

The universal character ring of some families of one-relator groups

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We study the universal character ring of some families of one-relator groups. As an application, we calculate the universal character ring of two-generator one-relator groups whose relators are palindromic and, in particular, of the $(-2, 2m + 1, 2n + 1)$ -pretzel knot for all integers m and n . For the $(-2, 3, 2n + 1)$ -pretzel knot, we give a simple proof of a result in [8] on its universal character ring, and an elementary proof of a result in [10] on the number of irreducible components of its character variety.

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1 Introduction

1.1 The character variety and the universal character ring

The set of representations of a finitely presented group G into $\mathrm{SL}_2(\mathbb{C})$ is an algebraic set defined over \mathbb{C} , on which $\mathrm{SL}_2(\mathbb{C})$ acts by conjugation. The set-theoretic quotient of the representation space by that action does not have good topological properties, because two representations with the same character may belong to different orbits of that action. A better quotient, the algebro-geometric quotient denoted by $X(G)$ (see Culler and Shalen [1] and Lubotzky and Magid [9]), has the structure of an algebraic set. There is a bijection between $X(G)$ and the set of all characters of representations of G into $\mathrm{SL}_2(\mathbb{C})$, hence $X(G)$ is usually called the *character variety* of G . It is determined by the traces of some fixed elements g_1, \dots, g_k in G . More precisely, one can find g_1, \dots, g_k in G such that for every element g in G there exists a polynomial P_g in k variables such that for any representation $\rho: G \rightarrow \mathrm{SL}_2(\mathbb{C})$ one has $\mathrm{tr}(\rho(g)) = P_g(x_1, \dots, x_k)$, where $x_j := \mathrm{tr}(\rho(g_j))$. The *universal character ring* of G is then defined to be the quotient of the polynomial ring $\mathbb{C}[x_1, \dots, x_k]$ by the ideal generated by all expressions of the form $\mathrm{tr}(\rho(u)) - \mathrm{tr}(\rho(v))$, where u and v are any two words in g_1, \dots, g_k which are equal in G ; see Lê and the author's [8]. The universal character ring of G is actually independent of the choice of g_1, \dots, g_k . The quotient of the universal character ring of G by its nilradical is equal to the ring of regular functions on the character variety $X(G)$.

1.2 Main results

Let $F_{a,w} := \langle a, w \rangle$ be the free group in 2 letters a and w . The character variety of $F_{a,w}$ is isomorphic to \mathbb{C}^3 by the Fricke–Klein–Vogt Theorem; see eg [9]. For every word u in $F_{a,w}$ there is a *unique* polynomial P_u in 3 variables such that for any representation $\rho: F_{a,w} \rightarrow \text{SL}_2(\mathbb{C})$ one has $\text{tr}(\rho(u)) = P_u(x, y, z)$, where $x := \text{tr}(\rho(a))$, $y := \text{tr}(\rho(w))$ and $z := \text{tr}(\rho(aw))$. Thus for every representation $\rho: G \rightarrow \text{SL}_2(\mathbb{C})$, where G is a group generated by a and w , we consider x, y , and z as functions of ρ .

For a word u in $F_{a,w}$, we denote by \overleftarrow{u} the word obtained from u by writing the letters in u in reversed order. The word u is called a *palindrome* if $\overleftarrow{u} = u$.

In this paper we calculate the universal character ring of some families of two-generator one-relator groups as follows.

Theorem 1 *The universal character ring of the group $\langle a, w \mid w^n \overleftarrow{r} = r^{-1} w^{n-1} \rangle$ is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the two polynomials $P_{\overleftarrow{r}} - P_{r^{-1}w^{-1}}$ and $P_{w^n \overleftarrow{r} a} - P_{r^{-1}w^{n-1}a}$.*

Theorem 2 *The universal character ring of the group $\langle a, w \mid w^n \overleftarrow{r} = r^{-1} w^{n-2} \rangle$ is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the two polynomials $P_{\overleftarrow{r}} - P_{r^{-1}w^{-2}}$ and $P_{w^n \overleftarrow{r} aw^{-1}} - P_{r^{-1}w^{n-2}aw^{-1}}$.*

As an application of [Theorem 1](#), we immediately obtain a simple proof of the following result in [8] on the universal character ring of the $(-2, 3, 2n + 1)$ –pretzel knot.

Theorem 3 [8] *The fundamental group of the $(-2, 3, 2n + 1)$ –pretzel knot is isomorphic to the group $\langle a, w \mid w^n \overleftarrow{r} = r^{-1} w^{n-1} \rangle$, where $r := a^{-1} w^{-1} a^{-1} w a$. Hence its universal character ring is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the two polynomials $Q := P_{\overleftarrow{r}} - P_{r^{-1}w^{-1}}$ and $R_n := P_{w^n \overleftarrow{r} a} - P_{r^{-1}w^{n-1}a}$. Explicitly,*

$$\begin{aligned}
 Q &= x - xy + (-3 + x^2 + y^2)z - xyz^2 + z^3, \\
 R_n &= S_{n-2}(y) + S_{n-3}(y) - S_{n-4}(y) - S_{n-5}(y) - S_{n-2}(y)x^2 \\
 &\quad + (S_{n-1}(y) + S_{n-3}(y) + S_{n-4}(y))xz - (S_{n-2}(y) + S_{n-3}(y))z^2,
 \end{aligned}$$

where the $S_k(y)$ are the Chebychev polynomials defined by $S_0(y) = 1$, $S_1(y) = y$ and $S_{k+1}(y) = yS_k(y) - S_{k-1}(y)$ for all integers k .

Applying [Theorem 3](#), we also give an elementary proof of the following result in Mattman [10] on the character variety of the $(-2, 3, 2n + 1)$ –pretzel knot.

