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Borsuk–Ulam theorems and their parametrized versions for spaces of type (a, b)

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Let X be a space of type (a,b) equipped with a free G–action, with $G=\mathbb{Z}_2$ or S^1. In this paper, we prove some theorems of Borsuk–Ulam-type and the corresponding parametrized versions for such G–spaces.

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1 Introduction

Following the second author, H K Singh and T Singh [11] and H K Singh [13], we define a space of type (a,b) as follow. Let X be a simply connected finite CW complex with $\mathbb{Z}$–cohomology groups satisfying $H^j(X;\mathbb{Z})=\mathbb{Z}$, if $j=0,n,2n$ or $3n$, and $H^j(X;\mathbb{Z})=0$, otherwise $(n>1)$. Let u_i generate $H^{in}(X;\mathbb{Z})$, for $i=0,1,2$ and 3. Then the structure of the $\mathbb{Z}$–cohomology ring of X is determined by the two integers a and b for which $u_1^2=au_2$ and $u_1u_2=bu_3$. In this case, X is said to be of type (a,b). These spaces include certain products of spheres and projective spaces, and were first studied by James [6] and Toda [16].

In [11], Pergher et al proved that $G=\mathbb{Z}_2$ cannot act freely on a space of type (a,b) if a is odd and b is even, and $G=S^1$ cannot act freely on a space of type (a,b) if $a\neq 0$. For the remaining (a,b), we may have free actions, for example, $S^3\times S^6$ is of type $(0,1)$ and admits free G–actions for $G=Z_2$ and S^1 (for other examples, see Dotzel and T Singh [4] and [13]), and also in [11] the possible $\mathbb{Z}_2$–cohomology rings of orbit spaces X/G of free actions of $G=\mathbb{Z}_2$ on spaces of type (a,b), where a and b are even, and of free actions of $G=S^1$ on spaces of type $(0,b)$, were determined. For $G=S^1$, one has two possibilities for the ring structure of the $\mathbb{Z}_2$–cohomology of X/G, which are described in Theorem 2.5. We denote by Λ_1, (respectively Λ_2), the collection of all free $G=S^1$–actions on X for which $H^*(X/G;\mathbb{Z}_2)$ has the structure described in Theorem 2.5(Λ_1), (respectively, in Theorem 2.5(Λ_2)).

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The first aim of this paper is to prove results of Borsuk–Ulam-type involving spaces X of type (a,b). For general information about the Borsuk–Ulam Theorem, including many of the concepts in this paper, the book [8] of Matoušek is recommended. In this direction, the results below concern to the existence of equivariant maps.

Theorem 1.1 (i) *Let X be a space of type (a,b), characterized by a natural number $n > 1$, where a and b are even, and let Y be a Hausdorff, pathwise connected and paracompact space. Suppose that X and Y are equipped with free $\mathbb{Z}_2$–actions and $H^{k+1}(Y/G;\mathbb{Z}_2) = 0$, for some k, $1 \le k < 3n$. Then, there is no $\mathbb{Z}_2$–equivariant map $f\colon X \to Y$.*

(ii) *Let X be a space of type $(0,b)$, characterized by a natural odd number $n > 1$, and let Y be a Hausdorff, pathwise connected and paracompact space. Suppose that X is equipped with a free S^1–action $\rho \in \Lambda_1$, (respectively $\rho \in \Lambda_2$), and Y is equipped with a free S^1–action; further, suppose $H^{k+1}(Y/G;\mathbb{Z}_2) = 0$, for some k, with $1 \le k < 3n$, (respectively $1 \le k < n$). Then, there is no S^1–equivariant map $f\colon X \to Y$.*

Remark In the above direction, some related results were obtained in [11], concerning the existence of equivariant maps $S^m \to X$, where S^m is equipped with standard G–actions ($G = \mathbb{Z}_2$ or S^1) and X is a space of type (a,b) equipped with arbitrary free G–actions.

Note that, in Theorem 1.1, Y can be taken as the k–dimensional sphere S^k.

In addition, the following Borsuk–Ulam-type theorems will be obtained.

Theorem 1.2 *Let X be a space of type (a,b), characterized by a natural number $n > 1$, where a and b are even. Suppose X is equipped with a free $\mathbb{Z}_2$–action, determined by a free involution $T\colon X \to X$.*

(i) *Then, for every continuous map $f\colon X \to \mathbb{R}^k$,*

$$\operatorname{cov.dim} A(f) \ge 3n - k \quad \text{if } 3n \ge k,$$

where $A(f)$ denotes the $\mathbb{Z}_2$–coincidence set of f (that is, $A(f) = \{x \in X \mid f(x) = f(T(x))\}$).

(ii) *If Y is a finite k–dimensional CW complex and $3n \ge 2k$, then for every continuous map $f\colon X \to Y$, $A(f)$ is nonempty.*

Remark Theorem 1.2(i) is the Yang version of the Borsuk–Ulam theorem for spaces of type (a,b). In particular, we will compute the $\mathbb{Z}_2$–index of Yang for these $\mathbb{Z}_2$–spaces. Theorem 1.2(ii) has the spirit of the results of Izydorek and Jaworowski [5], with spheres being replaced by spaces of type (a,b).

The second general goal of this paper is to prove parametrized Borsuk–Ulam theorems for spaces of type (a,b). Jaworowski [7], Dold [3], Nakaoka [10] and others extended the Borsuk–Ulam problem to a fibrewise setting, by considering continuous maps $f\colon S(E)\to E'$ which preserve fibres, where E and E' are total spaces of vector bundles over a space B and $S(E)$ means the associated sphere bundle. In this direction, related results were proved by the first and third authors in [9] (for bundles whose fibre has the same cohomology (mod p) of a product of spheres, with any free $\mathbb{Z}_p$–action, and for bundles whose fibre has the same rational cohomology ring as a product of spheres, with any free S^1–action), and in M Singh [14] (for bundles whose fibre has the mod 2 cohomology algebra of a real or complex projective space, with any free involution).

