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Mod p decompositions of gauge groups

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We give mod p decompositions of homotopy types of the gauge groups of principal bundles over spheres, which are compatible with mod p decompositions of Lie groups given by Mimura, Nishida and Toda. As an application, we also give some computations on the homotopy types of gauge groups. In particular, we show the p–local converse of the result of Sutherland on the classifications of the gauge groups of principal $\mathrm{SU}(n)$–bundles.

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1 Introduction

Let G be a topological group and let P be a principal G–bundle over a space K. The gauge group of P, denoted by $\mathcal{G}(P)$, is the group of all automorphisms of P endowed with the compact-open topology, where automorphisms of P are G–equivariant self-maps of P covering the identity map of K. By definition, one naively expects that $\mathcal{G}(P)$ inherits properties of G, and this is known to be true in some cases. For example, it is shown in Kishimoto and Kono [13] that higher homotopy commutativity of G yields a splitting of a certain exact sequence including $\mathcal{G}(P)$ and G. Localize at the odd prime p. If G is a Lie group, G is homotopy equivalent to a product of smaller spaces, which is called a mod p decomposition of G. Then as above, one can expect there is also a mod p decomposition of $\mathcal{G}(P)$. There are two results supporting this expectation: one is due to the first and the second authors [11] when G admits a finite order automorphism, and the other is due to Theriault [21] when G is of low rank and K is the 4–sphere. In this paper, generalizing the above results, we construct a mod p decomposition of $\mathcal{G}(P)$ when K is a sphere of certain dimension. Note that $\mathcal{G}(P)$ is not connected in general, implying the p–localization of $\mathcal{G}(P)$ does not make sense. So we set the p–localization of $\mathcal{G}(P)$ as

$$\mathcal{G}(P)_{(p)} = \Omega(B\mathcal{G}(P)_{(p)}).$$

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To set notation, let us recall mod p decompositions of Lie groups. Let X be a (p–local) finite H–space. By a classical result of Hopf, there is a rational homotopy equivalence

$$X \simeq_{(0)} S^{2n_1+1} \times \cdots \times S^{2n_\ell+1}.$$

We denote the index set $\{n_1, \dots, n_\ell\}$ by $\mathfrak{t}(X)$ and call it the type of X. Put $\mathfrak{t}_i(X) = \{k \in \mathfrak{t}(X) \mid k \equiv i \mod (p-1)\}$. We list the types of compact, simply connected, simple Lie groups.

$\mathrm{SU}(n)$	$1, 2, \dots, n-1$	G_2	$1, 5$
$\mathrm{Sp}(n)$	$1, 3, \dots, 2n-1$	F_4	$1, 5, 7, 11$
$\mathrm{Spin}(2n)$	$1, 3, \dots, 2n-3, n-1$	E_6	$1, 4, 5, 7, 8, 11$
$\mathrm{Spin}(2n+1)$	$1, 3, \dots, 2n-1$	E_7	$1, 5, 7, 9, 11, 13, 17$
		E_8	$1, 7, 11, 13, 17, 19, 23, 29$

Table 1

We state a result of Mimura, Nishida and Toda [17] on mod p decompositions of Lie groups.

Theorem 1.1 (Mimura, Nishida and Toda [17]) *Let G be a compact, simply connected, simple Lie group such that $H_*(G;\mathbb{Z})$ is p–torsion free. Then there is a homotopy equivalence*

$$G_{(p)} \simeq B_1 \times \cdots \times B_{p-1},$$

where $\mathfrak{t}(B_i) = \mathfrak{t}_i(G)$ for $i = 1, \dots, p-1$.

The above theorem is proved by Mimura, Nishida and Toda [17] by an explicit construction of B_i, and later, a simple proof is given by Wilkerson [24] using the unstable Adams operations. We remark here that Harris [9] gives a mod p decomposition of a Lie group G when G admits a finite order automorphism, which is included in Theorem 1.1 as a special case. It is shown in [11] that these mod p decompositions of Harris induce those of gauge groups.

For convenience, we regard a collection of sets $\{S_1, \dots, S_{p-1}\}$ as also indexed by $\mathbb{Z}/(p-1)$. Namely, for $n \in \mathbb{Z}$, S_n means S_i, where $1 \le i \le p-1$ and $n \equiv i \mod (p-1)$.

Let us return to gauge groups. Let G be a compact, simply connected, simple Lie group. We restrict the base space of principal G–bundles to S^{2d+2} with $d \in \mathfrak{t}(G)$. In constructing mod p decompositions of gauge groups, we exclude $\mathrm{Spin}(2n)$–bundles for a technical reason. However, we have the following construction. Let P be a principal $\mathrm{Spin}(2n)$–bundle over S^{2d+2}. Suppose that P is classified by an element of $\mathrm{Im}\,\iota_* \subset \pi_{2d+2}(B\mathrm{Spin}(2n))$ for the inclusion $\iota\colon \mathrm{Spin}(2n-1) \to \mathrm{Spin}(2n)$. Then there is a principal $\mathrm{Spin}(2n-1)$–bundle Q over S^{2d+2} such that P is the bundle associated with Q by ι. Let E be an S^{2n-1}–bundle over S^{2d+2} that is associated with P by the canonical action of $\mathrm{Spin}(2n)$ on S^{2n-1}. Then as a special case of a result of [11], we have a p–local homotopy equivalence

$$\mathcal{G}(P) \simeq_{(p)} \mathcal{G}(Q) \times \Gamma(E) \tag{1-1}$$

for any odd prime p, where $\Gamma(E)$ is the set of sections of E. By the standard fiber sequence

$$\mathrm{Spin}(2n-1) \xrightarrow{\iota} \mathrm{Spin}(2n) \to S^{2n-1}$$

together with Table 1, we see that $\iota_*\colon \pi_{2d+2}(B\mathrm{Spin}(2n-1)) \to \pi_{2d+2}(B\mathrm{Spin}(2n))$ is an isomorphism on free parts except for $d \neq n-1$. Thus we may say that in most cases, mod p decompositions of gauge groups of $\mathrm{Spin}(2n)$–bundles over S^{2d+2} are deduced from those of $\mathrm{Spin}(2n-1)$–bundles over S^{2d+2}.

Our main result is the following.

Theorem 1.2 *Let G be a compact, simply connected, simple Lie group such that $H_*(G;\mathbb{Z})$ is p–torsion free and $G \neq \mathrm{Spin}(2n)$, and let P be a principal G–bundle over S^{2d+2} for $d \in \mathfrak{t}(G)$. Then there is a homotopy equivalence*

$$\mathcal{G}(P)_{(p)} \simeq \mathcal{B}_1^P \times \cdots \times \mathcal{B}_{p-1}^P$$

and a homotopy fiber sequence

$$\Omega(\Omega_0^{2d+1} B_i) \to \mathcal{B}_i^P \to B_{i-d-1}$$

for $i \in \mathbb{Z}/(p-1)$.

The organization of the paper is as follows. In Section 2, we first give a proof of Theorem 1.1 adapted to our situation, which is a slight modification of the proof due to Wilkerson [24]. After that, we prove Theorem 1.2. In Section 3, we prove that the homotopy fiber sequence in Theorem 1.2 is principal when G is of low rank. As a consequence, we see that mod p decompositions of gauge groups due to Theriault [21] are included in Theorem 1.2 as a special case. In Section 4, we give two applications of mod p decompositions of gauge groups. The first application is to count the homotopy types of gauge groups as in Kono [15], and Crabb and Sutherland [2]. As a consequence, we see that the converse of a result of Sutherland [18] is true in the low rank case. The second application is on the adjoint bundles of given principal bundles. We prove that not all mod p decompositions of $\mathcal{G}(P)$ in Theorem 1.2 are induced from those of the adjoint bundle of P.