Theorem 4 [10] Suppose $n \neq 0, 1, 2$. Then character variety of the hyperbolic $(-2, 3, 2n + 1)$ -pretzel knot has 2 irreducible components if $2n + 1$ is not divisible by 3, and has 3 irreducible components if $2n + 1$ is divisible by 3.

As another application of Theorems 1 and 2, we calculate the universal character ring of the group $G = \langle a, w \mid R = 1 \rangle$, where R is a palindromic word in $F_{a,w}$. Since R is palindromic, it either has the form $R = \overleftarrow{r}gr$ or $R = \overleftarrow{r}g^2r$, where r is a word in $F_{a,w}$ and g is either a, a^{-1}, w , or w^{-1} . Without loss of generality, we consider the case $g = w$ only.

By setting $n = 0$ in Theorems 1 and 2, we obtain:

Theorem 5 The universal character ring of the group $\langle a, w \mid \overleftarrow{r}wr = 1 \rangle$, where r is a word in a and w , is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the two polynomials $P_{\overleftarrow{r}} - P_{r^{-1}w^{-1}}$ and $P_{\overleftarrow{r}a} - P_{r^{-1}w^{-1}a}$.

Theorem 6 The universal character ring of the group $\langle a, w \mid \overleftarrow{r}w^2r = 1 \rangle$, where r is a word in a and w , is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the two polynomials $P_{\overleftarrow{r}} - P_{r^{-1}w^{-2}}$ and $P_{\overleftarrow{r}aw^{-1}} - P_{r^{-1}w^{-2}aw^{-1}}$.

Remark 1.1 By Hilden, Tejada and Toro [5], tunnel number one knots have presentations with two generators and one relator, where the relator is palindromic in the two generators. Hence Theorems 5 and 6 can be applied to calculate the universal character ring of the knot group of tunnel number one knots.

In our and T Lê's work [6; 8] on the AJ conjecture of Frohman, Gelca and Lofaro [2], Garoufalidis [3] and Gelca [4], which relates the A-polynomial and the colored Jones polynomials of a knot, it is important to understand the universal character ring of the knot group. The universal character ring has been so far calculated for a few link groups, including two-bridge knot groups (Przytycki and Sikora [12] and Lê [13]), the $(-2, 3, 2n + 1)$ -pretzel knot groups [8] (see also Theorem 3 above), two-bridge link groups (Lê and the author [7]), and the $(-2, 2m + 1, 2n)$ -pretzel link groups (see the author's [14]).

In the present paper we consider the $(-2, 2m + 1, 2n + 1)$ -pretzel knot group, where m and n are integers. As an application of Theorem 5 we will show the following:

Theorem 7 *The fundamental group of the $(-2, 2m + 1, 2n + 1)$ -pretzel knot is isomorphic to the group $\langle a, w \mid \overleftarrow{r}wr = 1 \rangle$, where*

$$r = \begin{cases} su^{k-1}awaw^{-1}a^{-1}u^{-k} & \text{if } n = 2k, \\ su^kawa^{-1}w^{-1}a^{-1}u^{-k} & \text{if } n = 2k + 1, \end{cases}$$

$$u = (awaw^{-1})^{1-m}w,$$

$$s = \begin{cases} a(w^{-1}awa)^{-l} & \text{if } m = 2l, \\ (w^{-1}awa)^{-l} & \text{if } m = 2l + 1. \end{cases}$$

Hence its universal character ring is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the two polynomials $P_{\overleftarrow{r}} - P_{r^{-1}w^{-1}}$ and $P_{\overleftarrow{r}a} - P_{r^{-1}w^{-1}a}$.

1.3 Plan of the paper

In Section 1, we collect preliminary facts and lemmas that will be repeatedly used in the proofs of the main theorems of the paper. In Section 2, we consider the universal character ring of groups and prove Theorems 1 and 2. In Section 3, we study the universal character ring of pretzel knots and prove Theorems 4 and 7.

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2 Preliminary facts and lemmas

2.1 The backward operator

Recall from Section 1 that for a word u in $F_{a,w}$, we denote by \overleftarrow{u} the word obtained from u by writing the letters in u in reversed order.

Lemma 2.1 *One has $\overleftarrow{\overleftarrow{u}} = u$, $\overleftarrow{uv} = \overleftarrow{v}\overleftarrow{u}$ and $\overleftarrow{u^{-1}} = \overleftarrow{u}^{-1}$ for all words u, v in $F_{a,w}$. Hence $\overleftarrow{u^n} = \overleftarrow{u}^n$ for all integers n .*

Proof The first two identities follow directly from the definition of the backward operator $\overleftarrow{\cdot}$. The third identity follows from the second one by taking $v = u^{-1}$. \square

We will also use the following result of [13] and [14].

Lemma 2.2 *One has $P_{uv} = P_{\overleftarrow{u}\overleftarrow{v}}$ for all words u, v in $F_{a,w}$.*

2.2 Trace identities

For all matrices A, B, C in $SL_2(\mathbb{C})$, the following trace identities are well-known:

- (1) $\text{tr } A = \text{tr } A^{-1},$
- (2) $\text{tr } AB = \text{tr } BA,$
- (3) $\text{tr } BA + \text{tr } BA^{-1} = (\text{tr } A)(\text{tr } B),$
- (4) $\text{tr } BAC + \text{tr } BA^{-1}C = (\text{tr } A)(\text{tr } BC).$

Note that (3) and (4) follow from the Cayley–Hamilton Theorem $A + A^{-1} = P_A I_{2 \times 2}$, where $I_{2 \times 2}$ is the 2×2 identity matrix.

Lemma 2.3 One has

$$P_{ucd} + P_{udc} = -P_{cd^{-1}}P_u + P_cP_{ud} + P_dP_{uc}$$

for all words c, d, u in $F_{a,w}$.

