Detection of a nontrivial product in the stable homotopy groups of spheres

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In this paper, we prove that there exists a new family of nontrivial homotopy elements in the stable homotopy groups of spheres with dimension $q(p^n + sp + 2) - 4$. These nontrivial homotopy elements are represented by $\bar{\beta}_s h_0 h_n$ in the $E_2^{s,t}$-term of the Adams spectral sequence, where $p \geq 5$, $n > 4$, $p + 1 < s < 2p - 1$, $t = q(p^n + sp + 2) + s - 2$, $q = 2(p - 1)$.

55Q45; 55T15, 55S10

1 Introduction

Let $S$ be the sphere spectrum localized at an odd prime $p$ and let $A$ be the mod $p$ Steenrod algebra. To determine the stable homotopy groups of spheres $\pi_* S$ is one of the central problems in homotopy theory. One of the main tools for investigating this problem is the Adams spectral sequence (ASS) $E_2^{s,t} = \text{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow \pi_{t-s} S$, where the $E_2^{s,t}$-term is the cohomology of $A$ and the Adams differential is $d_r: E_r^{s,t} \to E_r^{s+r,t+r-1}$. In detecting nontrivial elements of $\pi_* S$ with the ASS, three problems arise: calculation of the $E_2$-terms $\text{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$, computation of the differentials and questions of extensions from $E_\infty$ to $\pi_* S$.

The known results on $\text{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ are as follows. From Liulevicius [8], we know that $\text{Ext}_A^{1,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has a $\mathbb{Z}_p$-basis consisting of $a_0 \in \text{Ext}_A^{1,1}(\mathbb{Z}_p, \mathbb{Z}_p)$ and $h_i \in \text{Ext}_A^{1,p^i q}(\mathbb{Z}_p, \mathbb{Z}_p)$ for all $i \geq 0$. $\text{Ext}_A^{2,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of $a_2$, $a_0 h_0 (i > 0)$, $g_i (i \geq 0)$, $k_i (i \geq 0)$, $b_i (i \geq 0)$ and $h_i h_j (j \geq i + 2, i \geq 0)$ whose internal degrees are $2q + 1$, $2$, $p^i q + 1$, $q(p^{i+1} + 2p^i)$, $q(2p^{i+1} + p^i)$, $p^{i+1} q$ and $q(p^i + p^j)$ respectively. $\text{Ext}_A^{3,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ for $p > 2$ has been computed by Aikawa [1].

Let $M$ be the Moore spectrum modulo an odd prime $p$ given by the cofibration

$$(1-1) \quad S \xrightarrow{p} S \xrightarrow{i_1} M \xrightarrow{j_1} \Sigma S.$$
Let $\alpha \colon \Sigma^q M \to M$ be the Adams map and $V(1)$ be its cofibre given by the cofibration
\[
\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i_2} V(1) \xrightarrow{j_2} \Sigma^{q+1} M.
\]
Let $\beta \colon \Sigma^{(p+1)q} V(1) \to V(1)$ be the $v_2$–mapping. It is well known that, in the ASS, the $\beta$–element $\beta_s = j_1 j_2 \beta^s i_2 i_1$ is a nontrivial element of order $p$ in $\pi_{(p+1)q}^* S$, where $p \geq 5$; see Miller, Ravenel and Wilson [10].

From Wang and Zheng [13], we know that $\beta_s \in \pi_{spq+(s-1)q-2} S$ is represented by the second Greek letter family element, $\tilde{\beta}_s \in \text{Ext}^q_{A} spq+(s-1)q+s-2,*(\mathbb{Z}_p, \mathbb{Z}_p)$, in the ASS and $\tilde{\beta}_s$ is represented by the element $s(s-1)a_2^{s-2}h_2.0h_{1,1}$ in the May spectral sequence (MSS).

Using the ASS, X Wang [12] proved in 1994 that the product $\tilde{\beta}_s h_0 b_n$ is a permanent cycle in the ASS and converges to a nontrivial element of order $p$ in $\pi_* S$. In 1998, X Wang and Q Zheng [13] proved the convergence of $\tilde{\beta}_s h_0 h_n$. Recently, X Liu [4; 5] and Liu and Li [6] proved the convergence of $\tilde{\beta}_s h_0 h_n b_0$, $\tilde{\gamma}_s+3 h_0 h_n h_m$ and $\tilde{\beta}_s h_0 h_n h_m$. However, all of them are working under the condition $s < p$. If $s > p$, the computation becomes much more complicated.

In this paper, we interest ourselves in the problem of convergence of the product $\tilde{\beta}_s h_0 h_n$ ($p + 1 < s < 2p - 1$), and get the following theorem.

**Theorem 1.1** If $n > 4$, $p \geq 5$ and $p + 1 < s < 2p - 1$, the product $\tilde{\beta}_s h_0 h_n$ survives to $E_\infty$ in the ASS and it converges to an element in $\pi_* S$.

**Remark 1.2** If $p + 1 < s < 2p - 2$, we believe that $\tilde{\beta}_s h_0 b_n$ also survives. However, this must be more complicated.

So far, not so many families of homotopy elements in $\pi_* S$ have been detected. In [2], Cohen detected a family $\xi_n \in \pi_{p^n q+q-3} S$, for $n \geq 1$, which has filtration 3 in the ASS and is represented by $h_0 b_n \in \text{Ext}^3_{A} p^n q+q(\mathbb{Z}_p, \mathbb{Z}_p)$. Lee [3] proved that $\beta_1^{p-1} \xi_n$ is nontrivial for all $n$, ie, $b_0^{p-1} h_0 b_n$ is a permanent cycle in the ASS and converges nontrivially to $\beta_1^{p-1} \xi_n$. This result gave another infinite family of elements in the stable homotopy of spheres. In [9], M Mahowald detected a family $\eta_j \in \pi_{pj+pq-2} S$, for $p = 2$, $j \neq 2$, which has filtration 2 in the ASS and is represented by $h_1 h_j \in \text{Ext}^2_{A} p^j q+pq(\mathbb{Z}_p, \mathbb{Z}_p)$.

For the convenience of the reader, let us briefly indicate the main idea in the proof of *Theorem 1.1*.

Note that $\tilde{\beta}_s$ and $h_0 h_n$ are both permanent cycles, so $\tilde{\beta}_s h_0 h_n$ is a permanent cycle, that is $d_r(\tilde{\beta}_s h_0 h_n) = 0$. Thus, to prove the convergence of the product $\tilde{\beta}_s h_0 h_n$, it is
enough to show that the product

\[ \tilde{\rho}_s h_0 h_n \neq 0 \in \text{Ext}^{s+2,t}_A(\mathbb{Z}_p, \mathbb{Z}_p) \]

and that it is not a \( d_r \) boundary in the ASS. For the latter, it is enough to show that

\[ \text{Ext}^{s+2-r,q(p^n+sp+s)+s-r+1}_A(\mathbb{Z}_p, \mathbb{Z}_p) = 0, \quad s + 2 > r \geq 2. \]

The MSS is a powerful tool to prove both of the above.

This paper is organized as follows. In Section 2, we introduce a good method used to compute the generators of the MSS \( E_1 \)-term. In Section 3, we use this method to prove some important results on Ext groups. The proof of Theorem 1.1 will be given in the last section.

