We prove a strong controlled generalization of a theorem of Bestvina and Walsh, which states that a \((k + 1)\)–connected map from a topological \(n\)–manifold to a polyhedron, \(2k + 3 \leq n\), is homotopic to a \(UV^k\)–map, that is, a surjection whose point preimages are, in some sense, \(k\)–connected. One consequence of our main result is that a compact \(ENR\) homology \(n\)–manifold, \(n \geq 5\), having the disjoint disks property satisfies the linear \(UV^{(n-3)/2}\)–approximation property for maps to compact \(ANR\)s. The method of proof is general enough to show that any compact \(ENR\) satisfying the disjoint \((k + 1)\)–disks property has the linear \(UV^k\)–approximation property.

57Q35, 57Q30, 57N99, 57P99

1 Introduction

A deformation theorem of Bestvina and Walsh \cite{2} states that, below middle and adjacent dimensions, a \((k + 1)\)–connected mapping of a compact topological manifold to compact polyhedron can be deformed to a \(UV^k\)–mapping; that is, a surjection whose fibers are in some sense \(k\)–connected. For example, if one has a map \(f\) from the \(n\)–sphere to the \(m\)–sphere, where \(n \leq m\), one generally expects a typical point inverse image to be a finite set (usually empty, if \(n < m\)), but the truth, however, may be rather the opposite: if \(n > 4\), then \(f\) is homotopic to a surjection with simply connected point inverses. This is predicted by the high connectivity of the homotopy fiber of the map. It is sometimes more useful to consider approximations by maps that behave like these “space-filling curves”, which are closer models of the underlying abstract homotopy theory, rather than adopt the usual strategy of approximating by smooth or piecewise linear maps. Controlled versions of this phenomenon were essential in the construction of nonresolvable homology manifolds (Bryant, Ferry, Mio and Weinberger \cite{5}; see also Pedersen, Quinn and Ranicki \cite{24} and Ferry \cite{13}) and in the “desingularization” of higher-dimensional homology manifolds (Bryant, Ferry, Mio and Weinberger \cite{6}).
In this paper we establish a strong controlled version of the Bestvina–Walsh Theorem for maps from an ENR homology $n$–manifold (with the disjoint disks property, or DDP, if $n \geq 5$) to an ANR. To our knowledge this is the first “controlled homotopy improvement” result for maps from spaces having no local linear structure. Among the main motivations for the paper are questions related to the cell-like approximation problem for homology manifolds explained below. The key observation is found in Theorem 6, a “UV–expansion theorem” for compact ENRs. One consequence of our main theorem is that a (controlled) homotopy equivalence $f : X \to Y$ between compact ENR homology $n$–manifolds is (controlled) homotopic to a $UV^\lfloor (n-3)/2 \rfloor$–map, provided $X$ has the disjoint disks property (DDP) if $n \geq 5$. If the Quinn [26] resolution obstruction for $X$, $\iota(X) \in 1 + 8\mathbb{Z}$, is different from that of $Y$, then $f$ is not homotopic to a $UV^\lfloor (n-1)/2 \rfloor$–map. This follows from arguments of Lacher (see remark below), which can be used to show that a $UV^\lfloor (n-1)/2 \rfloor$–homotopy equivalence between ENR homology $n$–manifolds, $n \geq 5$, is cell-like. If, however, $f$ is a sufficiently fine homotopy equivalence, depending on the metric on $Y$, then the results of Quinn [26] show that $\iota(X) = \iota(Y)$. Thus it is natural to suggest the following:

**CE-approximation conjecture**  
If $Y$ is a compact ENR homology $n$–manifold and $\epsilon > 0$ then there exists $\delta > 0$ such that if $X$ is a compact ENR homology $n$–manifold, $n \geq 5$, with the DDP and $f : X \to Y$ is a $\delta$–homotopy equivalence, then $f$ is $\epsilon$–homotopic to a cell-like map.

The CE-approximation conjecture can be used to establish a version of the Chapman–Ferry [9] $\alpha$–approximation theorem for homology manifolds.

**$\alpha$–approximation conjecture for homology manifolds**  
Given a compact ENR homology $n$–manifold $Y$, $n \geq 5$, with the DDP and $\epsilon > 0$ there is a $\delta > 0$ such that if $X$ is a compact ENR homology $n$–manifold (with the DDP if $n \geq 5$) and $f : X \to Y$ is a $\delta$–equivalence, the $f$ is $\epsilon$–homotopic to a homeomorphism.

It is not difficult to show that the $\alpha$–approximation conjecture implies that an ENR homology $n$–manifold, $n \geq 5$, with the disjoint disks property is topologically homogeneous, as conjectured in Bryant et al [5].

The methods we develop here are new, even in the case of PL or smooth manifolds, and provide an alternative proof of the Bestvina–Walsh Theorem referred to above. Other results of this type are due to Keldyš [16], Anderson [1], Frum-Ketkov [14], Wilson [31; 32], Walsh [30], Černavskii [10] and Ferry [12]. In fact, they apply to any ENR having sufficient general position properties, and the essential propositions and lemmas will be presented in this setting. These methods allow us to take a map that, in
Quinn’s terminology [25], is \((\varepsilon, k + 1)\)–connected, which we call a \(U^k\)–map, and “squeeze” it in a controlled fashion to be \((\mu, k + 1)\)–connected, for arbitrarily small \(\mu\). The desired \(U^k\)–map is obtained by taking a limit. The controls on the homotopies have sufficient uniformity to show that a compact ENR with the disjoint \((k + 1)\)–disks property \((\text{DDP}^{k+1})\) has the linear \(U^k\)–approximation property introduced in Bryant et al [6]. In the same work, Bryant et al constructed “resolutions”, which have the linear \(U^1\)–approximation property, and suggested that this latter property is stronger than the DDP. But a consequence of our main result is that every ENR homology \(n\)–manifold, \(n \geq 5\), with the DDP has the linear \(U^{(n-3)/2}\)–approximation property. This is a considerable strengthening of the disjoint disks property and indicates yet another way in which the exotic homology manifolds constructed in Bryant et al [5] resemble topological manifolds. Our techniques are strong enough to yield a relative theorem, which asserts that the homotopies of a given map to a \(U^k\)–map may be kept fixed on a sufficiently nicely embedded compact set. As a result we obtain a strong relative theorem for maps from a homology manifold with boundary.

Here is our main result. \((LCC^k\) subsets are defined in the next section. Informally, they are subsets that can be avoided by maps of a \((k + 1)\)–dimensional polyhedron into the ambient space.)

**Theorem 1** Suppose \(X\) is a compact, connected ENR satisfying the disjoint \((k + 1)\)–disks property, \(B\) is a compact ANR, \(Y\) is a metric space and \(p: B \to Y\) is a map. If \(f: X \to B\) is \(U^k(\varepsilon)\) over \(Y\), then \(f\) is \((C(k) \cdot \varepsilon)\)–homotopic (over \(Y\)) to a \(U^k\)–map, where \(C(k)\) is a positive constant depending only on \(k\).

Moreover, if \(Z\) is a compact, \(LCC^k\) subset of \(X\), then the homotopy of \(f\) to a \(U^k\)–map can be chosen to be fixed on \(Z\).

As Theorem 1 essentially defines the relative linear \(U^k\)–approximation property, we get as a corollary one of the results that motivated this paper.

**Theorem 2** Suppose \(X\) is a compact ENR homology \(n\)–manifold, \(n \geq 3\), with boundary \(\partial X\). If \(n \geq 5\) assume that \(X\) has the DDP and that \(\partial X\) is \(LCC^1\) in \(X\). Then \((X, \partial X)\) has the relative linear \(U^{(n-3)/2}\)–approximation property.

**Proof** It is well-known that a connected ANR of dimension \(\geq 1\) is arc-wise connected and locally arc-wise connected. In particular, any continuous map of \([0, 1]\) into \(X\) can be approximated by one whose image has dimension \(\leq 1\). If \(n = 3\) or 4, the \(\text{DDP}^1\) property of \(X\) follows from this fact together with Alexander duality: if \(U\) is any connected open subset of \(X\) and \(A\) is a closed, 1–dimensional subset of \(U\), then
\[ H_1(U, U - A) \cong \tilde{H}^{n-1}(A) = 0. \] (Integer coefficients are understood throughout this paper.) This, in turn, implies that the reduced homology group \( \tilde{H}_0(U - A) = 0. \) The \( LCC^0 \) property of \( \partial X \) in \( X \) follows immediately from the homology conditions given in the definition below.

If \( n \geq 5 \), then the results of Walsh [29] and Bryant [4] show that a homology \( n \)–manifold with the disjoint disks property also has the disjoint \( [(n - 1)/2] \)–disks property. (See the discussion in the next section.) If \( U \) is an open subset of \( X \), then, by definition (below), the inclusion \( U - \partial X \subseteq U \) induces an isomorphism on homology. If \( X - \partial X \) is locally simply connected at points of \( \partial X \), then, by the eventual Hurewicz Theorem [11], \( U - \partial X \subseteq U \) also induces an isomorphism on homotopy groups, hence, is a homotopy equivalence. (A subset of a space \( X \) with this property is called a \( Z \)–set.) □

A similar argument establishes a hybrid version.

**Theorem 3** Suppose \( X \) is a compact \( ENR \) homology \( n \)–manifold, \( n \geq 3 \), possibly with boundary, \( \partial X \) and \( Z \) is a compact, \( LCC^0 \) subset of \( X \) containing \( \partial X \). If \( n \geq 5 \) assume further that \( X \) has the \( DDP \) and that \( Z \) is \( LCC^{((n-3)/2)} \) in \( X \). Then \( (X, Z) \) has the relative linear \( UV^{((n-3)/2)} \)–approximation property.

As a special case (\( Y = \) a point) we recover the analogue of the theorem of Bestvina and Walsh for “nice” homology manifolds.

**Theorem 4** Suppose \( X \) is a compact, connected, \( ENR \) homology \( n \)–manifold, with boundary \( \partial X \), and suppose \( B \) is a compacted, connected \( ANR \). Suppose \( f: X \rightarrow B \) is a \( (k+1) \)–connected map for some \( k \geq 0 \), \( 2k + 3 \leq n \). If \( k \geq 1 \), we assume further \( X \) has the \( DDP \) and \( \partial X \) is \( LCC^1 \) in \( X \). Then \( f \) is homotopic, rel \( \partial X \), to a \( UV^k \)–map.

**Remark** Lacher [20, Sections 5 and 7] (see also Frum-Ketkov [14]), has shown that a \( UV^{((n-1)/2)} \)–map between compact \( n \)–manifolds must be cell-like if \( n \) is odd and, if \( n \) is even, it must be a spine map, in which spines of connected summands are collapsed to points. Thus, the result in Theorem 1 is best possible for maps from the \( n \)–sphere \( S^n \) to itself of degree \( d \neq \pm 1 \).

As a separate application we invoke a theorem of Krupski [17], who has shown that a homogeneous \( ENR \) of dimension \( \geq 3 \) has the \( DDP^1 \). Recall that a space \( X \) is homogeneous if, given points \( x, y \in X \), there is a homeomorphism of \( X \) onto itself taking \( x \) to \( y \). Combining Krupski’s result with Theorem 1 we obtain:

**Theorem 5** If \( X \) is a compact, connected, homogeneous \( ENR \) of dimension \( \geq 3 \), then \( X \) has the linear \( UV^0 \)–approximation property. In particular any map of \( X \) to a compact, simply-connected \( ANR \) is homotopic to a monotone map, that is, a surjection with connected point-inverses.
2 Definitions and preliminary results

A homology \( n \)-manifold is a space \( X \) having the property that for each \( x \in X \),

\[
H_k(X, X - x; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & k = n, \\
0 & k \neq n.
\end{cases}
\]

\( H_*(\cdot, \cdot; \mathbb{Z}) \) is understood to mean singular homology with integer coefficients. We say that \( X \) is an homology \( n \)-manifold with boundary if the condition \( H_n(X, X - x; \mathbb{Z}) \cong \mathbb{Z} \) is replaced by \( H_n(X, X - x; \mathbb{Z}) \cong \mathbb{Z} \) or \( 0 \) and, if \( \partial X = \{x \in X : H_n(X, X - x; \mathbb{Z}) \cong 0\} \), then \( \partial X \) is a homology \((n - 1)\)-manifold and \( H_*(U, U - \partial X; \mathbb{Z}) = 0 \) for every open subset \( U \) of \( X \). (Mitchell [23] shows that, if \( X \) is an ENR, \( \partial X \) is a homology \((n - 1)\)-manifold.)

