Milnor–Wood inequalities for products

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We prove Milnor–Wood inequalities for local products of manifolds. As a consequence, we establish the generalized Chern conjecture for products $M \times \Sigma^k$ of any manifold $M$ and $k$ copies of a surface $\Sigma$ for $k$ sufficiently large.

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1 Introduction

Let $M$ be an $n$–dimensional topological manifold. Consider the Euler class $\varepsilon_n(\xi)$ in $H^n(M, \mathbb{R})$ and Euler number $\chi(\xi) = \langle \varepsilon_n(\xi), [M] \rangle$ of an oriented $\mathbb{R}^n$–vector bundle $\xi$ over $M$. We say that the manifold $M$ satisfies a Milnor–Wood inequality with constant $c$ if for every flat oriented $\mathbb{R}^n$–vector bundle $\xi$ over $M$, the inequality

$$|\chi(\xi)| \leq c \cdot |\chi(M)|$$

holds. Recall that a bundle is flat if it is induced by a representation of the fundamental group $\pi_1(M)$. We denote by

$$MW(M) \in \mathbb{R} \cup \{+\infty\}$$

the smallest such constant.

If $X$ is a simply connected Riemannian manifold with closed quotients, we denote

$$\widehat{MW}(X) := \sup\{MW(M) : M \text{ is a closed quotient of } X\}.$$ 

Milnor’s seminal inequality [7] amounts to showing that the Milnor–Wood constant of the hyperbolic plane $\mathcal{H}$ is $\widehat{MW}(\mathcal{H}) = 1/2$, and in [3], we showed that $\widehat{MW}(\mathcal{H}^n) = 1/2^n$.

In this note we prove a product formula for the Milnor–Wood constants of general closed manifolds:

**Theorem 1.1** For any pair of compact manifolds $M_1$, $M_2$,

$$MW(M_1 \times M_2) = MW(M_1) \cdot MW(M_2).$$
For the product formula for universal Milnor–Wood constant, we restrict to Hadamard manifolds:

**Theorem 1.2** Let \( X_1, X_2 \) be Hadamard manifolds. Then

\[
\widehat{MW}(X_1 \times X_2) = \widehat{MW}(X_1) \cdot \widehat{MW}(X_2).
\]

One important application of Milnor–Wood inequalities is to make progress on the generalized Chern conjecture.

**Conjecture 1.3** (Generalized Chern conjecture) Let \( M \) be a closed oriented aspherical manifold. If the tangent bundle \( TM \) of \( M \) admits a flat structure then \( \chi(M) = 0 \).

This conjecture has been suggested by Milnor [7] and is a strong version of the famous Chern conjecture which merely predicts the vanishing of the Euler characteristic for affine manifolds, that is, for manifolds admitting a torsion-free flat connection.

As pointed out in [7], if \( MW(M) < 1 \) then the generalized Chern conjecture holds for \( M \). Indeed, if \( \chi(M) \neq 0 \) the inequality

\[
|\chi(M)| = |\chi(TM)| \leq MW(M) \cdot |\chi(M)| < |\chi(M)|
\]

leads to a contradiction to the assumption that \( M \) has a flat structure.

One can use Theorem 1.1 to extend the family of manifolds satisfying the generalized Chern conjecture. For instance, we prove a stable variant of the generalized Chern conjecture:

**Corollary 1.4** For any manifold \( M \), there exists \( k_0 \geq 0 \) such that the product \( M \times \Sigma^k \), where \( \Sigma \) is a surface of genus \( \geq 2 \), satisfies the generalized Chern conjecture for any \( k \geq k_0 \). If \( \chi(M) = 0 \), then \( k_0 = 0 \). If \( \chi(M) \neq 0 \), then one can take any \( k_0 > \log_2(MW(M)) \). In particular, in the latter case, the product \( M \times \Sigma^k \) does not admit an affine structure.