2 Proof of Theorem 1.2

We first reproduce a proof of Theorem 1.1 which is a slight modification of a proof by Wilkerson [24] adjusted to our situation.

Proof of Theorem 1.1 By assumption, one has

$$H_*(G;\mathbb{Z}_{(p)}) = \Lambda(x_{2n_1+1},\dots,x_{2n_\ell+1}), \quad |x_i| = i,$$

where $\mathfrak{t}(G) = \{n_1,\dots,n_\ell\}$ and each x_i is in the image of transgression of the universal bundle $G \to EG \to BG$. In particular, each x_i is primitive. For an integer q prime to p, let $\phi^q\colon BG_{(p)} \to BG_{(p)}$ be the unstable Adams operation of degree q as in [24]. Then we have $(\phi^q)^*(y) = q^k y$ for $y \in H^{2k}(BG;\mathbb{Z}_{(p)})$, implying

$$(\Omega\phi^q)_*(x_{2n_i+1}) = q^{n_i+1}x_{2n_i+1} \tag{2-1}$$

for $i = 1,\dots,\ell$. Choose an integer u whose modulo p reduction is a primitive $(p-1)^{\mathrm{st}}$ root of unity in $\mathbb{Z}/p$. Let $r\colon X \to X$ denote the r–power map of an H–space X. For integers b and m, we define a map $f(m,b)\colon G_{(p)} \to G_{(p)}$ as

$$f(m,b) = u^b \circ \Omega\phi^u - u^{b+m+1},$$

where the subtraction is given by the standard multiplication of $G_{(p)}$. Then by (2-1) and primitivity of x_{2n_i+1}, one has

$$f(m,b)_*(x_{2n_i+1}) = (u^{b+n_i+1} - u^{b+m+1})x_{2n_i+1} \tag{2-2}$$

for $i = 1,\dots,\ell$.

Define the index set $\mathfrak{t}_i^b$ as

$$\mathfrak{t}_i^b = \bigcup_{k\neq i}(\mathfrak{t}_k(G) \cup \{j+b+1 \mid j \in \mathfrak{t}_{k-b-1}(G)\})$$

for $i \in \mathbb{Z}/(p-1)$. If $\mathfrak{t}_i^b = \{i_1,\dots,i_s\}$, we put

$$F(i,b) = f(i_1,b)\circ\cdots\circ f(i_s,b)\colon G_{(p)} \to G_{(p)}.$$

Let $B(i,b)$ be the homotopy colimit of the diagram

$$G_{(p)} \xrightarrow{F(i,b)} G_{(p)} \xrightarrow{F(i,b)} G_{(p)} \to \cdots.$$

Then it follows from (2-2) that

$$H_*(B(i,b);\mathbb{Z}/p) = \Lambda(x_{2k+1} \mid k \in \mathfrak{t}_i(G))$$

and the natural map $G_{(p)} \to B(i,b)$ induces the projection in the mod p homology. Thus the natural map $\epsilon\colon G_{(p)} \to \prod_{i=1}^{p-1} B(i,b)$ induces an isomorphism on mod p homology.

Since the Hurewicz homomorphism induces an isomorphism from $\pi_*(G)\otimes\mathbb{Q}$ to the module of indecomposables of $H_*(G;\mathbb{Q})$, we see from (2-2) that

$$F(i,b)\colon \pi_{2k+1}(G_{(p)})/\text{torsion} \to \pi_{2k+1}(G_{(p)})/\text{torsion}$$

is an isomorphism for $k \notin \mathfrak{t}_i(G)$ and the zero map for $k \in \mathfrak{t}_i(G)$, implying that the map ϵ is a rational homotopy equivalence. Then since ϵ induces an isomorphism on mod p homology and is a rational homotopy equivalence between simply connected spaces, it is a homotopy equivalence. Thus the proof of Theorem 1.1 is completed by putting $B_i = B(i,b)$. □

Remark 2.1 Wilkerson's proof [24] of Theorem 1.1 is the case $b = 0$ in the above proof.

Remark 2.2 The homotopy type of $B(i,b)$ in the above proof does not depend on b. Indeed, the composite

$$B(i,b) \xrightarrow{\text{incl}} \prod_{i=1}^{p-1} B(i,b) \xrightarrow{\simeq} G_{(p)} \to B(i,0)$$

induces an isomorphism on mod p homology and then a homotopy equivalence, where the last arrow is the canonical map.

We next prove Theorem 1.2. The idea of the proof of Theorem 1.2 is to apply the construction in the above proof of Theorem 1.1 to fiber sequences including gauge groups. Then we need to know a relation of fiber sequences and homotopy colimits.

Lemma 2.3 (Farjoun [3]) *Suppose there is a commutative diagram*

$$\begin{tikzcd}
F_0 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & \cdots \\
E_0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \\
B_0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & \cdots
\end{tikzcd}$$

in which columns are homotopy fiber sequences. Then

$$\text{hocolim}\, F_n \to \text{hocolim}\, E_n \to \text{hocolim}\, B_n$$

is also a homotopy fiber sequence.

We describe the induced maps of the unstable Adams operations in homotopy groups.

Proposition 2.4 *Let G be a compact, simply connected, simple Lie group such that $H_*(G;\mathbb{Z})$ is p–torsion free and $G \neq \mathrm{Spin}(2n)$. Then for $d \in \mathfrak{t}(G)$, $\pi_{2d+1}(G_{(p)}) \cong \mathbb{Z}_{(p)}$.*

Proof In the case that G is $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$, we obtain the result by considering the inclusions into $\mathrm{SU}(\infty)$ and $\mathrm{Sp}(\infty)$, respectively. Since there is a homotopy equivalence $\mathrm{Spin}(2n+1) \simeq_{(p)} \mathrm{Sp}(n)$ for any odd prime p as in Friedlander [4], we have already proved the case of $\mathrm{Spin}(2n+1)$. For exceptional Lie groups, we can verify the claim using Theorem 1.1 together with a calculation of homotopy groups of B_i in [17, Proposition 6.3 and 6.6]. □

Corollary 2.5 *Let G be as in Proposition 2.4, and let q be an integer prime to p. For $d \in \mathfrak{t}(G)$, the induced map $(\Omega\phi^q)_*\colon \pi_{2d+1}(G_{(p)}) \to \pi_{2d+1}(G_{(p)})$ is the q^{d+1}–power map.*

Proof By Proposition 2.4, we have that the natural map

$$\pi_{2d+1}(G_{(p)}) \cong \pi_{2d+1}(G) \otimes \mathbb{Z}_{(p)} \to \pi_{2d+1}(G) \otimes \mathbb{Q}$$

is injective. Since the Hurewicz homomorphism induces an isomorphism of

$$\pi_{2d+1}(G) \otimes \mathbb{Q}$$

to the module of indecomposables of $H_{2d+1}(G;\mathbb{Q})$ as is mentioned in the proof of Theorem 1.1, the result follows from (2-1) and naturality of the Hurewicz homomorphism. □