In this paper, we obtain results of this nature, for bundles whose fibre is a space of type (a,b) with any free $\mathbb{Z}_2$–action and a,b even (or free S^1–action with $a=0$). Specifically, we will prove the following two theorems.

Theorem 1.3 *Let X be a space of type (a,b), characterized by a natural number $n>1$, where a and b are even. Given a paracompact space B, let $\pi\colon X\hookrightarrow E\to B$ be a fibre bundle equipped with a fibrewise free $\mathbb{Z}_2$–action, such that the quotient bundle $\hat{\pi}\colon \hat{E}\to B$ has the cohomology extension property; see Spanier [15, Chapter 5, Section 7] and Bredon [1, page 372]. Also, consider $\pi'\colon E'\to B$, a k–dimensional vector bundle, equipped with a fibrewise $\mathbb{Z}_2$–action on E', which is free on $E'-\{0\}$ ($\{0\}$ is the image of the zero-section). Let $f\colon E\to E'$ be a fibre preserving equivariant map and set $Z_f=f^{-1}(\{0\})$. If $3n\geq k$, then*

$$\operatorname{cohom.dim} Z_f \geq \operatorname{cohom.dim} B + 3n-k.$$

Theorem 1.4 *Let X be a space of type $(0,b)$, characterized by a natural odd number $n>1$. Given a paracompact space B, let $\pi\colon X\hookrightarrow E\to B$ be a fibre bundle with a fibrewise free S^1–action, such that the quotient bundle $\hat{\pi}\colon \hat{E}\to B$ has the cohomology extension property. Let $\pi'\colon E'\to B$ be a k–dimensional vector bundle, where k is even, with fibrewise S^1–action on E', which is free on $E'-\{0\}$. Consider $f\colon E\to E'$, a fibre preserving equivariant map and set $Z_f=f^{-1}(\{0\})$.*

(1) *If the free S^1–action ρ on X belongs to Λ_1 and $3n\geq k$, then*

$$\operatorname{cohom.dim} Z_f \geq \operatorname{cohom.dim} B + 3n-k.$$

(2) *If the free S^1–action ρ on X belongs to Λ_2 and $n \geq k$, then*

$$\text{cohom.}\dim Z_f \geq \text{cohom.}\dim B + n - k.$$

Finally, in the next result, we estimate the size of the $\mathbb{Z}_2$–coincidence set of a fibre preserving map.

Theorem 1.5 *Let X be a space of type (a,b), characterized by a natural number $n > 1$, where a and b are even. Given a paracompact space B, let $\pi\colon X \hookrightarrow E \to B$ a fibre bundle, equipped with a fibrewise free $\mathbb{Z}_2$–action, such that the quotient bundle $\widehat{\pi}\colon \widehat{E} \to B$ has the cohomology extension property. Consider $\pi''\colon E'' \to B$, a k–dimensional vector bundle, and $f\colon E \to E''$ be a fibre preserving map. As before, consider $A(f) = \{x \in E \mid f(x) = f(Tx)\}$, the $\mathbb{Z}_2$–coincidence set of f, where $T\colon E \to E$ is the generator of the free $\mathbb{Z}_2$–action on E. If $3n \geq k$, then*

$$\text{cohom.}\dim A(f) \geq \text{cohom.}\dim B + 3n - k.$$

Remark If B is a point, Theorem 1.5 reduces to Theorem 1.2(i).

The paper is organized as follows. In Section 2, we recall the required definitions and results, and establish notation. In Section 3, we compute a numerical index for spaces of type (a,b), which is related to the $\mathbb{Z}_2$–index of Yang. By using these indices, we prove Theorems 1.1 and 1.2. In Section 4, we present some lemmas involving the $H^*(B)$–algebra of $H^*(\widehat{E})$. In Section 4.2, we prove such Lemmas and Theorems 1.3, 1.4 and 1.5, using characteristic polynomials (these characteristic polynomials are presented in Section 4.1).

2 Preliminaries

We start by introducing some basic facts and establishing some notation. We assume that all spaces under consideration are paracompact and Hausdorff spaces. Here H^* denotes Čech cohomology, unless otherwise indicated. The symbol "$\cong$" denotes an appropriate isomorphism between algebraic objects.

Suppose that G is a compact Lie group. Write B_G, as usual, for the classifying space of G and $E_G \to B_G$ for the universal G–bundle. Given a G–space X, there is an associated fibration $p_X\colon X_G \to B_G$, with fibre X, where $X_G = (E_G \times X)/G$ is the Borel construction. There is also a natural map $\eta\colon X_G \to X/G$ which is a homotopy equivalence if G acts freely on X, and thus in this case the cohomology rings $H^*(X_G)$

and $H^*(X/G)$ are isomorphic. Associated to the fibration $p_X\colon X_G \to B_G$, one has the cohomological Leray–Serre spectral sequence. This spectral sequence has

$$E_2^{k,l} = H^k(B_G; \mathcal{H}^l(X; R)),$$

as its E_2–term and converges to $H^*(X_G; R)$ as an algebra in the sense of Bredon, where R is a commutative ring with unit; here, $H^k(B_G; \mathcal{H}^l(X; R))$ is the cohomology of B_G with local coefficients in the cohomology of X.

Suppose that X is connected. Then the local coefficients system $\mathcal{H}^0(X; R)$ over B_G is trivial and

$$E_2^{*,0} = H^*(B_G; H^0(X; R)) = H^*(B_G; R).$$

We say that the index of the G–space X is s, which depends on R, and we write $i(X; R) = s$, if the following condition is satisfied:

$$E_2^{*,0} = \cdots = E_s^{*,0} \neq E_{s+1}^{*,0}.$$

If $E_2^{*,0} = \cdots = E_\infty^{*,0}$, we say that $i(X; R) = \infty$.

