup L_r \cup X$ is then the desired r–component link. □

3.2 Generalised keyrings

We prove Theorem 1.2 by extending Flapan et al [3, Lemma 1] to higher dimensions in the following form.

Lemma 3.2 *Let K_N^n be embedded in $\mathbb{R}^{2n+1}$ such that it contains a link*

$$L \cup J_1 \cup \cdots \cup J_{m^2} \cup X_1 \cup \cdots \cup X_{m^2},$$

where L has at least m^2 n–simplices, and $\ell k_2(J_i, X_i) = 1$ for all i. Then there is an n–sphere Z in K_N^n with all its vertices on $L \cup J_1 \cup \cdots \cup J_{m^2}$, and an index set I with $|I| \geq m/2$, such that $\ell k_2(Z, X_j) = 1$ for all $j \in I$.

Proof The argument is that of Flapan et al [3], with the addition of the component L needed to create the analogue of their cycle C connecting the J_i.

Since L has at least m^2 simplices we may apply Lemma 2.5 to the (ordered) link $L \cup J_1 \cup \cdots \cup J_{m^2}$, obtaining an n–sphere $\mathcal{S}$ with all its vertices on $L \cup J_1 \cup \cdots \cup J_{m^2}$ and meeting each sphere J_i in an n–simplex δ_i. If at least $m/2$ of the mod two linking numbers $\ell k_2(\mathcal{S}, X_i)$ are nonzero then we are done by setting $Z = \mathcal{S}$, so we assume in what follows that fewer than $m/2$ of these mod two linking numbers are nonzero.

Following Flapan et al we define M to be the $m^2 \times m^2$ matrix over $\mathbb{Z}/2\mathbb{Z}$ with ij–entry $M_{ij} = \ell k_2(J_i, X_j)$. Let r_i be the i^{th} row of M. Then $M_{ii=1}$ for all i, and Flapan et al use this to show that there are indices $i_1, \ldots, i_k$ such that

$$V = r_{i_1} + \cdots + r_{i_k}$$

has at least m entries that are equal to 1. Let Z be the n–sphere obtained from $\mathcal{S}$ and $J_{i_1}, \ldots, J_{i_k}$ by omitting the interiors of the simplices $\delta_{i_1}, \ldots, \delta_{i_k}$. We claim that Z is the required n–sphere.

Indeed, for $j = 1, \ldots, m^2$ we have

$$\ell k_2(Z, X_j) = \ell k_2(\mathcal{S}, X_j) + \sum_{\ell=1}^{k} \ell k_2(J_{i_\ell}, X_j) = \ell k_2(\mathcal{S}, X_j) + V_j, \tag{2}$$

where $V_j = \sum_{\ell=1}^{k} \ell k_2(J_{i_\ell}, X_j)$ is the j^{th} entry of V. By construction at least m of the V_j are nonzero, and by assumption fewer than $m/2$ of the $\ell k_2(\mathcal{S}, X_j)$ are nonzero. Hence there are at least $m - m/2 = m/2$ indices j for which $V_j = 1$ while $\ell k_2(\mathcal{S}, X_j) = 0$. Consequently, the set $I = \{1 \le j \le m^2 \mid \ell k_2(\mathcal{S}, X_j) \neq V_j\}$ has at least $m/2$ elements. But $\ell k_2(Z, X_j) = 1$ if and only if $j \in I$, by (2), so we are done. □

We now obtain Theorem 1.2 as a corollary to Lemma 3.2 and Corollary 2.7.

Proof of Theorem 1.2 Recall that

$$\kappa_n(r) = 4r^2(2n+4) + n + \left\lceil \frac{4r^2 - 2}{n} \right\rceil + 1,$$

and for ease of notation let $\ell = \lceil (4r^2 - 2)/n \rceil$. Given an embedding of $K^n_{\kappa_n(r)}$ in $\mathbb{R}^{2n+1}$, choose $4r^2$ disjoint copies of K^n_{2n+4} contained in the embedding, together with a copy of $K^n_{n+\ell+1}$. By Taniyama [19] the i^{th} copy of K^n_{2n+4} contains a 2–component link $J_i \cup X_i$ such that $\ell k_2(J_i, X_i) = 1$, and by Corollary 2.7 the copy of $K^n_{n+\ell+1}$ contains an n–sphere L with at least $4r^2$ n–simplices. The result now follows by applying Lemma 3.2 with $m = 2r$ to the link

$$L \cup J_1 \cup \cdots \cup J_{4r^2} \cup X_1 \cup \cdots \cup X_{4r^2}. \qquad \square$$

4 Linking number in 2–component links

We now prove Theorems 1.3 and 1.5, concerning the linking number in a 2–component link. To prove each result we start with a suitable generalised keyring, and combine some of the "keys" to obtain the second component of the desired link.

4.1 Bounding the absolute value of the linking number from below

Proof of Theorem 1.3 We use a technique of Flapan et al from the proof of their [3, Lemma 2]. For simplicity of notation let $\ell = \lceil (2\lambda - 1)/n \rceil$, and choose disjoint copies of $K^n_{\kappa_n(2\lambda-1)}$ and $K^n_{n+\ell+1}$ contained in K^n_N. Given an embedding of K^n_N in $\mathbb{R}^{2n+1}$, the copy of $K^n_{\kappa_n(2\lambda-1)}$ contains a generalised keyring $R \cup L_1 \cup \cdots \cup L_{2\lambda-1}$ with $2\lambda - 1$ keys, by Theorem 1.2, while by Corollary 2.7 the copy of $K^n_{n+\ell+1}$ contains an n–sphere S with at least $2\lambda + 1$ n–simplices.

Orient S arbitrarily, and orient the L_i such that $\ell k(R, L_i) > 0$ for each i. Applying Lemma 2.5 to the oriented link $\mathcal{L} = S \cup L_1 \cup \cdots \cup L_{2\lambda-1}$ we obtain an n–sphere $\mathcal{S}$ with all its vertices on $\mathcal{L}$ and meeting each L_i in a single n–simplex δ_i, which receives opposite orientations from $\mathcal{S}$ and from L_i. Set $\mathcal{S}_0 = \mathcal{S}$, and for $i = 1, \dots, 2\lambda - 1$ let $\mathcal{S}_i$ be the complex obtained from $\mathcal{S}_{i-1}$ and L_i by omitting the interior of the disc δ_i. Then $\mathcal{S}_i$ is an n–sphere, because it is the result of gluing two discs along their common boundary $\partial\delta_i$, and as a chain we have

$$\mathcal{S}_i = \mathcal{S}_0 + \sum_{j=1}^{i} L_j \tag{3}$$

for $i \geq 1$.