Proof of Theorem 1.2 Let $\alpha\colon S^{2d+2} \to BG$ be the classifying map of P, and let $\mathrm{map}(X,Y;f)$ be the component of the space of maps from X to Y including f. Recall from Atiyah and Bott [1] that there is a natural homotopy equivalence

$$B\mathcal{G}(P) \simeq \mathrm{map}(S^{2d+2}, BG;\alpha). \tag{2-3}$$

Let u be an integer whose mod p reduction of a primitive $(p-1)^{\mathrm{st}}$ root of unity in $\mathbb{Z}/p$, and let $\underline{r}\colon S^{2d+2}_{(p)} \to S^{2d+2}_{(p)}$ denote a map of degree r. By Corollary 2.5, we can define a map

$$g_i\colon \Omega\mathrm{map}(S^{2d+2}_{(p)}, BG_{(p)};\alpha_{(p)}) \to \Omega\mathrm{map}(S^{2d+2}_{(p)}, BG_{(p)};\alpha_{(p)})$$

as

$$g_i = \Omega((\underline{u^{-d-1}})^* \circ \phi^u_*) - u^{i-d},$$

where $\phi^u\colon BG_{(p)} \to BG_{(p)}$ is the unstable Adams operation of degree u. Then there is a commutative diagram of homotopy fiber sequences:

$$\begin{tikzcd}
\Omega(\Omega_0^{2d+1}G_{(p)}) \arrow[r, "\Omega^{2d+2}f(i,-d-1)"] \arrow[d] & \Omega(\Omega_0^{2d+1}G_{(p)}) \arrow[d] \\
\Omega\mathrm{map}(S^{2d+2}_{(p)}, BG_{(p)}; \alpha_{(p)}) \arrow[r, "g_i"] \arrow[d] & \Omega\mathrm{map}(S^{2d+2}_{(p)}, BG_{(p)}; \alpha_{(p)}) \arrow[d] \\
G_{(p)} \arrow[r, "f(i-d-1,0)"] & G_{(p)}
\end{tikzcd} \tag{2-4}$$

Let $\mathfrak{t}_i^{-d-1}$ be as in the proof of Theorem 1.1. If $\mathfrak{t}_i^{-d-1} = \{i_1, \dots, i_s\}$, we define a map

$$G_i = g_{i_1} \circ \cdots \circ g_{i_s}\colon \Omega\mathrm{map}(S^{2d+2}_{(p)}, BG_{(p)}; \alpha_{(p)}) \to \Omega\mathrm{map}(S^{2d+2}_{(p)}, BG_{(p)}; \alpha_{(p)}).$$

Then (2-4) yields a commutative diagram:

$$\begin{tikzcd}
\Omega(\Omega_0^{2d+1}G_{(p)}) \arrow[r, "\Omega^{2d+2}F(i,-d-1)"] \arrow[d] & \Omega(\Omega_0^{2d+1}G_{(p)}) \arrow[d] \\
\Omega\mathrm{map}(S^{2d+2}_{(p)}, BG_{(p)}; \alpha_{(p)}) \arrow[r, "G_i"] \arrow[d] & \Omega\mathrm{map}(S^{2d+2}_{(p)}, BG_{(p)}; \alpha_{(p)}) \arrow[d] \\
G_{(p)} \arrow[r, "F(i-d-1,0)"] & G_{(p)}
\end{tikzcd} \tag{2-5}$$

For a self-map $h\colon X \to X$, we denote the homotopy colimit of a diagram

$$X \xrightarrow{h} X \xrightarrow{h} X \to \cdots$$

by $\mathrm{hocolim}\, h$. If $X = \Omega Y$ and $h = \Omega \bar{h}$ for some $\bar{h}\colon Y \to Y$, Lemma 2.3 implies a natural homotopy equivalence $\mathrm{hocolim}\, h \simeq \Omega(\mathrm{hocolim}\, \bar{h})$. Then we obtain

$$\mathrm{hocolim}\, \Omega^{2d+2}F(i,-d-1) \simeq \Omega(\Omega_0^{2d+1}(\mathrm{hocolim}\ F(i,-d-1))) \simeq \Omega(\Omega_0^{2d+1}B_i).$$

Put $\mathcal{B}_i^P = \mathrm{hocolim}\, G_i$. Then it follows from Lemma 2.3 and (2-5) that there is a homotopy fiber sequence

$$\Omega(\Omega_0^{2d+1}B_i) \to \mathcal{B}_i^P \to B_{i-d-1}$$

satisfying a homotopy commutative diagram of homotopy fiber sequences

$$\begin{tikzcd}
\Omega(\Omega_0^{2d+1}G_{(p)}) \arrow[r] \arrow[d] & \Omega\mathrm{map}(S^{2d+2}_{(p)}, BG_{(p)}; \alpha_{(p)}) \arrow[r] \arrow[d] & G_{(p)} \arrow[d] \\
\Omega(\Omega_0^{2d+1}B_i) \arrow[r] & \mathcal{B}_i^P \arrow[r] & B_{i-d-1}
\end{tikzcd}$$

in which vertical arrows are natural maps. Hence we get a homotopy commutative diagram of homotopy fiber sequences

$$\begin{tikzcd}
\Omega(\Omega_0^{2d+1}G_{(p)}) \arrow[r] \arrow[d] & \Omega\mathrm{map}(S^{2d+2}_{(p)}, BG_{(p)}; \alpha_{(p)}) \arrow[r] \arrow[d] & G_{(p)} \arrow[d] \\
\prod_{i=1}^{p-1}\Omega(\Omega_0^{2d+1}B_i) \arrow[r] & \prod_{i=1}^{p-1}\mathcal{B}_i^P \arrow[r] & \prod_{i=1}^{p-1}B_{i-d-1}
\end{tikzcd}$$

in which the right and the left vertical arrows are homotopy equivalences by Theorem 1.1. This implies that the middle vertical arrow is also a homotopy equivalence, and thus the proof is completed by a natural homotopy equivalence

$$\mathrm{map}(S^{2d+2}, BG; \alpha)_{(p)} \simeq \mathrm{map}(S^{2d+2}_{(p)}, BG_{(p)}; \alpha_{(p)})$$

and (2-3). □

3 The case of low rank Lie groups

In this section, we prove that the homotopy fiber sequence $\Omega(\Omega_0^{2d+1}B_i) \to \mathcal{B}_i^P \to B_{i-d-1}$ is principal for the low rank Lie groups in Table 2.

$\mathrm{SU}(n)$	$(p-1)(p-2) \geq n-1$
$\mathrm{Sp}(n), \mathrm{Spin}(2n+1)$	$(p-1)(p-2) \geq 2n-1$
G_2, F_4, E_6	$p \geq 5$
E_7, E_8	$p \geq 7$

Table 2

As a consequence, we see that the following result of Theriault [21] is included in Theorem 1.2. We set some notation. For a base point preserving map $f\colon S^n \to X$, we denote by δ^f the connecting map $\Omega X \to \Omega_0^n X$ of the evaluation fiber sequence $\Omega_0^n X \to \mathrm{map}(S^n, X; f) \to X$. Hereafter, if a Lie group G satisfies the conditions in Theorem 1.1, we identify $G_{(p)}$ with $\prod_{i=1}^{p-1} B_i$ by the homotopy equivalence of

Theorem 1.1. For $i = n, n+1$, it is known that $\pi_{2i}(SU(n))$ is cyclic, and then there is only one u_i such that $1 \le u_i \le p-1$ and $\pi_{2i}(B_{u_i}) \ne 0$. Similarly for $\mathrm{Sp}(n)$, it is also known that $\pi_{4n+2}(\mathrm{Sp}(n))$ is cyclic, implying there is only one v such that $1 \le v \le p-1$ and $\pi_{4n+2}(B_v) \ne 0$.

When the type of a (p–local) H–space is $\{n_1, \dots, n_\ell\}$ with $n_1 \le \dots \le n_\ell$, we put $\ell(X) = n_\ell$.