A Euclidean neighborhood retract (ENR) is a space homeomorphic to a closed subset of Euclidean space that is a retract of some neighborhood of itself, that is, a locally compact, finite-dimensional ANR. Topological manifolds and locally compact, finite-dimensional polyhedra are the most well-known examples of ENRs, but there are many other interesting types of examples, such as the exotic homology manifolds constructed in Bryant et al [5]. Perhaps the most important property of a topological manifold or locally compact polyhedra that generalizes to an arbitrary ENR \( X \) is the existence of mapping cylinder neighborhoods, which we have already mentioned above: If \( X \) is \( LCC^1 \) embedded in a topological manifold \( M \), \( \dim M - \dim X \geq 3 \), then \( X \) has a topological manifold neighborhood \( W \) with boundary in \( M \), which admits a retraction \( p : W \to X \), such that \( W \) is the mapping cylinder of \( p|\partial W \) and \( p \) is the mapping cylinder retraction (Miller [22] and Quinn [25]). This generalizes the notion of normal bundle neighborhoods for topological manifolds and regular neighborhoods for polyhedra. In fact, there are stable classification theorems for mapping cylinder neighborhoods of ENR homology manifolds analogous to those for normal bundle neighborhoods of topological manifolds (Bryant and Mio [7]).

A space \( X \) satisfies the disjoint disks property (DDP) if for every \( \epsilon > 0 \) and maps \( f, g : D^2 \to X \), there are maps \( f', g' : D^2 \to X \) so that \( d(f, f') < \epsilon, d(g, g') < \epsilon \) and \( f'(D^2) \cap g'(D^2) = \emptyset \). More generally, we say that a space \( X \) has the disjoint \( k \)-disks property, or DDP\(^k \), if any two maps of a \( k \)-cell into \( X \) can be approximated by maps with disjoint images. The DDP\(^k \) implies that maps \( f : D^i \to X \) and \( g : D^j \to X \) can be approximated by maps with disjoint images whenever \( i, j \leq k \).

Given \( \epsilon > 0 \) and a map \( p : B \to C \), a map \( f : A \to B \) is \( UV^k(\epsilon) \) over \( C \), if it has the \( \epsilon \)-homotopy lifting property over \( C \) for \((k + 1)\)-dimensional polyhedra. That is, if \((P, Q)\) is a polyhedral pair with \( \dim P \leq k + 1 \), \( \alpha_Q : Q \to A \) and \( \alpha : P \to B \), with \( f \circ \alpha_0 = \alpha|Q \), then there is a map \( \overline{\alpha} : P \to A \) extending \( \alpha_0 \) such that \( f \circ \overline{\alpha} \) is
\(\epsilon\)-homotopic over \(C\) to \(\alpha\) in \(B\), rel \(\alpha|Q\). The lift \(\overline{\alpha}\) of \(\alpha\) will be called an \(\epsilon\)-lift of \(\alpha\), rel \(\alpha_0\) (or, sometimes, rel \(Q\)), over \(C\), identical to the notion of Quinn [25, Definition 5.1] of a relatively \((\epsilon, k + 1)\)-connected map over \(C\).

There are two important special cases of this definition representing the two extremes on the degree of control. If \(p\) is a constant map or, equivalently, \(C\) is a point, then we have the usual notion of a \(k\)-connected map \(f: A \to B\). This is equivalent to \(f\) inducing isomorphisms on homotopy groups through dimension \(k\) and an epimorphism in dimension \(k + 1\). At the other extreme we have \(C = B\) and \(p = \text{id}_B\). In this case we will often omit reference to \(B\) as the control space and just say \(f: A \to B\) is \(UV^k(\epsilon)\).

A compact metric space \(C\) has property \(UV^k\), \(k \geq 0\), if for some (and, hence, any) embedding of \(C\) in an ANR \(X\) and every neighborhood \(U\) of \(C\) in \(X\), there is a neighborhood \(V\) of \(C\) lying in \(U\) such that the inclusion induced map \(\pi_i(V) \to \pi_i(U)\) is 0 for \(0 \leq i \leq k\). In the language of shape theory, the shape homotopy groups of \(C\) vanish through dimension \(k\) (see eg Mardešić [21]). A surjection \(f: A \to B\) between compact ANRs is \(UV^k\), \(k \geq 0\), if its point inverses have property \(UV^k\). A \(UV^{-1}\)-map is a surjection.

**Remark** For ANRs, property \(UV^k\) is equivalent to \(k\)-connectedness. For non-ANRs, especially nonlocally connected spaces, the situation is quite different. For example, the fundamental group of the dyadic solenoid, \(\Sigma = \text{proj lim}\{S^1, z \to z^2\}\), is trivial, but it fails to have property \(UV^1\). By contrast, the “topologists’ sine curve” in the plane and the Whitehead continuum, \(Wh \subseteq S^3\), are not locally connected, but satisfy property \(UV^k\) for all \(k \geq 0\).

The following basic result is due to Lacher [18; 20].

**Proposition 1** A surjection \(f: A \to B\) between compact ANRs is \(UV^k\) if and only if it is \(UV^k(\epsilon)\) for every \(\epsilon > 0\).

A compact metric pair \((X, Z)\) has the relative linear \(UV^k\)-approximation property if, for a given compact (metric) ANR \(B\) and map \(p: B \to Y\) of \(B\) to a metric space \(Y\), every map \(f: X \to B\) that is \(UV^k(\epsilon)\) over \(Y\), for some \(\epsilon > 0\), is \((C \cdot \epsilon)\)-homotopic over \(Y\), keeping \(f|Z\) fixed, to a \(UV^k\)-map, where \(C\) is a constant depending only on \(k\).

As the following proposition indicates, it is sufficient to prove Theorem 1, as well as the propositions and lemmas leading up to its proof, for mappings to compact polyhedra.

**Proposition 2** The (relative) \(UV^k\)-approximation property is equivalent to the (relative) \(UV^k\)-approximation property for maps to compact polyhedra.
Proof Suppose a compact metric pair \((X, Z)\) has the relative linear \(UV^k\)–approximation property for maps to compact polyhedra. Suppose \(B\) is a compact ANR, \(p: B \to Y\) is a map to a metric space, \(Y\), and suppose a map \(f: X \to B\) is \(UV^k(\epsilon)\) over \(Y\), for some \(\epsilon > 0\).

If \(B\) is finite-dimensional, then \(B\) has a mapping cylinder neighborhood \(N\) in \(\mathbb{R}^n\) with mapping cylinder projection \(\varphi: N \to B\). The composition \(f' = \iota \circ f: X \to N\) is \(UV^k(\epsilon)\) over \(Y\) with respect to the control map \(p' = p \circ \varphi: N \to B\). Since \(N\) is triangulable as a finite polyhedron, \(f'\) is \((C \cdot \epsilon)\)–homotopic over \(Y\), keeping \(f|Z\) fixed, to a \(UV^k\)–map, \(g': X \to N\), where \(C\) is a constant depending only on \(k\). Since \(\varphi\) is cell-like, \(g = \varphi \circ g': X \to B\) is \(UV^k\) and \(g\) is \((C \cdot \epsilon)\)–homotopic to \(f\) over \(Y\), keeping \(f|Z\) fixed.

If \(B\) is infinite-dimensional, we invoke a famous result of Chapman [8] which states that, if \(I^\infty\) is the Hilbert cube, then \(B \times I^\infty \cong K \times I^\infty\) for some finite polyhedron \(K\). The results of Chernavskii [10] and Ferry [12] show that, for \(n \geq 2k + 3\) and \(\delta_n > 0\), there is a \(UV^k\) map \(g_n: I^n \to I^{n+1}\) such that \(\pi g_n: I^n \to I^n\) is \(\delta_n\)–homotopic to the identity, where \(\pi: I^{n+1} \to I^n\) is the natural projection. For fixed \(j \geq 0\) and \(n \geq 2k + 3\) let \(g_{n,j} = g_{n+j} \circ \cdots \circ g_n: I^n \to I^{n+j+1}\) and let \(G_{n,j}: I^\infty \to I^\infty\) be defined by
\[
G_{n,j}(t_1, t_2, \ldots) = (g_{n,j}(t_1, \ldots, t_n), t_{n+j+2}, \ldots).
\]

Given \(\delta > 0\), we can choose the sequence \(\delta_{n+j}\) so that the sequence \(\{G_{n,j}\}\) converges to a map
\[
G_n: I^n \to I^\infty
\]
such that, if \(\pi_n: I^\infty \to I^n\) is the natural projection, \(d(\pi_n G_n, \text{id}_{I^n}) < \delta\). By a theorem of Lacher [20, Page 505], \(G_n\) is \(UV^k\), since each \(G_{n,j}\) is \(UV^k\), being the composition of a \(UV^k\)–map and the cell-like map \(K \times I^\infty \to K \times I^\infty\) that omits the \((n + j + 1)\)–coordinate of \(I^\infty\).

Assume the product metric on \(B \times I^\infty\), and provide \(K \times I^\infty\) with the metric induced by a homeomorphism
\[
B \times I^\infty \xrightarrow{\cong} K \times I^\infty.
\]
For any given \(\mu > 0\) we may choose \(n \geq 2k + 3\) and \(G_n: I^n \to I^\infty\) such that \(\text{id}_K \times G_n: K \times I^n \to K \times I^\infty\) is a \(UV^k\)–map and \(\text{id}_K \times (\pi_n \circ G_n): K \times I^n \to K \times I^n\) is \(\mu\)–homotopic to the identity over \(B \times I^\infty\) (with respect to inclusion).

Let \(p: B \times I^\infty \to Y\) be the composition of the projection \(\pi_B: B \times I^\infty \to B\) with \(p\). Given \(\delta > 0\) we can choose \(n \geq 2k + 3\) and \(G_n: I^n \to I^\infty\) as above so that \(\text{id}_K \times \pi_n: K \times I^\infty \to K \times I^n\) and \(\text{id}_K \times (\pi_n \circ G_n): K \times I^n \to K \times I^n\) are \(\delta\)–equivalences.
over $Y$. If $f: X \to B$ is $UV^k(\epsilon)$ over $Y$, as above, then, for appropriate $\delta$ and $n$, the map

$$\tilde{f} = (\text{id}_K \times \pi_n) \circ f: X \to K \times I^n$$

is $UV^k(2\epsilon)$ over $Y$. By hypothesis, $\tilde{f}$ is $(2\epsilon)$–homotopic, rel $Z$, over $Y$ to a $UV^k$–map

$$\tilde{g}: X \to K \times I^n.$$

The map

$$g = \pi_B \circ (\text{id}_K \times G_n) \circ \tilde{g}: X \to B$$

is $UV^k$, since $\tilde{g}$ and $(\text{id}_K \times G_n)$ are $UV^k$ and $\pi_B$ is cell-like. For suitably chosen $n$ and $\delta$ we can arrange it so that $g$ is $(3\epsilon)$–homotopic to $f$, and this will complete the nonrelative version.