**Remark** (1) One can replace \( \Sigma^k \) in Corollary 1.4 by any \( \mathcal{H}^k \)–manifold.

(2) The corollary is somehow dual to a question of Yves Benoist [1, Section 3, page 19] asking whether for every closed manifold \( M \) there exists \( m \) such that \( M \times (S^1)^m \) admits an affine structure. For example, if \( M \) is a hyperbolic manifold or a sphere, the product \( M \times S^1 \) admits an affine structure. On the other hand, if \( M \) admits a

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1In [7] Milnor suggested the generalized conjecture without the assumption that \( M \) is aspherical, however Smillie [9] gave counterexamples, in any even dimension \( \neq 2 \), when this assumption is omitted.
quaternionic hyperbolic structure then \( m = 1 \) will not suffice, since the holonomy representation of \( \pi_1(M) \) is superrigid in \( \text{Sp}(2, 1) \) by Corlette’s Theorem and the latter has no nontrivial 9–dimensional linear representations.

Note that since there are only finitely many isomorphism classes of oriented \( \mathbb{R}^n \)–bundles which admit a flat structure, it is immediate that the set

\[
\{ ||\chi(\xi)|| | \xi \text{ is a flat oriented } \mathbb{R}^n \text{–bundle over } M \}
\]

is finite for every \( M \). In particular, if \( \chi(M) \neq 0 \), there exists a finite Milnor–Wood constant \( MW(M) < +\infty \). However, in general, the Milnor–Wood constant can be infinite, since the implication

\[
\chi(M) = 0 \implies \chi(\xi) = 0,
\]

for a flat oriented \( \mathbb{R}^n \)–bundle \( \xi \), does not hold in general as we will show in Section 6. Our example is inspired by Smillie’s counterexample [9] of the generalized Chern conjecture for nonaspherical manifolds, and likewise this manifold is nonaspherical.

The following questions are quite natural:

1. Does there exist a finite constant \( c(n) \) depending on \( n \) only so that we have \( MW(M) \leq c(n) \) for every closed aspherical \( n \)–manifold?

2. Let \( X \) be a contractible Riemannian manifold such that there exists a closed \( X \)–manifold \( M \) with \( MW(M) < +\infty \). Is \( MW(X) \) necessarily finite?

3. Does \( \chi(M) = 0 \implies \chi(\xi) = 0 \) for flat oriented \( \mathbb{R}^n \)–bundles \( \xi \) over aspherical manifolds \( M \)?

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2 Proportionality principles and vanishing of the Euler class of tensor products

Lemma 2.1 Let \( X \) be a simply connected Riemannian manifold, \( G = \text{Isom}(M) \) and \( \rho: G \to \text{GL}_\mathbb{C}^n(\mathbb{R}) \) a representation. Then \( \chi(\xi_\rho)/\text{vol}(M) \), where \( M = \Gamma \backslash X \) is a closed \( X \)–manifold and \( \xi_\rho \) is the flat vector bundle induced on \( M \) by \( \rho \) restricted to \( \Gamma \), is a constant independent of \( M \).
There is a canonical isomorphism $H_c^*(G) \cong H^*(\Omega^*(X)^G)$ between the continuous cohomology of $G$ and the cohomology of the cocomplex of $G$–invariant differential forms $\Omega^*(X)^G$ on $X$ equipped with its standard differential. (For a semisimple Lie group $G$, every $G$–invariant form is closed, hence one further has $H_c^*(\Omega^*(X)^G) \cong \Omega^*(X)^G$.) In particular, in top dimension $n = \dim(X)$, the cohomology groups are 1–dimensional, $H_c^n(G) \cong H^n(\Omega^*(X)^G)) \cong \mathbb{R}$, and contain the cohomology class given by the volume form $\omega_X$.

Since the bundle $\xi_\rho$ over $M$ is induced by $\rho$, its Euler class $\varepsilon_n(\xi_\rho)$ is the image of $\varepsilon_n \in H_c^n(\GL^+(\mathbb{R}, n))$ under

$$H_c^n(\GL^+(\mathbb{R}, n)) \xrightarrow{\rho^*} H_c^n(G) \xrightarrow{\iota} H^n(\Gamma) \cong H^n(M),$$

where the middle map is induced by the inclusion $\Gamma \hookrightarrow G$. In particular,

$$\rho^*(\varepsilon_n) = \lambda \cdot [\omega_X] \in H_c^n(G)$$

for some $\lambda \in \mathbb{R}$ independent of $M$. It follows that $\chi(\xi_\rho)/\text{Vol}(M) = \lambda$. □