We now consider the linking numbers of the $\mathcal{S}_i$ with R, by considering Equation (3) in the group $H_n(\mathbb{R}^{2n+1} - R; \mathbb{Z})$. This gives

$$\ell k(R, \mathcal{S}_i) = [\mathcal{S}_i] = [\mathcal{S}_0] + \sum_{j=1}^{i} [L_j] = \ell k(R, \mathcal{S}_0) + \sum_{j=1}^{i} \ell k(R, L_j).$$

As in the proof of [3, Lemma 2] the sequence $(\ell k(R, \mathcal{S}_i))_{i=0}^{2\lambda-1}$ is strictly increasing, because the linking numbers $\ell k(R, L_i)$ are all positive. This sequence must therefore take 2λ distinct values, and the result now follows from the fact that there are only $2\lambda - 1$ integers k such that $|k| < \lambda$. □

4.2 The linking number modulo a prime

To prove Theorem 1.5 we will use the following lemma on sums of subsequences of finite integer sequences, considered modulo a prime p. Given an integer sequence $(\ell_1, \dots, \ell_m)$ we will say that $x \in \mathbb{Z}$ is a *subsequence sum* of $(\ell_1, \dots, \ell_m)$ if there is a subset $A \subseteq \{1, \dots, m\}$ such that

$$\sum_{i \in A} \ell_i = x.$$

We allow the possibility that A is empty, which implies that 0 is always a subsequence sum. Then:

Lemma 4.1 *Let $p \in \mathbb{N}$ be prime, and let $(\ell_1, \dots, \ell_{p-1})$ be a sequence of integers such that no ℓ_i is divisible by p. For any $s \in \mathbb{Z}$ there is a subsequence sum x of $(\ell_1, \dots, \ell_{p-1})$ such that $x \equiv s \bmod p$.*

We note that the sequence length $p-1$ is best possible, because a sequence of length $p-2$ that is constant mod p realises exactly $p-1$ mod p residue classes as subsequence sums.

Proof For $j = 1, \dots, p-1$ let Σ_j be the set of mod p residue classes that may be realised by a subsequence sum of $(\ell_1, \dots, \ell_j)$. Then $\Sigma_1 = \{\overline{0}, \overline{\ell_1}\}$, and our goal is to show that $\Sigma_{p-1} = \{\overline{0}, \overline{1}, \dots, \overline{p-1}\}$. We will do this by showing that $|\Sigma_{j+1}| \geq |\Sigma_j| + 1$ whenever $\Sigma_j \neq \{\overline{0}, \overline{1}, \dots, \overline{p-1}\}$. Since $\Sigma_j \subseteq \Sigma_{j+1}$ it suffices to show that there is an element of Σ_{j+1} that is not an element of Σ_j.

Suppose then that $\Sigma_j \neq \{\overline{0}, \overline{1}, \dots, \overline{p-1}\}$, and consider multiples of ℓ_{j+1} mod p. Since $\ell_{j+1} \not\equiv 0$ mod p we have

$$\{\overline{k\ell_{j+1}} \mid 0 \leq k \leq p-1\} = \{\overline{0}, \overline{1}, \dots, \overline{p-1}\} \supsetneq \Sigma_j \supseteq \{\overline{0}\},$$

so there is some $1 \leq k \leq p-1$ such that $\overline{k\ell_{j+1}} \notin \Sigma_j$. Consider the least such k. Then there is a (possibly empty, if $k = 1$) subset A of $\{1, \dots, j\}$ such that

$$\sum_{i \in A} \ell_i \equiv (k-1)\ell_{j+1} \text{ mod } p,$$

and setting $B = A \cup \{j+1\}$ we have

$$\sum_{i \in B} \ell_i \equiv k\ell_{j+1} \text{ mod } p.$$

Hence $\overline{k\ell_{j+1}}$ belongs to Σ_{j+1} but not Σ_j, and we are done. □

Proof of Theorem 1.5 The technique is similar to that used in the previous section to prove Theorem 1.3. Suppose that q is prime, and that N satisfies the inequality

$$N \geq \kappa_n(2q-1) + n + \left\lceil \frac{2q-3}{n} \right\rceil + 1$$

given in the statement of the theorem. By Theorem 1.2 and Corollary 2.7, N is so large that every embedding of K_N^n in $\mathbb{R}^{2n+1}$ contains a generalised keyring

$$R \cup L_1 \cup \dots \cup L_{2q-1}$$

with $2q-1$ keys, and an additional disjoint sphere S with at least $2q-1$ n–simplices. Orient the link $S \cup L_1 \cup \dots \cup L_{2q-1}$ as in the proof of Theorem 1.3, and let $\mathcal{S}$ be the n–sphere that results from applying Lemma 2.5 to this link.

Consider now the linking numbers $\ell k(R, \mathcal{S})$ and $\ell k(R, L_i)$ mod q. If $\ell k(R, L_i) \equiv 0$ for some i then we are done, so we may assume that all such linking numbers are nonzero mod q. Then by Lemma 4.1 there is a subset $A \subseteq \{1, \dots, q-1\}$ such that

$$\sum_{i \in A} [L_i] \equiv -[\mathcal{S}] \text{ mod } q,$$

and a subset $B \subseteq \{q+1, \ldots, 2q-1\}$ such that

$$\sum_{i \in B} [L_i] \equiv -[L_q] \bmod q.$$

Set $C = B \cup \{q\}$, to obtain a nonempty subset of $\{q, \ldots, 2q-1\}$ such that

$$\sum_{i \in C} [L_i] \equiv 0 \bmod q.$$

We now consider the chains

$$S_1 = \mathcal{S} + \sum_{i \in A} L_i \quad \text{and} \quad S_2 = S_1 + \sum_{i \in C} L_i.$$

In the homology group $H_n(\mathbb{R}^{2n+1} - R; \mathbb{Z})$ we have

$$[S_1] \equiv [S_2] \equiv 0 \bmod q,$$

and moreover $[S_1] \neq [S_2]$, because the linking numbers $[L_i]$ are all positive and C is nonempty. It follows that at least one of $[S_1]$ and $[S_2]$ is nonzero, and since both chains represent n–spheres we are done. □

We note that the argument used above does require q to be prime. For q composite, if $\ell k(R, \mathcal{S})$ is coprime to q and all linking numbers $\ell k(R, L_i)$ are equal to the same nontrivial divisor d of q, then no sphere formed from $\mathcal{S}$ and the L_i as above will link R with linking number divisible by q. We will therefore use a different strategy in Section 6 to prove the corresponding result when q may be composite.