## 2 Detecting generators in the May \( E_1 \)-term

From Ravenel [11], there is an MSS \( \{E^{s,t,*}_r, d_r\} \) which converges to \( \text{Ext}^{s,t}_A(\mathbb{Z}_p, \mathbb{Z}_p) \) with \( E_1 \)-term

\[ E^{s,t,*}_1 = E(h_{i,j} \mid i > 0, j \geq 0) \otimes P(b_{i,j} \mid i > 0, j \geq 0) \otimes P(a_i \mid i \geq 0), \]

where \( E(\cdot) \) denotes the exterior algebra, \( P(\cdot) \) denotes the polynomial algebra, and

\[ h_{i,j} \in E^{1,2(p^i-1)p^i,2i-1}_1, \quad b_{i,j} \in E^{2,2(p^i-1)p^{i+1},p(2i-1)}_1, \quad a_i \in E^{1,2p^i-1,2i+1}_1. \]

One has \( d_r : E^{s,t,M}_r \to E^{s+1,t,M-r}_r \) for \( r \geq 1 \), and if \( x \in E^{s,t,M}_r \) and \( y \in E^{s',t',M}_r \), then

\[ d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y). \]

From Liu and Wang [7, Proposition 2.5], there exists a graded commutativity in the May \( E_1 \)-term as follows:

\[
\begin{align*}
\left\{ \begin{array}{ll}
a_m h_{n,j} = h_{n,j} a_m, & h_{m,k} h_{n,j} = -h_{n,j} h_{m,k}, \\
a_m b_{n,j} = b_{n,j} a_m, & h_{m,k} b_{n,j} = b_{n,j} h_{m,k}, \\
a_m a_n = a_n a_m, & b_{m,n} b_{i,j} = b_{i,j} b_{m,n}.
\end{array} \right.
\]

The first May differential \( d_1 \) is given by

\[
\begin{align*}
\left\{ \begin{array}{l}
d_1(h_{i,j}) = -\sum_{0 \leq k < i} h_{i-k,k+j} h_{k,j}, \\
d_1(a_i) = -\sum_{0 \leq k < i} h_{i-k,k} a_k, \\
d_1(b_{i,j}) = 0.
\end{array} \right.
\]
For each element \( x \in E_1^{s,t,M} \), we define \( \dim x = s \), \( \deg x = t \). Then we have

\[
\begin{align*}
\dim h_{i,j} &= \dim a_i = 1, \quad \dim b_{i,j} = 2, \\
\deg h_{i,j} &= 2(p^i - 1)p^j = q(p^{i+j-1} + \cdots + p^j), \\
\deg b_{i,j} &= 2(p^i - 1)p^{j+1} = q(p^{i+j} + \cdots + p^{j+1}), \\
\deg a_i &= 2p^i - 1 = q(p^{i-1} + \cdots + 1) + 1, \\
\deg a_0 &= 1,
\end{align*}
\tag{2-3}
\]

where \( i \geq 1, \ j \geq 0 \).

We denote \( a_i, h_{i,j} \) and \( b_{i,j} \) by \( x, y \) and \( z \) respectively. By the graded commutativity of \( E_1^{*,*,*} \), we can consider a generator

\[
g = (x_1, \ldots, x_b)(y_1, \ldots, y_m)(z_1, \ldots, z_l) \in E_1^{b+m+2l,t+b,*},
\]

where \( t = (\bar{c}_0 + \bar{c}_1 p + \cdots + \bar{c}_n p^n)q \) with \( 0 \leq \bar{c}_i < p \) \((\bar{c}_n > 0)\), \( 0 < b < q \).

Note that the degrees of \( x_i, y_i \) and \( z_i \) can be uniquely expressed as

\[
\begin{align*}
\deg x_i &= q(x_{i,0} + x_{i,1}p + \cdots + x_{i,n}p^n) + 1, \\
\deg y_i &= q(y_{i,0} + y_{i,1}p + \cdots + y_{i,n}p^n), \\
\deg z_i &= q(0 + z_{i,1}p + \cdots + z_{i,n}p^n).
\end{align*}
\]

Furthermore, the sequence \((x_{i,0}, x_{i,1}, \ldots, x_{i,n})\) is of the form \((1, \ldots, 1, 0, \ldots, 0)\), while the sequences \((y_{i,0}, y_{i,1}, \ldots, y_{i,n})\) and \((0, z_{i,1}, \ldots, z_{i,n})\) are of the form \((0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)\). Denote the sequences by columns, then the generator \( g \) determines a matrix:

\[
A = \begin{pmatrix}
x_{1,0} & \cdots & x_{b,0} \\
x_{1,1} & \cdots & x_{b,1} \\
\vdots & \ddots & \vdots \\
x_{1,n} & \cdots & x_{b,n}
\end{pmatrix}, \quad
B = \begin{pmatrix}
y_{1,0} & \cdots & y_{m,0} & 0 & \cdots & 0 \\
y_{1,1} & \cdots & y_{m,1} & z_{1,1} & \cdots & z_{l,1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
y_{1,n} & \cdots & y_{m,n} & z_{1,n} & \cdots & z_{l,n}
\end{pmatrix}
\tag{2-4}
\]

The entries of the matrix (2-4) are 0 or 1. Because of the graded commutativity of \( E_1^{*,*,*} \), by interchanging columns in part \( A \) and \( B \) respectively, the matrix (2-4) can always be transformed into a new one whose entries \( x_{i,j}, y_{i,j}, z_{i,j} \) satisfy the
following conditions:

1. \( x_{1,j} \geq x_{2,j} \geq \cdots \geq x_{b,j}, x_{i,0} \geq x_{i,1} \geq x_{i,n} \) for \( i \leq b \) and \( j \leq n \).
2. If \( y_{i,j-1} = 0 \) and \( y_{i,j} = 1 \), then for all \( k < j \), \( y_{i,k} = 0 \).
3. If \( y_{i,j} = 1 \) and \( y_{i,j+1} = 0 \), then for all \( k > j \), \( y_{i,k} = 0 \).
4. \( y_{1,0} \geq y_{2,0} \geq \cdots \geq y_{m,0} \).
5. If \( y_{i,0} = y_{i+1,0}, y_{i,1} = y_{i+1,1}, \ldots, y_{i,j} = y_{i+1,j} \), then \( y_{i,j+1} \geq y_{i+1,j+1} \).
6. Conditions (2)–(5) apply also to \( z_{i,j} \).

For example, part \( A \) of the matrix (2-4) may be transformed into the following form:

\[
\begin{pmatrix}
1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 \\
1 & \cdots & 1 & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & \cdots & 1 \\
\end{pmatrix}
\]

(2-6)

The entries not displayed are all 0 and a column \((1, \ldots, 1, 0, \ldots, 0)^T\) denotes \( a_i \). Part \( B \) of the matrix (2-4) may be transformed into the following form:

\[
\begin{pmatrix}
I_1 \\
\vdots \\
I_n
\end{pmatrix}
\]

The entries of each \( I_i \) are all 1, the others are all 0. Unfortunately, we can not determine which columns \((0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)^T\) \((j 0 s, i 1 s, 0, \ldots, 0)\) in the above matrix denote \( h_{i,j} \) or \( b_{i,j-1} \). For this reason, we give the following definition.