To get a relative version we observe that, for any $\mu > 0$, we can arrange it so that the map $g$ constructed above will have the property that $g|Z$ is $\mu$–homotopic over $B$ to $f|Z$. Applications of the estimated homotopy extension will produce a map $g_\mu: X \to B$ such that $g_\mu$ is $\mu$–homotopic to $g$ over $B$ and $f$ is $(4\epsilon)$–homotopic to $g_\mu$, rel $Z$, over $Y$. By Lemma 2, below, $g_\mu$ is $UV^k(2\mu)$ over $B$. Applying this argument inductively, with control lifted to $B$, we can construct a sequence of $UV^k(\delta_n)$–maps $g_n: X \to B$ over $B$ that converge to a map $g: X \to B$, which is $(5\epsilon)$–homotopic to $f$ over $Y$. As a consequence of Proposition 1, $g$ is $UV^k$. □

A subset $A$ of an ANR $X$ is locally $k$–coconnected, or LCC$_k$, in $X$ if, for every connected open set $U \subseteq X$, $\pi_i(U, U - A) = 0$ for $0 \leq i \leq k + 1$. This is equivalent to the condition that the inclusion map $i: (X - A) \to X$ is $UV^k(\epsilon)$ for every $\epsilon > 0$. If $X$ is a topological $n$–manifold and $A$ is a closed subset of dimension $r$, $n - r \geq 3$, then $A$ is LCC$_{n-r-2}$ if and only if $A$ is LCC$^1$. This is essentially a consequence of Alexander duality and the Hurewicz Isomorphism Theorem (see Bryant [3] and Štan’ko [28]). This remains true if $X$ is an ENR homology $n$–manifold, $n \geq 5$, with the DDP (Bryant [4]; Walsh [29]).

**Proposition 3** If an ENR $X$ has the DDP$_k$ and $A$ is an LCC$_{k}^{k-1}$ subset of $X$, then any map $f$ of a $k$–dimensional polyhedron $K$ into $X$ can be approximated by an LCC$_{k-1}^{k-1}$ embedding that misses $A$. Moreover, if $f$ is already an LCC$_{k-1}^{k-1}$ embedding on a subpolyhedron $L$ (into the complement of $A$), then the approximation can be made to agree with $f$ on $L$.

**Outline of proof** This proposition is proved using techniques similar to those used to prove the main results of Bryant [4] and Walsh [30]. Since there are some differences, we outline a proof here.
Suppose \( K \) is a \( k \)–dimensional polyhedron and \( f: K \to X \) is a map. Let \( K_1, K_2, \ldots \) be a sequence of triangulations of \( K \) with mesh tending to 0. Use the \( DDP^k \) property of \( X \) to get a sequence \( f_j, j = 1, 2, \ldots \) of maps, where \( f_j \) is an approximation of \( f_{j-1}, j \geq 1, (f_0 = f) \), such that \( f_j(\sigma) \cap f_j(\tau) = \emptyset \) whenever \( \sigma \) and \( \tau \) are disjoint \( k \)–simplexes of \( K_j \). By taking extra care in choosing the sizes of subsequent approximations, we can guarantee that the limit map \( \bar{f}: K \to X \) satisfies this property for every \( j \) and, hence, is an embedding. Likewise, we can assume that the first and all subsequent approximations are chosen so that their images, as well as the image of \( \bar{f} \), misses \( A \). Arguments such as these may be found in Hurewicz and Wallman [15].

In order to get an \( LCC^{k-1} \) embedding we need an extra ingredient. Let \( N \) be a mapping cylinder neighborhood of \( X \) in some Euclidean space of dimension \( \geq 2k + 1 \) with mapping cylinder projection \( p: N \to X \). Let \( T_1 \subseteq T_2 \subseteq \cdots \) be the \( k \)–skeleta of a sequence of triangulations of \( N \) with mesh tending to 0. Given a map \( f: K \to X \) as above, we combine the process above with a sequence \( p_j: N \to X \), where \( p_j \) is an approximation of \( p_{j-1}, j \geq 1, (p_0 = p) \) so that \( p_j(T_j) \cap f_j(K) = \emptyset \) and the limit maps \( \bar{p} = \lim p_j \) and \( \bar{f} = \lim f_j \) satisfy \( \bar{p}(\bigcup T_j) \cap \bar{f}(K) = \emptyset \). If \( \alpha: (P, Q) \to (X, X - \bar{f}(K)) \) is a map of a \( k \)–dimensional polyhedral pair, then there is a small homotopy of \( \alpha \) to a map of \( P \) into \( T_j \) for some \( j \). We can choose \( j \) large enough and the homotopy small enough so that the image of the composition of the homotopy restricted to \( Q \) with \( p \) does not meet \( \bar{f}(K) \). After composing this map with \( p \) and using the estimated homotopy extension theorem (Bryant et al [5]), we can get a small homotopy of \( \alpha, \rel \alpha|Q \), to a map into \( X - \bar{f}(K) \).

This argument can easily be adapted to get the relative result.

The following property of \( UV^k \)–maps between compact ANRs is similar to results that may be found in Lacher [18; 19].

**Proposition 4** If a map \( f: X \to Y \) between compact ANRs is \( UV^k \), then, for each point \( y \) and each pair of open sets \( U \) and \( V \) in \( Y \) such that \( y \in \overline{V} \subseteq U \) and \( V \) is contractible to a point in \( U \), the induced homomorphisms \( \pi_i(f^{-1}(V)) \to \pi_i(f^{-1}(U)) \) are zero for \( 0 \leq i \leq k \).

We can use Proposition 4 inductively to prove:

**Proposition 5** Suppose \( f: X \to Y \) is a \( UV^k \)–map between compact ANRs and \( y \in Y \). Then for any neighborhood \( U \) of \( y \), there is a connected neighborhood \( V \) of \( y \) such that if \( g: (P, p_0) \to (f^{-1}(V), g(p_0)) \) is a map of a polyhedron \( P \) of dimension \( \leq k \) into \( f^{-1}(V) \), then \( g \) is homotopic, \rel \( p_0 \), in \( f^{-1}(U) \) to a constant map.
The following proposition illustrates a “Seifert–van Kampen”-type property of $U^k$-maps.

**Proposition 6** Suppose $X = A \cup B$, $A \cap B = C$, where $A$, $B$ and $C$ are compact ANRs, and suppose a mapping $f: X \to Y$ of $X$ to a compact ANR $Y$ has the property that $f|A$ and $f|B$ are $U^k$ and $f|C$ is $U^{k-1}$, $k \geq 0$. Then $f$ is $U^k$.

**Proof** Suppose $X = A \cup B$ and $f: X \to Y$ are given as above. For any $Z \subseteq Y$, set $Z^* = f^{-1}(Z)$. If $k = 0$, then, for each $y \in Y$, $y^* = (y^* \cap A) \cup (y^* \cap B)$ and $(y^* \cap A) \cap (y^* \cap B) = y^* \cap C \neq \emptyset$. Thus, $y^*$ is connected, and $f$ is $UV^0$ on $X$.

Assume that $k \geq 1$. Suppose $y \in Y$ and $W \subseteq V \subseteq U$ are neighborhoods of $y$ in $Y$ such that $W \subseteq V$ satisfy the conclusion of Proposition 5 for $f|C$ (for $U^{k-1}$-maps) and $V \subseteq U$ satisfy the conclusion of Proposition 5 for $f|A$ and $f|B$ (for $U^k$-maps). Suppose $g: S^i \to W^*$ is a mapping of the $i$-sphere, $1 \leq i \leq k$. Assume that $g^{-1}(A-B)$ and $g^{-1}(B-A)$ are nonempty. (Otherwise, $g$ is homotopic to a constant map in either $A$ or $B$ by Proposition 5.) Let $Q$ be a polyhedral neighborhood of $g^{-1}(C)$ in $S^i$, let $P = \mathcal{C}(S^i - Q)$ and let $P^A$ and $P^B$ be the union of the closures of the components of $S^i - Q$ lying in $g^{-1}(A - B)$ and $g^{-1}(B - A)$, respectively.

Using the ANR properties of $A$, $B$ and $C$, we can choose a fine enough neighborhood $Q$ of $g^{-1}(C)$ so that there is a small homotopy of $g$, rel $g^{-1}(C)$, in $W^*$ to a map we will still call $g: S^i \to W^*$ such that $g(Q) \subseteq C$ and the homotopy keeps $P^A$ and $P^B$ mapped into $A$ and $B$, respectively. As a separating polyhedron in a $PL$ $i$-manifold, $Q$ collapses to polyhedron $Q_0 \subseteq Q$ of dimension $i - 1$ using only collapses across $i$-dimensional simplexes. The associated deformation retraction of $Q$ onto $Q_0$ extends to a homotopy $R: S^i \times [0, 1] \to S^i$ of the identity map of $S^i$ to a map $r: S^i \to S^i$ such that $R((Q \cup P^A) \times [0, 1]) \subseteq Q \cup P^A$ and $R((Q \cup P^B) \times [0, 1]) \subseteq Q \cup P^B$. Let $P^A_0$ and $P^B_0$ be the union of the closures of the components of $S^i - Q_0$ that contain $P^A$ and $P^B$, respectively. Precomposing $g$ with $R$ gives a homotopy of $g$ to a map $h: S^i \to W^*$ such that $h(Q_0) \subseteq C$, $h(P^A_0) \subseteq A$ and $h(P^B_0) \subseteq B$.

By Proposition 5 $h|Q_0: Q_0 \to W^* \cap C$ is homotopic to a constant map $c$ in $V^* \cap C$. Use regular neighborhoods of $Q_0$ in $P^A_0$ and $P^B_0$ to extend this homotopy to a homotopy of $h$ in $W^* \cup (V^* \cap C)$ to a map $h_0: S^i \to W^* \cup (V^* \cap C) \subseteq V^*$ such that $h_0(P^A_0) \subseteq A$ and $h_0(P^B_0) \subseteq B$. Let $L$ be a quotient polyhedron of $S^i$ topologically homeomorphic to $S^i/Q_0$, containing subpolyhedra $L^A$ homeomorphic to $P^A_0/Q_0$ and $L^B$ homeomorphic to $P^B_0/Q_0$ so that $L^A \cap L^B$ is the point $[Q_0] \in S^i/Q_0$. Then $h_0$ induces a map $\tilde{h}: L \to V^*$ such that $\tilde{h}(L^A) \subseteq (V^* \cap A)$, $\tilde{h}(L^B) \subseteq (V^* \cap B)$ and $\tilde{h}([Q_0]) = c(Q_0)$ is a point of $C$. Proposition 5 then applies to each of $\tilde{h}|L^A$ and $\tilde{h}|L^B$ to provide a homotopy of $\tilde{h}$, rel $[Q_0]$, in $U^*$ to a constant. Thus, the inclusion homomorphism $\pi_i(W^*) \to \pi_i(U^*)$ is zero proving that $y^*$ is $U^k$ for each $y \in Y$. □
Proposition 7  Suppose $A$ and $B$ are compact, connected ANRs of dimension $\geq 1$, $\epsilon > 0$, $Z$ is an LCC$^{-1}$ subset of $A$, and $p: B \to C$ is a map, where $C$ is a metric space. If $f: A \to B$ is $UV^{-1}(\epsilon)$ over $C$, then $f$ is $2\epsilon$–homotopic (over $C$), rel $f|Z$, to a surjection.

Proof  Assume all measurements are made in $C$. Let $P$ be a finite subset of $B$ such that every point of $B$ can be joined to a point of $P$ by an arc of diameter $\leq \epsilon/2$ in both $B$ and $C$. By hypothesis, there is a map $\alpha: P \to A$ whose composition with $f$ is $\epsilon$–homotopic to the inclusion. Since $\dim A \geq 1$, we may assume $\alpha$ is one-to-one and, since $Z$ is LCC$^{-1}$, we may assume $\alpha(P) \cap Z = \emptyset$. Let $P' = \alpha(P)$. Using the homotopy extension theorem on a small neighborhood of $P'$ in $A$, which is disjoint from $Z$, we can get an extension of the $\epsilon$–homotopy of $f|P'$ to $\alpha^{-1}$ to an $\epsilon$–homotopy of $f$ to a map that sends $P'$ to $P$. Thus there is an $\epsilon$–homotopy of $f$, rel $f|Z$, to a map that is $UV^{-1}(\epsilon/2)$ over both $B$ and $C$. A sequence of such maps can be constructed so as to converge to a surjection that is $2\epsilon$–homotopic to $f$. \qed

The next lemma gives a criterion for determining when an extension of a $UV^k(\epsilon)$–map is (almost) $UV^k(\epsilon)$.