5 Technical preliminaries II: Triangulations of an *M*–simplex

We now establish some additional technical preliminaries needed to prove Theorem 1.4. For this theorem we will need to work with links containing identically triangulated discs D^n with many n–simplices, and to this end we will construct a triangulation of an M–simplex into many M–simplices.

5.1 The triangulations

For $\ell \in \mathbb{N}$ let Δ_ℓ^M be the M–simplex

$$\Delta = [\ell \boldsymbol{e}_1, \ell \boldsymbol{e}_2, \ldots, \ell \boldsymbol{e}_{M+1}] \subseteq \mathbb{R}^{M+1},$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{M+1}$ are the standard basis vectors. Then:

Lemma 5.1 *The family of planes*

$$\left\{ \sum_{k=i}^{j} x_k \in \mathbb{Z} \;\middle|\; 1 \le i \le j \le M \right\} \tag{4}$$

subdivides Δ_ℓ^M into ℓ^M M–simplices. The symmetry group of this triangulation is the dihedral group D_{M+1} of order $2(M+1)$, with the action given by permutations of the basis vectors $\boldsymbol{e}_i$ that preserve or reverse the cyclic ordering $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_{M+1}$.

We will call an M–simplex triangulated as in Lemma 5.1 a *triangulated M–simplex of side length ℓ*, and denote it by $\Delta^M(\ell)$.

Remark 5.2 The triangulation $\Delta^2(\ell)$ is simply the standard division of an equilateral triangle of side length ℓ into ℓ^2 equilateral triangles of side length 1. In this case all simplices of the triangulation are isometric. However, for $M \ge 3$ the simplices of the triangulation may no longer all be isometric. This may be seen in the case $\Delta^3(2)$, where four of the 3–simplices are regular tetrahedra, and the remaining four are obtained by cutting an octahedron along two of the three planes of symmetry that pass through four vertices.

Remark 5.3 The $(M+1)$–cycle $(1\,2\,\dots\,M+1)$ in D_{M+1} reverses the orientation of Δ_l^M if and only if M is odd, and when M is even the order two elements of D_{M+1} reverse orientation if and only if $M \equiv 2 \bmod 4$. So $\Delta^M(\ell)$ has an orientation reversing symmetry if and only if $M \not\equiv 0 \bmod 4$.

Proof We proceed by subdividing the simplex

$$\Sigma_\ell^M = \{\boldsymbol{x} \in \mathbb{R}^M \mid 0 \le x_1 \le x_2 \le \dots \le x_M \le \ell\}$$

into ℓ^M simplices, and then pull this subdivision back to Δ_l^M. The chief reason for working with Δ_ℓ^M rather than Σ_ℓ^M is that the symmetries of the triangulation are more readily seen.

We first observe that for each permutation $\sigma \in S_M$, the set

$$\delta_\sigma = \{\boldsymbol{x} \in \mathbb{R}^M \mid 0 \le x_{\sigma(1)} \le x_{\sigma(2)} \le \dots \le x_{\sigma(M)} \le 1\}$$

is an M–simplex, and that the collection of such simplices gives a subdivision of I^M into $M!$–simplices. These simplices are defined by the family of planes

$$\{x_i = 0\} \cup \{x_i = 1\} \cup \{x_j - x_i = 0\},$$

and translating these according to $\mathbb{Z}^M \leq \mathbb{R}^M$ we see that the family

$$\{x_i \in \mathbb{Z} \mid 1 \leq i \leq M\} \cup \{x_j - x_i \in \mathbb{Z} \mid 1 \leq i < j \leq M\} \tag{5}$$

gives a subdivision of all of $\mathbb{R}^M$ into isometric simplices. The planes bounding Σ_ℓ^M belong to this family, and it follows that the subdivision of $\mathbb{R}^M$ restricts to a subdivision of Σ_ℓ^M. This subdivision must have ℓ^M simplices, on purely volumetric grounds.

We now pull this triangulation back to Δ_ℓ^M via the linear map that sends the vertex $\boldsymbol{e}_i$ of Δ_ℓ^M to the vertex $\boldsymbol{e}_i + \cdots + \boldsymbol{e}_M$ of Σ_ℓ^M for $i \leq M$, and the vertex $\boldsymbol{e}_{M+1}$ to the vertex $\boldsymbol{0}$. Let $\{\phi_i\}$ be the dual basis to $\{\boldsymbol{e}_i\}$. Then ϕ_i pulls back to $\phi_1 + \cdots + \phi_i$, and we see that the family (5) pulls back to the family (4). This linear map induces an affine homeomorphism between Σ_ℓ^M and Δ_ℓ^M, and so these planes give us the desired triangulation.

To see that the symmetry group is D_{M+1}, we observe that on the plane $\sum x_i = \ell$ containing Δ_ℓ^M, the conditions

$$\sum_{k=i}^{j} x_k \in \mathbb{Z} \quad \text{and} \quad \sum_{k=1}^{i-1} x_k + \sum_{k=j+1}^{M+1} x_k \in \mathbb{Z}$$

are equivalent. Thus, each family of planes defining the subdivision may be viewed as a division of a necklace of $M+1$ beads into two connected components, and conversely. Symmetries of the triangulation therefore correspond to precisely those permutations of the beads that preserve adjacency, giving us D_{M+1}. □

Construction 5.4 For $M \geq n+1$ we define $K_M^n(\ell)$ to be the subcomplex of $\Delta^{M-1}(\ell)$ consisting of precisely those simplices lying entirely within the n–skeleton $(\Delta_\ell^{M-1})^{(n)} \cong K_M^n$. Each n–simplex of $(\Delta_\ell^{M-1})^{(n)}$ lies in an n–dimensional coordinate plane, and is isometric to Δ_ℓ^n; intersecting the family of planes (4) with this subspace subdivides this simplex into a $\Delta^n(\ell)$. Thus $K_M^n(\ell)$ is a space homeomorphic to K_M^n, with each n–simplex of K_M^n mapping onto a copy of $\Delta^n(\ell)$. As such we will call it a *triangulated complete n–complex on M vertices of side length ℓ*.

5.2 Counting the vertices

The number of vertices in a $\Delta^k(\ell)$ is equal to the number of nonnegative integer solutions to the equation

$$x_1 + x_2 + \cdots + x_{k+1} = \ell,$$

and the number of vertices in the interior of a $\Delta^k(\ell)$ is the number of positive integer solutions to this equation. These numbers are $\binom{k+\ell}{k}$ and $\binom{\ell-1}{k} = \binom{\ell-1}{\ell-k-1}$ respectively.