**Definition 2.1** Define the polynomial algebra

\[
\mathbb{E}_{s,t,*}^s = \mathbb{P}[h_{i,j} \mid i > 0, j \geq 0] \otimes \mathbb{P}[b_{i,j} \mid i > 0, j \geq 0] \otimes \mathbb{P}[a_i \mid i \geq 0]
\]

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and note the obvious identification $E_{1}^{s,t,*} = \tilde{E}_{1}^{s,t,*}/(h_{i,j}^2)$. If, in the above, $b_{i,j}$ is replaced by $h_{i,j+1}$, then we get

$$F_{1}^{s,t,*} := P[a_i \mid i \geq 0] \otimes P[h_{i,j} \mid i > 0, j \geq 0].$$

By the graded commutativity of $F_{1}^{s,t,*}$, we can consider a generator

$$g = (x_1, \ldots, x_b)(y_1, \ldots, y_m) \in F_{1}^{b+m,t+b,*},$$

where $t = (\bar{c}_0 + \bar{c}_1 p + \cdots + \bar{c}_n p^n) q$ with $0 \leq \bar{c}_i < p(\bar{c}_n > 0), 0 < b < q$. Similarly, the generator $g$ determines a matrix:

$$\begin{pmatrix}
  x_{1,0} & \cdots & x_{b,0} & y_{1,0} & \cdots & y_{m,0} \\
  x_{1,1} & \cdots & x_{b,1} & y_{1,1} & \cdots & y_{m,1} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  x_{1,n} & \cdots & x_{b,n} & y_{1,n} & \cdots & y_{m,n}
\end{pmatrix}
\begin{bmatrix}
  c_0 \\
  c_1 \\
  \vdots \\
  c_n
\end{bmatrix}$$

(2-7)

By interchanging columns in parts $A$ and $B$ respectively, the matrix (2-7) can be transformed into a new matrix of the form

$$\begin{pmatrix}
  I_1 & \cdots & I_n
\end{pmatrix}
\begin{pmatrix}
  x_{1,0} & \cdots & x_{b,0} & y_{1,0} & \cdots & y_{m,0} \\
  x_{1,1} & \cdots & x_{b,1} & y_{1,1} & \cdots & y_{m,1} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  x_{1,n} & \cdots & x_{b,n} & y_{1,n} & \cdots & y_{m,n}
\end{pmatrix}
\begin{bmatrix}
  I_1 \\
  I_2 \\
  \vdots \\
  I_n
\end{bmatrix}$$

where a column $(1, \ldots, 1, 0, \ldots, 0)^T$ (with $i \, 1 s$) in part $A$ denotes $a_i$ and a column $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)^T$ (with $j \, 0 s, \, i \, 1 s, \, 0, \ldots, 0$) in part $B$ denotes $h_{i,j}$. By the properties of the $p$–adic numbers, we have the following system of equations:

$$\begin{cases}
  x_{1,0} + \cdots + x_{b,0} + y_{1,0} + \cdots + y_{m,0} = \bar{c}_0 + k_1 p = c_0, \\
  x_{1,1} + \cdots + x_{b,1} + y_{1,1} + \cdots + y_{m,1} = \bar{c}_1 - k_1 + k_2 p = c_1, \\
  \vdots \\
  x_{1,n-1} + \cdots + x_{b,n-1} + y_{1,n-1} + \cdots + y_{m,n-1} = \bar{c}_{n-1} - k_{n-1} + k_n p = c_{n-1}, \\
  x_{1,n} + \cdots + x_{b,n} + y_{1,n} + \cdots + y_{m,n} = \bar{c}_n - k_n = c_n,
\end{cases}$$

(2-8)

where $k_i \geq 0$. 

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**Definition 2.2** In (2-8), the integer sequence \( k = (k_1, k_2, \ldots, k_n) \) is called the carry sequence.

**Definition 2.3** In (2-8), the integer sequence \( c = (c_0, c_1, \ldots, c_n) \) which is determined by \((\tilde{c}_0, \tilde{c}_1, \ldots, \tilde{c}_n)\) and the carry sequence \( k = (k_1, k_2, \ldots, k_n) \) is called the sum of the row sequence.

**Definition 2.4** For the sum of the row sequence \( c \), we denote \( m_0 = \max\{c_0 - b, 0\} \), \( m_i = \max\{c_i - c_{i-1}, 0\} \) for \( i > 0 \) and \( \tilde{m} = m_0 + m_1 + \cdots + m_n \).

We have the following simple method for constructing matrix solutions of (2-8) which satisfy the conditions (2-5)(1–5).

**Simple method 2.5** Without loss generality, we suppose the first \( i \) rows are as follows:

\[
\begin{pmatrix}
1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 \\
1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 \\
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{i-1}
\end{pmatrix}
\]

Then the \((i + 1)^{th}\) row is constructed as follows:

1. If \( c_i \geq c_{i-1} \), put 1s in the next neighboring \( c_i - c_{i-1} \) columns, like so:

\[
\begin{pmatrix}
1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 \\
1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 \\
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{i-1} - c_{i-1}
\end{pmatrix}
\]

2. If \( c_i < c_{i-1} \), delete 1s from some former \( c_{i-1} - c_i \) columns:

\[
\begin{pmatrix}
1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 \\
1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 & \ldots & 1 \\
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{i-1} - c_i
\end{pmatrix}
\]

where \( r + h = c_{i-1} - c_i, r, h \geq 0 \).
Sometimes, by Simple method 2.5, we can not detect the generators of $F_1^{b+m,t+b,*}$. For example, assume that the first two rows are as follows:

$$
\begin{pmatrix}
1 & \cdots & 1 & 1 \\
1 & \cdots & 1 & 0
\end{pmatrix}
$$

By Simple method 2.5(1), we have the following matrix:

$$
\begin{pmatrix}
1 & \cdots & 1 & 1 \\
1 & \cdots & 1 & 0
\end{pmatrix}
$$

It detects the generator $a_3^{s-2}a_1h_{3,0} \in F_1^{b+m,t+b,*}$, but we can not get the following matrix by Simple method 2.5(1):

$$
\begin{pmatrix}
1 & \cdots & 1 & 1 \\
1 & \cdots & 1 & 0
\end{pmatrix}
$$

However, the above matrix actually exists, and it will in fact detect the generator $a_3^{s-2}a_1h_{2,0}h_{1,2} \in F_1^{b+m+1,t+b,*}$. The following discussion gives us a good idea to solve this problem.