**Lemma 2.2** Let $\rho:\GL^+(n, \mathbb{R}) \times \GL^+(m, \mathbb{R}) \rightarrow \GL^+(nm, \mathbb{R})$ denote the tensor representation. If $n, m \geq 2$, then

$$\rho^*_\otimes(\varepsilon_{nm}) = 0 \in H_c^{nm}(\GL(n, \mathbb{R}) \times \GL(m, \mathbb{R})).$$

**Proof** The case $n = m = 2$ was proven in [3, Lemma 4.1], based on the simple observation that interchanging the two $\GL^+(2, \mathbb{R})$ factors does not change the sign of the top dimensional cohomology class in $H_c^4(\GL(2, \mathbb{R}) \times \GL(2, \mathbb{R})) \cong \mathbb{R}$, but it changes the orientation on the tensor product, and hence the sign of the Euler class in $H_c^4(\GL^+(4, \mathbb{R}))$.

Let us now suppose that at least one of $n, m$ is strictly greater than 2, or equivalently, that $n + m < nm$. The Euler class is in the image of the natural map

$$H^{nm}(\BGL(nm, \mathbb{R})) \rightarrow H_c^{nm}(\GL(n, \mathbb{R}) \times \GL(m, \mathbb{R})).$$

By naturality, we have a commutative diagram

$$\begin{array}{ccc}
H^{nm}(\BGL^+(nm, \mathbb{R})) & \xrightarrow{\rho^*_\otimes} & H_c^{nm}(\GL^+(nm, \mathbb{R})) \\
\downarrow \rho^*_\otimes & & \downarrow \rho^*_\otimes \\
H^{nm}(\B(\GL^+(n, \mathbb{R}) \times \GL^+(m, \mathbb{R}))) & \rightarrow & H_c^{nm}(\GL^+(n, \mathbb{R}) \times \GL^+(m, \mathbb{R}))).
\end{array}$$

Since the image of the lower horizontal arrow is contained in degree $\leq n + m$, it follows that $\rho^*_\otimes(\varepsilon_{nm}) = 0$. □
3 Representations of products

Lemma 3.1 Let \( H_1, H_2 \) be groups and \( \rho: H_1 \times H_2 \to \text{GL}_n(\mathbb{R}) \) a representation of the direct product and suppose that \( \rho(H_i) \) is nonamenable for both \( i = 1, 2 \). Then, up to replacing the \( H_i \) by finite index subgroups, either

- \( V = \mathbb{R}^n \) decomposes as an invariant direct sum \( V = V' \oplus V'' \), where the restriction \( \rho|V' = \rho'_1 \otimes \rho'_2 \) is a nontrivial tensor representation, or
- \( V = V_1 \oplus V_2 \), where \( \rho(H_i) \) is scalar on \( V_i \).

Proof This can be easily deduced from the proof of [3, Proposition 6.1]. \( \square \)

Proposition 3.2 Let \( H = \prod_{i=1}^k H_i \) be a direct product of groups and let \( \rho: H \to \text{GL}_n^+(\mathbb{R}) \) be an orientable representation, where \( n = \sum_{i=1}^k m_i \). Suppose that \( \rho(H_i) \) is nonamenable for every \( i \). Then, up to replacing the \( H_i \) by finite index subgroups \( H' = \prod_{i=1}^k H'_i \), either

1. there exists \( 1 \leq i_0 < k \) such that \( V = \mathbb{R}^n \) decomposes nontrivially to an invariant direct sum \( V = V' \oplus V'' \) and the restricted representation

\[
\rho|_{(H'_{i_0} \times \prod_{i > i_0} H'_i, V')} : H'_{i_0} \times \prod_{i > i_0} H'_i \to \text{GL}(V')
\]

is a nontrivial tensor, or

2. the representation \( \rho' \) factors through

\[
\rho' : \prod_{i=1}^k H'_i \to \left( \prod_{i=1}^k \text{GL}_{m'_i}(\mathbb{R}) \right)^+ \to \text{GL}_n^+(\mathbb{R}),
\]

where the latter homomorphism is, up to conjugation, the canonical diagonal embedding, and \( \rho'(H'_i) \) restricts to a scalar representation on each \( \text{GL}_{m'_i}(\mathbb{R}) \), for \( i \neq j \).