Counting the vertices of a $\Delta^M(\ell)$ according to the open simplex of Δ^M_ℓ that they belong to we find that it has

$$\sum_{k=0}^{M} \binom{M+1}{k+1}\binom{\ell-1}{\ell-k-1} = \binom{\ell+M}{M} \tag{6}$$

vertices (the two sides are the coefficient of x^ℓ in $(1+x)^{M+1}(1+x)^{\ell-1} = (1+x)^{\ell+M}$).

Of particular interest is the number of vertices belonging to $K^n_{2n+4}(\ell)$, as this complex is homeomorphic to K^n_{2n+4}, and may be used to construct links in which each component has many n–simplices. Setting $M = 2n+3$ in Equation (6), and truncating the sum at $k = n$, we therefore find that $K^n_{2n+4}(\ell)$ has a total of

$$V(n,\ell) = \sum_{k=0}^{n} \binom{2n+4}{k+1}\binom{\ell-1}{k} \tag{7}$$

vertices.

For a more tractable bound, observe that the triangulated simplex $\Delta^n(\ell)$ has ℓ^n n–simplices, each with $n+1$ vertices, and so has at most $(n+1)\ell^n$ vertices. The complex $K^n_{2n+4}(\ell)$ contains $\binom{2n+4}{n+1}$ such triangulated simplices, and therefore

$$V(n,\ell) \le (n+1)\binom{2n+4}{n+1}\ell^n$$

(this also follows from the inequalities

$$\binom{2n+4}{k+1} \le \binom{2n+4}{n+1} \quad \text{and} \quad \binom{\ell-1}{k} \le \ell^n$$

for $k \le n$). Stirling's formula $m! \sim \sqrt{2\pi m}(m/e)^m$ leads to the asymptotic formula $\binom{2m}{m} \sim 4^m/\sqrt{\pi m}$, and hence

$$(n+1)\binom{2n+4}{n+1} = \frac{(n+1)(n+2)}{n+3}\binom{2(n+2)}{n+2} \sim \sqrt{\frac{n}{\pi}}\, 4^{n+2} = C\sqrt{n}4^n.$$

Consequently, asymptotically $V(n,\ell)$ grows no faster than $C\sqrt{n}(4\ell)^n$.

6 Linking number mod q

The goal of this section is to prove Theorem 1.4, which we recall states that given $q \in \mathbb{N}$, embeddings of sufficiently large complete n–complexes in $\mathbb{R}^{2n+1}$ contain 2–component links with linking number a nonzero multiple of q. Before proving this theorem we need one more technical lemma:

Lemma 6.1 *Let R be a positive integer. For ℓ sufficiently large $\Delta^n(\ell)$ contains a triangulated disc D with $r \geq R$ n–simplices $\Delta_1, \dots, \Delta_r$, which may be labelled such that*

$$D_{ij} = \bigcup_{k=i}^{j} \Delta_k$$

is a disc for any $1 \leq i \leq j \leq r$. The conclusion holds for $\ell \geq R$, so the side length required grows at most linearly with R.

Proof Write $\Sigma^n(\ell)$ for the n–simplex Σ^n_ℓ subdivided by the family of planes given by Equation (5). Then $\Sigma^n(\ell)$ and $\Delta^n(\ell)$ are simplicially isomorphic, so it suffices to construct a suitable disc D in $\Sigma^n(\ell)$. We will construct D as the union of the n–simplices of $\Sigma^n(\ell)$ that meet a suitably chosen line L in $\mathbb{R}^n$. The case $n = 2$, $\ell = 4$ is illustrated in Figure 4.

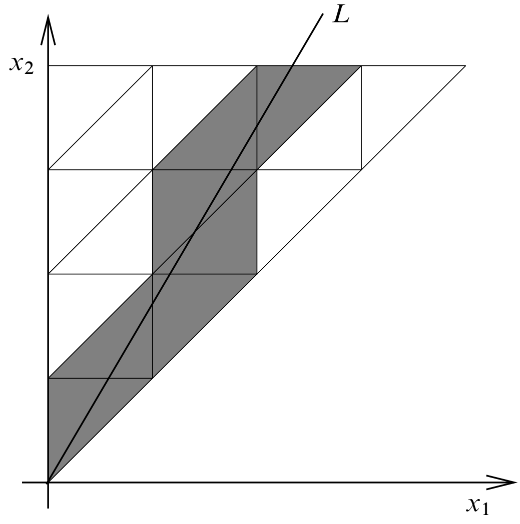

Figure 4: Illustrating the construction of the disc D of Lemma 6.1 in the case $n = 2$, $\ell = 4$. A line L with irrational slope $\alpha > 1$ meets each line defining the triangulation exactly once, and except at $\mathbf{0}$ never passes through the intersection of two such lines. We take D to be the union of the 2–simplices intersecting L (shaded grey). The disc D contains at least ℓ n–simplices (here at least 4), since it must include at least one from each horizontal slice.

Since $\mathbb{R}$ is infinite-dimensional as a vector space over $\mathbb{Q}$, we may choose

$$0 < \alpha_1 < \cdots < \alpha_n = 1$$

such that $\{\alpha_1, \ldots, \alpha_n\}$ is linearly independent over $\mathbb{Q}$. Write $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, and let L be the line $L = \{t\boldsymbol{\alpha} \mid t \in \mathbb{R}\}$. Each plane in the family (5) may be written in the form $\boldsymbol{c}^T \boldsymbol{x} = u$, where $\boldsymbol{c} \in \mathbb{Z}^n$ and $u \in \mathbb{Z}$, and the linear independence of $\{\alpha_1, \ldots, \alpha_n\}$ over $\mathbb{Q}$ may be used to show that

(1) L meets each plane in the family (5) transversely; and

(2) each point of L other than $\mathbf{0}$ lies on at most one plane in this family.

Together these facts imply that, with the exception of simplices containing $\mathbf{0}$, L can meet only n– and $(n-1)$–simplices of $\Sigma^n(\ell)$, and that if it intersects an n–simplex at all it must intersect it in its interior.

Observe that the line segment $\{t\boldsymbol{\alpha} \mid 0 \le t \le \ell\}$ is contained in Σ^n_ℓ, and cuts each plane $x_n = k$ for $k = 1, \ldots, \ell$. Consequently L must pass through at least one n–simplex of $\Sigma^n(\ell)$ lying in the slice $\{\boldsymbol{x} \mid k-1 \le x_n \le k\}$ for each $1 \le k \le \ell$, and so passes through at least ℓ n–simplices of $\Sigma^n(\ell)$. Suppose that L passes through exactly r n–simplices of $\Sigma^n(\ell)$, and label them consecutively $\Delta_1, \ldots, \Delta_r$ in the order in which they are encountered when tracing L in the direction $\boldsymbol{\alpha}$. We claim that

$$D_{ij} = \bigcup_{k=i}^{j} \Delta_k$$

is a disc for any $1 \le i \le j \le \ell$, from which the result follows.