For an element $g = x_1x_2 \cdots x_b \cdot y_1y_2 \cdots y_m \in F_1^{b+m,t+b,*}$, we denote the set of terms in $d_1\{g\}$ by $D_1\{g\}$. Then $D_1\{g\}$ generates a submodule of $F_1^{b+m+1,t+b,*}$ and $D_1^k\{g\} = D_1\{\cdots D_1\{g\} \cdots \}$ generates a submodule of $F_1^{b+m+k,t+b,*}$. Furthermore, we have

\[
D_1\{a_i\} = \{a_0h_{i,0}, a_1h_{i-1,1}, \ldots, a_{i-1}h_{1,i-1}\},
\]

\[
D_1\{h_{i,j}\} = \{h_1, jh_{i-1,j+1}, h_2, jh_{i-2,j+2}, \ldots, h_{i-1},jh_{1,i+j-1}\}.
\]

From [7, Lemmas 5.5 and 5.7], we get the following diagram:
where \( *_1 \) denotes the resolution \( h_{i,j} \rightarrow h_{i-k,j+k}h_{k,j} \) and \( a_i \rightarrow a_{i-j}h_{j,i-j}, k \geq 0 \), and \( *_2 \) denotes the replacement \( h_{i,j+1} \rightarrow b_{i,j} \).

From the discussion above, the determination of \( E_{s,t}^{s,t+b,*} \) is reduced to the following steps:

**Step 1** Express \( t/q \) as a \( p \)-adic number so that \( t = (c_0 + c_1p + \cdots + c_np^n)q \).

**Step 2** List all possible carry sequences \( k \) such that in the corresponding sum of row sequence \( c \), \( \tilde{m} \leq s - b \) (Definition 2.4).

**Step 3** For each sum of a row sequence \( c \), we can solve the associated system of equations by Simple method 2.5. Thus, we get all generators of \( \tilde{F}_{1}^{b} + \tilde{m}^{,t+b,*} \).

**Step 4** Through the replacement and resolution

\[
(2-9) \quad h_{i,j+1} \rightarrow b_{i,j}, \quad h_{i,j} \rightarrow h_{i-k,j+k}h_{k,j}, \quad a_i \rightarrow a_{i-j}h_{j,i-j},
\]

we get all generators of \( E_{1}^{s,t+b,*} \).

# 3 Some results on Ext groups

In this section, we will prove some results on Ext groups which will be used in the proof of the main theorem.

**Theorem 3.1** If \( p \geq 5 \), \( n > 4 \) and \( p + 1 < s < 2p - 1 \), then we have that the product

\[
\tilde{\beta}_s h_0 h_n \neq 0 \in \text{Ext}_A^{s+2,t,*}(\mathbb{Z}_p, \mathbb{Z}_p), \text{ where } t = q(p^n + sp + s) + s - 2.
\]

**Proof** Let \( s = s' + p \). Then

\[
t = q(p^n + p^2 + (s' + 1)p + s') + s' + p - 2,
\]

where \( 0 < s' < p - 1 \).

In the MSS, the product \( \tilde{\beta}_s h_0 h_n \) is represented by

\[
s(s-1)a_2^{s-2} h_{2,0} h_{1,1} h_{1,0} h_{1,n} \in E_{1}^{s+2,t,M},
\]

where \( M = 5s - 4 = 5s' + 5p - 4 \). We need to prove that \( E_{2}^{s+1,t,M+r} = 0 \) (\( r \geq 1 \)).

By using the method which was introduced in Section 2, the generator \( g \in F_{1}^{s,t,*} \) can be represented by \( a_{k_1}a_{k_2} \cdots a_{k_l}h_{i_1,j_1} \cdots h_{i_m,j_m} \). For convenience, we write \( g = x_1 \cdots x_l y_1 \cdots y_m \in F_{1}^{s,t,*} \), where \( x_i = a_{k_i}, y_i = h_{i_m,j_m}, k_1 \geq k_2 \geq \cdots \geq k_l \),
\[ j_1 \leq j_2 \leq \cdots \leq j_m, \quad i_m \geq i_{m+1} \] if \( j_i = j_{i+1} \). Since \( l = s' + p - 2 \) and \( m \leq s + 1 \), then \( g = x_1 \cdots x_{s'} + p - 2y_1 \cdots y_m \in F_{s'}^{s', \{1\}} \) and we have the following system of equations:

\[
\begin{align*}
&x_{1,0} + \cdots + x_{s'} + p - 2, 0 + y_{1,0} + \cdots + y_{m,0} = s' + k_1 p = c_0, \\
&x_{1,1} + \cdots + x_{s'} + p - 2, 1 + y_{1,1} + \cdots + y_{m,1} = s' + 1 - k_1 + k_2 p = c_1, \\
&x_{1,2} + \cdots + x_{s'} + p - 2, 2 + y_{1,2} + \cdots + y_{m,2} = 1 - k_2 + k_3 p = c_2, \\
&x_{1,3} + \cdots + x_{s'} + p - 2, 3 + y_{1,3} + \cdots + y_{m,3} = 0 - k_3 + k_4 p = c_3, \\
&\vdots \\
&x_{1,n-1} + \cdots + x_{s'} + p - 2, n - 1 + y_{1,n-1} + \cdots + y_{m,n-1} = 0 - k_{n-1} + k_n p = c_{n-1}, \\
&x_{1,n} + \cdots + x_{s'} + p - 2, n + y_{1,n} + \cdots + y_{m,n} = 1 - k_n = c_n. \\
\end{align*}
\]

\[(3-1)\]

**Case 1:** \( 0 < s' < p - 2 \) From (3-1), the carry sequence \( k = (k_1, k_2, \ldots, k_n) \) can only be of the following forms:

\[
\begin{align*}
& (0, \ldots, 0), \\
& (1, 0, \ldots, 0), \\
& (1, 1, 0, \ldots, 0), \\
& (1, 1, \ldots, 1). \\
\end{align*}
\]

**Subcase 1.1** When \( k = (0, \ldots, 0) \), the corresponding \( c = (s', s' + 1, 1, 0, \ldots, 0, 1) \). We see that the first two rows are:

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix} \begin{pmatrix}
s' \\
1
\end{pmatrix} \begin{pmatrix}
s' + 1
\end{pmatrix}
\]

Then the possible third rows are:

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix} \begin{pmatrix}
s' \\
1
\end{pmatrix} \begin{pmatrix}
s' + 1 \\
\cdots (1) \\
\cdots (2)
\end{pmatrix}
\]

If we choose (1) as the third row, then we get the following solution:

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix} \begin{pmatrix}
s' \\
1
\end{pmatrix} \begin{pmatrix}
s' + 1
\end{pmatrix}
\]

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Thus, we can obtain the following matrix:

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\end{pmatrix}
\]

It detects the generator \(a_3a_2^{s'}a_0^{p-2}h_{1,1}h_{1,n} \in F_1^{s,t,*}\), and by the replacement and resolution (2-9), we have

\[
\{a_2^{s'-1}a_0^{p-1}h_{3,0}h_{1,1,n} \quad a_2^{s'-1}a_1a_0^{p-2}h_{2,1}h_{1,1,n} \quad a_2^{s'-1}a_2^{p-2}h_{1,1}h_{1,1,n} \}
\in E_1^{s+1,t, M_1}
\]

with May filtration \(M_1 = 5s' + p + 1 < M\), and

\[
\{a_3a_2^{s'-1}a_0^{p-2}b_{1,0}h_{1,n} \quad a_3a_2^{s'-1}a_0^{p-2}h_{1,1,b_{1,n-1}} \}
\in E_1^{s+1,t, M_2}
\]

with May filtration \(M_2 = 5s' + 2p + 1 < M\).