This index has the following property.

Proposition 2.1 (Volovikov [17, Property(iii), page 917]) *If $G = \mathbb{Z}_2$ and X is a free G–space, then $i(X;\mathbb{Z}_2) = i(X)$ exceeds the $\mathbb{Z}_2$–index of Yang of [18] by unity, ie,*

(1) $$i(X) = 1 + \mathbb{Z}_2\text{–Yang-index } (X).$$

Others results related to $i(X)$ include the following.

Theorem 2.2 [17, Theorem 2.2, page 918, $G = \mathbb{Z}_2$] *Let X be a compact and connected $\mathbb{Z}_2$–space such that $i(X;\mathbb{Z}_2) \geq 2m+1$. Let Y be a CW complex of dimension m and $f\colon X \to Y$ a continuous map. In addition, if $i(X;\mathbb{Z}_2) = 2m+1$, assume that $f^*\colon H^m(Y) \to H^m(X)$ is trivial. Then $A(f)$ is nonempty.*

Theorem 2.3 (Coelho and the second and third authors [2, Theorem 1.1]) *Let G be a compact Lie group and X, Y be Hausdorff, pathwise connected and paracompact free G–spaces. Suppose that for some natural $m \geq 1$, $i(X; R) \geq m+1$ and $H^{k+1}(Y/G; R) = 0$, for some $1 \leq k \leq m$.*

- (i) *If $k = m$ and $\beta_m(X; R) < \beta_{m+1}(B_G; R)$, there is no G–equivariant map $f\colon X \to Y$.*
- (ii) *If $1 \leq k < m$ and $0 < \beta_{k+1}(B_G; R)$, there is no G–equivariant map $f\colon X \to Y$.*

Here, $\beta_i(\cdot\,; R)$ denotes the i^{th} Betti number.

We recall the following well known facts:

$$H^*(B_G;\mathbb{Z}_2)=\begin{cases}\mathbb{Z}_2[s] & \deg s=1,\ G=\mathbb{Z}_2,\\ \mathbb{Z}_2[t] & \deg t=2,\ G=S^1.\end{cases}$$

2.1 The cohomology rings of some orbit spaces

In [11], Pergher et al determined the possible $\mathbb{Z}_2$–cohomology rings of orbit spaces X/G of free actions of $G=\mathbb{Z}_2$ on spaces of type (a,b), where a and b are even, and of free actions of $G=S^1$ on spaces of type $(0,b)$. This is described below.

Theorem 2.4 [11, Theorem 4.1] *Let $G=\mathbb{Z}_2$ act freely on a space X of type (a,b), characterized by a natural number $n>1$, where both a and b are even. Then, as a graded commutative algebra,*

$$H^*(X/G;\mathbb{Z}_2)=\mathbb{Z}_2[x,z]/\langle x^{3n+1},z^2+\alpha x^n z+\beta x^{2n},x^{n+1}z\rangle,$$

where $\alpha,\beta\in\mathbb{Z}_2$, $\deg x=1$ and $\deg z=n$.

Theorem 2.5 [11, Theorem 4.2] *Let $G=S^1$ act freely on a space X of type $(0,b)$, characterized by a natural number $n>1$. Then $H^*(X/G;\mathbb{Z}_2)$ is isomorphic to one of the following graded commutative algebras:*

(Λ_1) $\mathbb{Z}_2[y,z]/\langle y^{(3n+1)/2},z^2+\alpha y^n,y^{(n+1)/2}z\rangle$, *where $\alpha\in\mathbb{Z}_2$, $\deg y=2$, and $\deg z=n$.*

(Λ_2) $\mathbb{Z}_2[y,z]/\langle y^{(n+1)/2},z^2\rangle$, *where $\deg y=2$, $\deg z=2n$, and b is odd.*

3 Proofs of the theorems of Borsuk–Ulam-type

In this section, we prove Theorems 1.1 and 1.2. We need the following lemma, where the G–spaces X are understood as those described in the statements of these two theorems.

Lemma 3.1 (i) *If $G=\mathbb{Z}_2$ or $G=S^1$, with $\rho\in\Lambda_1$, then*

$$i(X;\mathbb{Z}_2)=3n+1.$$

(ii) *If $G=S^1$, with $\rho\in\Lambda_2$, then*

$$i(X;\mathbb{Z}_2)=n+1.$$

Proof In the case that $G=\mathbb{Z}_2$, for the generators of $H^*(X;\mathbb{Z}_2)$, we have the relations $u_1^2=0$ and $u_1u_2=0$. For the corresponding spectral sequence, one has that $E_2^{k,l}\cong H^k(BG)\otimes H^l(X)$, the sequence does not collapse at the E_2–term and no line can survive to infinity (see [11, proof of Theorem 4.1]). By the multiplicative properties of the spectral sequence, we have $d_{n+1}(1\otimes u_1)=0$, $d_{n+1}(1\otimes u_3)=0$ and $d_{n+1}(1\otimes u_2)\neq 0$. Therefore, we get that $E_{n+2}^{k,l}=\mathbb{Z}_2$, for every k, if $l=0$ or $l=3n$. Also, we have $E_{n+2}^{k,l}=\mathbb{Z}_2$, for $k=0,1,2,\dots,n$, if $l=n$. In the remaining cases, $E_{n+2}^{k,l}=0$. Again, the multiplicative properties show that $d_{n+2}(1\otimes u_i)=0$, for $i=1,2,3$, and $d_{n+3}(1\otimes u_3)\neq 0$.