Theorem 3.1 (Theriault [21]) (1) *Let G be a compact, connected, simple Lie group, and let P be a principal G–bundle over S^4. If $\ell(G)+2 < p$, there is a p–local homotopy equivalence*

$$\mathcal{G}(P) \simeq_{(p)} \prod_{i \in \mathrm{t}(G)} (S^{2i+1} \times \Omega_0^4 S^{2i+1}).$$

(2) *Let P be a principal $\mathrm{SU}(n)$–bundle over S^4 classified by $\alpha \in \pi_4(B\mathrm{SU}(n))$, and for $i = n, n+1$, let Z_i^P be the homotopy fibers of the composite*

$$B_{i-2} \xrightarrow{\mathrm{incl}} \mathrm{SU}(n)_{(p)} \xrightarrow{\delta^\alpha_{(p)}} \Omega_0^3 \mathrm{SU}(n)_{(p)} \xrightarrow{\mathrm{proj}} \Omega_0^3 B_{u_i},$$

where u_i is as above. Then there is a homotopy equivalence

$$\mathcal{G}(P)_{(p)} \simeq Z_n^P \times Z_{n+1}^P \times \prod_{\substack{i \not\equiv n, n+1 \\ \mathrm{mod}\ (p-1)}} B_i \times \prod_{\substack{i \not\equiv u_n, u_{n+1} \\ \mathrm{mod}\ (p-1)}} \Omega_0^4 B_i.$$

(3) *Let P be a principal $\mathrm{Sp}(n)$–bundle over S^4 classified by $\alpha \in \pi_4(B\mathrm{Sp}(n))$, and let W^P be the homotopy fiber of the composite*

$$B_{2n-1} \xrightarrow{\mathrm{incl}} \mathrm{Sp}(n)_{(p)} \xrightarrow{\delta^\alpha_{(p)}} \Omega_0^3 \mathrm{Sp}(n)_{(p)} \xrightarrow{\mathrm{proj}} \Omega_0^3 B_v,$$

where v is as above. Then there is a homotopy equivalence

$$\mathcal{G}(P)_{(p)} \simeq W^P \times \prod_{\substack{i \not\equiv 2n-1 \\ \mathrm{mod}\ (p-1)}} B_i \times \prod_{\substack{i \not\equiv v \\ \mathrm{mod}\ (p-1)}} \Omega_0^4 B_i.$$

Remark 3.2 Since $B\mathrm{Spin}(2n+1) \simeq_{(p)} B\mathrm{Sp}(n)$ for any odd prime p, (3) implies a result for $\mathrm{Spin}(2n+1)$. Theriault [21] also gave a partial result for $\mathrm{Spin}(2n)$. We get a complete result by the construction in Section 1.

Remark 3.3 The above definition of Z_i^P and W^P is equivalent to that of X_k, Y_k in [21] by [19, Lemma 5.1]. As we see below, the definition of Z_i^P and W^P is better suited to our situation.

Recall from [19] the universality of homotopy associative and homotopy commutative H–spaces. Let X be a homotopy associative and homotopy commutative H–space, and let A be a subspace of X. X is called universal for A if for any homotopy associative and homotopy commutative H–space Y and a map $f\colon A \to Y$, there is a unique H–map $\overline{f}\colon X \to Y$ extending f, up to homotopy. By an elementary observation, one gets the following.

Lemma 3.4 *For $i = 1, 2$, let X_i be a homotopy associative and homotopy commutative H–space and let A_i be a subspace of X_i for which X_i is universal. Then $X_1 \times X_2$ is universal for $A_1 \vee A_2$.*

We use the following result of Theriault [19].

Theorem 3.5 (Theriault [19]) *Let (G, p) be as in Table 2. Then for $i \in \mathbb{Z}/(p-1)$, there are a homotopy associative and homotopy commutative H–structures μ_i on B_i and a subspace A_i of B_i satisfying the following.*

(1) *The inclusion $A_i \to B_i$ induces an isomorphism*

$$\Lambda(\overline{H}_*(A_i;\mathbb{Z}/p)) \xrightarrow{\cong} H_*(B_i;\mathbb{Z}/p).$$

(2) *B_i is universal for A_i.*

We can generalize [21, Theorem 1.1] simply by replacing 4 with $2d+2$; here we omit the proof.

Lemma 3.6 (cf [21, Theorem 1.1]) *Let (G, p) be as in Table 2 and let μ_i be as in Theorem 3.5. For $\alpha \in \pi_{2d+2}(BG)$, the connecting map $\delta^\alpha_{(p)}\colon G_{(p)} \to \Omega_0^{2d+1} G_{(p)}$ is an H–map with respect to the H–structure $\mu_1 \times \cdots \times \mu_{p-1}$ on $G_{(p)}$ and the loop multiplication on $\Omega_0^{2d+1} G_{(p)}$.*

Remark 3.7 The H–structure $\mu_1 \times \cdots \times \mu_{p-1}$ on $G_{(p)}$ is homotopy commutative. Then by a result of McGibbon [16], it is not equivalent to the standard H–structure on $G_{(p)}$ in general.

Let us decompose the connecting map δ^α as a product of certain maps.

Proposition 3.8 *Let G be a compact, simply connected, simple Lie group such that $G \neq \mathrm{Spin}(2n)$ and $H_*(G;\mathbb{Z})$ is p–torsion free. Then for $0 \le k \le i + p(p-1)$, $\pi_{2k}(B_i) = 0$ unless $k \equiv i \mod (p-1)$ with $k > \ell(G)$.*

Proof By Table 2 and Theorem 3.5, we have

$$H_*(A_i;\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}\langle x_{2i+1+2m(p-1)},\dots,x_{2i+1+2(m+n)(p-1)}\rangle,\quad |x_j| = j \tag{3-1}$$

for some $m,n \geq 0$. In [17], B_i is shown to be spherically resolvable. Namely, there is a fiber sequence

$$B(j) \to B(j+1) \to S^{2i+1+2(m+j)(p-1)}$$

for $j = 0,\dots,n$ such that $B(0) = S^{2i+1+2m(p-1)}$ and $B(n+1) = B_i$. Recall from Toda [22] that $\pi_{2\ell+2k}(S^{2\ell+1}) = 0$ for $\ell > 0$, $0 \leq k \leq 2p(p-1)+3$ and $k \not\equiv 0 \bmod (p-1)$. Thus by an inductive calculation of the homotopy exact sequence of the above fiber sequence, we obtain the result for $k > \ell(G)$. The remaining case is proved quite similarly to Proposition 2.4. □

Corollary 3.9 *Let (G,p) be as in Table 2. Then for $\alpha \in \pi_{2d+2}(BG)$ with $d+\ell(G)+1 \leq p(p-1)$, the restriction of $\delta^\alpha_{(p)}$ to A_i can be compressed into*

$$\Omega_0^{2d+1}B_{i-d-1} \subset \Omega_0^{2d+1}G.$$

Proof By Proposition 3.8 and (3-1), we see that the homotopy set $[A_{i-d-1},\Omega_0^{2d+1}B_j]$ is trivial for $j \not\equiv i \bmod (p-1)$ if $d+\ell(G)+1 \leq p(p-1)$, implying the result. □

Remark 3.10 By Tables 1 and 2, one can easily check that the inequality $d+\ell(G)+1 \leq p(p-1)$ always holds for $d = 1$.