Since the open ray $\{t\boldsymbol{\alpha} \mid t > 0\}$ only meets n– and $(n-1)$–simplices of $\Sigma^n(\ell)$, consecutive n–simplices Δ_k and Δ_{k+1} must intersect in an $(n-1)$–simplex. In addition, for $d \ge 2$ the simplices Δ_k and Δ_{k+d} are separated by at least two planes from the family (5), and so meet in at most an $(n-2)$–simplex. Since $D_{ii} = \Delta_i$ is a disc, and $D_{i,k+1}$ is the result of gluing D_{ik} and Δ_{k+1} along the $(n-1)$–simplex $\Delta_k \cap \Delta_{k+1}$, it follows by induction that D_{ij} is a disc, as claimed. □

We now prove Theorem 1.4. The argument again proceeds by converting a suitably large generalised keyring to a 2–component link, but now we require additionally that the keys of the keyring are copies of $K^n_{n+2}(q)$. Our underlying approach is similar to that of Fleming [5, Theorem 3.1], but differs from his in the size of the keys and the method used to combine them to form the second component of the link.

Proof of Theorem 1.4 We show that the result holds for

$$N = 4q^2 V(n,q) + n + \left\lceil \frac{4q^2 - 2}{n} \right\rceil + 1,$$

where $V(n,q)$ is given by (7) and equals the number of vertices belonging to $K^n_{2n+4}(q)$. Since $V(n,q) \le (n+1)\binom{2n+4}{n+1}q^n$, we conclude that N is no greater than

$$C(n+1)\binom{2n+4}{n+1}q^{n+2}$$

for some constant C.

Given an embedding of K^n_N in $\mathbb{R}^{2n+1}$, we let $C_1,\dots,C_{4q^2}$ be disjoint copies of $K^n_{2n+4}(q)$ contained in K^n_N, and use the remaining $n+\lceil(4q^2-2)/n\rceil+1$ vertices and Corollary 2.7 to construct an n–sphere L with at least $4q^2$ n–simplices. The complex C_i is homeomorphic to K^n_{2n+4}, and so by Taniyama [19] contains a two component link $J_i \cup X_i$ such that $\ell k(J_i, X_i) \ne 0$, and each component is a copy of $K^n_{n+2}(q)$. Applying Lemma 3.2 to the link $L \cup J_1 \cup \dots \cup J_{4q^2} \cup X_1 \cup \dots \cup X_{4q^2}$ we obtain a generalised keyring $R \cup L_1 \cup \dots \cup L_q$, where $\ell k(R, L_i) \ne 0$ for each i, and each L_i is a copy of $K^n_{n+2}(q)$. We will use R as one component of our link, and we will seek to construct the second as a connected sum of some of the L_i. In what follows we therefore consider homology classes in $H_n(\mathbb{R}^{2n+1}-R;\mathbb{Z})$.

Orient the L_i such that $\ell k(R, L_i) = [L_i]$ is positive for each i, and for $1 \le k \le q$ consider the values of the sums $\sum_{i=1}^{k}[L_i] \bmod q$. Since there are q sums and q possible values modulo q, by the pigeonhole principle there must either be a sum that is zero mod q, or else two sums that are equal modulo q. In either case we obtain integers a, b satisfying $1 \le a \le b \le q$ such that

$$\sum_{i=a}^{b}[L_i] \equiv 0 \bmod q.$$

From now on we restrict our attention to the spheres $L_a,\dots,L_b$.

Our construction now departs from that of Fleming. Each component L_i is a copy of $K^n_{n+2}(q)$, and as such has $n+2$ faces which are triangulated n–simplices of side length q. We claim that it is possible to choose distinct faces δ_i, δ'_i of L_i, each a copy of $\Delta^n(q)$, and orientation reversing simplicial isomorphisms $\psi_i\colon \delta_i \to \delta'_{i+1}$. For $n \not\equiv 0 \bmod 4$ this may be done by choosing distinct faces δ_i, δ'_i of L_i arbitrarily, since in this case $\Delta^n(q)$ has both orientation preserving and reversing symmetries, by Remark 5.3. However, for $n \equiv 0 \bmod 4$ we must choose them inductively, beginning with δ_a and using the fact that $\Delta^n(q)$ has at least one face of each orientation to choose δ'_{i+1} based on the choice of δ_i. The face δ_{i+1} of L_{i+1} may then be chosen arbitrarily from those left.

By Lemma 6.1 each face $\delta_i \cong \Delta^n(q)$ contains a triangulated disc D_i with $r \geq q$ n–simplices $\Delta_{i1}, \ldots, \Delta_{ir}$, such that

$$(D_i)_{cd} = \bigcup_{k=c}^{d} \Delta_{ik}$$

is a disc for each $1 \leq c \leq d \leq r$. Let ϕ_i be the restriction of ψ_i to D_i, let $D'_{i+1} = \phi_i(D_i)$, and for $1 \leq j \leq r$ let P_{ij} be the oriented sphere satisfying

$$P_{ij} \cap L_i = \Delta_{ij}, \quad P_{ij} \cap L_{i+1} = \phi_i(\Delta_{ij})$$

that results from applying Corollary 2.2 to the pairs (L_i, D_i) and (L_{i+1}, D'_{i+1}).

For $1 \leq k \leq r$ we now consider the sums $\sum_{j=1}^{k}[P_{ij}]$ modulo q. Since there are q possible values mod q and at least q sums we may again choose integers c_i, d_i satisfying $1 \leq c_i \leq d_i \leq r$ such that

$$\sum_{j=c_i}^{d_i}[P_{ij}] \equiv 0 \bmod q.$$

Let $Q_i = \sum_{j=c_i}^{d_i} P_{ij}$. Then Q_i represents an n–sphere with all its vertices on $L_i \cup L_{i+1}$ and satisfying

$$Q_i \cap L_i = (D_i)_{c_i d_i}, \quad Q_i \cap L_{i+1} = \phi_i((D_i)_{c_i d_i}), \quad \ell k(R, Q_i) \equiv 0 \bmod q.$$

If $\ell k(R, Q_i) \neq 0$ for some i then we are done by setting $S = Q_i$, so we may assume that in fact $\ell k(R, Q_i) = 0$ for all i. In that case we let S be the complex obtained from $L_a, \ldots, L_b$ and $Q_a, \ldots, Q_{b-1}$ by omitting the interiors of the discs $Q_a \cap L_a, \ldots, Q_{b-1} \cap L_{b-1}$ and $Q_a \cap L_{a+1}, \ldots, Q_{b-1} \cap L_b$. Then S is a connected sum of n–spheres, hence an n–sphere, and as a chain we have

$$S = \sum_{i=a}^{b} L_i + \sum_{i=a}^{b-1} Q_i.$$

It follows that

$$[S] = \sum_{i=a}^{b}[L_i] + \sum_{i=a}^{b-1}[Q_i] = \sum_{i=a}^{b}[L_i] > 0,$$

and since also $\sum_{i=a}^{b}[L_i] \equiv 0 \bmod q$ we are done. □

Remark 6.2 For $n=1$ the auxiliary sphere $\mathcal{S}$ of Lemma 2.5 is not needed to construct the keyring, reducing the number of vertices required in this case to

$$4q^2V(1,q) = 4q^2(6+15(q-1)) = 12q^2(5q-3),$$

as given after the statement of the theorem.