Then, the differential

$$d_{3n+1}\colon E_{3n+1}^{k,3n}\to E_{3n+1}^{k+3n+1,0}$$

is an isomorphism and for all $k\geq 0$,

$$E_{3n+2}^{k+3n+1,0}=\frac{\ker d_{3n+1}}{\operatorname{im} d_{3n+1}}=\frac{E_{3n+1}^{k+3n+1,0}}{E_{3n+1}^{k+3n+1,0}}=0\neq E_{3n+1}^{k+3n+1,0}.$$

Thus,

$$E_2^{*,0}=\cdots=E_{3n+1}^{*,0}\neq E_{3n+2}^{*,0},$$

which implies $i(X;\mathbb{Z}_2)=3n+1$.

For $G=S^1$, in both cases, the proof is analogous by using the properties of the corresponding spectral sequence given in [11, proof of Theorem 4.2]. □

Proof of Theorem 1.1 For (i), by Lemma 3.1(i), we have that $i(X;\mathbb{Z}_2)=3n+1$. Since $H^{k+1}(Y/\mathbb{Z}_2;\mathbb{Z}_2)=0$ for some $1\leq k<3n$ and $\beta_{k+1}(B_{\mathbb{Z}_2})=1$, it follows from Theorem 2.3(ii) that there is no equivariant map $X\to Y$. The argument is analogous for (ii). □

Proof of Theorem 1.2 By Lemma 3.1(i), $i(X;\mathbb{Z}_2)=3n+1$. It follows from Proposition 2.1 that the $\mathbb{Z}_2$–index of X is $3n$. Thus, from [18, Theorem 4.1, page 270],

$$\operatorname{cov.dim} A(f)\geq 3n-k,$$

which proves (i). For (ii), since $i(X;\mathbb{Z}_2)=3n+1\geq 2k+1$ and additionally if $i(X;\mathbb{Z}_2)=2k+1$, $f^*\colon H^k(Y)\to H^k(X)$ is trivial, it follows from Theorem 2.2 that $A(f)$ is nonempty. □

4 Proof of parametrized Borsuk–Ulam theorems for spaces of type (a, b)

In this section, we prove Theorems 1.3, 1.4 and 1.5. First we develop a technical discussion on the objects involved in these theorems, for which will be assumed the hypotheses described in their statements. We will need some lemmas involving the $H^*(B)$–algebra of $H^*(\widehat{E})$, where $\widehat{E}$ is the total space of quotient bundle $\widehat{\pi}\colon \widehat{E} \to B$.

Given a topological space X of type (a, b), where a and b are even (respectively, a topological space X of type $(0, b)$), let $\pi : X \hookrightarrow E \to B$ be a fibre bundle equipped with a fibrewise free $\mathbb{Z}_2$–action (respectively, fibrewise free S^1–action) such that the quotient bundle $\widehat{\pi}\colon \widehat{E} \to B$ has the cohomology extension property. Consider $\pi'\colon E' \to B$ a k–dimensional vector bundle equipped with a fibrewise G–action ($G = \mathbb{Z}_2$ or S^1), which is free on $E' - \{0\}$. If $f\colon E \to E'$ is a fibre preserving G–equivariant map, write $Z_f = f^{-1}(\{0\})$ and $\widehat{Z}_f = Z_f/G$.

Let $H^*(B)[x, z]$ be the polynomial ring over $H^*(B)$ in the indeterminates x and z. For $G = \mathbb{Z}_2$, in Section 4.1 we will introduce certain characteristic polynomials belonging to $H^*(B)[x, z]$, denoted by $W_1(x, z)$, $W_2(x, z)$ and $W_3(x, z)$, and will show that $H^*(\widehat{E})$ and $H^*(B)[x, z]/\langle W_1(x, z), W_2(x, z), W_3(x, z)\rangle$ are isomorphic as $H^*(B)$–modules. Therefore, each polynomial $q(x, z)$ in $H^*(B)[x, z]$ determines an element of $H^*(\widehat{E})$, which will be denoted by $q(x, z)|_{\widehat{E}}$. We will write $q(x, z)|_{\widehat{Z}_f}$ for the image of $q(x, z)|_{\widehat{E}}$ by the $H^*(B)$–homomorphism

$$i^*\colon H^*(\widehat{E}) \to H^*(\widehat{Z}_f),$$

where i^* is the homomorphism induced by the natural inclusion.

Similarly, for $G = S^1$, we will show that if the free S^1–action on X is in Λ_1 (respectively in Λ_2), then $H^*(B)[y, z]/\langle W_1(y, z), W_2(y, z), W_3(y, z)\rangle$ (respectively $H^*(B)[y, z]/\langle W_1(y), W_2(y, z)\rangle$) and $H^*(\widehat{E})$ are isomorphic as $H^*(B)$–modules; again, $W_1(y, z), W_2(y, z)$ and $W_3(y, z)$ will be certain special characteristic polynomials belonging to $H^*(B)[y, z]$. Therefore, each polynomial $q(y, z)$ in $H^*(B)[y, z]$ yields elements $q(y, z)|_{\widehat{E}}$ and $q(y, z)|_{\widehat{Z}_f}$ in $H^*(\widehat{E})$ and $H^*(\widehat{Z}_f)$, respectively.

Also, we will recall the known characteristic polynomial $W'(x)$, used by Dold [3] (and called there the Stiefel–Whitney polynomial), which is a characteristic polynomial in the indeterminate x of degree 1, associated to the vector bundle $\pi'\colon E' \to B$. With these objects in hand, we have the following lemmas.

Lemma 4.1 (Case $G=\mathbb{Z}_2$) *Suppose that* $q(x,z)\in H^*(B)[x,z]$ *is a polynomial satisfying* $q(x,z)|\hat{Z}_f=0$. *Then, there are polynomials* $r_1(x,z)$, $r_2(x,z)$ *and* $r_3(x,z)$ *in* $H^*(B)[x,z]$ *so that*

$$q(x,z)W'(x)=r_1(x,z)W_1(x,z)+r_2(x,z)W_2(x,z)+r_3(x,z)W_3(x,z).$$

Lemma 4.2 (Case $G=S^1$) *Suppose that* $q(y,z)\in H^*(B)[y,z]$ *is a polynomial satisfying* $q(y,z)|\hat{Z}_f=0$.