For $\alpha \in \pi_{2d+2}(BG)$ and $i \in \mathbb{Z}/(p-1)$, let δ^α_i denote the composite

$$B_{i-d-1} \xrightarrow{\text{incl}} G_{(p)} \xrightarrow{\delta^\alpha_{(p)}} \Omega_0^{2d+1}G_{(p)} \xrightarrow{\text{proj}} \Omega_0^{2d+1}B_i.$$

Lemma 3.11 *Let (G,p) be as in Table 2 and let $\alpha \in \pi_{2d+2}(BG)$. If $d+\ell(G)+1 \leq p(p-1)$, it holds that $\delta^\alpha_{(p)} \simeq \delta^\alpha_1 \times \cdots \times \delta^\alpha_{p-1}$.*

Proof Consider the H–structure $\mu_1 \times \cdots \times \mu_{p-1}$ on $G_{(p)}$ as in Lemma 3.6. Then by Lemma 3.6, $\delta^\alpha_{(p)}$ and δ^α_i are H–maps. By Corollary 3.9, $\delta^\alpha_{(p)}|_{A_1\vee\cdots\vee A_{p-1}}$ and $\delta^\alpha_1|_{A_1} \vee \cdots \vee \delta^\alpha_{p-1}|_{A_{p-1}}$ are homotopic, and thus for Lemma 3.4 and Theorem 3.5, the proof is completed. □

We now state the main result of this section.

Theorem 3.12 *Let (G,p) be as in Table 2 and let P be a principal G–bundle over S^{2d+2} for $d \in \mathfrak{t}(G)$ classified by $\alpha \in \pi_{2d+2}(BG)$. If $d+\ell(G)+1 \le p(p-1)$, the homotopy fiber sequence*

$$\Omega(\Omega_0^{2d+1} B_i) \to \mathcal{B}_i^P \to B_{i-d-1}$$

is principal and is classified by δ_i^α for $i \in \mathbb{Z}/(p-1)$.

Proof Consider the diagram

$$\begin{tikzcd}
A_{i-d-a} \arrow[r, "\mathrm{incl}"] \arrow[d, equal] & B_{i-d-1} \arrow[r, "\delta_i^\alpha"] \arrow[d, "\mathrm{incl}"] & \Omega_0^{2d+1} B_i \arrow[d, "\mathrm{incl}"] \\
A_{i-d-a} \arrow[r, "\mathrm{incl}"] & G_{(p)} \arrow[r, "\delta_{(p)}^\alpha"] & \Omega_0^{2d+1} G_{(p)}.
\end{tikzcd}$$

By Corollary 3.9, the outer rectangle is homotopy commutative, and then so is also the right square by Theorem 3.5 and Lemma 3.6. Let F_i^α be the homotopy fiber of δ_i^α. Since $\Omega\mathrm{map}(S^{2d+2}, BG)_{(p)}$ is the homotopy fiber of $\delta_{(p)}^\alpha$, the above observation shows that there exists a map $\theta\colon F_i^\alpha \to \Omega\mathrm{map}(S^{2d+2}, BG)_{(p)}$ satisfying the following homotopy commutative diagram.

$$\begin{tikzcd}
\Omega(\Omega_0^{2d+1} B_i) \arrow[r] \arrow[d] \arrow[dd, bend right, equal] & F_\alpha(i) \arrow[r] \arrow[d, "\theta"] & B_{i-d-1} \arrow[d] \arrow[dd, bend left, equal] \\
\Omega(\Omega_0^{2d+1} G_{(p)}) \arrow[r] \arrow[d] & \Omega\mathrm{map}(S^{2d+2}, BG;\alpha)_{(p)} \arrow[r] \arrow[d] & G_{(p)} \arrow[d] \\
\Omega(\Omega_0^{2d+1} B_i) \arrow[r] & \mathcal{B}_i^P \arrow[r] & B_{i-d-1}
\end{tikzcd}$$

Thus the top and the bottom homotopy fiber sequences are equivalent, completing the proof. □

Let (G,p) be as in Table 2 and choose $d \in \mathfrak{t}(G)$. Define the index set $I_p(G,d)$ as

$$I_p(G,d) = \{1 \le i \le p-1 \mid [A_{i-d-1}, \Omega_0^{2d+1} B_i] = 0\}.$$

We give a rough description of the homotopy types of p–localized gauge groups using $I_p(G,d)$.

Corollary 3.13 *Let (G,p) be as in Table 2 and let P be a principal G–bundle over S^{2d+2} with $d \in \mathfrak{t}(G)$. If $d+\ell(G)+1 \le p(p-1)$, there is a homotopy equivalence:*

$$\mathcal{G}(P)_{(p)} \simeq \prod_{i \notin I_p(G,d)} \mathcal{B}_i \times \prod_{i \in I_p(G,d)} (B_{i-d-1} \times \Omega(\Omega_0^{2d+1} B_i))$$

Proof Let $\alpha \in \pi_{2d+2}(BG)$ be the classifying map of P. If $i \in I_p(G,d)$, the restriction of δ_i^α to A_{i-d-1} is trivial, implying triviality of δ_i^α by Theorem 3.5 and Lemma 3.6. Thus the proof is completed by Theorems 1.2 and 3.12. □

As in the proof of Corollary 3.9, we can calculate $I_p(G,d)$ in some cases.

Proposition 3.14 *For (G,p) in Table 2, we have the following.*

(1) *If $\ell(G)+d+1 < p$ or (G,p,d) is as in the following table, $I_p(G,d) = \{1,\dots,p-1\}$.*

G	(d,p)
G_2	$(1,5),(5,5)$
F_4	$(1,5),(1,7),(1,11),(5,5),(7,7)$
E_6	$(1,11),(7,17),(8,19)$
E_7	$(1,17),(7,23),(13,29)$
E_8	$(1,17),(1,23),(1,29),(7,23),(7,29),(11,37),$ $(13,29),(13,41),(17,43),(19,41),(19,47)$

(2) *If $d+1<p$, $I_p(\mathrm{SU}(n),d)$ includes $1 \le i \le p-1$ satisfying*

$$i \not\equiv n, n+1, \dots, n+d \mod (p-1).$$

(3) *If $d+1<p$, $I_p(\mathrm{Sp}(n),d)$ includes $1 \le 2i-1 \le p-1$ satisfying*

$$i \not\equiv n, n+1, \dots, n+\left\lfloor \frac{d}{2} \right\rfloor \mod \left(\frac{p-1}{2}\right).$$

Proof (1) If $\ell(G)+d+1<p$, B_i is $S^{2i+1}_{(p)}$ or a point. Then the first assertion follows from Toda's calculation of the homotopy groups of spheres [22] as mentioned above. The second assertion can be verified by a calculation of the homotopy groups of B_i in [17].

(2) If $\ell(B_{i-d-1})+d+1<n$, we see from Proposition 3.8 that i belongs to $I_p(\mathrm{SU}(n),d)$. (3) is quite analogous to (2). □

Proof of Theorem 3.1 (1) follows from Corollary 3.13 and Proposition 3.14(1). Quite similarly, for (2) and (3), we only need to show $u_i \equiv i+2 \mod (p-1)$ for $i=n,n+1$ and $v \equiv 2n+1 \mod (p-1)$. Indeed, by Proposition 2.4, we have the desired congruences. □

4 Applications

In this section, we give two applications of the mod p decompositions of gauge groups. One is to count the homotopy types and the other is on a relation of gauge groups with adjoint bundles.

4.1 Counting homotopy types

It is of interest to count the number of the homotopy types of $\mathcal{G}(P)$ when P ranges over all principal G–bundles over K, where G and K are fixed. This problem was first studied by the second author in [15] for $G = \mathrm{SU}(2)$ and $K = S^4$. Later, many results were obtained concerning this problem. See Hamanaka and Kono [7], Hamanaka, Kaji and Kono [6], Crabb and Sutherland [2], Kamiyama, Kishimoto, Kono and Tsukuda [10], Theriault [20], and Sutherland [18] for example. Here, we determine the homotopy types of $\mathcal{G}(P)_{(p)}$ when P ranges over all principal $\mathrm{SU}(n)$–bundles over S^4 and $n-1 \le (p-1)(p-2)$. As a consequence, we see that the converse of a result of Sutherland [18] holds at the above prime p. We notice here that one can easily generalize the following calculation to $\mathrm{SU}(n)$–bundles over S^{2d+2} for $d \le p-2$.