Lemma 1  Suppose $X_1 \subseteq X_2$ and $B$ are compact ANRs, $\delta > 0$ and $\epsilon > 0$, suppose $p: B \to Y$ is a map of $B$ to a metric space $Y$, and suppose that for some integer $k \geq 0$, $f: X_2 \to B$ is a map such that

(i) $f|X_1$ is $UV^k(\epsilon)$ over $Y$, and

(ii) if $g$ is a map of a $k$–dimensional polyhedron $R$ into $X_2$, then $g$ is $\delta$–homotopic over $Y$ to a map of $R$ into $X_1$.

Then $f$ is $UV^k(2\delta + \epsilon)$ over $Y$.

Proof  Suppose $(P, Q)$ is a polyhedral pair, $\dim P \leq k + 1$, and suppose $\alpha: P \to B$ and $\alpha_0: Q \to X_2$ satisfy $f \circ \alpha_0 = \alpha|Q$. For any $\mu > 0$ there is a $\mu$–homotopy over $B$ of the identity on $P$ to a map $r: P \to P$, which is fixed on $Q$ and outside a neighborhood of $Q$, that deformation retracts a small regular neighborhood $N$ of $Q$ onto $Q$. By precomposing $\alpha$ with such a map, we can get a $\mu$–homotopy (over $B$) of $\alpha$ to a map $\alpha_1: P \to B$, whose restriction to $N$ can be lifted by $\alpha_0 \circ r|N$. Thus, if $\mu$ is sufficiently small, $\alpha: P \to B$ is $\delta$–homotopic to $\alpha_1: P \to B$, rel $\alpha|Q$, such that $\alpha_1|N$ can be lifted to an extension of $\alpha_0: Q \to X_2$ to a map we will still call $\alpha_0$: $N \to X_2$ such that $f \circ \alpha_0 = \alpha_1|N$.

Let $P_0 = C\ell(P - N)$ and let $Q_0 = N \cap P_0 = \text{bd}(N)$. Since $\dim Q_0 \leq k$, there is a $\delta$–homotopy (over $Y$) of $\alpha_0|Q_0$ that takes $Q_0$ into $X_1$. Since $Q_0$ is collared in $N$,
An argument virtually identical to the one just given also proves the following lemma.

**Lemma 2** Suppose $X$ and $B$ are compact ANRs, $\delta > 0$ and $\epsilon > 0$. If $f: X \to B$ is $U^k(\epsilon)$ over a metric space $Y$ and $g$ is $\delta$–homotopic to $f$ over $Y$, then $g$ is $U^k(2\delta + \epsilon)$ over $Y$.

The proof of the next lemma is an easy application of the definition.

**Lemma 3** Suppose $A$, $B$ and $C$ are compact metric spaces and $f: B \to C$ is a $U^k(\epsilon)$–map for some $\epsilon > 0$. Then there exists $\delta > 0$ such that if $g: A \to B$ is $U^k(\delta)$ over $B$, then $f \circ g: A \to C$ is $U^k(2\epsilon)$ (over $C$).

## 3 Simple homotopies and ENRs

The proof of our main result involves a double induction argument in which the principal inductive step is established in two parts. Suppose $X$ is a compact ENR satisfying the $DDP^{k+1}$, $B$ is a finite complex and $f: X \to B$, is a map such that $f$ is $U^k - 1$ and $U^k(\epsilon)$ over $B$, for some $\epsilon > 0$. We will want to prove that, for arbitrary $\mu > 0$, there is a $(D \cdot \epsilon)$–homotopy of $f$ to a $U^k(\mu)$–map for some constant $D$, depending only on $k$.

One part is to show that there is a “simple homotopy solution” to this shrinking problem. That is, we will show:
We are generalizing here the notion of collapse and expansion from PL topology to ENRs in the obvious manner. That is, an elementary $k$–collapse ($k$–expansion) $Y \subseteq X \quad (X \not\subseteq Y)$ means $Y$ is obtained from $X$ by affixing a $k$–cell $D$ to $X$ by a map $a: C \rightarrow X$, where $C$ is a (nice) $(k−1)$–cell in $\partial C$. Alternatively, we can think of $Y = X \cup_a D$ as the mapping cylinder of $a$, rel $\partial C$. (We do not require, in general, that the attaching map be an embedding.) This provides a natural deformation retraction of $Y$ to $X$. A collapse (expansion) $Y \subseteq X \quad (X \not\subseteq Y)$ is a sequence of elementary collapses (expansions).

The main goal of this section is to prove Theorem 6, below, which is a special case of part (2) for elementary expansions in which the attaching map is a “nice” embedding. But first we make some elementary observations about maps defined on balls and spheres.

Let $S^m \subseteq \mathbb{R}^{m+1}$ denote the PL $m$–sphere, defined as the join of $(m + 1)$ copies of $S^0 = \{-1, 1\} \subseteq \mathbb{R}$, and let $B^m = S^{m−1} \ast 0 \subseteq \mathbb{R}^m$, with natural inclusions $S^{m−1} = S^{m−1} \times 0 \subseteq S^m$ and $B^m = B^m \times 0 \subseteq B^{m+1}$. In $\mathbb{R}^{n+1} = \mathbb{R}^k \times \mathbb{R}^{n-k} \times \mathbb{R}$, consider the $(n,k),(n+1,k+1)$ ball pairs

$$(B^n, B^k) \subseteq (B^n, B^k) \ast \{1\} = (E, D).$$

If $p: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is the natural projection, then $p^{-1}(x) \cap \partial B^k$ is an $(n−k−1)$–sphere, a point or the empty set, accordingly as $x \in \text{int} B^k$, $x \in \partial B^k$ or $x \notin B^k$. Similarly, if $p_+: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^k \times 0 \times \mathbb{R}$ is the natural projection, $p_+^{-1}(x) \cap (\partial E – \text{int } B^n)$ is an $(n−k−1)$–sphere, a point or the empty set, accordingly as $x \in \text{int } D$, $x \in \partial D$ or $x \notin D$. The projection $\mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ maps $\partial_+ E = (\partial E – \text{int } B^n)$ homeomorphically onto $B^n$. The inverse of this homeomorphism composed with the projection $p_+$ give a $UV^{n-k-2}$–map

$q: B^n \rightarrow D.$

In terms of join structures, $q$ is the composition

$$B^n = S^{n-k-1} \ast S^{k-1} \ast 0 \xrightarrow{\cong} S^{n-k-1} \ast S^{k-1} \ast 1 \xrightarrow{p_+} 0 \ast S^{k-1} \ast 1 = D.$$  

Let

$$B^n_+ = B^k \ast S^{n-k-1} \supseteq B^n.$$
Using the product structure on \((B^n_+ - \text{int } B^n)\) (pinched on \(\partial B^k\)) we can extend \(q\) to a \(UV^{n-k-2}\)–map we will still call

\[
q: B^n_+ \rightarrow B^n_+ \cup D,
\]

which is the identity on \(\partial B^n_+\) and one-to-one on \((B^n_+ - B^n)\). Using join structures, one can construct a homotopy

\[
h: B^n_+ \times [0, 1] \rightarrow B^n_+ \cup D
\]

such that

\[
h_0 = \iota: B^n_+ \hookrightarrow B^n_+ \cup D, \quad h_1 = q,
\]

and, for each \(t \in [0, 1]\) and each \(y \in B^n_+ \cup D\), \(h_t^{-1}(y)\) is either empty, a point or an \((n-k-1)\)–sphere.

If \(M\) is a topological \(n\)–manifold and \(D\) is a \((k+1)\)–cell attached to \(M\) along a \(k\)–cell \(A\) that is nice in both \(M\) and \(\partial D\), then we can use the model above to construct homotopy from the inclusion map \(M \hookrightarrow M \cup_A D\) to a map \(q: M \rightarrow M \cup_A D\), which is the identity outside a relative regular neighborhood of \(A\), rel \(\partial A\), whose point inverses are either points or \((n-k-1)\)–spheres. This implies \(q\) is \(UV^{n-k-2}\). If \(2k+1 \leq n\), then \(M\) has the \(DDP^k\) and \(q\) will be \(UV^{k-1}\). The main theorem of this section asserts that arbitrarily close approximate versions of this result hold for an \(ENR\) \(X\) with the \(DDP^k\). (Ultimately, the main results will apply to show that the inclusion \(X \subseteq X \cup_A D\) is homotopic to a \(UV^{k-1}\)–map.)

Given a closed pair \((A, B)\) in a space \(X\), a relative neighborhood of \(A\), rel \(B\), in \(X\) is a subset \(U\) of \(X\) containing \(A\) such that \(U - B\) is a neighborhood of \(A - B\) in \(X - B\).

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Theorem 6  Suppose $X$ is an ENR with the $DDP^k$, $k \geq 0$ and suppose $\gamma: C \to X$ is an embedding of a $k$–cell $C$ onto an $LCC^{k-1}$ $k$–cell $A \subseteq X$, and $\overline{X} = X \cup_A D$ is the relative mapping cylinder of $\gamma$, rel $\partial C$, with mapping cylinder retraction $d: \overline{X} \to X$. Assume a metric on $\overline{X}$ extending a given one on $X$. Then for every relative neighborhood $U$ of $A$, rel $A$, in $X$ and every $\eta > 0$, there is an $\eta$–homotopy $h: X \times I \to \overline{X}$ over $X$ of the inclusion $i: X \to \overline{X}$ such that

(i) each $h_t$ is the identity outside $U$, 

(ii) $d \circ h: X \times I \to X$ is an $\eta$–homotopy that deformation retracts a relative neighborhood of $A$, rel $\partial A$, onto $A$ inside $U$, 

(iii) $h_1: X \to \overline{X}$ is $UV^{k-1}(\eta)$ over $\overline{X}$.

Proof  Assume that $X$ is tamely embedded in $\mathbb{R}^m$, $m > 2 \dim X$, so as to have a mapping cylinder neighborhood $N$ with mapping cylinder projection $\pi: N \to X$ (Miller [22]). Given any triangulation of $N$, $\pi$ restricted to its $k$–skeleton can be approximated arbitrarily closely by an $LCC^{k-1}$ embedding whose image misses $A$. For any $\epsilon > 0$, there is a triangulation of $N$, with $k$–skeleton $T$ such that any map of a $k$–dimensional polyhedral pair $(P, Q)$ into $(N, T)$ can be $\epsilon$–homotoped, rel $Q$, into $T$. Thus, for a given sequence $\epsilon_0, \epsilon_1, \ldots$ of positive numbers, there is a sequence $T_0 \subseteq T_1 \subseteq \cdots$ of $k$–dimensional polyhedra, $LCC^{k-1}$ embedded in $X - A$, such that any map of a $k$–dimensional polyhedral pair $(P, Q)$ into $(X, T_j)$, $j < i$, can be $\epsilon_i$–homotoped, rel $Q$, into $T_i$.

Suppose we are given $\overline{X} = X \cup_A D$. Let $X' = (X \times 0) \cup (A \times I) \subseteq X \times I$, $I = [0, 1]$ and let $p: X' \to X$ be projection to the first factor. Let $g: X' \to \overline{X}$ be the map that sends each of the vertical intervals in $\partial A \times I$ to a point, but is otherwise one-to-one. We may assume $d: \overline{X} \to X$ is the map induced by $p$. Equip $X'$ with the metric $\rho$ inherited from the embedding into $\mathbb{R}^m \times [0, 1]$ with the product metric, where $X \subseteq \mathbb{R}^m \times 0$ as above and $(x, t) \mapsto (x, t)$ if $x \in A$. Since the quotient map $g: X' \to \overline{X}$ is cell-like, it is sufficient, by Lemma 3, to prove the theorem with $\overline{X}$ replaced by $X'$ and $d: \overline{X} \to X$ replaced by $p: X' \to X$.