Similarly, if we choose (2) as the third row, then we get the following solution:

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\end{pmatrix}
\]

Thus we can obtain the following matrix:

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\end{pmatrix}
\]

It detects the generator \(a_2^{s'}a_0^{p-2}h_{2,1}h_{1,n} \in F_1^{s,t,*}\), and by the replacement and resolution (2-9), we have
\begin{align*}
\{a_2^{s'}a_0^{p-2}h_{1,2}h_{1,1}h_{1,n} a_2^{s'-1}a_0^{p-1}h_{2,1}h_{2,0}h_{1,n} a_2^{s'-1}a_1a_0^{p-2}h_{2,1}h_{1,1}h_{1,n}\} &\in E_1^{s+1,t,M_1} \\
\text{with May filtration } M_1 = 5s' + p + 1 < M, \\
\{a_2^{s'}a_0^{p-2}h_{2,1}b_{1,n-1}\} &\in E_1^{s+1,t,M_2} \\
\text{with May filtration } M_2 = 5s' + 2p + 1 < M, \text{ and} \\
\{a_3a_2^{s'-1}a_0^{p-2}h_{1,1}b_{1,n-1}\} &\in E_1^{s+1,t,M_3} \\
\text{with May filtration } M_3 = 5s' + 4p - 1 < M.
\end{align*}

**Subcase 1.2** When \( k = (1, 0, \ldots, 0) \), the corresponding \( c = (s' + p, s', 1, 0, \ldots, 0, 1) \). Similar to **Subcase 1.1**, we have

\[
a_3a_2^{s'-1}a_1^{p-2}h_{1,0}h_{1,0}h_{1,n} = 0 \quad \text{and} \quad a_3a_2^{s'-1}a_1^{p}h_{2,0}h_{2,0}h_{1,n} = 0.
\]

**Subcase 1.3** When \( k = (1, 1, \ldots, 0) \), the corresponding \( c = (s' + p, s' + p, 1, 0, \ldots, 0, 1) \). Then we have

\[
a_2^{s'+p-2}h_{2,0}h_{2,0}h_{1,n} = 0.
\]

**Subcase 1.4** When \( k = (1, 1, 1, \ldots, 1) \), then we have that the corresponding \( c = (s, s, p, p - 1, p - 1, \ldots, p - 1, 0) \). We can construct the matrix solutions of (3-1) as follows.

Note that the first two rows are:

\[
\begin{pmatrix}
1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1
\end{pmatrix}
\]

Then the possible third rows are:

\[
\begin{pmatrix}
1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & \cdots & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\
1 & \cdots & 1 & 1 & 0 & \cdots & 0 & 1 & 0 \\
1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{s}{s} \\
\frac{s}{s} \\
\frac{p}{p} \cdots (1) \\
\frac{\cdots (2) }{\cdots (3)}
\end{pmatrix}
\]
If we choose (1), (2) and (3) as the third row, respectively, then we get the following three solutions:

\[
\begin{pmatrix}
1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}
\]

\[p, \quad p, \quad p, \quad p,\]

If we choose (1), then the four rows can be expressed as:

\[
\begin{pmatrix}
1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}
\]

\[s, \quad s, \quad p, \quad p - 1\]

We obtain the following matrix:

\[
\begin{pmatrix}
1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}
\]

\[s, \quad s, \quad p, \quad p - 1\]

So we can get the generator \(a_n^{p-1}a_2^{s' - 2}h_{2,0}h_{2,0} \in F^{s',t,M_5}_1\) with May filtration \(M_5 = (2n + 1)(p - 1) + 5s' + 3\).

If we choose (2), then the four rows can be expressed as:

\[
\begin{pmatrix}
1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 0 \\
1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\
1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix}
\]

\[s, \quad s, \quad p, \quad p - 1, \ldots (1) \]

\[\ldots (2)\]
Similarly, we can get \( a_n^{p-1} a_2^{s'-1} h_{3,0} h_{2,0}, a_n^{p-2} a_3 a_2^{s'-1} h_{n,0} h_{2,0} \in F_1^{s,t,M_5} \) with May filtration \( M_5 = (2n + 1)(p - 1) + 5s' + 3 \).

If we choose (3), then the four rows can be expressed as:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & \ldots & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & \ldots & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
s \\
p \\
p - 1 & \ldots & 1 \\
p - 1 & \ldots & 1
\end{pmatrix}
\]

Similarly, we can get \( a_n^{p-2} a_2^{s'} h_{n,0} h_{3,0}, a_n^{p-3} a_3 a_2^{s'} h_{n,0} h_{n,0} \in F_1^{s,t,M_5} \) with May filtration \( M_5 = (2n + 1)(p - 1) + 5s' + 3 \).

From the above discussion, we get the following possible generators in \( F_1^{s,t,M_5} : a_n^{p-1} a_3 a_2^{s'-2} h_{2,0} h_{2,0}, a_n^{p-1} a_3 a_2^{s'-1} h_{3,0} h_{2,0}, a_n^{p-2} a_3 a_2^{s'-1} h_{n,0} h_{2,0}, a_n^{p-2} a_2^{s'} h_{n,0} h_{3,0} \) and \( a_n^{p-3} a_3 a_2^{s'} h_{n,0} h_{n,0} \) which will be denoted by \( x_0, x_1, x_2, x_3 \) and \( x_4 \) respectively, where \( x_0 = 0 \) and \( x_4 = 0 \). It is easy to see that \( x_0, x_1, x_2, x_3 \) and \( x_4 \) belong to \( E_1^{s,t,M_5} \). At the same time, we also get generators of \( E_1^{s+1,t,M_5} \) by (2-9). Then we list all the possibilities in Table 1.