Proof We have

$$\begin{aligned} P_{u(dc)} &= P_{dc}P_u - P_{uc^{-1}d^{-1}} && \text{by (3)} \\ &= P_{dc}P_u - (P_dP_{uc^{-1}} - P_{uc^{-1}d}) && \text{by (3)} \\ &= P_{dc}P_u - P_dP_{uc^{-1}} + P_{uc^{-1}d} \\ &= P_{dc}P_u - P_d(P_cP_u - P_{uc}) + (P_cP_{ud} - P_{ucd}) && \text{by (3) and (4)} \\ &= (P_{cd} - P_cP_d)P_u + P_cP_{ud} + P_dP_{uc} - P_{ucd} && \text{by (2)} \\ &= -P_{cd^{-1}}P_u + P_cP_{ud} + P_dP_{uc} - P_{ucd} && \text{by (3)}. \end{aligned}$$

The lemma follows. □

2.3 Chebyshev polynomials

Let $S_k(t)$ be the Chebyshev polynomials defined by $S_0(t) = 1, S_1(t) = t$ and $S_{k+1}(t) = tS_k(t) - S_{k-1}(t)$ for all integers k .

It is easy to see that $S_k(2) = k + 1$ and $S_k(-2) = (-1)^k(k + 1)$ for all integers k .

Lemma 2.4 One has $S_k^2(t) - tS_k(t)S_{k-1}(t) + S_{k-1}^2(t) = 1$.

Proof Let $g_k(t) = S_k^2(t) - tS_k(t)S_{k-1}(t) + S_{k-1}^2(t)$. Then

$$\begin{aligned} g_k(t) &= (S_k(t) - tS_{k-1}(t))S_k(t) + S_{k-1}^2(t) \\ &= -S_{k-2}(t)S_k(t) + (tS_{k-2} - S_{k-3}(t))S_{k-1}(t) \\ &= S_{k-2}(t)(tS_{k-1}(t) - S_k(t)) - S_{k-3}(t)S_{k-1}(t) \\ &= S_{k-2}^2(t) - (tS_{k-2}(t) - S_{k-1}(t))S_{k-1}(t) \\ &= g_{k-1}(t). \end{aligned}$$

It means that $g_k(t)$ does not depend on k and so $g_k(t) = g_0(t) = 1$. Hence

$$S_k^2(t) - tS_k(t)S_{k-1}(t) + S_{k-1}^2(t) = g_k(t) = 1. \quad \square$$

3 Proof of Theorems 1 and 2

3.1 The universal character ring of two-generator one-relator groups

Proposition 3.1 Let $G := \langle a, w \mid u = v \rangle$, where u and v are two words in $F_{a,w}$. Then the universal character ring of G is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the four polynomials $P_u - P_v$, $P_{ua} - P_{va}$, $P_{uw} - P_{vw}$ and $P_{uwa} - P_{vwa}$.

Proof By [14, Proposition 1.1], the universal character ring of G is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the five polynomials $P_u - P_v$, $P_{ua} - P_{va}$, $P_{uw} - P_{vw}$, $P_{uaw} - P_{vaw}$ and $P_{uwa} - P_{vwa}$. From Lemma 2.3 it follows that

$$\begin{aligned} (P_{uaw} - P_{vaw}) + (P_{uwa} - P_{vwa}) \\ = -P_{aw^{-1}}(P_u - P_v) + P_a(P_{uw} - P_{vw}) + P_w(P_{ua} - P_{va}). \end{aligned}$$

Hence the universal character ring of G is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the four polynomials $P_u - P_v$, $P_{ua} - P_{va}$, $P_{uw} - P_{vw}$ and $P_{uwa} - P_{vwa}$. □

Remark 3.2 From the proof of [14, Proposition 1.1], it is easy to see that the polynomial $P_{uwa} - P_{vwa}$ in Proposition 3.1 can be replaced by any polynomial of the form $P_{ug_1^{\epsilon_1} g_2^{\epsilon_2}} - P_{vg_1^{\epsilon_1} g_2^{\epsilon_2}}$, where $\{g_1, g_2\} = \{a, w\}$ and $\epsilon_1, \epsilon_2 \in \{\pm 1\}$.

3.2 Proof of Theorem 1

The group in Theorem 1 is $\langle a, w \mid w^n \overleftarrow{r} = r^{-1} w^{n-1} \rangle$.

To prove Theorem 1 we will need the following propositions.

Proposition 3.3 One has

$$P_{w^n \overleftarrow{r} \overleftarrow{u}} - P_{r^{-1} w^{n-1} \overleftarrow{u}} = (P_{w^n \overleftarrow{r} u w^{-1}} - P_{r^{-1} w^{n-1} u w^{-1}}) - P_{u w^{n-1}} (P_{\overleftarrow{r}} - P_{r^{-1} w^{-1}}),$$

for all words u in $F_{a,w}$.

Proof We have

$$\begin{aligned} P_{w^n \overleftarrow{r} \overleftarrow{u}} &= P_{\overleftarrow{w^n \overleftarrow{r} \overleftarrow{u}}} && \text{by Lemma 2.2} \\ &= P_{w^n u r} && \text{by Lemma 2.1} \\ &= P_{(w^{n-1} u)(r w)} \\ &= P_{w^{n-1} u} P_{r w} - P_{(w^{n-1} u)(r w)^{-1}} && \text{by (3)} \\ &= P_{u w^{n-1}} P_{r^{-1} w^{-1}} - P_{r^{-1} w^{n-1} u w^{-1}} && \text{by (1) and (2)}. \end{aligned}$$