Moreover, if all \( m_i \) are even then either \( m'_i < m_i \) for some \( i \) or one can replace \( \text{GL} \) with \( \text{GL}_n^+ \) everywhere.

The notation \( (\prod_{i=1}^k \text{GL}_{m'_i}(\mathbb{R}))^+ \) stands for the intersection of \( \prod_{i=1}^k \text{GL}_{m'_i}(\mathbb{R}) \) with the positive-determinant matrices.
Proof We argue by induction on $k$. For $k = 2$ the alternative is immediate from Lemma 3.1. Suppose $k > 2$. If (1) does not hold, it follows from Lemma 3.1 that, up to replacing the $H_i$ by some finite index subgroups, $V$ decomposes invariantly to $V = V_1 \oplus V'_1$ where $\rho(H_1)$ is scalar on $V'_1$ and $\rho(\prod_{i>1} H_i)$ is scalar on $V_1$. We now apply the induction hypothesis for $\prod_{i>1} H_i$ restricted to $V'_1$.

Finally, in case (2), since $\sum m_i = n$, either $m'_i < m_i$ for some $i$ or equality holds everywhere. In the latter case, if all the $m_i$ are even, given $g \in H_i$, since the restriction of $\rho(g)$ to each $V_j \neq i$ is scalar, it has positive determinant. We deduce that also $\rho(g)|_{V_i}$ has positive determinant.

\[\square\]

4 Multiplicativity of the Milnor–Wood constant for product manifolds: A proof of Theorem 1.1

Let $M_1, M_2$ be two arbitrary manifolds. We prove that

$$MW(M_1 \times M_2) = MW(M_1) \cdot MW(M_2).$$

First note that the inequality $MW(M_1 \times M_2) \geq MW(M_1) \cdot MW(M_2)$ is trivial. Indeed, let $\xi_1, \xi_2$ be flat oriented bundles over $M_1$ and $M_2$, respectively, of the right dimension such that $|\chi(\xi_i)| = MW(M_i) \cdot |\chi(M_i)|$ for $i = 1, 2$. Then $\xi_1 \times \xi_2$ is a flat bundle over $M_1 \times M_2$ with

$$|\chi(\xi_1 \times \xi_2)| = |\chi(\xi_1)||\chi(\xi_2)| = MW(M_1) \cdot MW(M_2) \cdot |\chi(M_1 \times M_2)|.$$

For the other inequality, let $\xi$ be a flat oriented $\mathbb{R}^n$–bundle over $M_1 \times M_2$, where $n = \dim(M_1) + \dim(M_2)$. We need to show that

$$|\chi(\xi)| \leq MW(M_1) \cdot MW(M_2) \cdot |\chi(M_1 \times M_2)|.$$

Observe that if we replace $M$ by a finite cover, and the bundle $\xi$ by its pullback to the cover, then both sides of the previous inequality are multiplied by the degree of the covering.

The flat bundle $\xi$ is induced by a representation

$$\rho: \pi_1(M_1 \times M_2) \cong \pi_1(M_1) \times \pi_1(M_2) \to \text{GL}_n^+(\mathbb{R}).$$

If $\rho(\pi_1(M_i))$ is amenable for $i = 1$ or $2$, then $\rho^*(\epsilon_n) = 0$ [3, Lemma 4.3] and hence $\chi(\xi) = 0$ and there is nothing to prove. Thus, we can without loss of generality suppose that, upon replacing $\pi_1(M_1 \times M_2)$ by a finite index subgroup, the representation $\rho$ factors as in Proposition 3.2.
In case (1) of the proposition, we obtain that $\rho^*(\varepsilon_n) = 0$ by Lemma 2.2 and [3, Lemma 4.2]. In case (2) we get that $\rho$ factors through