(i) *If the free* S^1*–action on* X *is in* Λ_1, *then there are polynomials* $r_1(y,z)$, $r_2(y,z)$ *and* $r_3(y,z)$ *in* $H^*(B)[y,z]$ *so that*

$$q(y,z)W'(y)=r_1(y,z)W_1(y,z)+r_2(y,z)W_2(y,z)+r_3(y,z)W_3(y,z).$$

(ii) *If the free* S^1*–action on* X *is in* Λ_2, *then there are polynomials* $r_1(y,z)$ *and* $r_2(y,z)$ *in* $H^*(B)[y,z]$ *so that*

$$q(y,z)W'(y)=r_1(y,z)W_1(y)+r_2(y,z)W_2(y,z).$$

4.1 Characteristic polynomials

As announced above and using the Dold technique, in this section we introduce the characteristic polynomials associated to the fibre bundle (X,E,π,B). Since the quotient bundle $(X/G,\hat{E},\hat{\pi},B)$ (G is $\mathbb{Z}_2$ or S^1) has the cohomology extension property, the Leray–Hirsch Theorem can be applied (see [1, Chapter VII, Theorem 1.4]). There are two cases to consider.

4.1.1 Case $G=\mathbb{Z}_2$ From Theorem 2.4, one has that $H^*(X/G;\mathbb{Z}_2)$ is a free graded module generated by the elements

$$1,a,a^2,\dots,a^{3n-1},a^{3n},c,ac,\dots,a^nc,$$

subject to the relations $a^{3n+1}=0$, $c^2+\alpha a^nc+\beta a^{2n}=0$ and $a^{n+1}c=0$, where $a\in H^1(X/G;\mathbb{Z}_2)$, $c\in H^n(X/G;\mathbb{Z}_2)$ and $\alpha,\beta\in\mathbb{Z}_2$.

It follows from the Leray–Hirsch theorem that there exist elements $\mathbf{a}\in H^1(\hat{E})$ and $\mathbf{c}\in H^n(\hat{E})$ such that the natural homomorphism $j^*\colon H^*(\hat{E})\to H^*(X/G)$ maps $\mathbf{a}$ into a and $\mathbf{c}$ into c. Further, via the induced homomorphism $\hat{\pi}^*$, $H^*(\hat{E})$ is an $H^*(B)$–module generated by

$$1,\mathbf{a},\mathbf{a}^2,\dots,\mathbf{a}^{3n-1},\mathbf{a}^{3n},\mathbf{c},\mathbf{ac},\dots,\mathbf{a}^n\mathbf{c}. \tag{2}$$

Then, we can express the elements $\mathbf{a}^{3n+1} \in H^{3n+1}(\widehat{E})$, $\mathbf{a}^{n+1}\mathbf{c} \in H^{2n+1}(\widehat{E})$ and $\mathbf{c}^2 + \alpha\mathbf{a}^n\mathbf{c} + \beta\mathbf{a}^{2n} \in H^{2n}(\widehat{E})$ in terms of the basis (2), that is, there exist unique elements $\omega_i, \overline{\omega}_i, \nu_i, \overline{\nu}_i, \mu_i, \overline{\mu}_i \in H^i(B)$ such that

$$\begin{aligned}
\mathbf{a}^{3n+1} &= \omega_{3n+1} + \omega_{3n}\mathbf{a} + \cdots + \omega_1\mathbf{a}^{3n} + \overline{\omega}_{2n+1}\mathbf{c} + \overline{\omega}_{2n}\mathbf{ac} + \cdots + \overline{\omega}_{n+1}\mathbf{a}^n\mathbf{c},\\
\mathbf{a}^{n+1}\mathbf{c} &= \nu_{2n+1} + \nu_{2n}\mathbf{a} + \cdots + \nu_1\mathbf{a}^{2n} + \gamma\mathbf{a}^{2n+1} + \overline{\nu}_{n+1}\mathbf{c} + \overline{\nu}_n\mathbf{ac} + \cdots + \overline{\nu}_1\mathbf{a}^n\mathbf{c},\\
\mathbf{c}^2 + \alpha\mathbf{a}^n\mathbf{c} + \beta\mathbf{a}^{2n} &= \mu_{2n} + \mu_{2n-1}\mathbf{a} + \cdots + \mu_1\mathbf{a}^{2n-1} + \beta'\mathbf{a}^{2n} + \overline{\mu}_n\mathbf{c} + \cdots\\
&\qquad + \overline{\mu}_1\mathbf{a}^{n-1}\mathbf{c} + \alpha'\mathbf{a}^n\mathbf{c},
\end{aligned}$$

where $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{Z}_2$. The announced characteristic polynomials in the indeterminates x and z of degrees 1 and n, respectively, associated to the fibre bundle (X, E, π, B), are then defined by the following formulas:

$$\begin{aligned}
W_1(x,z) &= \omega_{3n+1} + \omega_{3n}x + \cdots + \omega_1x^{3n} + x^{3n+1} + \overline{\omega}_{2n+1}z + \cdots + \overline{\omega}_{n+1}x^nz,\\
W_2(x,z) &= \nu_{2n+1} + \nu_{2n}x + \cdots + \nu_1x^{2n} + \gamma x^{2n+1} + \overline{\nu}_{n+1}z + \cdots + \overline{\nu}_1x^nz + x^{n+1}z,\\
W_3(x,z) &= \mu_{2n} + \mu_{2n-1}x + \cdots + \mu_1x^{2n-1} + (\beta+\beta')x^{2n} + \overline{\mu}_nz + \cdots\\
&\qquad + \overline{\mu}_1x^{n-1}z + (\alpha+\alpha')x^nz + z^2.
\end{aligned}$$