Similarly,

$$\begin{aligned} P_{r^{-1} w^{n-1} \overleftarrow{u}} &= P_{\overleftarrow{r^{-1} w^{n-1} \overleftarrow{u}}} && \text{by Lemma 2.2} \\ &= P_{\overleftarrow{r}^{-1} u w^{n-1}} && \text{by Lemma 2.1} \\ &= P_{u w^{n-1}} P_{\overleftarrow{r}} - P_{\overleftarrow{r} u w^{n-1}} && \text{by (3)} \\ &= P_{u w^{n-1}} P_{\overleftarrow{r}} - P_{w^n \overleftarrow{r} u w^{-1}} && \text{by (1) and (2)}. \end{aligned}$$

Hence

$$P_{w^n \overleftarrow{r} \overleftarrow{u}} - P_{r^{-1} w^{n-1} \overleftarrow{u}} = (P_{w^n \overleftarrow{r} u w^{-1}} - P_{r^{-1} w^{n-1} u w^{-1}}) - P_{u w^{n-1}} (P_{\overleftarrow{r}} - P_{r^{-1} w^{-1}}). \quad \square$$

Proposition 3.4 One has

$$P_{w^n \overleftarrow{r}} - P_{r^{-1} w^{n-1}} = -(S_{n-1}(y) + S_{n-2}(y))(P_{\overleftarrow{r}} - P_{r^{-1} w^{-1}}).$$

Proof Let $g_n = P_{w^n \overleftarrow{r}} - P_{r^{-1} w^{n-1}}$. By applying Identity (3), it is easy to show that $g_{n+1} = y g_n - g_{n-1}$ for all integers n (note that $P_w = y$). By definition, $g_0 = P_{\overleftarrow{r}} - P_{r^{-1} w^{-1}}$. Applying Lemmas 2.1, 2.2 and (1), we get

$$g_1 = P_{w \overleftarrow{r}} - P_{r^{-1}} = P_{w r} - P_r = P_{r^{-1} w^{-1}} - P_{\overleftarrow{r}} = -(P_{\overleftarrow{r}} - P_{r^{-1} w^{-1}}).$$

Hence, by induction on n , we can easily show that

$$g_n = -(S_{n-1}(y) + S_{n-2}(y))(P_{\overleftarrow{r}} - P_{r^{-1}w^{-1}}).$$

Proposition 3.4 follows. □

We now prove Theorem 1. Let $f_n(u) = P_{w^n \overleftarrow{r} u} - P_{r^{-1}w^{n-1}u}$ for $u \in F_{a,w}$. Then, by Proposition 3.1, the universal character ring of the group $G = \langle a, w \mid w^n \overleftarrow{r} = r^{-1}w^{n-1} \rangle$ is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal I generated by the four polynomials $f_n(1)$, $f_n(a)$, $f_n(w)$ and $f_n(wa)$, where $x = P_a$, $y = P_w$ and $z = P_{aw}$.

Let $Q = P_{\overleftarrow{r}} - P_{r^{-1}w^{-1}}$. Since \overleftarrow{r} and $r^{-1}w^{-1}$ are conjugate in G (by w^n), it is clear from the definition of the universal character ring that Q is contained in the ideal I .

By Proposition 3.3,

$$f_n(\overleftarrow{u}) = f_n(uw^{-1}) - P_{uw^{n-1}}Q.$$

In particular, we have $f_n(w) = f_n(1) - P_{w^n}Q$ and $f_n(wa) = f_n(a) - P_{aw^n}Q$. By Proposition 3.4, $f_n(1) = -(S_{n-1}(y) + S_{n-2}(y))Q$. Hence the ideal I is generated by the two polynomials $Q = P_{\overleftarrow{r}} - P_{r^{-1}w^{-1}}$ and $f_n(a) = P_{w^n \overleftarrow{r} a} - P_{r^{-1}w^{n-1}a}$. Theorem 1 follows.

3.3 Proof of Theorem 2

The group in Theorem 2 is $\langle a, w \mid w^n \overleftarrow{r} = r^{-1}w^{n-2} \rangle$.

To prove Theorem 2 we will need the following propositions.

Proposition 3.5 *One has*

$$P_{w^n \overleftarrow{r} \overleftarrow{u}} - P_{r^{-1}w^{n-2} \overleftarrow{u}} = P_{w^n \overleftarrow{r} (wuw^{-1})} - P_{r^{-1}w^{n-2} (wuw^{-1})}$$

for all words u in $F_{a,w}$.

Proof From the proof of Proposition 3.3 we have

$$P_{w^n \overleftarrow{r} \overleftarrow{u}} = P_{rw}P_{uw^{n-1}} - P_{r^{-1}w^{n-2} (wuw^{-1})}.$$

Similarly,

$$\begin{aligned} P_{r^{-1}w^{n-2} \overleftarrow{u}} &= P_{\overleftarrow{r^{-1}w^{n-2} \overleftarrow{u}}} && \text{by Lemma 2.2} \\ &= P_{\overleftarrow{r^{-1}} uw^{n-2}} && \text{by Lemma 2.1} \\ &= P_{(\overleftarrow{r}w)^{-1} (uw^{n-1})} && \text{by (2)} \\ &= P_{\overleftarrow{r}w}P_{uw^{n-1}} - P_{w^n \overleftarrow{r} (wuw^{-1})} && \text{by (1) and (2)}. \end{aligned}$$

Hence

$$P_{w^n \overleftarrow{r} u} - P_{r^{-1} w^{n-2} u} = P_{w^n \overleftarrow{r} (wuw^{-1})} - P_{r^{-1} w^{n-2} (wuw^{-1})} + P_{uw^{n-1}} (P_{rw} - P_{\overleftarrow{r} w}).$$

The proposition follows, since $P_{rw} - P_{\overleftarrow{r} w} = 0$ by Lemma 2.2. □

Proposition 3.6 *One has*

$$\begin{aligned} (P_{w^n \overleftarrow{r} u} - P_{r^{-1} w^{n-2} u}) &+ (P_{w^n \overleftarrow{r} (wuw^{-1})} - P_{r^{-1} w^{n-2} (wuw^{-1})}) \\ &= -P_{uw^{-2}} (P_{w^n \overleftarrow{r} u} - P_{r^{-1} w^{n-2} u}) + P_{uw^{-1}} (P_{w^n \overleftarrow{r} w} - P_{r^{-1} w^{n-2} w}) \\ &\quad + P_w (P_{w^n \overleftarrow{r} (uw^{-1})} - P_{r^{-1} w^{n-2} (uw^{-1})}) \end{aligned}$$

for all words u in $F_{a,w}$.