We first recall the following description of the connecting map δ^α.

Proposition 4.1 (Whitehead [23]) *For $\alpha \in \pi_d(X)$, the connecting map $\delta^\alpha\colon \Omega X \to \Omega_0^d X$ corresponds to the Samelson product of $\langle \bar\alpha, 1_{\Omega X}\rangle$ in ΩX through the adjoint congruence*

$$[\Omega X, \Omega_0^d X] \cong [S^{d-1} \wedge \Omega X, \Omega X],$$

where $\bar\alpha\colon S^{d-1} \to \Omega X$ is the adjoint of α.

Fix an isomorphism $\pi_4(B\mathrm{SU}(n)) \cong \mathbb{Z}$. For $n-1 \le (p-1)(p-2)$, let us determine the order of δ_i^k for $k \in \mathbb{Z} \cong \pi_4(B\mathrm{SU}(n))$. Let $\lambda_i\colon B_i \to \mathrm{SU}(n)_{(p)}$ and $\bar\lambda_i\colon A_i \to \mathrm{SU}(n)_{(p)}$ denote the inclusions. Since $(\lambda_i)_*\colon [S^3 \wedge B_{i-2}, B_i] \to [S^3 \wedge B_{i-2}, \mathrm{SU}(n)_{(p)}]$ is injective, it follows from Proposition 4.1 that the desired order is equal to that of the Samelson product $\langle k, \lambda_{i-2}\rangle$ in the group of the homotopy set $[S^3 \wedge B_{i-2}, \mathrm{SU}(n)_{(p)}]$, where we denote the adjoint $S^3 \to \mathrm{SU}(n)$ of k also by k. Let $\iota\colon \mathrm{SU}(n) \to \mathrm{U}(n)$ be the inclusion. Then since $S^3 \wedge B_{i-2}$ is simply connected, the order of the Samelson product $\langle k, \lambda_{i-2}\rangle$ is equal to that of $\langle \iota\circ k, \iota\circ\lambda_{i-2}\rangle$. Note that the group structures of $[S^3 \wedge B_{i-2}, \mathrm{U}(n)_{(p)}]$ given by a suspension parameter of the source space and the multiplication of $\mathrm{U}(n)_{(p)}$ are the same. Here, we choose the latter one. In particular, by $\iota\circ k = \underline{k}\circ\iota\circ 1$ and linearity of Samelson products, we have

$$\langle \iota\circ k, \iota\circ\lambda_{i-2}\rangle = \underline{k}\circ\langle \iota\circ 1, \iota\circ\lambda_{i-2}\rangle, \tag{4-1}$$

where $\underline{k}\colon \mathrm{U}(n)_{(p)} \to \mathrm{U}(n)_{(p)}$ is the k–power map as above. So we determine the order of $\langle \iota\circ 1, \iota\circ\lambda_{i-2}\rangle$ in the group $[S^3\wedge B_{i-2}, \mathrm{U}(n)_{(p)}]$. As in Kishimoto and Nagao [14], the order of $\langle \iota\circ 1, \iota\circ\lambda_{i-2}\rangle$ is equal to that of $\langle \iota\circ 1, \iota\circ\bar{\lambda}_{i-2}\rangle$. Obviously, the projection $\pi\colon S^3\times A_{i-2}\to S^3\wedge A_{i-2}$ induces an injection of groups $\pi^*\colon [S^3\wedge A_{i-2}, \mathrm{U}(n)_{(p)}]\to [S^3\times A_{i-2}, \mathrm{U}(n)_{(p)}]$. Moreover, $\pi^*(\langle \iota\circ 1\circ\pi_1, \iota\circ\bar{\lambda}_{i-2}\rangle)$ is the commutator of $\iota\circ 1\circ\pi_1$ and $\iota\circ\bar{\lambda}_{i-2}\circ\pi_2$ in the group $[S^3\times A_{i-2}, \mathrm{U}(n)_{(p)}]$, where π_i is the i^{th} projection. Thus we determine the order of the commutator $[\iota\circ 1\circ\pi_1, \iota\circ\bar{\lambda}_{i-2}\circ\pi_2]$ in the group $[S^3\times A_{i-2}, \mathrm{U}(n)_{(p)}]$. To this end, we apply the following result of Hamanaka [5]. Let $x_{2i-1}\in H^{2i-1}(\mathrm{U}(n);\mathbb{Z})$ be the suspension of the i^{th} universal Chern class $c_i\in H^{2i}(B\mathrm{U}(n);\mathbb{Z})$.

Theorem 4.2 (Hamanaka [5]) *Let X be a CW–complex of dimension $\le 2n+2p-4$. For $f,g\colon X\to\mathrm{U}(n)$, we put*

$$\gamma_k = \sum_{\substack{i+j+1=k\\ 0\le i,j\le n-1}} f^*(x_{2i+1})g^*(x_{2j+1}).$$

Then the order of the commutator $[f,g]$ in the group $[X,\mathrm{U}(n)_{(p)}]$ is equal to the order of $(\gamma_n,\dots,\gamma_{n+p-2})$ in the cokernel of

$$\bigoplus_{i=0}^{p-2}(n+i)!\,\mathrm{ch}_{n+i}\colon \widetilde{K}(X)_{(p)}\to\bigoplus_{i=0}^{p-2}H^{2n+2i}(X;\mathbb{Z}_{(p)}), \tag{4-2}$$

where ch_k is the $2k$–dimensional part of the Chern character.

For an integer m, let $\nu_p(m)$ be the maximum k such that m is divisible by p^k.

Proposition 4.3 *Let $n-1\le(p-1)(p-2)$ and let $\iota,\bar{\lambda}_i$ be as above. Then the order of the commutator $[\iota\circ1\circ\pi_1, \iota\circ\bar{\lambda}_{i-2}\circ\pi_2]$ in the group $[S^3\times A_{i-2},\mathrm{U}(n)_{(p)}]$ is $p^{\nu_p(i(i-1))}$ for $i=n,n+1$.*

Proof It is shown in [14] that $\widetilde{K}(\Sigma A_m)_{(p)}$ is generated by $\xi_{m,0},\dots,\xi_{m,\ell_m}$ for some $\ell_m>0$ such that

$$\mathrm{ch}_{n-1}(\xi_{m,s}) = \begin{cases}\dfrac{a_s}{(n-1)!}\,\Sigma x_{2n-3} & m\equiv n-2 \bmod (p-1),\\ 0 & m\not\equiv n-2 \bmod (p-1),\end{cases}$$

$$\mathrm{ch}_{n}(\xi_{m,s}) = \begin{cases}\dfrac{b_s}{n!}\,\Sigma x_{2n-3} & m\equiv n-1 \bmod (p-1),\\ 0 & m\not\equiv n-1 \bmod (p-1),\end{cases}$$

and

$$a_0, b_0 \in \mathbb{Z}_{(p)}^\times,$$

where x_j is as in (3-1). Then since $\widetilde{K}(S^3 \times A_s) \cong \widetilde{K}^{-1}(S^3) \otimes \widetilde{K}^{-1}(A_s)$, we obtain that the image of the map (4-2) for $X = A_{n-2}, A_{n-1}$ is generated by $n(n-1)(u_3 \times x_{2n-3})$ and $n(n+1)(u_3 \times x_{2n-1})$, respectively, where u_3 is a generator of $H^3(S^3; \mathbb{Z})$. Thus the proof is completed by Theorem 4.2. □

Corollary 4.4 *Let $n-1 \le (p-1)(p-2)$. Then for $\mathrm{SU}(n)$ and $i = n, n+1$, the order of $\delta_i^1 \colon B_{i-2} \to \Omega_0^3 B_i$ is $p^{\nu_p(i(i-1))}$.*

Let P_k be a principal $\mathrm{SU}(n)$–bundle over S^4 classified by $k \in \mathbb{Z} \cong \pi_4(B\mathrm{SU}(n))$. Sutherland [18] obtains the following necessary condition for $\mathcal{G}(P_k)$ and $\mathcal{G}(P_\ell)$ having the same homotopy type with some indeterminacy, which is fixed later by Hamanaka and the second author [7].