Consider the homomorphism of $H^*(B)$–algebras,

$$\sigma\colon H^*(B)[x,z] \to H^*(\widehat{E}), \quad \text{determined by } (x,z) \mapsto (\mathbf{a},\mathbf{c}). \tag{3}$$

We have that $\ker(\sigma)$ is the ideal generated by the characteristic polynomials $W_1(x,z)$, $W_2(x,z)$ and $W_3(x,z)$ and, consequently,

$$H^*(B)[x,z]/\langle W_1(x,z), W_2(x,z), W_3(x,z)\rangle \cong H^*(\widehat{E}). \tag{4}$$

The characteristic polynomial for the bundle $\pi'\colon E' \to B$ Following [3; 10], we first recall the characteristic polynomial associated to a k–dimensional vector bundle $\pi'\colon E' \to B$, equipped with a fibrewise $\mathbb{Z}_2$–action which is free on $E' - (\{0\})$. Write SE' for the total space of the sphere bundle associated to $\pi'\colon E' \to B$. Since $\mathbb{Z}_2$ acts freely on SE', we obtain the projective bundle $(\mathbb{R}P^{k-1}, \widehat{SE'}, \widehat{\pi}', B)$ and the principal $\mathbb{Z}_2$–bundle $SE' \to \widehat{SE'}$. We have that

$$H^*(\mathbb{R}P^{k-1}; \mathbb{Z}_2) \cong \mathbb{Z}_2[a']/\langle a'^k\rangle,$$

where $a' = (i')^*(s)$, $s \in H^1(B\mathbb{Z}_2)$ is the generator and $i'\colon \mathbb{R}P^{k-1} \to B\mathbb{Z}_2$ is a classifying map for the principal $\mathbb{Z}_2$–bundle $S^{k-1} \to \mathbb{R}P^{k-1}$. Consider the class $\mathbf{a}' = h^*(s) \in H^1(\widehat{SE'})$, where $h\colon \widehat{SE'} \to B\mathbb{Z}_2$ is a classifying map for the principal $\mathbb{Z}_2$–bundle $SE' \to \widehat{SE'}$. The $\mathbb{Z}_2$–module homomorphism $\theta\colon H^*(\mathbb{R}P^{k-1}) \to H^*(\widehat{SE'})$ defined by $a' \mapsto \mathbf{a}'$ is a cohomology extension of the fibre. Then, it follows from the

Leray–Hirsch theorem that $H^*(\widehat{SE'})$ is an $H^*(B)$–module, via the induced homomorphism $(\hat{\pi}')^*$, generated by the elements

$$1, \mathbf{a}', (\mathbf{a}')^2, \dots, (\mathbf{a}')^{k-1}.$$

We can express $(\mathbf{a}')^k \in H^k(\widehat{SE}')$ as

$$(\mathbf{a}')^k = \omega'_k + \omega'_{k-1}\mathbf{a}' + \dots + (\mathbf{a}')^{k-1},$$

for unique elements $\omega'_i \in H^i(B)$. Following the usual pattern, the characteristic polynomial in the indeterminate x of degree 1, associated to the vector bundle $\pi'\colon E' \to B$, is defined as

$$W'(x) = \omega'_k + \omega'_{k-1}x + \dots + \omega'_1 x^{k-1} + x^k.$$

As before, we then have the isomorphism of $H^*(B)$–algebras

$$H^*(B)[x]/\langle W'(x)\rangle \cong H^*(\widehat{SE}'),$$

which comes from the rule $x \mapsto \mathbf{a}'$.

4.1.2 Case $G = S^1$ Taking the previously considered fibre bundle (X, E, π, B), let us now consider the quotient bundle $(X/G, \hat{E}, \hat{\pi}, B)$. It follows from Theorem 2.5 and Leray–Hirsch Theorem that $H^*(\hat{E})$ is $H^*(B)$–isomorphic to one of the following $H^*(B)$–algebras:

(i) If the free S^1–action ρ on X is in Λ_1,

$$H^*(B)[y,z]/\langle W_1(y,z), W_2(y,z), W_3(y,z)\rangle, \tag{5}$$

where the characteristic polynomials associated to the fibre bundle (X, E, π, B), in the indeterminates y and z, of degrees 2 and n, respectively, are given by

$$\begin{aligned}
W_1(y,z) &= \omega_{3n+1} + \omega_{3n-1}y + \dots + \omega_2 y^{(3n-1)/2} + y^{(3n+1)/2} + \overline{\omega}_{2n+1}z + \cdots \\
&\qquad + \overline{\omega}_{n+2}y^{(n-1)/2}z, \\
W_2(y,z) &= \nu_{2n+1} + \nu_{2n-1}y + \dots + \nu_2 y^{(2n-1)/2} + \overline{\nu}_{n+1}z + \overline{\nu}_{n-1}yz + \cdots \\
&\qquad + \overline{\nu}_2 y^{(n-1)/2}z + y^{(n+1)/2}z, \\
W_3(y,z) &= \mu_{2n} + \mu_{2n-2}y + \dots + \mu_2 y^{n-1} + \alpha' y^n + \overline{\mu}_n z + \overline{\mu}_{n-2}yz \\
&\qquad + \overline{\mu}_1 y^{(n-1)/2}z + z^2,
\end{aligned}$$

with $\omega_i, \overline{\omega}_i, \nu_i, \overline{\nu}_i, \mu_i, \overline{\mu}_i \in H^i(B)$ and $\alpha' \in \mathbb{Z}_2$.