Let

\[
\begin{align*}
k_1 &= a_n^{p-1} a_3 a_2^{s'-2} h_{2,0} h_{1,1} h_{1,0}, \\
k_2 &= a_n^{p-1} a_2^{s'-1} a_1 h_{3,0} h_{2,0} h_{1,1}, \\
k_3 &= a_n^{p-1} a_2^{s'-1} h_{2,1} h_{2,0} h_{1,0}, \\
k_4 &= a_n^{p-1} a_3 a_2^{s'-2} h_{2,0} h_{1,1} h_{1,0}, \\
k_5 &= a_n^{p-1} a_3 a_2^{s'-2} h_{3,0} h_{2,0} h_{1,1}, \\
k_6 &= a_n^{p-1} a_3 a_2^{s'-1} h_{2,1} h_{2,0} h_{1,0}, \\
k_7 &= a_n^{p-2} a_2^{s'-2} a_1 h_{3,0} h_{2,0} h_{1,1}, \\
k_8 &= a_n^{p-2} a_3 a_2^{s'-2} h_{n,0} h_{2,0} h_{1,2}, \\
k_9 &= a_n^{p-2} a_2^{s'-1} a_1 h_{n,0} h_{3,0} h_{1,1}, \\
k_{10} &= a_n^{p-2} a_2^{s'-1} a_1 h_{n,0} h_{2,1} h_{2,0}, \\
k_{11} &= a_n^{p-2} a_2^{s'-1} a_1 h_{n,0} h_{3,0} h_{2,0}, \\
k_{12} &= a_n^{p-2} a_3 a_2^{s'-1} h_{n,0} h_{2,0} h_{1,0}, \\
k_{13} &= a_n^{p-2} a_3 a_2^{s'-1} h_{n,0} h_{1,1} h_{1,0}, \\
k_{14} &= a_n^{p-2} a_3 a_2^{s'-1} h_{n,0} h_{3,0} h_{2,0}, \\
k_{15} &= a_n^{p-2} a_3 a_2^{s'-1} h_{n,0} h_{3,0} h_{2,0}, \\
k_{16} &= a_n^{p-2} a_3 a_2^{s'-1} h_{n,0} h_{2,0} h_{2,0}, \\
k_{17} &= a_n^{p-2} a_3 a_2^{s'-1} h_{n,0} h_{2,0} h_{1,1}, \\
k_{18} &= a_n^{p-2} a_3 a_2^{s'-1} h_{n,0} h_{2,0} h_{1,1}, \\
k_{19} &= a_n^{p-2} a_3 a_2^{s'-1} h_{n,0} h_{2,0} h_{1,1}, \\
k_{20} &= a_n^{p-2} a_3 a_2^{s'-1} h_{n,0} h_{2,0} h_{1,1}, \\
k_{21} &= a_n^{p-3} a_3 a_2^{s'-1} a_1 h_{n,0} h_{n-1,1} h_{2,0}, \\
k_{22} &= a_n^{p-3} a_3 a_2^{s'-1} a_1 h_{n,0} h_{n-1,1} h_{2,0}, \\
k_{23} &= a_n^{p-3} a_3 a_2^{s'-1} a_1 h_{n,0} h_{n-2,2} h_{2,0}, \\
k_{24} &= a_n^{p-3} a_3 a_2^{s'-1} a_1 h_{n,0} h_{n-2,2} h_{2,0}, \\
k_{25} &= a_n^{p-3} a_3 a_2^{s'-1} a_1 h_{n,0} h_{n-3,3} h_{3,0}, \\
k_{26} &= a_n^{p-3} a_3 a_2^{s'-1} a_1 h_{n,0} h_{n-3,3} h_{2,0}, \\
k_{27} &= a_n^{p-3} a_3 a_2^{s'-1} a_1 h_{n,0} h_{n-3,3} h_{2,0}, \\
k_{28} &= a_n^{p-3} a_3 a_2^{s'-1} a_1 h_{n,0} h_{n-3,3} h_{2,0}, \\
k_{29} &= a_n^{p-3} a_3 a_2^{s'-1} a_1 h_{n,0} h_{n-3,3} h_{2,0}.
\end{align*}
\]
The generators of $F^{s,t, M_5}_{1}$ | The generators of $E^{s+1,t, M_5}_{1}$
---|---
$x_0 = a_n^{p-1}a_3a_2^{s-2}h_2,0h_{2,0}$ | $a_n^{p-1}a_3a_2^{s-2}h_{2,0}h_{1,1}h_{1,0}$
$x_1 = a_n^{p-1}a_2^{s-1}h_{3,0}h_{2,0}$ | $a_n^{p-1}a_2^{s-1}h_{3,0}h_{1,1}h_{1,0}$
| $a_n^{p-1}a_2^{s-1}h_{2,1}h_{2,0}h_{1,0}$
| $a_n^{p-1}a_2^{s-1}a_1h_{3,0}h_{2,0}h_{1,1}$
| $a_n^{p-2}a_2^{s-1}a_0h_{n,0}h_{3,0}h_{2,0}$
| $a_n^{p-2}a_2^{s-1}a_1h_{n-1,1}h_{3,0}h_{2,0}$
| $a_n^{p-2}a_2^{s-1}a_2h_{n-2,2}h_{3,0}h_{2,0}$
| $a_n^{p-2}a_2^{s-1}h_{n-3,3}h_{3,0}h_{2,0}$
| $a_n^{p-2}a_2^{s-1}h_{n-i,i}h_{3,0}h_{2,0}$ (4 ≤ i ≤ n − 1)

$x_2 = a_n^{p-2}a_3a_2^{s-1}h_{n,0}h_{2,0}$ | $a_n^{p-2}a_3a_2^{s-1}h_{n,0}h_{1,1}h_{1,0}$
| $a_n^{p-2}a_3a_2^{s-1}a_0h_{n,0}h_{3,0}h_{2,0}$
| $a_n^{p-2}a_3a_2^{s-1}a_1h_{n,0}h_{2,1}h_{2,0}$
| $a_n^{p-2}a_3a_2^{s-1}a_0h_{n,0}h_{2,0}h_{1,2}$
| $a_n^{p-2}a_3a_2^{s-2}a_1h_{n,0}h_{2,0}h_{1,1}$
| $a_n^{p-2}a_3a_2^{s-1}h_{n-1,1}h_{2,0}h_{1,0}$
| $a_n^{p-2}a_3a_2^{s-1}h_{n-3,3}h_{3,0}h_{2,0}$
| $a_n^{p-2}a_3a_2^{s-1}h_{n-i,i}h_{3,0}h_{2,0}$ (4 ≤ i ≤ n − 1)

$x_3 = a_n^{p-2}a_2^{s}h_{n,0}h_{3,0}$ | $a_n^{p-2}a_2^{s}h_{n,0}h_{2,1}h_{1,0}$
| $a_n^{p-2}a_2^{s}a_0h_{n,0}h_{2,0}h_{1,2}$
| $a_n^{p-2}a_2^{s}a_0h_{n,0}h_{3,0}h_{2,0}$
| $a_n^{p-2}a_2^{s}a_0h_{n,0}h_{3,0}h_{1,1}$
| $a_n^{p-2}a_2^{s}h_{n-1,1}h_{3,0}h_{1,0}$
| $a_n^{p-2}a_2^{s}h_{n-2,2}h_{3,0}h_{2,0}$
| $a_n^{p-2}a_2^{s}h_{n-i,i}h_{3,0}h_{3,0}$ (4 ≤ i ≤ n − 1)
| $a_n^{p-3}a_2^{s}a_1h_{n,0}h_{n-1,1}h_{3,0}$
| $a_n^{p-3}a_2^{s}a_2h_{n,0}h_{n-2,2}h_{3,0}$
| $a_n^{p-3}a_3a_2^{s}h_{n,0}h_{n-3,3}h_{3,0}$
| $a_n^{p-3}a_i a_3a_2^{s}h_{n,0}h_{n-i,i}h_{3,0}$ (4 ≤ i ≤ n − 1)

$x_4 = a_n^{p-3}a_3a_2^{s}h_{n,0}h_{n,0}$ | $a_n^{p-3}a_3a_2^{s}h_{n,0}h_{n-1,1}h_{1,0}$
| $a_n^{p-3}a_3a_2^{s}h_{n,0}h_{n-2,2}h_{2,0}$
| $a_n^{p-3}a_3a_2^{s}h_{n,0}h_{n-3,3}h_{3,0}$
| $a_n^{p-3}a_3a_2^{s}h_{n,0}h_{n-i,i}h_{1,0}$ (4 ≤ i ≤ n − 1)

Table 1: The generators of $E^{s+1,t, M_5}_{1}$
where \( 4 \leq i \leq n - 1 \), and then consider the first May differential of \( x_1, x_2, x_3 \), \( d_1(x_1) = -k_1 + \cdots, d_1(x_2) = -k_2 + \cdots, d_1(x_3) = -k_3 + \cdots \). We can see that the leading terms \( k_1, k_2, k_3 \) are not contained in the first May differential of the other generators and are also not equal to \( a_2^{p-2}h_{2,0}h_{1,1},h_{1,0}h_{1,n} \) up to sign. From the above results we know that \( x_1, x_2, x_3 \) in \( E_r^{s,t} \) \( (r \geq 2) \) is not bounded.