Suppose then that we are given $\eta > 0$. Let $\{0 = t_0 < t_1 < \cdots < t_\ell = 1\}$ be a subdivision of $I$ of mesh $< \eta/3$. Given a relative neighborhood $U$ of $A$, rel $A$, positive numbers $\epsilon_0, \ldots, \epsilon_\ell$ and $k$–dimensional polyhedra $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_\ell$ in $X - A$, as above, construct a sequence of relative neighborhoods

$V_\ell \subseteq \cdots \subseteq V_1 \subseteq V_0 \subseteq U$

of $A$, rel $A$ and an $\epsilon_0$–homotopy $R: X \times I \to X$ as follows:
(1) \( R_0 = \text{id}_X \).
(2) \( R_t|[\{(X-U) \cup A\} = \text{id} \text{ for all } t \in I \).
(3) \( R(U \times I) \subseteq U \).
(4) \( R(\mathcal{C}l(V_0) \times I) \subseteq U \).
(5) \( R_I^{-1}(A) = \mathcal{C}l(V_0) \).
(6) \( R(\mathcal{C}l(V_i) \times I) \subseteq (V_{i-1} - T_{i-1}) \) for \( 1 \leq i \leq \ell \).
(7) \( R|V_i \times I \) is an \( \epsilon_i \)-homotopy, \( 0 \leq i \leq \ell \).

That is, \( R \) is an \( \epsilon_0 \)-deformation retraction of a neighborhood \( V_0 \) of \( A \) onto \( A \) inside a neighborhood \( U \) of \( A \), which has been extended to \( X \) by the estimated homotopy extension theorem of Bryant et al [5]. (For the remainder of this proof, all neighborhoods of \( A \) are relative neighborhoods of \( A \), rel \( \partial A \).)

Having constructed \( R \) satisfying (1)–(5), the neighborhoods \( V_1, \ldots, V_\ell \) satisfying properties (6) and (7) are obtained from continuity of \( R \). The positive number \( \epsilon_0 \) will be chosen so that subsets of \( X \) of diameter \( < \epsilon_0 \) will have diameter \( < \eta/2 \) throughout the homotopy \( R \). The numbers \( \epsilon_i \) (and the polyhedra \( T_i \), \( i \geq 1 \)), will be chosen inductively so that, for any polyhedral pair \( (P, Q) \) of dimension \( \leq k + 1 \) and map \( \alpha: (P, Q) \to (V_{i-1}, T_{i-1}) \), there is an \( \epsilon_i \)-homotopy of \( \alpha \), rel \( \alpha|Q \), in \( V_{i-2} \) to a map of \( P \) into \( T_i \) (where \( U = V_{-1} \)). We assume, furthermore, that \( \epsilon_i < \min \{ \epsilon_0/3, \text{dist}(A, X-V_i) \} \), for \( i > 0 \).

For each \( i = 1, \ldots, \ell \), let \( \lambda_i: (\mathcal{C}l(V_{i-1}) - V_i) \to [t_{i-1}, t_i] \) be an Urysohn function that takes \( \text{bd}(V_{i-1}) \) to \( t_{i-1} \) and \( \text{bd}(V_i) \) to \( t_i \). Combine these maps to get a map \( \lambda: X \to I \) that takes \( X-V_0 \) to 0 and \( V_\ell \) to 1.

A map \( q: X \to X' \) can then be defined by setting

(a) \( q(x) = (R_1(x), 0) \), if \( x \in (X-V_0) \),
(b) \( q(x) = (R_1(x), \lambda_i(x)) \), if \( x \in (V_{i-1} - V_i) \), \( i = 1, \ldots, \ell \), and
(c) \( q(x) = (R_1(x), 1) \), if \( x \in V_\ell \).

Then \( R_1 = p \circ q \), and the homotopy \( \text{id}_{X'} \simeq p \) composed with \( q \) gives a homotopy of \( q \) to \( p \circ q = R_1 \). Piecing this homotopy together with \( R \) gives a homotopy \( h': X \times I \to X' \) from the inclusion \( X \subseteq X' \) to \( q \).

The claim now is that \( q: X \to X' \) is \( UV^{k-1} \)(\( \eta \)).

To this end, suppose we are given a polyhedral pair \( (P, Q) \) of dimension \( \leq k \) and maps \( \alpha: P \to X' \) and \( \alpha_0: Q \to X \) with \( q \circ \alpha_0 = \alpha|Q \). As in the proof of Lemma 1 we may assume that, after a small perturbation of \( \alpha \), rel \( \alpha|Q \), there is a small regular neighborhood \( W \) of \( Q \) in \( P \) and an extension of \( \alpha_0 \) to \( W \) lifting \( \alpha|W \). This perturbation is
obtained by precomposing $\alpha$ with a perturbation of the identity on $P$ that deformation retracts $W$ to $Q$. Let $Q_0 = \text{bd}(W)$ and let $P_0 = P - \text{int}(W)$. After a second small perturbation of $\alpha$ we may assume each $S_i = \alpha^{-1}(A \times [t_{i-1}, t_i]) \cap P_0$ $(i \geq 1)$ and each $B_i = \alpha^{-1}(A \times t_i) \cap P_0$ $(i \geq 0)$ are subpolyhedra of $P_0$. Set $S_0 = \alpha^{-1}(X) \cap P_0$. Thus, we have
\[
P = W \cup P_0 = W \cup S_0 \cup S_1 \cup \cdots \cup S_{\ell},
\]
where $W \cap P_0 = \text{bd}(W)$ and $S_{i-1} \cap S_i = B_{i-1}$ for $1 \leq i \leq \ell$.

Observe that $h': X \times I \to X'$ provides a homotopy from $\alpha_0': W \to X$ to $\alpha|W: W \to X'$. Set $\alpha' = p \circ \alpha$ and observe that $\alpha'(P_0 - S_0) \subseteq A$.

Proceed inductively to move $\alpha'(P_0 - \text{int}(S_{\ell}))$ off of $A$ using the moves below:

- An $\epsilon_0$–homotopy of $\alpha'$ to a map $\alpha'_0$, that takes $B_0$ into $T_0$ and is constant outside a small neighborhood of $B_0$ in $P_0$ that misses $S_i$, $i \geq 2$.
- An $\epsilon_1$–homotopy of $\alpha'_0$ to a map $\alpha'_1$ that takes $S_1$ into $T_1$ and is constant on $S_0$ and outside a small neighborhood of $S_1$ that misses $S_i$, $i \geq 3$. Since $\alpha'_0(S_2) \subseteq A$, our choice of $\epsilon_1$ ensures that $\alpha'_1(B_1) \subseteq \alpha'_1(S_2) \subseteq V_1$.
- An $\epsilon_2$–homotopy of $\alpha'_1$ to a map $\alpha'_2$ that takes $S_2$ into $T_2$ and is constant on $S_0 \cup S_1$ and outside a small neighborhood of $S_2$ that misses $S_i$, $i \geq 4$. Since $\alpha'_1(S_3) \subseteq A$, our choice of $\epsilon_2$ ensures that $\alpha'_2(B_2) \subseteq \alpha'_2(S_3) \subseteq V_2$.

Continuing this process produces a homotopy of $\alpha'$ to $\alpha'_{\ell-1}: P_0 \to X$, which moves no point of $P_0$ more than twice, such that $\alpha'_{\ell-1}(S_i) \subseteq V_{i-2} - V_{i+1}$ for $1 \leq i \leq \ell$ (where $V_{\ell+1} = \emptyset$). Since $W$ is a (small) regular neighborhood of $Q$ in $P$, this homotopy, restricted to $\text{bd}(W)$, can be extended over $W$ to a $2\epsilon_0$–homotopy of $\alpha'|W$ that is constant on $Q$ by the estimated homotopy extension theorem. The resulting map $\overline{\alpha}: P \to X$ satisfies $\overline{\alpha}|Q = \alpha_0$.

Our choice of $\epsilon_0$ ensures that $p \circ \overline{\alpha}$ is $\eta/2$–homotopic to $p \circ \alpha$. Since $\overline{\alpha}(S_i) \subseteq V_{i-2} - V_{i+1}$, $q \circ \overline{\alpha}$ is $\eta$–homotopic to $\alpha$. \qed

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Addendum If $Z \subseteq X - A$ is a closed subset of $X$, then the homotopy $h: X \times I \to \overline{X}$ can be chosen to be fixed on $Z$.

4 Special case

We will establish Theorem 1 by first proving the special case in which $C = B$ and $p = \text{id}_B$.

**Theorem 7** Suppose $X$ is a compact, connected ENR satisfying the disjoint $(k + 1)$–disks property, $B$ is a compact, connected ANR and $f: X \to B$ is $UV^k(\epsilon)$ for some $\epsilon > 0$. Then $f$ is $(C(k) \cdot \epsilon)$–homotopic to a $UV^k$–map, where $C(k)$ is a positive constant depending only on $k$.

Moreover, if $Z$ is an $\text{LCC}^k$ subset of $X$, then the homotopy of $f$ to a $UV^k$–map can be chosen to be fixed on $Z$.

In Section 5 we indicate how the proof of Theorem 7 can be modified to obtain our main result. We shall separate the proof of Theorem 7 into two cases: $k = 0$ and $k \geq 1$. The intent is to present the main ideas first in a somewhat less cluttered setting, so that they may be a bit more transparent. This approach has, of course, introduced redundancies into the exposition, but we hope they prove to be of value to the reader.

4.1 $UV^0$

In this section we assume only that $X$ is a compact ENR satisfying the DDP$^1$, also known as the disjoint arcs property, and that $Z$ is a compact, $\text{LCC}^0$ subset of $X$. Recall from Proposition 2 that we may also assume throughout that $B$ is a compact polyhedron. We start by proving a simple homotopy analogue of our main result in the base case $k = 0$. Keep in mind that all measurements are made in $B$ unless specifically indicated otherwise.

**Proposition 8** Suppose a surjection $f: X \to B$ is a $UV^0(\delta)$–map and $\mu > 0$. Then there is an ENR $\overline{X}$ obtained by adding 1– and 2–cells to $X - Z$ and an extension $\overline{f}: \overline{X} \to B$ such that $\overline{f}$ is $UV^0(\mu)$ and $\overline{X}$ $2\delta$–collapses to $X$.

**Proof** Triangulate $B$ so that the diameter of the star of each simplex is less than $\mu' < \mu/3$, where $\mu'$ is chosen so that maps into $B$ that are $\mu'$–close are $\mu/3$–homotopic. The inverse image under $f$ of each simplex $\sigma \in B$ is compact. If $U_\sigma$ is a small path-connected open neighborhood of $\sigma$ in $B$, then $f^{-1}(\sigma)$ is contained in finitely many
components of $f^{-1}(U_\sigma)$. Attach finitely many 1–cells to $X – Z$ connecting the components of $f^{-1}(U_\sigma)$ that contain points of $f^{-1}(\sigma)$ so that their boundaries are mutually exclusive, and extend the map $f$ over each of these 1–cells so that their images lie in $U_\sigma$. Doing this for each $\sigma \in B$ produces a space $X_1$ and an extension $f_1: X_1 \to B$ of $f$. If the neighborhood $U_\sigma$ of each $\sigma \in B$ is sufficiently small, $f_1$ is $UV^0(\mu/3)$: For each simplex $\sigma$ in $B$, choose a neighborhood $V_\sigma$ of $\sigma$ lying in $U$ so that $f^{-1}(V_\sigma)$ meets only components of $f^{-1}(U_\sigma)$ which meet $f^{-1}(\sigma)$. A path in $B$ can be broken into finitely many segments, each lying in one of these sets $V_\sigma$. It suffices to $\mu'$–lift one such segment relative to given lifts on the ends. But this is easily accomplished using the 1–cells of $X_1$.