(ii) If the free S^1–action ρ on X is in Λ_2,

$$H^*(B)[y,z]/\langle W_1(y), W_2(y,z)\rangle, \tag{6}$$

where the characteristic polynomials in the indeterminates y and z, of degrees 2 and $2n$, respectively, are given by:

$$\begin{aligned} W_1(y) &= \omega_{n+1} + \omega_{n-1}y + \cdots + \omega_2 y^{(n-1)/2} + y^{(n+1)/2} \\ W_2(y,z) &= \nu_{4n} + \nu_{4n-2}y + \cdots + \nu_{3n+1}y^{(n-1)/2} + \overline{\nu}_{2n}z + \overline{\nu}_{2n-2}yz + \cdots \\ &\qquad + \overline{\nu}_{n+1}y^{(n-1)/2}z + z^2, \end{aligned}$$

with $\omega_i, \nu_i, \overline{\nu}_i \in H^i(B)$.

Characteristic polynomial for the bundle π': $E' \to B$ with S^1–action Similarly to the $\mathbb{Z}_2$–case, we recall the characteristic polynomial associated to a k–dimensional vector bundle π': $E' \to B$, equipped with a fibrewise S^1–action which is free on $E' - (\{0\})$, with k even. Denote by SE' the total space of the sphere bundle associated to π': $E' \to B$. Since S^1 acts freely on SE', we obtain the complex projective bundle $(P^{(k-2)/2}(\mathbb{C}), \widehat{SE}', \widehat{\pi}', B)$ and the principal S^1–bundle $SE' \to \widehat{SE}'$, where $P^{(k-2)/2}(\mathbb{C}) = S^{k-1}/S^1$ denotes the $(k-2)$–dimensional complex projective space. We have

$$H^*(P^{(k-2)/2}(\mathbb{C}); \mathbb{Z}_2) \cong \mathbb{Z}_2[b']/((b')^{k/2}),$$

with $b' = i^*(t) \in H^2(P^{(k-2)/2}(\mathbb{C}); \mathbb{Z}_2)$, where $t \in H^2(BS^1; \mathbb{Z}_2)$ is the generator and i: $P^{(k-2)/2}(\mathbb{C}) \to BS^1$ is a classifying map for the principal S^1–bundle $S^{k-1} \to P^{(k-2)/2}(\mathbb{C})$.

Following the same argument of the $\mathbb{Z}_2$–case, we have

$$\frac{H^*(B)[y]}{\langle W'(y)\rangle} \cong H^*(\widehat{SE'}),$$

where

$$W'(y) = \omega'_{m+1}1 + \omega'_{m-1}y + \cdots + \omega'_2 y^{(k-2)/2} + y^{k/2}$$

is the characteristic polynomial associated to $E' \to B$.

4.2 Proofs of the announced results

Proof of Lemma 4.1 The arguments follow the pattern developed by Dold [3]. Let $q(x,z)$ be a polynomial in $H^*(B)[x,z]$ with $q(x,z)|_{\widehat{Z}_f} = 0$. From the continuity property of the Čech cohomology, there is an open subset $V \subset \widehat{E}$, with $V \supset \widehat{Z}_f$ and $q(x,z)|_V = 0$. From the exact sequence

$$\cdots \longrightarrow H^*(\widehat{E}, V) \xrightarrow{j_1^*} H^*(\widehat{E}) \longrightarrow H^*(V) \longrightarrow \cdots,$$

there exists $\mu \in H^*(\widehat{E}, V)$ such that $j_1^*(\mu) = q(x,z)|_{\widehat{E}}$, where $j_1\colon \widehat{E} \to (\widehat{E}, V)$ is the natural inclusion. Now consider the map

$$\widehat{f}\colon \widehat{E} - \widehat{Z}_f \to \widehat{E}' - \{0\}$$

induced by the equivariant map $f\colon E \to E'$. Since $W'(\mathbf{a}') = 0$ and $\widehat{f}^*$, the homomorphism induced in cohomology, is a $H^*(B)$–homomorphism, we get

$$W'(x)|_{\widehat{E}-\widehat{Z}_f} = W'(\mathbf{a}) = W'(\widehat{f}^*(\mathbf{a}')) = \widehat{f}^*(W'(\mathbf{a}')) = 0.$$

On the other hand, from the exact sequence

$$\cdots \longrightarrow H^*(\widehat{E}, \widehat{E} - \widehat{Z}_f) \xrightarrow{j_2^*} H^*(\widehat{E}) \longrightarrow H^*(\widehat{E} - \widehat{Z}_f) \longrightarrow \cdots,$$

there is $\theta \in H^*(\widehat{E}, \widehat{E} - \widehat{Z}_f)$ such that $j_2^*(\theta) = W'(x)|_{\widehat{E}}$, where $j_2\colon \widehat{E} \to (\widehat{E}, \widehat{E} - \widehat{Z}_f)$ is the inclusion. Hence,

$$q(x,z)W'(x)|_{\widehat{E}} = j_1^*(\mu) j_2^*(\theta) = j^*(\mu \smile \theta)$$

by the naturality of the cup product. Note that

$$\mu \smile \theta \in H^*(\widehat{E}, V \cup (\widehat{E} - \widehat{Z}_f)) = H^*(\widehat{E}, \widehat{E}),$$

which implies $\mu \smile \theta = 0$. Thus, $q(x,z)W'(x)|_{\widehat{E}} = 0$, and from (4) we conclude that there exist polynomials $r_1(x,z), r_2(x,z)$ and $r_3(x,z)$ in $H^*(B)[x,z]$ such that

$$q(x,z)W'(x) = r_1(x,z)W_1(x,z) + r_2(x,z)W_2(x,z) + r_3(x,z)W_3(x,z)$$

in the ring $H^*(B)[x,z]$. This completes the proof. □

Proof of Theorem 1.3 Let $q(x) \in H^*(B)[x,z]$ be a nonzero polynomial such that $\deg q(x) < 3n - k + 1$. If $q(x)|_{\widehat{Z}_f} = 0$, consider the equality