Proof By Lemma 2.3, for any word v in $F_{a,w}$, we have

$$\begin{aligned} P_{vu} + P_{v(wuw^{-1})} &= P_{v(uw^{-1})w} + P_{vw(uw^{-1})} \\ &= -P_{uw^{-2}} P_v + P_{uw^{-1}} P_{vw} + P_w P_{v(uw^{-1})}. \end{aligned}$$

In particular,

$$\begin{aligned} P_{w^n \overleftarrow{r} u} + P_{w^n \overleftarrow{r} (wuw^{-1})} &= -P_{uw^{-2}} P_{w^n \overleftarrow{r} u} + P_{uw^{-1}} P_{w^n \overleftarrow{r} w} \\ &\quad + P_w P_{w^n \overleftarrow{r} (uw^{-1})}, \\ P_{r^{-1} w^{n-2} u} + P_{r^{-1} w^{n-2} (wuw^{-1})} &= -P_{uw^{-2}} P_{r^{-1} w^{n-2} u} + P_{uw^{-1}} P_{r^{-1} w^{n-2} w} \\ &\quad + P_w P_{r^{-1} w^{n-2} (uw^{-1})}. \end{aligned}$$

The proposition follows by taking the difference of the two identities above. □

Proposition 3.7 *One has*

$$\begin{aligned} P_{w^n \overleftarrow{r} u} - P_{r^{-1} w^{n-2} u} &= -S_{n-2}(y) (P_{\overleftarrow{r} u} - P_{r^{-1} w^{-2} u}), \\ P_{w^n \overleftarrow{r} w} - P_{r^{-1} w^{n-2} w} &= -S_{n-1}(y) (P_{\overleftarrow{r} w} - P_{r^{-1} w^{-2} w}). \end{aligned}$$

Proof The proof is similar to that of Proposition 3.4, so we omit the details. □

We now prove Theorem 2. Let $f_n(u) = P_{w^n \overleftarrow{r} u} - P_{r^{-1} w^{n-2} u}$. Then, by Proposition 3.1 and Remark 3.2, the universal character ring of the group $G = \langle a, w \mid w^n \overleftarrow{r} = r^{-1} w^{n-2} \rangle$ is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal I generated by the four polynomials $f_n(1)$, $f_n(a)$, $f_n(w)$ and $f_n(aw^{-1})$, where $x = P_a$, $y = P_w$ and $z = P_{aw}$.

By Proposition 3.5, $f_n(\overleftarrow{u}) = f_n(uuw^{-1})$. Moreover, we have

$$f_n(u) + f_n(uuw^{-1}) = -P_{uw^{-2}}f_n(1) + P_{uw^{-1}}f_n(w) + P_w f_n(uw^{-1})$$

by Proposition 3.6. Hence

$$f_n(u) + f_n(\overleftarrow{u}) = -P_{uw^{-2}}f_n(1) + P_{uw^{-1}}f_n(w) + P_w f_n(uw^{-1}).$$

In particular,

$$(5) \quad 2f_n(a) = -P_{aw^{-2}}f_n(1) + P_{aw^{-1}}f_n(w) + P_w f_n(aw^{-1}).$$

Let $Q = P_{\overleftarrow{r}} - P_{r^{-1}w^{-2}}$. Since \overleftarrow{r} and $r^{-1}w^{-2}$ are conjugate in G (by w^n), it is clear that from the definition of the universal character ring that Q is contained in the ideal I .

By Proposition 3.7, $f_n(1) = -S_{n-2}(y)Q$ and $f_n(w) = -S_{n-1}(y)Q$. These identities and (5) imply that the ideal I is generated by Q and $f_n(aw^{-1})$, and so the universal character ring of G is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the two polynomials $P_{\overleftarrow{r}} - P_{r^{-1}w^{-2}}$ and $P_{w^n \overleftarrow{r} aw^{-1}} - P_{r^{-1}w^{n-2}aw^{-1}}$.

4 Pretzel knots

4.1 Proof of Theorem 7

The fundamental group of the $(-2, 2m + 1, 2n + 1)$ -pretzel knot is

$$\pi = \langle a, b, c \mid bab^{-1} = (ac)^{-m}c(ac)^m, a^{-1}ba = (cb)^n c(cb)^{-n} \rangle.$$

The first relation in the group is $(ac)^m ba = c(ac)^m b$, ie $a(ca)^{m-1}cba = ca(ca)^{m-1}cb$. Let $w = (ca)^{m-1}cb$, then $awa = caw$. It implies that $ca = awaw^{-1}$ and $cb = (ca)^{1-m}w = (awaw^{-1})^{1-m}w$. Let $u = (awaw^{-1})^{1-m}w$. Then $cb = u$ and so

$$b = c^{-1}u = awa^{-1}w^{-1}a^{-1}(awaw^{-1})^{1-m}w = a(awaw^{-1})^{-m}w.$$

The second relation in the group π becomes $(awaw^{-1})^{-m}wa = u^n awaw^{-1}a^{-1}u^{-n}$, which is equivalent to

$$(awaw^{-1}a^{-1}u^{-n})^{-1} = (u^{-n}(awaw^{-1})^{-m}wa)^{-1}.$$

Therefore

$$\pi = \langle a, w \mid u^n awa^{-1}w^{-1}a^{-1} = a^{-1}w^{-1}awaw^{-1} \rangle.$$

To proceed, we will need the following lemma.