Since \( M_5 = (2n + 1)(p - 1) + 5s' + 3 \), we take

\[
\begin{align*}
    g_4 &= a_n^{p-1}a_2^{p-2}a_0h_{3,0}h_{2,0}h_{1,1}h_{1,0}, \\
    g_6 &= a_n^{p-1}a_2^{p-1}h_{2,0}h_{1,2}h_{1,1}h_{1,0}, \\
    g_8 &= a_n^{p-2}a_2^{p-1}a_1h_{n,0}h_{2,0}h_{1,2}h_{1,1}, \\
    g_{10} &= a_n^{p-2}a_2^{p-1}a_1h_{n,0}h_{2,1}h_{1,1}h_{1,0}, \\
    g_{12} &= a_n^{p-2}a_2^{p-1}a_0h_{n-1,1}h_{3,0}h_{2,0}h_{1,0}, \\
    g_{14} &= a_n^{p-2}a_2^{p-1}a_1h_{n-1,1}h_{3,0}h_{1,1}h_{1,0}, \\
    g_{16} &= a_n^{p-2}a_3a_2^{p-1}h_{n-3,3}h_{3,0}h_{1,1}h_{1,0}, \\
    g_{18} &= a_n^{p-2}a_3a_2^{p-1}h_{n-1,i}h_{i,0}h_{1,1}h_{1,0}, \\
    g_{20} &= a_n^{p-3}a_2^{p-1}h_{n,0}h_{n-1,1}h_{2,1}h_{1,0}, \\
    g_{22} &= a_n^{p-3}a_2^{p-1}a_1a_0h_{n,0}h_{n-1,1}h_{3,0}h_{2,0}, \\
    g_{24} &= a_n^{p-3}a_3a_2^{p-1}a_0h_{n,0}h_{n-2,2}h_{3,0}h_{2,0}, \\
    g_{26} &= a_n^{p-3}a_3a_2^{p-1}a_0h_{n,0}h_{n-3,3}h_{3,1}h_{1,1}, \\
    g_{28} &= a_n^{p-3}a_3a_2^{p-1}a_0h_{n,0}h_{n-4,2}h_{2,0}h_{1,0}, \\
    g_5 &= a_n^{p-1}a_2^{p-2}a_1h_{2,1}h_{2,0}h_{1,1}h_{1,0}, \\
    g_7 &= a_n^{p-2}a_3a_2^{p-2}a_0h_{n,0}h_{2,0}h_{1,1}h_{1,0}, \\
    g_9 &= a_n^{p-2}a_2^{p-1}a_0h_{n,0}h_{3,0}h_{1,1}h_{1,0}, \\
    g_{11} &= a_n^{p-2}a_2^{p-1}a_0h_{n,0}h_{2,1}h_{2,0}h_{1,0}, \\
    g_{13} &= a_n^{p-2}a_2^{p-1}h_{n-1,1}h_{2,0}h_{1,2}h_{1,0}, \\
    g_{15} &= a_n^{p-2}a_2^{p-1}a_1a_2^{p-1}h_{n-i,1}h_{2,1}h_{2,0}h_{1,0}, \\
    g_{17} &= a_n^{p-2}a_2^{p-1}a_1h_{n-2,2}h_{2,1}h_{2,0}h_{1,0}, \\
    g_{19} &= a_n^{p-2}a_2^{p-1}h_{n-i,1}h_{i,0}h_{2,1}h_{1,0}, \\
    g_{21} &= a_n^{p-3}a_3a_2^{p-1}a_0h_{n,0}h_{n-1,1}h_{2,0}h_{1,0}, \\
    g_{23} &= a_n^{p-3}a_3a_2^{p-1}a_0h_{n,0}h_{n-2,2}h_{2,0}h_{1,1}, \\
    g_{25} &= a_n^{p-3}a_3a_2^{p-1}a_0h_{n,0}h_{n-3,3}h_{2,1}h_{1,0}, \\
    g_{27} &= a_n^{p-3}a_3a_2^{p-1}a_0h_{n,0}h_{n-4,2}h_{2,0}h_{1,1}, \\
    g_{29} &= a_n^{p-3}a_3a_2^{p-1}a_0h_{n,0}h_{n-4,2}h_{i,0}h_{2,0},
\end{align*}
\]

where \( 4 \leq i \leq n - 1 \). By (2-2), an easy computation shows that

\[
\begin{align*}
    d_1(k_4) &= g_4 + g_5 - g_6 + \cdots, \\
    d_1(k_5) &= -g_4 + g_5 + \cdots, \\
    d_1(k_6) &= g_5 - g_6 + \cdots, \\
    d_1(k_7) &= -g_7 + \cdots, \\
    d_1(k_8) &= -g_8 + \cdots, \\
    d_1(k_9) &= g_9 - g_{10} + \cdots, \\
    d_1(k_{10}) &= g_{10} - g_{11} + \cdots, \\
    d_1(k_{11}) &= g_9 + g_{11} + \cdots, \\
    d_1(k_{12}) &= -g_{12} - g_{13} + g_{14} + \cdots, \\
    d_1(k_{13}) &= -g_{12} + g_{13} + g_{14} + \cdots, \\
    d_1(k_{14}) &= g_{13} + g_{14} + \cdots.
\end{align*}
\]
$d_1(k_{15}) = g_{15} + \cdots,$

$d_1(k_{16}) = g_{16} + \cdots,$

$d_1(k_{17}) = g_{17} + \cdots,$

$d_1(k_{18}) = g_{18} + \cdots,$

$d_1(k_{19}) = g_{19} + \cdots,$

$d_1(k_{20}) = g_{20} + \cdots,$

$d_1(k_{21}) = -g_{21} + \cdots,$

$d_1(k_{22}) = -s'g_{22} + \cdots,$

$d_1(k_{23}) = g_{23} + \cdots,$

$d_1(k_{24}) = -(s' + 1)g_{24} + \cdots,$

$d_1(k_{25}) = g_{25} + \cdots,$

$d_1(k_{26}) = g_{26} + \cdots,$

$d_1(k_{27}) = g_{27} + \cdots,$

$d_1(k_{28}) = -g_{28} + \cdots,$

$d_1(k_{29}) = -s'g_{29} + \cdots.$

Through the computation of the first May differentials of the generators and the rank of its coefficient matrix, we see that their first May differentials of generators are linearly independent.