7 An alternate proof of Theorem 1.3

To further illustrate the applications of the triangulations of Section 5 we give a second proof of Theorem 1.3, without the polynomial bound on the number of vertices required. Namely, we show that given $\ell \in \mathbb{N}$, for N sufficiently large every embedding of K_N^n in $\mathbb{R}^{2n+1}$ contains a 2–component link with linking number at least ℓ in absolute value.

The proof we give is modelled on Flapan's original proof [2] of the corresponding result for $n=1$. Her argument is based on combining 2–component links with "sufficiently many vertices", and for $n \geq 2$ we will replace this condition on the number of vertices with a requirement that the components contain triangulated n–simplices of sufficient side length. The side length available will typically shrink when two components are combined (unlike the number of vertices, which typically goes up), and consequently this change leads to a significant change in the growth of the number of vertices required.

7.1 Splicing links

In this section we establish higher-dimensional analogues of Flapan [2, Lemmas 2 and 1]. These are Lemmas 7.2 and 7.3 below, respectively. In preparation for this we need an additional technical lemma on triangulated n–simplices.

Lemma 7.1 *Deleting an arbitrary M–simplex from a triangulated M–simplex of side length ℓ leaves a triangulated M–simplex of side length at least $\lfloor M\ell/(M+1)\rfloor$.*

Proof Let δ be the deleted simplex, and let $\boldsymbol{x}$ be a point in the interior of δ. In barycentric coordinates on Δ_ℓ^M we have

$$\boldsymbol{x} = \ell \sum_{i=1}^{M+1} t_i \boldsymbol{e}_i,$$

and since $\sum t_i = 1$ we must have $t_i \leq 1/(M+1)$ for some i. Let Δ be the intersection of $\Delta^M(\ell)$ with the halfspace $x_i \geq \lceil \ell/(M+1) \rceil$. Then Δ is a triangulated M–simplex contained in Δ_ℓ^M, and Δ does not contain δ because Δ does not contain $\boldsymbol{x}$. Moreover, Δ has side length

$$\ell - \left\lceil \frac{\ell}{M+1} \right\rceil = \left\lfloor \ell - \frac{\ell}{M+1} \right\rfloor = \left\lfloor \frac{M\ell}{M+1} \right\rfloor,$$

so we are done. □

Lemma 7.2 *Let $X_1 \cup Y_1 \cup X_2 \cup Y_2$ be a 4–component link contained in some embedding of K_N^n in $\mathbb{R}^{2n+1}$. Suppose that for some orientation of $X_1 \cup Y_1 \cup X_2 \cup Y_2$ we have $\ell k(X_1, Y_1) \geq 1$ and $\ell k(X_2, Y_2) = p \geq 1$, and suppose also that each component contains a triangulated n–simplex of side length ℓ with $\ell^n \geq p$. Then K_N^n contains disjoint n–spheres L, Z and W such that*

1. *$\ell k(L, Z) = p_1 \geq 1$ and $\ell k(L, W) = p_2 \geq p$ for some orientation of the link $L \cup Z \cup W$;*
2. *L contains a triangulated n–simplex of side length at least $\lfloor n\ell/(n+1) \rfloor$;*
3. *Z is equal to either X_1 or Y_1;*
4. *W is equal to either X_2 or Y_2.*

Proof As in Flapan [2], if $\ell k(X_2, Y_1)$ is nonzero we may set $L = X_2$, $Z = Y_1$, and $W = Y_2$; and if $\ell k(Y_2, X_1)$ is nonzero we may set $L = Y_2$, $Z = X_1$, and $W = X_2$. So in what follows we may assume that $\ell k(X_1, Y_2) = \ell k(X_2, Y_1) = 0$.

Let D_i be a $\Delta^n(\ell)$ contained in X_i, for each i, and let $\phi\colon D_1 \to D_2$ be a simplicial isomorphism. After reversing orientation on both X_1 and Y_1 if necessary we may assume that ϕ reverses orientation, and so we may apply Corollary 2.2 to the pairs (X_1, D_1) and (X_2, D_2). We label the resulting spheres $P_0, \dots, P_{\ell^n}$ as in the statement of the corollary, and following Flapan the equation

$$[X_1] + [X_2] + \sum_{j=0}^{\ell^n} [P_j] = 0$$

holds in the n^{th} homology group $H_n(\mathbb{R}^{2n+1} - Y_2; \mathbb{Z})$.

By our assumption that $\ell k(X_1, Y_2) = 0$ we have $[X_1] = 0$ in $H_n(\mathbb{R}^{2n+1} - Y_2; \mathbb{Z})$, so

$$0 < p = [X_2] = -\sum_{j=0}^{\ell^n} [P_j].$$

The right hand side consists of $\ell^n + 1 > p$ terms, so for some index q we must have $[P_q] \geq 0$. We consider two cases, according to whether or not $[P_q] = 0$ in $H_n(\mathbb{R}^{2n+1} - Y_1; \mathbb{Z})$.

If $[P_q]$ is nonzero in $H_n(\mathbb{R}^{2n+1} - Y_1; \mathbb{Z})$ then we construct L from P_q and X_2 by deleting the interior of the disc $X_2 \cap P_q$. L is the connected sum of the n–spheres P_q and X_2, and so is itself an n–sphere. As a chain we have $L = P_q + X_2$, and therefore

$$\begin{aligned} [L] &= [P_q] + [X_2] \geq p && \text{in } H_n(\mathbb{R}^{2n+1} - Y_2; \mathbb{Z}), \\ [L] &= [P_q] + [X_2] = [P_q] \neq 0 && \text{in } H_n(\mathbb{R}^{2n+1} - Y_1; \mathbb{Z}). \end{aligned}$$

So we obtain the desired link by letting $Z = Y_1$ and $W = Y_2$, and reorienting Z if necessary so that $\ell k(L, Z)$ is positive.