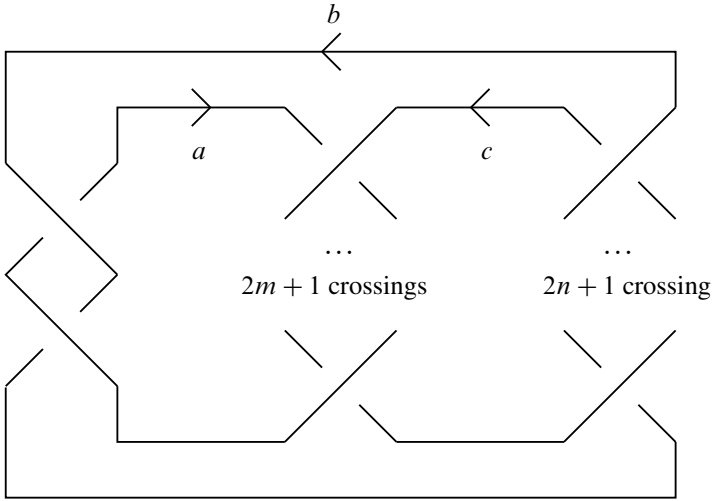


Figure 1: The $(-2, 2m + 1, 2n + 1)$ -pretzel knot

Lemma 4.1 One has $u = \overleftarrow{s}ws$, where

$$s = \begin{cases} a(w^{-1}awa)^{-l} & \text{if } m = 2l, \\ (w^{-1}awa)^{-l} & \text{if } m = 2l + 1. \end{cases}$$

In particular, u is palindromic, ie $u = \overleftarrow{u}$.

Proof We first note that $(uv)^{k+1} = u(vu)^k v$ for all integers k . If $m = 2l$ then

$$\begin{aligned} u &= (awaw^{-1})^{-l}(awaw^{-1})^{1-l}w \\ &= (awaw^{-1})^{-l}[(awa)(w^{-1}awa)^{-l}w^{-1}]w \\ &= [(awaw^{-1})^{-l}a]w[a(w^{-1}awa)^{-l}]. \end{aligned}$$

By Lemma 2.1, $\overleftarrow{s} = (awaw^{-1})^{-l}a$. Hence $u = \overleftarrow{s}ws$.

If $m = 2l + 1$ then similarly

$$\begin{aligned} u &= (awaw^{-1})^{-l-1}(awaw^{-1})^{1-l}w \\ &= (awaw^{-1})^{-l-1}(awa)(w^{-1}awa)^{-l} \\ &= (awaw^{-1})^{-l}w(w^{-1}awa)^{-l}. \end{aligned}$$

Hence $u = \overleftarrow{s}ws$, where $s = (w^{-1}awa)^{-l}$. □

Proposition 4.2 One has $\pi = \langle a, w \mid \overleftarrow{r}wr = 1 \rangle$, where

$$r = \begin{cases} su^{k-1}awaw^{-1}a^{-1}u^{-k} & \text{if } n = 2k, \\ su^kawa^{-1}w^{-1}a^{-1}u^{-k} & \text{if } n = 2k + 1. \end{cases}$$

Proof Recall that $\pi = \langle a, w \mid u^n awa^{-1}w^{-1}a^{-1} = a^{-1}w^{-1}awau^{n-1} \rangle$.

If $n = 2k$ then the relation in π is $u^{2k}awa^{-1}w^{-1}a^{-1} = a^{-1}w^{-1}awau^{2k-1}$, which is equivalent to $u^{-k}a^{-1}w^{-1}awau^{2k-1}awaw^{-1}a^{-1}u^{-k} = 1$.

If $n = 2k + 1$ then the relation in π is $u^{2k+1}awa^{-1}w^{-1}a^{-1} = a^{-1}w^{-1}awau^{2k}$, which is equivalent to $u^{-k}a^{-1}w^{-1}a^{-1}wau^{2k+1}awa^{-1}w^{-1}a^{-1}u^{-k} = 1$.

The proposition then follows from Lemma 4.1. □

We now complete the proof of Theorem 7. Proposition 4.2 and Theorem 5 imply that the universal character of π is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the two polynomials $P_{\overleftarrow{r}} - P_{r^{-1}w^{-1}}$ and $P_{\overleftarrow{r}a} - P_{r^{-1}w^{-1}a}$.

4.2 Proof of Theorem 4

Let V be the character variety of the $(-2, 3, 2n + 1)$ -pretzel knot. Then by Theorem 3, V is the zero locus of the two polynomials Q and R_n , where

$$Q = x - xy + (x^2 + y^2 - 3)z - xyz^2 + z^3,$$

$$R_n = (y + 2)S_{n-2}(y) - (y^2 + y - 2)S_{n-3}(y) - S_{n-2}(y)x^2$$

$$+ ((y - 1)S_{n-2}(y) + yS_{n-3}(y))xz - (S_{n-2}(y) + S_{n-3}(y))z^2.$$

It is known that 3-strand pretzel knots are small knots (see [11]), hence by [1] their character varieties have irreducible components of dimension 1 only. Therefore, all irreducible components of V have dimension exactly 1.

Note that if $n = 0, 1$ or 2 then the $(-2, 3, 2n + 1)$ -pretzel knot is a torus knot, otherwise it is hyperbolic. From now on we suppose that $n \neq 0, 1, 2$.

Lemma 4.3 *Suppose $\gcd(2n + 1, 3) = 1$. Then $z \neq 0$ on V except a finite number of points.*

Proof Fix $z = 0$. Then $Q = x(1 - y)$ and

$$R_n = (y + 2)S_{n-2}(y) - (y^2 + y - 2)S_{n-3}(y) - S_{n-2}(y)x^2.$$

Note that $S_k(2) = k + 1$ for all integers k .