Let $C$ be a 1–cell in $C\ell(X_1 – X)$. Since $f: X \to B$ is $UV^0(\delta)$, $f_1|C$ has a $\delta$–lift to $X$, rel $\partial C$, which we may assume is an embedding into $X – Z$. Call the image arc $A$. Attach a 2–cell $D$ to $X_1$ by identifying its boundary with $A \cup C$. Call the result $X_2$, and use the $\delta$–homotopy from $f_1(C)$ to $A$ to extend $f_1$ to $f_2: X_2 \to B$. Unfortunately, the map $f_2$ is no longer $UV^0(\mu/3)$, since all we know about the image of $D$ is that it has size $\delta$ in $B$.

We remedy this as follows. Parameterize $D$ as the quotient of $A \times I$ with the intervals over $\partial A$ identified to points, and identify $A$ with $A \times 0$ and $C$ with $A \times 1$. Let $A_0$ be a finite subset of $A$ such that every point of $D$ is within $\mu/3$ (measured in $B$) of a point of $A_0 \times I \subseteq D$. Let $y$ be a point of $A_0$, let $\beta = y \times I \subseteq D$ and let $x = y \times 1 \in C$. Since $f$ is surjective, there is a point $x'$ in $X$ such that $f_2(x) = f(x')$. By changing $f$ by a small homotopy, if necessary, we can assume $x' \notin Z$. Since $f_1$ is $UV^0(\mu/3)$, there is a path $\beta'$ in $X_1 – Z$ connecting $y$ to $x'$ such that $f_2 \circ \beta$ is $\mu/3$–homotopic to $f_1 \circ \beta'$ (rel $\{x, y\}$). We have a map from $\beta$ to $\beta'$ sending $x$ to $x'$ and $y$ to $y$, so we can attach its mapping cylinder (rel $y$) to $X_2$. We can extend the map $f_2$ to this mapping cylinder, using the $(\mu/3)$–homotopy above, so that mapping cylinder fibers have size $< \mu/3$ in $B$. Thus, all points on the new 2–cell are $(\mu/3)$–close to $X$, as well. Performing this construction for all $y \in A_0$ produces a relative 2–complex $X_3$, and a map $f_3: X_3 \to B$, which, by Lemma 1, is $UV^0(\mu)$. $X_3$ $\delta$–collapses to $X_2$, which, in turn, $\delta$–collapses to $X_1 – \text{int } C$.

Repeat this construction for each 1–cell, $C \subseteq C\ell(X_1 – X)$, making sure that the corresponding family of attaching arcs is mutually exclusive in $X$. The resulting space $\overline{X}$ $2\delta$–collapses to $X$ and admits a $UV^0(\mu)$–map $\overline{f}: \overline{X} \to B$. \hfill \Box

The figure below illustrates a single 2–cell attached to $X_2$ and a single point $y \in A_0$. The placement of the path $\beta'$ is misleading, however, since it can wind about the other 1–cells we attached to $X$ when we formed $X_1$.

The following proposition provides the key to proving Theorem 7 for the case $k = 0$. 
**Proposition 9** Suppose \( f : X \to B \) is \( UV^0(\epsilon) \), and \( \mu > 0 \). Then \( f \) is \( 10\epsilon \)-homotopic, rel \( f|Z \), to a \( UV^0(\mu) \)-map.

**Proof** Suppose \( X \) and \( B \) are given as in the hypothesis, and suppose \( \mu > 0 \). By Proposition 7, we can get a \( 2\epsilon \)-homotopy of \( f \) to a surjection. By Lemma 2 the resulting map, which we shall still call \( f \), is \( UV^0(5\epsilon) \). Set \( \delta = 5\epsilon \).

Proceed as in the proof of Proposition 8. Obtain \( X_1 \subseteq X_2 \) from \( X \) by attaching 1–cells to \( X - Z \) to get \( X_1 \) and 2–cells to \( X_1 - Z \) to get \( X_2 \), together with extensions \( f_1 \subseteq f_2 \) of \( f : X \to B \) to \( X_1 \) and \( X_2 \), respectively. These were constructed so that \( f_1 \) is \( UV^0(\mu') \) and \( f_2 \) is \( UV^0(\delta) \), where \( \mu' > 0 \) will be determined later. We may assume that the arcs in \( X \) along which the 2–cells are attached to form \( X_2 \) are mutually exclusive. Enclose the attaching arcs in neighborhoods whose closures are mutually exclusive and miss \( Z \). Let \( D \) be a 2–cell of \( X_2 - X_1 \) attached to \( X \) along an arc \( A \). (The complementary arc \( C \subseteq \partial D \) was added when \( X_1 \) was constructed.) The arc \( \beta \subseteq D \) and path \( \beta' \subseteq X_1 \) from points \( x \in C \) and \( x' \in X \), respectively, to a point \( y \) in \( A \), were chosen so that \( f_2(x) = f(x') \) and \( f_2|\beta \) and \( f_1|\beta' \) are \( \mu' \)-homotopic in \( B \).

For a given \( \eta_2 > 0 \), Theorem 6 provides us with a homotopy \( h : X \times I \to X_2 \) of the inclusion \( \iota : X \to X_2 \) to a \( UV^0(\eta_2) \)-map \( q_2 : X \to X_2 \) over \( X_2 \) such that \( h \) is fixed at the identity on the complement of the union of the neighborhoods of the attaching arcs and \( h \) composed with the collapse \( X_2 \setminus X \) is an \( \eta_2 \)-homotopy on \( X \). In particular, \( h \) is fixed on \( Z \). Let \( y_1, x_1, x'_1 \) be points of \( X \) that map to \( y, x, x' \), respectively. Then there are arcs \( \beta_1 \) and \( \beta'_1 \) in \( X - Z \) joining \( y_1 \) to \( x_1 \) and \( y_1 \) to \( x'_1 \), respectively, such that \( q_2(\beta_1) \) and \( q_2(\beta'_1) \) are \( \eta_2 \)-homotopic to \( \beta \) and \( \beta' \), respectively. We may assume
that $\beta_1$ and $\beta'_1$ are embedded and that $\beta_1 \cap \beta'_1 = y_1$. We may also assume that the collection of all the arcs $\beta_1 \cup \beta'_1$ is mutually exclusive. It is possible to arrange it so that $q_2(\beta_1) = \beta$ and $q_2(\beta'_1) = \beta'$ at the expense of ending up with a map $q_2$ that is $UV^0(6\eta_2)$ over $X_2$: Given $\beta_1 \cup \beta'_1$ in $X$, let $X'$ be the space obtained by attaching $(\beta_1 \cup \beta'_1) \times I$ to $X$ so that $(\beta_1 \cup \beta'_1) \times 0$ is identified with $(\beta_1 \cup \beta'_1)$ and the intervals over the endpoints of $\beta_1$ and $\beta'_1$ are identified to points. Construct a map $X' \to X_2$ extending $q_2$ using the $\eta_2$–homotopy from $q_2(\beta_1 \cup \beta'_1) \to \beta \cup \beta'$, rel the endpoints of $\beta$ and $\beta'$. Then, by Lemma 1, this map is $UV^0(3\eta_2)$ over $X_2$. By Lemma 3 and Theorem 6, we can find a map from $X$ to $X'$ so that the composition $X \to X' \to X_2$ is $UV^0(6\eta_2)$ over $X_2$. Thus, after rescaling, we may assume that $q_2$ is $UV^0(\eta_2)$ over $X_2$, $q_2(\beta_1) = \beta$ and $q_2(\beta'_1) = \beta'$. In Proposition 8 this construction is performed a finite number of times for each of the 2–cells added to $X$ to form $X_2$. Since the collection of arcs $\beta_1 \cup \beta'_1$ is mutually exclusive, we can perform this construction for all of the arcs simultaneously; hence, we can assume that we have a $UV^0(\eta_2)$–map $q_2$: $X \to X_2$ over $X_2$ that works as above for all of the $(\beta, \beta')$ arc-path pairs. The next step in the proof of Proposition 8 was to add mapping cylinders of the maps $\beta \to \beta'$ (rel $y$) to $X_2$. The ENR $X$ is obtained from $X_2$ by attaching 2–cells (the mapping cylinders) along the family of arcs $\beta \cup \beta'$. We also obtain an extension $\tilde{f}: X \to B$ of $f_2$ that is $UV^0(\mu')$ and $\delta$–homotopic to the collapse from $X$ to $X_2$ composed with $f_2$. Form the space $X_3$ by attaching 2–cells to $X$ along the arcs $\beta_1 \cup \beta'_1$, and get a $UV^0(\eta_2)$–map $q_3$: $X_3 \to X$ over $X$ by combining $q_2$: $X \to X_2$ with a map between corresponding attaching 2–cells that realizes the mapping cylinder identification. That is, the 2–cell attached along $\beta_1 \cup \beta'_1$ should be thought of as the product $\beta_1 \times I$, with $\beta_1 \times 0$ identified with $\beta_1$, $\beta_1 \times 1$ identified with $\beta'_1$ and $y_1 \times I$ identified to the point $y_1$. Given an $\eta_3 > 0$ apply Theorem 6 to get a $UV^0(\eta_3)$–map $q_3$: $X \to X_3$ over $X_3$, along with accompanying homotopies. Lemma 3 tells us that we can choose $\mu'$, $\eta_2$ and $\eta_3$ sequentially so that, after performing the constructions above, the composition

$$X \xrightarrow{q_3} X_3 \xrightarrow{q'} X \xrightarrow{\tilde{f}} B$$

is $UV^0(\mu)$. During this process, $f$ has undergone two $\delta$– or one $10\epsilon$–homotopy, and each of these homotopies can be chosen to fixed on $Z$. \hfill \Box

**Proof of Theorem 7 in the case $k = 0$** To get a $UV^0$–map from a $UV^0(\epsilon)$–map, simply apply Proposition 9 inductively to get a sequence of homotopies of maps from $X$ to $B$, which starts with $f$ and converges to a homotopy of $f$ to a map that is $UV^0(\delta)$ for every $\delta > 0$ and is fixed on $Z$. We may make the positive number $\mu$ in Proposition 9 small enough so that the homotopy from the $UV^0(\mu)$–map to a $UV^0$–map has size $< \epsilon$; hence, $f$ is $11\epsilon$–homotopic to a $UV^0$–map, rel $f|Z$. \hfill \Box
4.2 $UV^k, k \geq 1$

Throughout this section we will assume that $X$ is a compact ENR with the $DDP^{k+1}$, $k \geq 1$, $Z$ is a compact, $LCC^k$ subset of $X$ and $B$ is a compact polyhedron. To proceed, we need the following finite generation lemma.

**Lemma 4** Suppose $f: X \to B$ is $UV^{k-1}$, where $k \geq 1$. Given $\mu > 0$, we can attach finitely many $(k+1)$–cells to $X - Z$ along their boundaries to obtain an ENR $X_1$ and an extension of $f$ to an $UV^k(\mu)$–map $f_1: X_1 \to B$.

**Proof** First observe that, if $A$ and $B$ are connected open subsets of $X$ with $\mathcal{C}(A) \subseteq B$, then there is a compact polyhedron $P$ and maps $\iota: A \to P$ and $\rho: P \to B$ such that $\rho \iota: A \to B$ is the inclusion map. This follows from the fact that $X$ is compact and is a retract of an open set in some Euclidean space. Thus, the images of the inclusion-induced maps $\pi_1(A) \to \pi_1(B)$ and $H_*(A) \to H_*(B)$ are finitely generated.

Triangulate $B$ so that each open vertex star $U$ has diameter $\ll \mu$, and its closure lies in a contractible open set $V$. Given $\alpha: I^{k+1} \to B$ with a lift $\alpha_0: \partial I^{k+1} \to X$, choose a subdivision of $I^{k+1}$ so that the image of each simplex lies in a vertex star of the triangulation of $B$. Since $f$ is $UV^{k-1}$, we can lift the $k$–skeleton of this subdivision and assume the lifts to be embeddings into $X - Z$. Attaching $(k + 1)$–cells to $X$ allows us to extend the lift over the $(k + 1)$–skeleton of $I^{k+1}$. We could then produce the desired $UV^k(\mu)$–map, provided the image of $\pi_k(f^{-1}(U))$ in $\pi_k(f^{-1}(V))$ is finitely generated. This is certainly true if $k = 1$, as observed above.