$$\rho: \pi_1(M_1) \times \pi_1(M_2) \longrightarrow (\text{GL}_{m_1}'(\mathbb{R}) \times \text{GL}_{m_2}'(\mathbb{R}))^+ \longrightarrow \text{GL}_n^+(\mathbb{R}),$$

where the latter embedding $i$ is up to conjugation the canonical embedding. Furthermore, up to replacing $\rho$ by a representation in the same connected component of

$$\text{Rep}\left(\pi_1(M_1) \times \pi_1(M_2), (\text{GL}_{m_1}'(\mathbb{R}) \times \text{GL}_{m_2}'(\mathbb{R}))^+\right)$$

which will have no influence on the pullback of the Euler class, we can without loss of generality suppose that the scalar representations of $\pi_1(M_1)$ on $\text{GL}_{m_2}'$ and $\pi_1(M_2)$ on $\text{GL}_{m_1}'$ are trivial, so that $\rho$ is a product representation. If $m_1'$ or $m_2'$ is odd, then $i^*(\varepsilon_n) = 0 \in H^n_c((\text{GL}_{m_1}'(\mathbb{R}) \times \text{GL}_{m_2}'(\mathbb{R}))^+)$. If $m_1'$ and $m_2'$ are both even then Proposition 3.2 further tells us that either $m_i' < m_i$ for $i = 1$ or $2$, or the image of $\rho$ lies in $\text{GL}_{m_1}'(\mathbb{R}) \times \text{GL}_{m_2}'(\mathbb{R})$. In the first case, the Euler class vanishes [3, Lemma 4.2], while in the second case, we immediately obtain the desired inequality. This finishes the proof of Theorem 1.1.

\section{Multiplicativity of the universal Milnor–Wood constant for Hadamard manifolds: A proof of Theorem 1.2}

Theorem 1.2 can be reformulated as follows:

\textbf{Theorem 5.1} \textit{Let $X$ be a Hadamard manifold with de Rham decomposition $X = \prod_{i=1}^k X_i$, then $\widetilde{MW}(X) = \prod_{i=1}^k \widetilde{MW}(X_i)$.}

\textbf{Proof} The inequality “\geq” is obvious. Let $M = \Gamma \backslash X$ be a compact $X$–manifold. We must show that $MW(M) \leq \prod_{i=1}^k \widetilde{MW}(X_i)$. Note that $\Gamma$ is torsion-free. Let us also assume that $k \geq 2$. If $M$ is reducible one can argue by induction using Theorem 1.1. Thus we may assume that $M$ is irreducible. Observe that this implies that $\text{Isom}(X)$ is not discrete. If $\Gamma$ admits a nontrivial normal abelian subgroup then by the flat torus theorem (see [2, Chapter 7]), $X$ admits an Euclidean factor which implies the vanishing of the Euler class. Assuming that this is not the case we apply Farb–Weinberger [4, Theorem 1.3] to deduce that $X$ is a symmetric space of noncompact type. Thus, up to replacing $M$ by a finite cover (equivalently, replace $\Gamma$ by a finite index subgroup), we may assume that $\Gamma$ lies in

$$G = \text{Isom}(X) = \prod_{i=1}^k \text{Isom}(X_i) = \prod_{i=1}^k G_i,$$

where $G_i$ is the stabilizer of $X_i$. Hence

$$\text{Isom}(X)/\Gamma \cong \prod_{i=1}^k \text{Isom}(X_i)/\Gamma_i \cong \prod_{i=1}^k G_i,$$

and

$$\text{Isom}(X)/\Gamma \cong \prod_{i=1}^k \text{Isom}(X_i)/\Gamma_i \cong \prod_{i=1}^k G_i.$$
where the $G_i$ are adjoint simple Lie groups without compact factors and $\Gamma \leq G$ is irreducible in the sense that its projection to each factor is dense. Denote by $\widetilde{G}_i$ the universal cover of $G_i$, and by $\widetilde{\Gamma} \leq \prod_{i=1}^{k} \widetilde{G}_i$ the pullback of $\Gamma$.