If $x = 0$ then $R_n = p(y)$, where $p(y) := (y + 2)S_{n-2}(y) - (y^2 + y - 2)S_{n-3}(y)$. Note that $p(2) = 4S_{n-2}(2) - 4S_{n-3}(2) = 4(n - 1) - 4(n - 2) = 4$, hence $p(y)$ is a nonzero polynomial in y and so it has a finite number of roots.

If $y = 1$ then $R_n = (3 - x^2)S_{n-2}(y)$. Note that $S_{3k+2}(1) = 0, S_{3k}(1) = S_{3k+1}(1) = (-1)^k$. Since $\gcd(2n + 1, 3) = 1$, we have $S_{n-2}(y) = S_{n-2}(1) = \pm 1$. Hence $R_n = 0$ if and only if $x = \pm\sqrt{3}$. The lemma follows. □

By Lemma 4.3, we separate the proof of Theorem 4 into 2 cases: $\gcd(2n + 1, 3) = 1$ and $\gcd(2n + 1, 3) = 3$.

4.3 The case $\gcd(2n + 1, 3) = 1$

Then by Lemma 4.3, $z \neq 0$ on V except a finite number of points. Without loss of generality, we may suppose $z \neq 0$ on V . Let $Q' = Qz^{-1}$ and $R'_n = R_n + S_{n-2}(y)Q'$. Then we have

$$Q' = x^2 - (yz^2 + y - 1)z^{-1}x + y^2 + z^2 - 3 \quad \text{and} \quad R'_n = -\alpha z^{-1}x + \beta,$$

where

$$\begin{aligned} \alpha &= (z^2 + y - 1)S_{n-2}(y) - yz^2S_{n-3}(y), \\ \beta &= (y^2 + y - 1)S_{n-2}(y) - (y^2 + y - 2 + z^2)S_{n-3}(y). \end{aligned}$$

To proceed, we will need the following lemma.

Lemma 4.4 *One has $\alpha \neq 0$ on $V \cap \{z \neq 0\}$ except a finite number of points.*

Proof Assume $\alpha = 0$ on $V \cap \{z \neq 0\}$. Then $\alpha = \beta = 0$, which implies that

$$\begin{aligned} (y - 1)S_{n-2}(y) &= (yS_{n-3}(y) - S_{n-2}(y))z^2, \\ (y^2 + y - 1)S_{n-2}(y) - (y^2 + y - 2)S_{n-3}(y) &= z^2S_{n-3}(y). \end{aligned}$$

Hence

$$\begin{aligned} (6) \quad (yS_{n-3}(y) - S_{n-2}(y))[(y^2 + y - 1)S_{n-2}(y) - (y^2 + y - 2)S_{n-3}(y)] \\ - (y - 1)S_{n-2}(y)S_{n-3}(y) = 0. \end{aligned}$$

Let $q(y)$ be the left-hand side of (6). Note that $q(2) = (n - 3)(n + 3) - (n - 1)(n - 2) = 3n - 11 \neq 0$. It implies that $q(y)$ is a nonzero polynomial in y and so it has a finite number of roots. For each root y of $q(y)$, the system

$$\begin{aligned} (y - 1)S_{n-2}(y) &= (yS_{n-3}(y) - S_{n-2}(y))z^2, \\ (y^2 + y - 1)S_{n-2}(y) - (y^2 + y - 2)S_{n-3}(y) &= z^2S_{n-3}(y), \end{aligned}$$

has at most 2 solutions z , since either $yS_{n-3}(y) - S_{n-2}(y)$ or $S_{n-3}(y)$ is nonzero (by Lemma 2.4). For each solution (y, z) of the system $\alpha = \beta = 0$, the equation $Q' = 0$ has at most 2 solutions x . Therefore $\alpha \neq 0$ on $V \cap \{z \neq 0\}$ except on a finite number of points. □

Since $\gcd(2n + 1, 3) = 1$, by Lemmas 4.3 and 4.4 we may assume that $\alpha \neq 0$ and $z \neq 0$ on V . The equation $R'_n = 0$ is then equivalent to $x = z\beta/\alpha$. Hence

$$\begin{aligned} \alpha^2 Q' &= z^2 \beta^2 - (yz^2 + y - 1)\beta\alpha + (y^2 + z^2 - 3)\alpha^2 \\ &= (-2 + 3y - y^3 + z^2)\{S_{n-2}(y)^2 - (y - 1)S_{n-2}(y)S_{n-3}(y) \\ &\quad - (3S_{n-2}(y)^2 - (2y + 1)S_{n-2}(y)S_{n-3}(y) + 2S_{n-3}(y)^2)z^2 \\ &\quad + (S_{n-2}(y)^2 - yS_{n-2}(y)S_{n-3}(y) + S_{n-3}(y)^2)z^4\}. \end{aligned}$$

By Lemma 2.4, we have $S_{n-2}(y)^2 - yS_{n-2}(y)S_{n-3}(y) + S_{n-3}(y)^2 = 1$. It follows that

$$\begin{aligned} \alpha^2 Q' &= (-2 + 3y - y^3 + z^2)\{1 + S_{n-2}(y)S_{n-3}(y) - S_{n-3}(y)^2 \\ &\quad - (2 + S_{n-2}(y)^2 - S_{n-2}(y)S_{n-3}(y))z^2 + z^4\} \\ &= (-2 + 3y - y^3 + z^2)T(y, z), \end{aligned}$$

where

$$\begin{aligned} T(y, z) &:= t_0(y) + t_2(y)z^2 + z^4, \\ t_0(y) &:= 1 + S_{n-2}(y)S_{n-3}(y) - S_{n-3}(y)^2, \\ t_2(y) &:= -(2 + S_{n-2}(y)^2 - S_{n-2}(y)S_{n-3}(y)). \end{aligned}$$

Lemma 4.5 *Suppose $n \neq 1, 2$. Then $t_0(y) \in \mathbb{C}[y]$ is a polynomial of positive degree and it does not have any repeated factors.*

Proof Note that $S_k(2) = k + 1$ and $S_k(-2) = (-1)^k(k + 1)$ for all integers k . It follows that $h_0(y) = 1 + S_{n-2}(y)S_{n-3}(y) - S_{n-3}(y)^2$ is equal to $n - 1$ if $y = 2$; and is equal to $(n - 1)(5 - 2n)$ if $y = -2$. Hence $h_0(y) \in \mathbb{C}[y]$ is a polynomial of positive degree since $n \neq 1, 2$.