Theorem 4.5 (Sutherland [18], Hamanaka and Kono [7]) *Let P_k be a principal $\mathrm{SU}(n)$–bundle over S^4 classified by $k \in \mathbb{Z} \cong \pi_4(B\mathrm{SU}(n))$. If $\mathcal{G}(P_k) \simeq \mathcal{G}(P_\ell)$, it holds that $(n(n^2-1), k) = (n(n^2-1), \ell)$.*

The converse of the above theorem is true for $n = 3$ [7] but not for $n = 2$ [15]. However, for $n = 2$, the converse is still true if we look at odd prime components, which is generalized by the following result.

Theorem 4.6 *Let P_k be a principal $\mathrm{SU}(n)$–bundle over S^4 classified by $k \in \mathbb{Z} \cong \pi_4(B\mathrm{SU}(n))$ and suppose $n-1 \le (p-1)(p-2)$. Then $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_\ell)_{(p)}$ if and only if $\min\{\nu_p(n(n^2-1)), \nu_p(k)\} = \min\{\nu_p(n(n^2-1)), \nu_p(\ell)\}$.*

Proof Notice that the equality (4-1) implies that $\delta_i^k \simeq k \circ \delta_i^1$ and that since p is odd, at most one of $n-1, n, n+1$ is divisible by p. Then if $(\nu_p(n(n^2-1)), \nu_p(k)) = (\nu_p(n(n^2-1)), \nu_p(\ell))$, δ_i^k and δ_i^l have the same order for both $i = n, n+1$ by Corollary 4.4, implying that there exists $a_i \in \mathbb{Z}_{(p)}^\times$ such that $\delta_i^k \simeq a_i \circ \delta_i^\ell$ for $i = n, n+1$. Define a map $a \colon \Omega_0^3 \mathrm{SU}(n)_{(p)} \to \Omega_0^3 \mathrm{SU}(n)_{(p)}$ as

$$a = a_n \times a_{n+1} \times 1\colon \qquad \Omega_0^3 B_n \times \Omega_0^3 B_{n+1} \times \prod_{\substack{i \not\equiv n, n+1 \\ \bmod (p-1)}} \Omega_0^3 B_i \to \Omega_0^3 B_n \times \Omega_0^3 B_{n+1} \times \prod_{\substack{i \not\equiv n, n+1 \\ \bmod (p-1)}} \Omega_0^3 B_i.$$

It follows from Lemma 3.11 that $\delta_{(p)}^k \simeq a \circ \delta_{(p)}^\ell$. Since the map a is a homotopy equivalence, we obtain that $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_\ell)_{(p)}$, completing the proof. □

Remark 4.7 Sutherland [18] proved a result for $\mathrm{Sp}(n)$–bundles over S^4 that is analogous to Theorem 4.5. We can also show its converse as in Theorem 4.6 using naturality of the mod p decompositions of $\mathrm{SU}(2n)$ and $\mathrm{Sp}(n)$ with respect to the inclusion $\mathrm{Sp}(n) \to \mathrm{SU}(2n)$.

4.2 Adjoint bundles

Let us first set some terminology and notation for fiberwise spaces. A fiberwise space over B is a space X equipped with a map $\pi_X\colon X \to B$. Let $\pi_X\colon X \to B$ and $\pi_Y\colon Y \to B$ be fiberwise spaces over B. A fiberwise map from X to Y is a map $f\colon X \to Y$ satisfying $\pi_Y \circ f = \pi_X$. A fiberwise homotopy is a homotopy $h_t\colon X \times [0,1] \to Y$ such that h_t is a fiberwise map for each $t \in [0,1]$. We then define a fiberwise homotopy equivalence in the obvious way. The fiberwise product of X and Y, denoted by $X \times_B Y$ is $\{(x,y) \in X \times Y \mid \pi_X(x) = \pi_Y(y)\}$, where $X \times_B Y \to B$ is the restriction of $\pi_X \times \pi_Y$. X is called a trivial fiberwise space if π_X is a homotopy equivalence. If X is fiberwise homotopy equivalent to a fiberwise product of non-trivial fiberwise spaces, we say that X is fiberwise decomposable.

For a topological group G, let P be a principal G–bundle over K. The adjoint bundle of P, denoted by $\mathrm{ad}\,P$, is the associated fiber bundle $P \times G/\sim$, where $(p,g) \sim (q,h)$ if $q = pk$ and $h = k^{-1}gk$ for $p,q \in P$ and $g,h,k \in G$. Then as in [1], there is a natural isomorphism

$$\mathcal{G}(P) \cong \Gamma(\mathrm{ad}\,P), \tag{4-3}$$

where $\Gamma(\mathrm{ad}\,P)$ is the set of all sections of $\mathrm{ad}\,P$. It has been a standard method to study the gauge group $\mathcal{G}(P)$ by investigating $\mathrm{ad}\,P$ as a fiberwise space over K. See [2; 11; 13], for example. However, the converse might not work well in general. Namely, there are properties of the space $\mathcal{G}(P)$ that are not deduced from fiberwise properties of $\mathrm{ad}\,P$. Combining results of Crabb and Sutherland [2] and the second author [15], we get an example for this converse problem, and this example is the only one we know so far. By (4-3), a fiberwise product decomposition of $(\mathrm{ad}\,P)^{\mathrm{fib}}_{(p)}$ yields a product decomposition of $\mathcal{G}(P)_{(p)}$, where $(-)^{\mathrm{fib}}_{(p)}$ is the fiberwise p–localization. The first two authors [11] construct mod p decompositions of gauge groups by constructing fiberwise decompositions of adjoint bundles. We show not all mod p decompositions of gauge groups are achieved in this way.

We start by recalling a result of Wilkerson [25] on the uniqueness of mod p decompositions. A space is called indecomposable if it is not homotopy equivalent to a product of two contractible spaces.

Theorem 4.8 (Wilkerson [25]) *Let X be a simply connected, finite H–space. If $X_{(p)} \simeq X_1 \times \cdots \times X_n$ where each X_i is indecomposable, then the homotopy types of $X_1, \ldots, X_n$ are unique up to permutations.*

If G is a compact, simply connected, simple Lie group such that $H_*(G;\mathbb{Z})$ is p–torsion free and $G \neq \mathrm{Spin}(2n)$, then the non-contractible product factor B_i of $G_{(p)}$ is indecomposable. See [17]. Then by Theorem 4.8, we have the following.

Corollary 4.9 *Let G be a compact, simply connected, simple Lie group such that $G \neq \mathrm{Spin}(2n)$ and $H_*(G;\mathbb{Z})$ is p–torsion free. Then for any decomposition $G_{(p)} \simeq X_1 \times \cdots \times X_n$, there is a partition $I_1 \sqcup \cdots \sqcup I_n = \{1, \ldots, p-1\}$ such that*

$$X_k \simeq \prod_{i \in I_k} B_i$$

for $k = 1, \ldots, n$.

We next calculate the cohomology of the adjoint bundle $\mathrm{ad}EG$. It is well known that $\mathrm{ad}EG$ is fiberwise homotopy equivalent to the free loop space $\mathcal{L}BG$ over BG. Then we consider the cohomology of free loop spaces. In [12], the following useful operation is constructed. Since there is a section of the projection $\mathcal{L}X \to X$, we assume $H^*(X;R) \subset H^*(\mathcal{L}X;R)$.