If $[P_q] = 0$ in $H_n(\mathbb{R}^{2n+1} - Y_1; \mathbb{Z})$ then we construct L from X_1, X_2 and P_q by deleting the interiors of the discs $X_i \cap P_q$. Clearly, L is again an n–sphere. As a chain we have $L = X_1 + P_q + X_2$, and therefore

$$\begin{aligned} [L] &= [X_1] + [P_q] + [X_2] = [P_q] + [X_2] \geq p && \text{in } H_n(\mathbb{R}^{2n+1} - Y_2; \mathbb{Z}), \\ [L] &= [X_1] + [P_q] + [X_2] = [X_1] \geq 1 && \text{in } H_n(\mathbb{R}^{2n+1} - Y_1; \mathbb{Z}). \end{aligned}$$

So we obtain the desired link by letting $Z = Y_1$ and $W = Y_2$.

In every case above Z was equal to either X_1 or Y_1, and W was equal to either X_2 or Y_2. To complete the proof we must show that L contains a triangulated n–simplex of side length at least $\lfloor n\ell/(n+1)\rfloor$. If $q = 0$ then L contains D_2 and we are done, and otherwise L contains $D_2 \setminus (X_2 \cap P_q)$ and we are done by Lemma 7.1. □

Lemma 7.3 *Let $L \cup Z \cup W$ be a 3–component link contained in some embedding of K_N^n in $\mathbb{R}^{2n+1}$, and suppose that for some orientation of $L \cup Z \cup W$ we have $\ell k(L, Z) = p_1 > 0$, $\ell k(L, W) = p_2 > 0$. Suppose that Z and W contain triangulated simplices Δ_Z and Δ_W of side length ℓ, with $\ell^n \geq p_1 + p_2$, and that there is an orientation reversing simplicial isomorphism $\phi \colon \Delta_Z \to \Delta_W$. Then K_N^n contains an n–sphere J disjoint from L such that*

1. *$\ell k(L, J) \geq p_1 + p_2$ for some orientation of $L \cup J$;*
2. *J contains a triangulated n–simplex of side length at least $\lfloor n\ell/(n+1)\rfloor$.*

Proof As in the proof of Lemma 7.2 we apply Corollary 2.2 to the pairs (Z, Δ_Z) and (W, Δ_W), obtaining spheres $P_0, \ldots, P_{\ell^n}$. In the homology group $H_n(\mathbb{R}^{2n+1} - L; \mathbb{Z})$ we have the equation

$$[Z] + [W] + \sum_{j=0}^{\ell^n} [P_j] = 0,$$

so that

$$p_1 + p_2 = [Z] + [W] = -\sum_{j=0}^{\ell^n} [P_j].$$

As in the proof of Lemma 7.2 above, the right-hand side has $\ell^n + 1 > p_1 + p_2$ terms, so there must be an index q such that $[P_q] \geq 0$. Let J be the n–sphere obtained from Z, P_q and W by deleting the interiors of the discs $P_q \cap Z$ and $P_q \cap W$. Then J is disjoint from L by Remark 2.3, and as a chain $J = Z + P_q + W$, so

$$[J] = [Z] + [P_q] + [W] \geq p_1 + p_2$$

in $H_n(\mathbb{R}^{2n+1} - L; \mathbb{Z})$. Condition (2) above holds by the same argument as in Lemma 7.2, and the result follows. □

Combining Lemmas 7.2 and 7.3 we obtain the following:

Corollary 7.4 *Let $X_1 \cup Y_1 \cup X_2 \cup Y_2$ be a 4–component link contained in some embedding of K_N^n in $\mathbb{R}^{2n+1}$. Suppose that*

1. *for some orientation of $X_1 \cup Y_1 \cup X_2 \cup Y_2$ we have $\ell k(X_1, Y_1) \geq 1$ and $\ell k(X_2, Y_2) = p \geq 1$;*
2. *each component contains a triangulated n–simplex of side length ℓ with $\ell^n \geq 2p$;*
3. *either $n \not\equiv 0 \bmod 4$, or X_1 and Y_1 each contain two such triangulated n–simplices, one of each possible orientation.*

Then K_N^n contains disjoint n–spheres L and J, each containing a triangulated n–simplex of side length at least $\lfloor n\ell/(n+1) \rfloor$, and such that $\ell k(L, J) \geq p + 1$.

Proof The hypotheses of Lemma 7.2 are satisfied, so we obtain a three component link $L \cup Z \cup W$ satisfying the conditions given in that Lemma. These conditions imply the hypotheses of Lemma 7.3, except perhaps the condition that $\ell^n \geq p_1 + p_2$ and the condition that ϕ may be chosen to reverse orientation.

If the hypothesis $\ell^n \geq p_1 + p_2$ does not hold then we must have $p_1 + p_2 > 2p$, which implies $p_i \geq p + 1$ for some i. So if this occurs we are done by simply letting J be either Z or W, as appropriate.

To see that the condition on ϕ is satisfied we use our third hypothesis above. If $n \not\equiv 0 \bmod 4$ then $\Delta^n(\ell)$ has an orientation reversing symmetry, and otherwise Z is equal to either X_1 or Y_1, and so contains a $\Delta^n(\ell)$ of each orientation. We may therefore choose Δ_Z and Δ_W to have opposite orientations, and apply Lemma 7.3 to get the desired result. □

7.2 Theorem 1.3, revisited

Using the results of the previous section we reprove Theorem 1.3 in the following weakened form.

Theorem 7.5 *Given $\lambda \geq 2$, let $\mu = \lceil \sqrt[n]{2(\lambda-1)} \rceil$, and suppose that N is sufficiently large that K^n_N contains disjoint copies of $K^n_{2n+4}(2^i\mu)$ for $i = 0, \dots, \lambda-2$, and an additional disjoint copy of $K^n_{2n+4}(2^{\lambda-2}\mu)$. Then every embedding of K^n_N in $\mathbb{R}^{2n+1}$ contains a two-component link $L \cup J$ such that, for some orientation of the components, $\ell k(L, J) \geq \lambda$.*

Proof Given an embedding of K^n_N in $\mathbb{R}^{2n+1}$, let $C_1, \dots, C_\lambda$ be disjoint subcomplexes of K^n_N such that C_1 is a $K^n_{2n+4}(2^{\lambda-2}\mu)$, and C_i is a $K^n_{2n+4}(2^{\lambda-i}\mu)$ for $i = 2, \dots, \lambda$. Each C_i is homeomorphic to K^n_{2n+4}, and so by Taniyama [19] contains a two component link $S_i \cup T_i$ which we may orient such that $\ell k(S_i, T_i) \geq 1$. We will use these to inductively construct links $L_i \cup J_i$ such that

(1) $\ell k(L_i, J_i) \geq i$;

(2) all vertices of $L_i \cup J_i$ lie in $C_1 \cup \dots \cup C_i$ (and so $L_i \cup J_i$ is disjoint from C_j for $j > i$);

(3) for $i < \lambda$ the spheres L_i and J_i each contain a triangulated n–complex of side length at least $2^{\lambda-i-1}\mu$.