$$q(x)W'(x) = r_1(x,z)W_1(x,z) + r_2(x,z)W_2(x,z) + r_3(x,z)W_3(x,z),$$

given by Lemma 4.1. Note we have that $\deg W'(x) = k$, $\deg W_1(x,z) = 3n+1$, $\deg W_2(x,z) = 2n+1$ and $\deg W_3(x,z) = 2n$. Since

$$\deg q(x) + k = \max_i\{\deg r_i(x,y) + \deg W_i(x,y)\},$$

we get

$$\deg q(x) + k \geq \deg r_1(x, y) + 3n + 1 \geq 3n + 1.$$

Therefore, $\deg q(x) \geq 3n + 1 - k$, which is a contradiction. Hence $q(x)|_{\hat{Z}_f} \neq 0$. Equivalently, the $H^*(B)$–homomorphism

$$\bigoplus_{i=0}^{3n-k} H^*(B).x^i \to H^*(\hat{Z}_f),$$

given by $x \mapsto x|_{\hat{Z}_f}$, is a monomorphism. Thus, if $3n \geq k$,

$$\text{cohom.dim}\, Z_f \geq \text{cohom.dim}\, B + 3n - k,$$

since $\text{cohom.dim}\, Z_f \geq \text{cohom.dim}\, \hat{Z}_f$ by Quillen [12, Proposition A.11]. □

Next, we prove the results for the case $G = S^1$.

Proof of Lemma 4.2 For (i), let $q(y, z)$ be a polynomial in $H^*(B)[y, z]$ such that $q(y, z)|_{\hat{Z}_f} = 0$. From arguments similar to those used in the proof of Lemma 4.1, we conclude that $q(y, z)W'(y)|_{\hat{E}} = 0$. Therefore, by (5), there are polynomials $r_1(y, z)$, $r_2(y, z)$ and $r_3(y, z)$ in $H^*(B)[y, z]$ such that

$$q(y, z)W'(y) = r_1(y, z)W_1(y, z) + r_2(y, z)W_2(y, z) + r_3(y, z)W_3(y, z).$$

Using (6), the proof for (ii) is completely analogous. □

Proof of Theorem 1.4 For (1), let $q(y) \in H^*(B)[y, z]$ be a nonzero polynomial such that $\deg q(y) < 3n - k + 1$. If $q(y)|_{\hat{Z}_f} = 0$, one has from Lemma 4.2(i) that

$$q(y)W'(y) = r_1(y, z)W_1(y, z) + r_2(y, z)W_2(y, z) + r_3(y, z)W_3(y, z),$$

where we have $\deg W'(y) = k$, $\deg W_1(y, z) = 3n + 1$, $\deg W_2(y, z) = 2n + 1$ and $\deg W_2(y, z) = 2n$. Thus, we conclude that $\deg q(y, z) \geq 3n - k + 1$, which is a contradiction. Hence $q(y)|_{\hat{Z}_f} \neq 0$. As above, the $H^*(B)$–homomorphism

$$\bigoplus_{i=0}^{(3n-k-1)/2} H^*(B).y^i \to H^*(\hat{Z}_f),$$

given by $y^i \mapsto y^i|_{\hat{Z}_f}$, is a monomorphism. Thus, if $3n \geq k$,

$$\text{cohom.dim}\, Z_f \geq \text{cohom.dim}\, B + 3n - k.$$

Using Lemma 4.2(ii), the proof for (2) is completely analogous. □

Finally, we prove the announced parametrized result.

Proof of Theorem 1.5 Let α denote the vector bundle $E'' \to B$, and V denote the total space of $\alpha \oplus \alpha$. Then, $\mathbb{Z}_2$ acts on V by permuting coordinates in each fibre. This action has the diagonal $\Delta \subset V$ as its fixed point set. Note that Δ is the total space of a k–dimensional subbundle of $\alpha \oplus \alpha$, and the orthogonal complement $\Delta^{\perp}$ is also the total space of a k–dimensional subbundle of $\alpha \oplus \alpha$, which is called the *diagonal bundle*. Note that $\Delta^{\perp}$ is invariant under the $\mathbb{Z}_2$–action on V, and this restricted $\mathbb{Z}_2$–action on $\Delta^{\perp}$ is free outside the zero section. Consider the fibre preserving $\mathbb{Z}_2$–equivariant map $F\colon E \to V$ given by

$$F(x) = (f(x), f(Tx)).$$

The linear projection along the diagonal defines an equivariant fibre preserving map $r\colon (V, V - \Delta) \to (\Delta^{\perp}, \Delta^{\perp} - 0)$, where 0 is the zero section of $\Delta^{\perp}$. Let $h = r \circ F$ be the composition

$$(E, E - A(f)) \xrightarrow{F} (V, V - \Delta) \xrightarrow{r} (\Delta^{\perp}, \Delta^{\perp} - 0).$$

Note that $Z_h = h^{-1}(0) = F^{-1}(\Delta) = A(f)$ and $h\colon E \to \Delta^{\perp}$ is a fibre preserving $\mathbb{Z}_2$–equivariant map. Applying Theorem 1.3 to the map h, if $3n \geq k$ we obtain

$$\operatorname{cohom.dim} A(f) = \operatorname{cohom.dim} Z_h \geq \operatorname{cohom.dim} B + 3n - k. \qquad \square$$

Remark In Theorem 1.5, we observe that the fibre preserving map $f\colon E \to E''$ is not necessarily $\mathbb{Z}_2$–equivariant with respect to the standard fibrewise $\mathbb{Z}_2$–action on $E'' \to B$, where the generating involution of the $\mathbb{Z}_2$–action is taken to be the antipodal map (in each fibre) $x \mapsto -x$, which is free away from the zero section. In the case that $f\colon E \to E''$ is equivariant with respect to the antipodal action on $E'' \to B$, one has an explicit formula in the proof of Theorem 1.5; indeed, one has $r(x,y) = ((x-y)/2, (y-x)/2)$ and thus $h = r \circ F(x) = (f(x), -f(x))$.

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