If $k \geq 2$, then $\pi_i(f^{-1}(U)) = 0$ for $0 \leq i < k$, since $f$ is $UV^{k-1}$ (see Lacher [18, Theorem 3.2]). Thus, $\pi_k(f^{-1}(U)) = H_k(f^{-1}(U); \mathbb{Z})$, by the Hurewicz Isomorphism Theorem; hence, the image of $\pi_k(f^{-1}(U))$ in $\pi_k(f^{-1}(V))$ is finitely generated. Choosing a finite set of generators for each such image and attaching $(k + 1)$–cells to kill the images completes the construction. \hfill \Box

The next result is the analogue of Proposition 8 for $k \geq 1$.

**Proposition 10** Suppose $f: X \to B$ is $UV^{k-1}$ and $UV^k(\delta)$. For every $\mu > 0$ there exists an ENR $\overline{X}$ obtained by adding cells of dimension $\leq k + 2$ to $X - Z$ and an extension $\overline{f}: \overline{X} \to B$ so that $\overline{f}$ is $UV^k(\mu)$ and $\overline{X}$ $2\delta$–collapses to $X$.

**Proof** Since $f$ is $UV^{k-1}$, Lemma 4 ensures that we can attach finitely many $(k + 1)$–cells to $X$ along their boundaries, forming $X_1$, and extend $f$ to $f_1: X_1 \to B$ so that $f_1$ is $UV^k(\mu')$, where $0 < \mu' \ll \mu$. By Proposition 3, we may assume the attaching spheres are $LCC^k$ embedded and mutually exclusive in $X - Z$. Let $C$ be one such
(k+1)–cell. Since f is $U^k \delta$, $f_1|C$ has a $\delta$–lift to $X$, rel $\partial$, which we may assume to be an $LCC^k$ embedding into $X - Z$. Call the image $A$. Attach a $(k + 2)$–cell $D$ to $X_1$ along $A \cup C$, obtaining $X_2$. The $\delta$–homotopy of $f|A$ to $f_1|C$, rel $f|\partial A (= \partial C)$, gives us an extension of $f_1$ to $f_2$: $X_2 \to B$ so that $f_2(D)$ has size $\delta$ in $B$.

Unfortunately, $f_2$ is only $U^k \delta$. We modify the proof of Proposition 8 so that we can recover property $U^k (\mu')$.

Use the $\delta$–homotopy of $f|A$ to $f_1|C$, rel $f|\partial A$, to parameterize $D$ as the quotient of $A \times I$ with the intervals in $\partial A \times I$ identified to points. Here, $A$ is identified with $A \times 0$ and $C$ is identified with $A \times 1$. Suppose $0 < \eta \ll \mu'$. Introduce the following notation:

- $J$ is the $k$–skeleton of a fine triangulation of $A$.
- $K \subseteq J$ is the $(k - 1)$–skeleton of $A$.
- $R = J \times [0, 1] \subseteq D$.
- $S = K \times [0, 1] \subseteq R \subseteq D$.
- $L = S \cup (J \times \{0, 1\}) \subseteq R \subseteq D$.

Choose the triangulation of $A$ fine enough so that if $P$ is an $i$–dimensional polyhedron, $0 \leq i \leq k$, then any map of $P$ into $D$ can be $\eta$–homotoped into $R$ (over $B$).

By the inductive hypothesis we can $\eta'$–lift the map $f_2|L$ to $X$ (rel $f_2|L \cap A$), for any preassigned $\eta' > 0$. This gives a map $\alpha_0: L \to X$, which is the identity on $L \cap A$, and which we may assume results in an $LCC^k$ embedding of $L \cup A$ into $X - Z$ (Proposition 3). Let $L'$ be the image of this map. Since $\eta'$ can be made arbitrarily small, we may use the estimated homotopy extension theorem to deform $f_2|D$ (rel $f_2|A$) slightly so that this lift is exact. Thus, we also have a map of the mapping cylinder $M$ of the map $\alpha_0: L \to L'$ (rel $J \times 0$) into $B$ with mapping cylinder fibers projecting to points in $B$. Attach $M$ to $X_2$ along $L \cup L'$ to get $X'_2$ and an extension $f'_2: X'_2 \to B$ that sends mapping cylinder fibers of $M$ to points. Observe that if $M'$ is the portion of this mapping cylinder under $S$, then $X_2 \cup M'$ $\delta$–collapses to $X_2$.

We now have a map $\alpha = f_2|R$: $(R, L) \to B$ and a lift $\alpha_0$ of $\alpha|L$ to $X - Z$. Thus, there is a $\mu'$–lift $\overline{\alpha}: R \to X_1 - Z$, and the $\mu'$–homotopy between $f_1 \circ \overline{\alpha}$ and $\alpha$ is fixed on $L$. This $\mu'$–homotopy provides an extension of $f'_2$ to the mapping cylinder $M_1 \supseteq M$ (rel $R \cap A$) of $\overline{\alpha}$ so that mapping cylinder fibers have size $\mu'$ in $B$. Attach this mapping cylinder to $X'_2$ along $M \cup R \cup \overline{\alpha}(R)$ to get $\overline{X}$, which $\delta$–collapses to $X_2$, and extend $f'_2$ to $\overline{f}: \overline{X} \to B$.

The result of this construction is to produce a relative $(k + 2)$–complex $(\overline{X}, X)$, which $2\delta$–collapses to $X$, such that every map of a $k$–dimensional polyhedron into $\overline{X}$ can
be \((\eta + \mu')\)–homotoped into \(X\) (over \(B\)). Lemma 1 guarantees that, if \(\eta\) and \(\mu'\) are sufficiently small, then \(\tilde{f}\) is \(UV^k(\mu)\).

One should observe that, although \(\tilde{f}\) is \(UV^{k-1}\) on \(X\), it is not \(UV^{k-1}\) on \(\overline{X}\). \(\square\)

Here is the key proposition for the proof of Theorem 7 when \(k \geq 1\).

**Proposition 11** Suppose \(f : X \to B\) is \(UV^k(\epsilon)\). Then there is a constant \(D(k)\), depending only on \(k\), such that, for every \(\mu > 0\), \(f\) is \((D(k) \cdot \epsilon)\)–homotopic, rel \(f|Z\), to a \(UV^k(\mu)\)–map. Moreover, the constants \(D(k)\), \(k \geq 0\), are related to the constants \(C(k)\), \(k \geq 0\) of Theorem 7 by the formula \(D(k) = 2(2C(k-1) + 1)\).

**Proof of Theorem 7 for \(k \geq 1\) assuming Proposition 11** Suppose \(f : X \to B\) is \(UV^k(\epsilon)\). Given arbitrary \(\mu > 0\), Proposition 11 assures us of the existence of a \((2(2C(k-1) + 1))\)–homotopy of \(f\), rel \(f|Z\), to a \(UV^k(\mu)\)–map. If \(\mu\) is sufficiently small, we can repeat this process to get an \(\epsilon\)–homotopy, rel \(Z\), of the resulting map to one that is \(UV^k(\eta)\) for every \(\eta > 0\), hence, \(UV^k\) by Proposition 1. Thus, \(C(k) = 4C(k-1) + 3\). Since \(C(-1) = 2\) (Proposition 7), we get the explicit formula \(C(k) = 3 \cdot 4^{k+1} - 1\). \(\square\)

**Proof of Proposition 11** We use induction on \(k\), the case \(k = 0\) having already been established. The proof of the inductive step follows closely the proof for the case \(k = 0\). We will assume Theorem 7 in dimensions \(< k\). Keep in mind throughout that, unless otherwise indicated, all measurements are made in \(B\).

Assume that \(k \geq 1\) and \(f : X \to B\) is \(UV^k(\epsilon)\) for some \(\epsilon > 0\). Assume, inductively, that \(f\) is \((C(k-1) \cdot \epsilon)\)–homotopic, rel \(f|Z\), to a \(UV^{k-1}\)–map, which we shall still call \(f\). Then, by Lemma 3 the “new” \(f\) is now \(UV^k((2C(k-1) + 1)\epsilon)\). Set \(\delta = (2C(k-1) + 1)\epsilon\).

As in the proof of Proposition 10 build a relative \((k + 2)\)–complex \((\overline{X}, X)\), which \(2\delta\)–collapses to \(X\) and on which the map \(f\) extends to a \(UV^k(\mu)\)–map \(\tilde{f} : \overline{X} \to B\) for a given \(\mu > 0\). As in the proof for \(k = 0\) we need to retrace the steps in the construction of \(\overline{X}\).

We start by constructing \(X_1 \subseteq X_2\) from \(X\) by attaching \((k + 1)\)–cells to \(X - Z\) to get \(X_1\) and \((k + 2)\)–cells to \(X_1 - Z\) to get \(X_2\). These relative complexes come with extensions \(f_1 \subseteq f_2\) of \(f : X \to B\) to \(X_1\) and \(X_2\), respectively, such that \(f_1\) is \(UV^k(\mu')\) and \(f_2\) is \(UV^k(\delta)\), where \(0 < \mu' \ll \mu\), and \(X_2\) \(\delta\)–collapses to \(X\). Each \((k + 2)\)–cell \(D\) is attached to \(X_1\) along \(\partial D = A \cup C\), where \(C\) is a \((k + 1)\)–cell attached to \(X\) while forming \(X_1\), and \(A \subsetneq X\) is the complementary \((k + 1)\)–cell.
in $\partial D$. We may assume, by Proposition 3, that the collection of cells $A$ is mutually exclusive and lies in $X - Z$.

In each $(k + 2)$–cell $D$ attached to $X$ (along a $(k + 1)$–cell $A$ in its boundary) we identify a $(k + 1)$–complex $R = J \times [0, 1]$, where $J$ is the $k$–skeleton of a fine triangulation of $A$. The next step is to attach the mapping cylinder $M$ of a map $R \to R' \subset X_1 \subset X_2$ (rel $R \cap A$) to $X_2$, and, after doing this for each $(k + 2)$–cell $D$, we obtain the space $\overline{X} \supseteq X_2$ and an extension of $f_2$ to $\overline{f}: \overline{X} \to B$ that is $UV^k(\mu')$. The space $\overline{X}$ $2\delta$–collapses to $X$: the first $\delta$–collapse comes from the collapses $M \downarrow (R \cup R')$ of the relative mapping cylinders, and the second comes from the collapses $D \downarrow A$.

For a given $\eta_2 > 0$, apply Theorem 6 to get a map $q_2: X \to X_2$ that is $UV^k(\eta_2)$ over $X_2$ and equal to the identity on $Z$. We can $\eta_2$–lift each of the complexes $R \cup R'$ to $R_1 \cup R'_1 \subset X - Z$ and assume by Proposition 3 that each of $R_1$ and $R'_1$ is homeomorphic to $R$, that each $R_1 \cup R'_1$ is embedded, and that the collection of all such lifts is mutually exclusive. By an argument similar to the one in the proof for $k = 0$, we may assume that the lifts are exact. Thus, for each complex $R_1 \cup R'_1$, there is a homeomorphism $r: R_1 \to R'_1$, which is the identity on $R_1 \cap R'_1$, that commutes with $q_2$. For each $(R_1, R'_1)$–pair attach the mapping cylinder $M_1$ of $r$ to $X$ forming $X_3$, and extend the map $q_2: X \to X_2$ to a map $q': X_3 \to \overline{X}$, which is $UV^k(\eta_2)$ over $\overline{X}$ and the identity on $Z$, using the mapping cylinder identifications $M_1 \to M$.

For a given $\eta_3$, apply Theorem 6 again to get an $UV^k(\eta_3)$–map $q_3: X \to X_3$ over $X_3$, which is the identity on $Z$. Lemma 3 tells us that we can choose $\mu'$, $\eta_2$ and $\eta_3$ sequentially so that, after performing the constructions above, the composition

$$X \xrightarrow{q_3} X_3 \xrightarrow{q'} \overline{X} \xrightarrow{\overline{f}} B$$

is $UV^k(\mu)$ over $B$.