The link $L_\lambda \cup J_\lambda$ is then the required link.

Each component S_i, T_i is isomorphic to the boundary of a triangulated $(n+1)$–simplex of side length equal to that of C_i, and as such has $n+2$ faces which are each a triangulated n–simplex of this same side length. For the base case we may therefore simply let $L_1 \cup J_1 = S_1 \cup T_1$.

Given $1 \leq i \leq \lambda - 1$, suppose that we have constructed $L_i \cup J_i$ but not yet $L_{i+1} \cup J_{i+1}$. Let $\ell k(S_i, T_i) = p \geq i$. If $p \geq \lambda$ then we simply set $L_j \cup J_j = S_i \cup T_i$ for $j \geq i$ and the construction is complete, so suppose that $p < \lambda$. Then every component of the link $S_{i+1} \cup T_{i+1} \cup L_i \cup J_i$ contains a triangulated n–simplex of side length at least $\ell = 2^{\lambda-i-1}\mu \geq \mu$, and ℓ satisfies $\ell^n \geq \mu^n \geq 2(\lambda-1) \geq 2p$. Moreover, as the boundary of a $K^n_{n+1}(\ell)$, each component of $S_{i+1} \cup T_{i+1}$ must contain at least one $\Delta^n(\ell)$ face of each orientation. Working entirely within the K^n_M spanned by the vertices of $C_1 \cup \dots \cup C_{i+1}$ we may therefore apply Corollary 7.4 to obtain a 2–component link $L_{i+1} \cup J_{i+1}$ satisfying $\ell k(L_{i+1}, J_{i+1}) \geq p + 1 \geq i + 1$.

Each component of $L_{i+1} \cup J_{i+1}$ contains a triangulated n–simplex of side length at least

$$\left\lfloor \frac{n\ell}{n+1} \right\rfloor = \left\lfloor \frac{2^{\lambda-i-1} n\mu}{n+1} \right\rfloor.$$

Now $n/(n+1) \geq \frac{1}{2}$, so for $i < \lambda - 1$ the quantity $2^{\lambda-i-2}\mu$ is an integer satisfying

$$\frac{2^{\lambda-i-1} n\mu}{n+1} \geq \frac{2^{\lambda-i-1}\mu}{2} = 2^{\lambda-i-2}\mu,$$

and therefore

$$\left\lfloor \frac{n\ell}{n+1} \right\rfloor = \left\lfloor \frac{2^{\lambda-i-1} n\mu}{n+1} \right\rfloor \geq 2^{\lambda-i-2}\mu = 2^{\lambda-(i+1)-1}\mu.$$

This establishes condition (3) above when $i + 1 < \lambda$, completing the inductive step. □

References

[1] **J H Conway**, **C M Gordon**, *Knots and links in spatial graphs*, J. Graph Theory 7 (1983) 445–453 MR722061

[2] **E Flapan**, *Intrinsic knotting and linking of complete graphs*, Algebr. Geom. Topol. 2 (2002) 371–380 MR1917057

[3] **E Flapan**, **B Mellor**, **R Naimi**, *Intrinsic linking and knotting are arbitrarily complex*, Fund. Math. 201 (2008) 131–148 MR2448416

[4] **E Flapan**, **J Pommersheim**, **J Foisy**, **R Naimi**, *Intrinsically n–linked graphs*, J. Knot Theory Ramifications 10 (2001) 1143–1154 MR1871222

[5] **T Fleming**, *Intrinsically linked graphs with knotted components*, J. Knot Theory Ramifications 21 (2012) 1250065, 10 MR2911081

[6] **T Fleming**, **A Diesl**, *Intrinsically linked graphs and even linking number*, Algebr. Geom. Topol. 5 (2005) 1419–1432 MR2171815

[7] **M H Freedman**, **V S Krushkal**, **P Teichner**, *Van Kampen's embedding obstruction is incomplete for 2–complexes in* $\mathbf{R}^4$, Math. Res. Lett. 1 (1994) 167–176 MR1266755

[8] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR1867354

[9] **L Lovász**, **A Schrijver**, *A Borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs*, Proc. Amer. Math. Soc. 126 (1998) 1275–1285 MR1443840

[10] **S A Melikhov**, *Combinatorics of embeddings* `arXiv:1103.5457`

[11] **S A Melikhov**, *The van Kampen obstruction and its relatives*, Tr. Mat. Inst. Steklova 266 (2009) 149–183 MR2603266

[12] **S Negami**, *Ramsey-type theorems for spatial graphs and good drawings*, J. Combin. Theory Ser. B 72 (1998) 53–62 MR1604681

[13] **S Negami**, *Note on Ramsey theorems for spatial graphs*, Theoret. Comput. Sci. 263 (2001) 205–210 MR1846928

[14] **J L Ramírez Alfonsín**, *Knots and links in spatial graphs: a survey*, Discrete Math. 302 (2005) 225–242 MR2179645

[15] **H Sachs**, *On a spatial analogue of Kuratowski's theorem on planar graphs — an open problem*, from: "Graph theory", (M Borowiecki, J W Kennedy, M M Sysło, editors), Lecture Notes in Math. 1018, Springer, Berlin (1983) 230–241 MR730653

[16] **J Segal**, **S Spież**, *Quasi embeddings and embeddings of polyhedra in* $\mathbf{R}^m$, from: "Proceedings of the Tsukuba Topology Symposium", (Y Kodama, editor), volume 45 (1992) 275–282 MR1180814

[17] **A B Skopenkov**, *Embedding and knotting of manifolds in Euclidean spaces*, from: "Surveys in contemporary mathematics", (N Young, Y Choi, editors), London Math. Soc. Lecture Note Ser. 347, Cambridge Univ. Press (2008) 248–342 MR2388495

[18] **M Skopenkov**, *Embedding products of graphs into Euclidean spaces*, Fund. Math. 179 (2003) 191–198 MR2029321

[19] **K Taniyama**, *Higher dimensional links in a simplicial complex embedded in a sphere*, Pacific J. Math. 194 (2000) 465–467 MR1760794

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