We have $t_0(y) = S_{n-2}(y)(S_{n-2}(y) - (y - 1)S_{n-3}(y)) = S_{n-2}(y)(S_{n-3}(y) - S_{n-4}(y))$. If $n \geq 4$ then

$$\begin{aligned} S_{n-2}(y) &= \prod_{j=1}^{n-2} \left(y - 2 \cos \frac{j\pi}{n-1} \right), \\ S_{n-3}(y) - S_{n-4}(y) &= \prod_{j=1}^{n-3} \left(y - 2 \cos \frac{(2j-1)\pi}{2n-5} \right), \end{aligned}$$

(see, for example, [8, Lemma 4.13]), hence

$$t_0(y) = \prod_{j=1}^{n-2} \left(y - 2 \cos \frac{j\pi}{n-1} \right) \prod_{j=1}^{n-3} \left(y - 2 \cos \frac{(2j-1)\pi}{2n-5} \right)$$

does not have any repeated factors.

Similarly, if $n \leq -1$ then by letting $n' = -(n+1) \geq 0$, we have

$$\begin{aligned} t_0(y) &= S_{n'+1}(y)(S_{n'+2}(y) - S_{n'+3}(y)) \\ &= - \prod_{j=1}^{n'+1} \left(y - 2 \cos \frac{j\pi}{n'+2} \right) \prod_{j=1}^{n'+3} \left(y - 2 \cos \frac{(2j-1)\pi}{2n'+7} \right) \end{aligned}$$

since $S_k(y) = -S_{-k-2}(y)$ for all integers k . Hence $h_0(y)$ does not have any repeated factors. If $n = 0$ then $t_0(y) = -(y^2 - y - 1)$. If $n = 3$ then $t_0(y) = y$. The lemma follows. □

Proposition 4.6 *Suppose $n \neq 0, 1, 2$. Then $T(y, z)$ is irreducible in $\mathbb{C}[y, z]$.*

Proof If $T(y, z)$ has a factor $z + f(y)$, where $f(y) \neq 0$, then $z - f(y)$ is also a factor of $T(y, z)$. Hence $T(y, z)$ has a factor $z^2 - f^2(y)$.

If $T(y, z)$ has a factor $z^2 + f(y)z + g(y)$, where $f(y), g(y) \neq 0$, then it is easy to see that $z^2 - f(y)z + g(y)$ is also a factor of $T(y, z)$. In this case, we have

$$\begin{aligned} z^4 + h_2(y)z^2 + h_0(y) &= (z^2 + f(y)z + g(y))(z^2 - f(y)z + g(y)) \\ &= z^4 + (2g(y) - f^2(y))z^2 + g^2(y). \end{aligned}$$

Hence $g^2(y) = t_0(y)$, which is impossible since $t_0(y) \in \mathbb{C}[y]$ is a polynomial of positive degree and it does not have any repeated factors by Lemma 4.5.

Assume that the polynomial $T(y, z)$ is reducible. Then by the above arguments, we may suppose that $T(y, z) = (z^2 + g(y))(z^2 + t_2(y) - g(y))$, where $g(y) \neq 0$. In this case $g(y)(t_2(y) - g(y)) = t_0(y)$ which implies that $t_2(y)^2/4 - t_0(y) = (g(y) - t_2(y)/2)^2$. Note that $t_2(y)^2 - 4t_0(y) = (4 + S_{n-2}(y)^2)(S_{n-2}(y) - S_{n-3}(y))^2$. Hence

$$4 + S_{n-2}(y)^2 = (2g(y) - t_2(y))^2 / (S_{n-2}(y) - S_{n-3}(y))^2.$$

Let $h(y) = (2g(y) - t_2(y)) / (S_{n-2}(y) - S_{n-3}(y)) \in \mathbb{C}[y]$ then $4 + S_{n-2}(y)^2 = h(y)^2$, ie $(h(y) - S_{n-2}(y))(h(y) + S_{n-2}(y)) = 4$. It follows that both $h(y) - S_{n-2}(y)$ and $h(y) + S_{n-2}(y)$ are constant polynomials, and so is $S_{n-2}(y)$. This can not occur since $n \neq 0, 1, 2$. Therefore $T(y, z)$ is irreducible in $\mathbb{C}[y, z]$. □

We now complete the proof of [Theorem 4](#). Note that $T(2, z) = z^4 - (n+1)z^2 + n - 1$ is not divisible by $(-2 + 3y - y^3 + z^2)|_{y=2} = z^2 - 4$, since $T(2, \pm 2) = 11 - 3n \neq 0$. Hence $T(y, z)$ is not divisible by $-2 + 3y - y^3 + z^2$ and so, by [Proposition 4.6](#), $\alpha^2 Q' = (-2 + 3y - y^3 + z^2)T(y, z)$ has exactly 2 irreducible factors. Therefore V has exactly 2 irreducible components.

4.4 The case $\gcd(2n + 1, 3) = 3$

From the proof of [Lemma 4.3](#) and the proof of [Theorem 4](#) for the case $\gcd(2n+1, 3) = 1$, it is easy to see that in this case V has exactly 3 irreducible components, where one of them is $\{z = 0, y = 1\}$.

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