Let $\rho: \Gamma \to \text{GL}^+_n(\mathbb{R})$ be a representation inducing a flat oriented vector bundle $\xi$ over $M$. Up to replacing $\Gamma$ by a finite index subgroup, we may suppose that $\rho(\Gamma)$ is Zariski connected. Let $S \leq \text{GL}^+_n(\mathbb{R})$ be the semisimple part of the Zariski closure of $\rho(\Gamma)$, and let $\rho': \Gamma \to S$ be the quotient representation. By superrigidity, the map $\text{Ad} \circ \rho': \Gamma \to \text{Ad}(S)$ extends to

$$\phi: \Gamma \leq \prod_{i=1}^{k} G_i \longrightarrow \text{Ad}(S)$$

(see [5], [6] and [8]). This map can be pulled back to $\widetilde{\phi}: \widetilde{\Gamma} \to S$. Recall also that $\prod_{i=1}^{k} \widetilde{G}_i$ is a central discrete extension of $\prod_{i=1}^{k} G_i$ and, likewise, $\widetilde{\Gamma}$ is a central extension of $\Gamma$. If

$$n_i = \dim X_i \quad \text{and} \quad n = \sum_{i=1}^{k} n_i$$

we deduce from Proposition 3.2 and Lemma 2.2 that either the Euler class vanishes or the image of $\widetilde{\phi}$ lies (up to decomposing the vector space $\mathbb{R}^n$ properly) in $(\prod_{i=1}^{k} \text{GL}_{n_i})^\perp$.

Suppose that $\overline{M}W(X_i)$ is finite for all $i = 1, \ldots, k$ and let $M_i$ be closed $X_i$–manifolds. Let $\xi'_i$ be the flat vector bundle on $\prod_{i=1}^{k} M_i$ coming from $\widetilde{\rho}$ reduced to $\prod_{i=1}^{k} M_i$, and let $\xi'_i$ be the vector bundle on $M_i$ induced by $\widetilde{\rho}_i$, $i = 1, \ldots, k$. By Lemma 2.1, we have

$$\frac{\chi(\xi)}{\text{vol}(M)} = \frac{\chi(\xi'_i)}{\text{vol}(\prod_{i=1}^{k} M_i)} \leq \prod_{i=1}^{k} \frac{\chi(\xi'_i)}{\text{vol}(M_i)} \leq \prod_{i=1}^{k} \overline{M}W(X_i),$$

which finishes the proof. \hfill \Box

6 Example: a flat bundle with nonzero Euler number over a manifold with zero Euler characteristic

Recall that given two closed manifolds of even dimension, the Euler characteristic of connected sums behaves as

$$\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2.$$  

The idea is to find $M = M_1 \# M_2$ such that $M_1$ admits a flat bundle with nontrivial Euler number which in turn induces such a bundle on the connected sum, and to choose
then $M_2$ in such a way that the Euler characteristic of the connected sum vanishes. Take thus

$$M_1 = \Sigma_2 \times \Sigma_2, \quad M_2 = (S^1 \times S^3) \# (S^1 \times S^3) \quad \text{and} \quad M = M_1 \# M_2.$$ 

These manifolds have the following Euler characteristics:

$$\chi(M_1) = 4, \quad \chi(M_2) = 2 \cdot \chi(S^1 \times S^3) - 2 = -2, \quad \chi(M) = 0.$$ 

Let $\eta$ be a flat bundle over $\Sigma_2$ with Euler number $\chi(\eta) = 1$. (Note that we know that such a bundle exists by [7].) Let $f: M \to M_1$ be a degree 1 map obtained by sending $M_2$ to a point, and consider

$$\xi = f^*(\eta \times \eta).$$ 

Obviously, since $\eta$ is flat, so is the product $\eta \times \eta$ and its pullback by $f$. Moreover, the Euler number of $\xi$ is

$$\chi(\xi) = \chi(\eta \times \eta) = 1.$$ 

Indeed, the Euler number of $\eta \times \eta$ is the index of a generic section of the bundle, which we can choose to be nonzero on $f(M_2)$, so that we can pull it back to a generic section of $\xi$ which will clearly have the same index as the initial section on $\eta \times \eta$.

\section*{References}


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