Theorem 4.10 (Kishimoto and Kono [12]) *There is a linear map $\hat{\sigma}\colon \overline{H}^*(X;R) \to H^{*-1}(\mathcal{L}X;R)$ satisfying the following, where R is a commutative ring.*

(1) *For the inclusion $\iota\colon \Omega X \to \mathcal{L}X$, it holds that*

$$\iota^* \circ \hat{\sigma} = \sigma,$$

where $\sigma\colon \overline{H}^*(X;R) \to H^{*-1}(\Omega X;R)$ is the cohomology suspension.

(2) *For $x, y \in \overline{H}^*(X;R)$, we have*

$$\hat{\sigma}(xy) = \hat{\sigma}(x)y + (-1)^{|x||y|}x\hat{\sigma}(y).$$

(3) *If $R = \mathbb{Z}/p$, we have*

$$\hat{\sigma}\mathcal{P}^1 = \mathcal{P}^1\hat{\sigma} \quad \text{and} \quad \hat{\sigma}\beta = \beta\hat{\sigma}.$$

Let G be a compact, simply connected, simple Lie group such that $G \neq \mathrm{Spin}(2n)$ and $H_*(G;\mathbb{Z})$ is p–torsion free. It holds that

$$H^*(BG;\mathbb{Z}/p) = \mathbb{Z}/p[y_{2i+2} \mid i \in \mathfrak{t}(G)] \tag{4-4}$$

and $1 \in \mathfrak{t}(G)$. Then it follows from the Borel transgression theorem that

$$H^*(G;\mathbb{Z}/p) = \Lambda(\sigma(y_{2i+2}) \mid i \in \mathfrak{t}(G)).$$

Thus by Theorem 4.10, we can apply the Leray–Hirsch theorem to the evaluation fiber sequence $G \simeq \Omega BG \to \mathcal{L}BG \to BG$ and get

$$H^*(\mathcal{L}BG;\mathbb{Z}/p) = \mathbb{Z}/p[y_{2i+2} \mid i \in \mathfrak{t}(G)] \otimes \Lambda(\hat{x}_{2i+1} \mid i \in \mathfrak{t}(G)), \quad \hat{x}_{2i+1} = \hat{\sigma}(y_{2i+2}).$$

Let P be a principal G–bundle over S^4 classified by $1 \in \mathbb{Z} \cong \pi_4(BG)$. Then since $\mathcal{L}BG$ is fiberwise homotopy equivalent to $\mathrm{ad}EG$ over BG, there is a homotopy pullback diagram

$$\begin{tikzcd} \mathrm{ad}P \arrow[r, "j"] \arrow[d] & \mathcal{L}BG \arrow[d] \\ S^4 \arrow[r, "i_1"] & BG, \end{tikzcd}$$

implying that

(4-5)
$$H^*(\mathrm{ad}P;\mathbb{Z}/p) = \mathbb{Z}/p[u_4]/(u_4^2) \otimes \Lambda(\bar{x}_{2i+1} \mid i \in \mathfrak{t}(G)),$$
$$u_4 = i_1^*(y_4), \quad \bar{x}_{2i+1} = j^*(\hat{x}_{2i+1}),$$

where i_1 represents $1 \in \mathbb{Z} \cong \pi_4(BG)$. Let $r_p(G)$ be the number of $i \in \mathbb{Z}/(p-1)$ such that $\mathfrak{t}_i(G) \neq \varnothing$. We give a criterion for fiberwise indecomposability of $(\mathrm{ad}P)^{\mathrm{fib}}_{(p)}$.

Theorem 4.11 *Let G be a compact, simply connected, simple Lie group such that $G \neq \mathrm{Spin}(2n)$ and $H_*(G;\mathbb{Z})$ is p–torsion free, and let P be a principal G–bundle over S^4 classified by $1 \in \mathbb{Z} \cong \pi_4(BG)$. Suppose there exists a prime $p > 3$ satisfying the following conditions.*

(1) $p-2 \in \mathfrak{t}(G)$.

(2) *$\mathcal{P}^1 y_4$ includes the term $a y_4 y_{2p-2}$ for $a \in (\mathbb{Z}/p)^\times$.*

Then $(\mathrm{ad}P)^{\mathrm{fib}}_{(p)}$ is not fiberwise decomposable into $r_p(G)$ non-trivial fiberwise spaces over S^4.

Proof By the condition (2) together with Theorem 4.10 and (4-5), we get

(4-6)
$$\mathcal{P}^1 \bar{x}_3 = a u_4 \otimes \bar{x}_{2p-3} + \cdots .$$

Assume that $(\mathrm{ad}P)^{\mathrm{fib}}_{(p)}$ is fiberwise homotopy equivalent to $X_1 \times_{S^4} \cdots \times_{S^4} X_{r_p(G)}$, where X_i is a non-trivial fiberwise space over S^4 for $i = 1, \dots, r_p(G)$. Then in particular, we have that the homotopy fiber F_i of $X_i \to S^4$ is not contractible for each i. By Corollary 4.9, we may assume B_1 is included in F_1. Then it follows from (4-6) that $B_1 \times B_{p-2}$ must be included in F_1. Thus by Corollary 4.9, at least one F_i is contractible, a contradiction. □

Let G, P be as in Theorem 1.2. Then $\mathcal{G}(P)_{(p)}$ is decomposable into $r_p(G)$ non-contractible spaces. Then the following shows that not all these decompositions are induced from those of $(\mathrm{ad}\,P)^{\mathrm{fib}}_{(p)}$.

Corollary 4.12 *Let G be a compact, simply connected, simple Lie group such that $G \neq \mathrm{Spin}(2n), \mathrm{Sp}(1)\ (\cong \mathrm{SU}(2)), \mathrm{SU}(3)$ and $H_*(G;\mathbb{Z})$ is p–torsion free, and let P be a principal G–bundle over S^4 classified by $1 \in \mathbb{Z} \cong \pi_4(BG)$. Then there exists an odd prime p such that $(\mathrm{ad}\,P)^{\mathrm{fib}}_{(p)}$ is not fiberwise decomposable into $r_p(G)$ non-trivial fiberwise spaces over S^4.*

Proof When G is exceptional, $p = \ell(G)+2$ satisfies all conditions in Theorem 4.11 by Hamanaka and Kono [8].

For $G = \mathrm{SU}(n)$ with $n > 3$, we can take a prime $p > 3$ satisfying $p-1 \leq n \leq 2p-1$, implying the conditions (1) and (2) in Theorem 4.11 are satisfied. By an elementary calculation, we have

$$\mathcal{P}^1 c_2 = -(p+1)c_2c_{p-1} + \cdots$$

and then the condition (3) is satisfied, where $c_i \in H^{2i}(B\mathrm{SU}(n);\mathbb{Z})$ is the i^{th} universal Chern class. Thus the proof is done by Theorem 4.11 in this case.

The case $G = \mathrm{Sp}(n)$ with $n > 1$ follows quite similarly from an equality

$$\mathcal{P}^1 q_1 = (-1)^{\frac{p+1}{2}}(p+1)q_1q_{\frac{p-1}{2}} + \cdots,$$

where $q_i \in H^{4i}(B\mathrm{Sp}(n);\mathbb{Z})$ is the i^{th} universal symplectic Pontrjagin class. □

Remark 4.13 By [11], we have that for $G = \mathrm{SU}(3)$, $(\mathrm{ad}\,P)^{\mathrm{fib}}_{(p)}$ is fiberwise decomposable into two fiberwise spaces over S^4.

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