During this process $f$ has undergone two $\delta$–homotopies (each of which fixed $Z$) so that $D(k) = 2(2C(k - 1) + 1)$. Although the resulting map is $UV^k(\mu)$, it may no longer be $UV^i$ for any $i = 0, \ldots, k$.

5 Proof of Theorem 1

We now show how to alter the proof of Theorem 7 to prove Theorem 1. The key is in establishing an analogous simple homotopy version corresponding to Propositions 8 and 10. We maintain our basic assumption that $X$ is a compact $ENR$ with the $DDP^{k+1}$ and $B$ is a compact polyhedron.
Proposition 12  Suppose \( Y \) is a metric space, \( p: B \to Y \) is a map, \( k \geq 0 \) and \( f: X \to B \) is a \( UV^{k-1} \)– and a \( UV^k(\delta) \)–map over \( Y \) for some \( \delta > 0 \) and \( Z \) is a compact, \( LCC^k \) subset of \( X \). Then for every \( \mu > 0 \), there is an ENR \( \overline{X} \) obtained by adding cells of dimension \( \leq k + 2 \) to \( X - Z \) and an extension \( \overline{f}: \overline{X} \to B \) so that \( \overline{f} \) is \( UV^k(\mu) \) over \( B \) and \( \overline{X} \) \( 2\delta \)–collapses to \( X \) over \( Y \).

Proof  Since \( f \) is \( UV^{k-1} \), we can attach finitely many \((k + 1)\)–cells to \( X - Z \) along their boundaries, forming \( X_1 \), and extend \( f \) to \( f_1: X_1 \to B \) so that \( f_1 \) is \( UV^k(\mu') \) (over \( B \)), where \( 0 < \mu' \ll \mu \) (Lemma 4). Let \( C \) be one such \((k + 1)\)–cell. Since \( f \) is \( UV^k(\delta) \) over \( Y \), the map \( f_1|C: C \to B \) has a \( \delta \)–lift \( g: C \to X \) (over \( Y \)), rel \( g|\partial C \). Let \( A = g(C) \) and assume, by Proposition 3, that \( A \) is \( LCC^{k-1} \) embedded in \( X - Z \). Using the \( \delta \)–homotopy of \( f_1|C \) to \( f \circ g \) (over \( Y \)), we may attach the mapping cylinder \( D \) of \( g \), rel \( \partial C \), to \( X_1 \) and extend \( f_1 \) to \( X_1 \cup D \). Then \( X_1 \cup D \) \( \delta \)–collapses to \( X \) over \( Y \).

The rest of the proof now follows as in the proofs of Propositions 8 and 10. As in the proofs of these two propositions, the map \( f_2 \) is no longer \( UV^k(\mu') \) over \( B \). The construction that remedies this defect, however, is exactly the same. \( \square \)

Proof of Theorem 1  After constructing \( \overline{X} \) using Proposition 12, we can apply Theorem 6 to get a homotopy of \( f \), fixing \( Z \) and controlled over \( Y \), to map that is \( UV^k(\mu) \)–map over \( B \) and over \( Y \), for some preassigned \( \mu > 0 \). The resulting map satisfies the hypotheses of Theorem 7, which takes over to complete the proof. We need only ensure that subsequent homotopies are small enough in \( B \) so that their sizes add up to \( < \epsilon \) in \( Y \). \( \square \)

6  Pseudoisotoping codimension 1 submanifolds to \( UV^k \)–maps

In this section we establish a theorem in the spirit of the early results of Keldyš [16], Cernavskii [10] and Ferry [12]. We start with the following observation.

Proposition 13  Suppose \( M \) is a compact topological \((n + 1)\)–manifold, \( N \) is a locally flat \( n \)–dimensional closed submanifold, separating \( M \) into submanifolds \( M_1 \) and \( M_2 \), such that the inclusion \( N \subseteq M_i, i = 1, 2, \) is \( UV^k(\mu) \), for some \( \mu > 0 \). Then the inclusion \( N \subseteq M \) is \( UV^k(\mu) \).

Proof  The proof of the proposition is fairly straightforward. Given a map \( \alpha: (P, Q) \to (M, N) \), where \( P \) is a polyhedron of dimension \( \leq k + 1 \), deform \( \alpha \) slightly, keeping \( \alpha|Q \) fixed, so that \( \alpha \) sends a small regular neighborhood \( W \) of \( Q \) in \( P \) into \( N \)
Moreover, if $W$ can be done simultaneously, and the resulting ambient isotopy $w$-mappings, and the associated shellings have mutually exclusive supports, then they can respect to the projection $N$ of $P_0$. Now apply the $UV^k(\mu)$ assumptions on the inclusions $N \subseteq M_i$, $i = 1, 2$, to the separate pieces of $P_0$. □

**Theorem 8** Suppose $M$ is a compact topological $(n + 1)$–manifold and $N$ is a locally flat, closed $n$–dimensional submanifold separating $M$ into submanifolds $M_1$ and $M_2$ such that each inclusion $N \subseteq M_i$, $i = 1, 2$, is $UV^k(\epsilon)$, for some $\epsilon > 0$ and $k$, $2k + 3 \leq n$. Then there is constant $D(k) > 0$, depending only on $k$, such that, for every $\mu > 0$, there is an ambient $(D(k) \cdot \epsilon)$–isotopy on $M$, supported in an arbitrarily preassigned neighborhood of a $(k + 2)$–dimensional polyhedron, to a homeomorphism $h: M \to M$ such that each of the inclusions $h(N) \subseteq M_i$, $i = 1, 2$, is $UV^k(\mu)$. Thus, there is a constant $C(k)$, depending only on $k$, such that the inclusion $N \subseteq M$ is ambient $(C(k) \cdot \epsilon)$–pseudoisotopic to a $UV^k$–map.

Moreover, if $W \subseteq N$ is a compact, $(n - 1)$–dimensional, locally flat submanifold of $N$, separating $N$ into submanifolds $N_1$ and $N_2$, such that each of the inclusions $N_j \subseteq M_i$, $i, j = 1, 2$, is $UV^k(\epsilon)$, for some $\epsilon > 0$ and $k$, $2k + 3 \leq n$, then the isotopies can be chosen to be fixed on $\partial W$.

By an *ambient pseudoisotopy* on $M$ we mean a level-preserving map $H: M \times I \to M \times I$ such that $H_0 = \text{id}_M$ and $H|M \times [0, 1): M \times [0, 1) \to M \times [0, 1)$ is a homeomorphism.

Suppose $N$ is a locally flat $n$–dimensional submanifold of a topological $(n + 1)$–manifold $M$. Suppose $g: I^k \to M_1$ is a locally flat embedding of the $k$–cell into $M$ such that $A = g(I^{k-1} \times 0) = g(I^k) \cap N$ is a locally flat $(k - 1)$–cell in $N$. Let $E \subseteq N$ be a locally flat $n$–cell in $N$ containing $A$ as a properly embedded $(k - 1)$–cell. The embedding $g$ extends to a locally flat embedding, which we shall still call $g: E \times I \to M$, such that $g(E \times 0) = E$. Using a local collar structure of $N$ in $M$ in a neighborhood of $E$, one can find an ambient isotopy $H$ on $M$, fixed outside any preassigned neighborhood of $g(E \times I)$ and on $N - \text{int } E$ taking $N$ to $(N - \text{int } E) \cup g(E)$. Moreover, $H|N$ can be made arbitrarily small with respect to the projection $N \cup E \times I \to N$. We will refer to the isotopy $H$ as a $k$–shelling. The discussion following **Lemma 3** shows that there is a $UV^{n-k-2}$–map $h: (N - \text{int } E) \cup g(E) \to N \cup g(I^k)$.

If $g_j: I^k \to M$, $j = 1, \ldots, r$, is a finite collection of mutually exclusive such embeddings, and the associated shellings have mutually exclusive supports, then they can be done simultaneously, and the resulting ambient isotopy $H$ on $M$ will be called a multi-$k$–shelling.
Proof of Theorem 8  Proceed inductively following the proof of Theorem 1. Suppose $N \subseteq M$, $M_1$, and $M_2$, are given as in the statement of the theorem with the inclusion $N \subseteq M_i$ $UV^k(\epsilon), i = 1, 2$, for some $\epsilon > 0$, $2k + 3 \leq n$. Assume $G: M \times I \to M \times I$ is an ambient $(C(k - 1) \cdot \epsilon)$–pseudoisotopy on $M$ such that $g = G_1|N: N \to M$ is $UV^{k-1}$ and, for $0 \leq t < 1$, each of the inclusions $G_t(N) \subseteq M_i, i = 1, 2$, is $UV^{k-1}(\epsilon_t)$, where $\epsilon_t \to 0$ as $t \to 1$.

Given $\mu > 0$, apply Proposition 10 (or Proposition 8) to get an ENR $N_i, i = 1, 2$, obtained by adding cells of dimension $\leq k + 2$ to $N$ and an extension $g_i: N_i \to M_i$ of $g$ so that $g_i$ is $UV^k(\mu)$ and $N_i$ $2\epsilon$–collapses to $N$. Since $2k + 3 \leq n$, we may assume the cells added to $N$ to get $N_1$ are disjoint from those added to get $N_2$.

By the estimated homotopy extension theorem there is a $t$, $0 \leq t < 1$, such that, if $g' = G_t|N$, then $g'$ can be extended to a $UV^k(\mu)$–map $g'_i: N_i \to M_i, i = 1, 2$, as well. If $2k + 3 < n$, we can assume the maps $g'_i: N_i \to M_i$ are embeddings. If $2k + 3 = n$, then we can use a standard “piping” construction to make each $g'_i$ an embedding at the possible expense of doubling the size of the collapses $N_i \subseteq N$ over $M$. We can now appeal to Rourke and Sanderson [27, Theorem 3.26] to get multi-$(k + 2)$–shelling $G'$ on $M$ such that $G_1 G_t|N: N \to M$ is $UV^k(\mu)$. Controls needed to accomplish this as we expand along the cells in the collapse are the same as in the proof of Theorem 1.

The only difference is that the constructions are performed inside of $M$.

Proposition 13 shows that the limit map $N \to M$ is $UV^k$.

The “moreover” part of the theorem is easily established using the methods above. □

The following is a corollary to the proof of Theorem 8.

Theorem 9  Suppose $X$ is a compact ENR and $Y \subseteq X$ is a closed subset such that $X - Y$ is an open topological $(n + 1)$–manifold and $Y$ is $LCC^{k+1}$ in $X$. Suppose $N \subseteq X - Y$ is a locally flat, closed $n$–dimensional submanifold separating $X$ into closed components $X_1$ and $X_2$ such that each inclusion $N \subseteq X_i, i = 1, 2$, is $UV^k(\epsilon)$, for some $\epsilon > 0$ and $2k + 3 \leq n$. Then there is constant $D(k) > 0$, depending only on $k$, such that, for every $\mu > 0$, there is an ambient $(D(k) \cdot \epsilon)$–isotopy on $X$, supported in an arbitrarily preassigned neighborhood of a $(k + 2)$–dimensional polyhedron lying in $X - Y$, to a homeomorphism $h: X \to X$ such that each of the inclusions $h(N) \subseteq X_i, i = 1, 2$, is $UV^k(\mu)$. Thus, there is a constant $C(k)$, depending only on $k$, such that the inclusion $N \subseteq X$ is ambient $(C(k) \cdot \epsilon)$–pseudoisotopic to a $UV^k$–map.

Moreover, if $W \subseteq N$ is a compact, $(n - 1)$–dimensional, locally flat submanifold of $N$, separating $N$ into submanifolds $N_1$ and $N_2$, such that each of the inclusions $N_j \subseteq X_i, i, j = 1, 2$, is $UV^k(\epsilon)$, for some $\epsilon > 0$ and $k, 2k + 3 \leq n$, then the isotopies can be chosen to be fixed on $\partial W$.  

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