**Case 2: $s' = p - 2$** From (3-1), we can get the carry sequence $k = (0, 0, 1, \ldots, 1)$, the corresponding $c = (s', s' + 1, p + 1, p - 1, \ldots, p - 1, 0)$. Similar to Subcase 1.4, we can get the five generators in the $E_1^{s+1,t,*}$ as follows:

\[
a_4^{s'}a_0^{p-2}h_{3,1}h_{1,2}h_{1,2} = 0, \quad a_4^{s'}a_0^{p-2}h_{2,2}h_{2,1}h_{1,2},
\]

\[
a_4^{s'-1}a_3a_0^{p-2}h_{2,2}h_{2,1}h_{2,1} = 0, \quad a_4^{s'-1}a_3a_0^{p-2}h_{3,1}h_{2,2}h_{1,2},
\]

\[
a_4^{s'-2}a_3^2a_0^{p-2}h_{3,1}h_{2,2}h_{2,2} = 0.
\]

The first May differential of the above generators are as follows:

\[
d_1(a_4^{s'}a_0^{p-2}h_{2,2}h_{2,1}h_{1,2}) = s'a_4^{s'-1}a_0^{p-1}h_{4,0}h_{2,2}h_{2,1}h_{1,2} + \cdots \neq 0,
\]

\[
d_1(a_4^{s'-1}a_3a_0^{p-2}h_{3,1}h_{2,2}h_{1,2}) = -a_4^{s'-1}a_0^{p-1}h_{3,1}h_{3,0}h_{2,2}h_{1,2} + \cdots \neq 0.
\]

From the above results, we get $E_2^{s+1,t,M+r} = 0$ ($r \geq 1$) in the MSS, so it follows that $\beta_{s}h_{0}h_{n} \neq 0$ in $\operatorname{Ext}_{A}^{s+2,t,*}(\mathbb{Z}_p, \mathbb{Z}_p)$. \qed
Theorem 3.2  Let $p \geq 5$, $n > 4$ and $p + 1 < s < 2p - 1$. Then we have that
\[ \operatorname{Ext}_A^{s+2-r,t'-r+1,*}(\mathbb{Z}_p, \mathbb{Z}_p) = 0, \]
where $t' = q(s + sp + p^n) + s$ and $2 \leq r < s + 2$.

Proof  We need to prove $E_2^{s+2-r,t'-r+1,*} = 0$. Let $s = s' + p$, then $t' - r + 1 = q(p^n + p^2 + (s' + 1)p + s') + s' + p - r - 1$. We claim that $s' + p - r - 1 \geq 0$. Otherwise, if $s' + p - r - 1 < 0$ and $p \geq 5$, then $p > q + (s' + p - r - 1) \geq p$, it is a contradiction. Consider $g = x_1 \cdots x_{s' + p - r - 1} y_1 \cdots y_m \in F_1^{s' + p - r + 2, t' - r - 1,*}$, where $x_i = a_{k_i}, y_i = h_{i_m i_m}, k_1 \geq k_2 \geq \cdots \geq k_l, j_1 \leq j_2 \leq \cdots \leq j_m, i_m \geq i_m + 1$ if $j_m = j_{m+1}$.

Case 1: $0 < s' < p - 2$

Subcase 1.1  When $k = (0, \ldots, 0)$, the corresponding $c = (s', s' + 1, 1, 0, \ldots, 0, 1)$. Then we get that the generators are
\[ a_2 a_2^{s'-1} a_0^{p-r-1} h_{1,1} h_{1,n} \in F_1^{s' + p - r + 1, t' - r - 1, M}, \]
\[ a_2 a_2^{s'-1} a_0^{p-r-1} h_{2,1} h_{1,n} \in F_1^{s' + p - r + 1, t' - r - 1, M}, \]
with May filtration $M = 5s' + p - r + 3$. By (2-9), we have
\[
\begin{pmatrix}
 a_2^{s'-1} a_0^{p-r} h_{3,0} h_{1,1} h_{1,n} & a_2^{s'-1} a_1 a_0^{p-r-1} h_{2,1} h_{1,1} h_{1,n} \\
 a_2^{s'-1} a_0^{p-r-1} h_{1,2} h_{1,1} h_{1,n} & a_3 a_2^{s'-2} a_0^{p-r} h_{2,0} h_{1,1} h_{1,n} \\
 a_3 a_2^{s'-1} a_0^{p-r-1} h_{1,0} h_{1,1} h_{1,n} & a_3 a_2^{s'-1} a_0^{p-r-1} h_{1,1} h_{1,n} \\
 a_2^{s'-1} a_0^{p-r} h_{1,1} h_{1,1} h_{1,n} & a_2^{s'-1} a_0^{p-r} h_{2,1} h_{2,0} h_{1,n} \\
 a_2^{s'-1} a_1 a_0^{p-r} h_{2,1} h_{1,1} h_{1,n} & a_2^{s'-1} a_0^{p-r-1} h_{2,1} h_{1,n} \\
 a_3 a_2^{s'-1} a_0^{p-2} h_{1,1} h_{1,n} & a_3 a_2^{s'-1} a_0^{p-2} h_{1,1} h_{1,n} \\
 a_3 a_2^{s'-1} a_0^{p-2} h_{1,1} h_{1,n} & a_3 a_2^{s'-1} a_0^{p-2} h_{1,1} h_{1,n} \\
 & a_3 a_2^{s'-1} a_0^{p-2} h_{1,1} h_{1,n}
\end{pmatrix}
\in F_1^{s' + p - r + 2, t' - r - 1, M'}.
\]

Subcase 1.2  When $k = (1, 0, \ldots, 0)$ or $k = (1, 1, 0, \ldots, 0)$, it is easy to show that such a $g$ cannot exist.

Subcase 1.3  When we have $k = (1, \ldots, 1)$, then we have that the corresponding $c = (s' + p, s' + p, p, p - 1, \ldots, p - 1, 0)$. The generators exist if and only if $r = 2$. We list all the possibilities in the following:
\[ a_n^{p-2} a_2^{s'-1} h_{4,0} h_{3,0} h_{2,0} \in F_1^{s' + p - r + 2, t' - r + 1,*}, \]
\[ a_n^{p-1} a_3 a_2^{s'-3} h_{2,0} h_{2,0} h_{2,0} = 0, \]
\[ a_n^{p-2} a_3 a_2^{s'-2} h_{4,0} h_{2,0} h_{2,0} = 0, \]
\[ a_n^{p-4} a_3 a_2^{s'} h_{3,0} h_{3,0} h_{3,0} = 0, \]
\[ a_n^{p-6} a_3 a_2^{s'} h_{4,0} h_{4,0} h_{3,0} = 0, \]
\[ a_n^{p-7} a_3 a_2^{s'} h_{4,0} h_{4,0} h_{4,0} = 0. \]
Furthermore,

\begin{align*}
d_1(a_2^{s'-1}a_0^{p-r}h_{3,0}h_{1,1}h_{1,1}) &= -a_2^{s'-1}a_0^{p-r}h_{2,1}h_{1,1}h_{0}h_{1,1} + \cdots \neq 0, \\
d_1(a_2^{s'-1}a_1a_0^{p-r-1}h_{2,1}h_{1,1}h_{1,1}) &= -(s' - 2)a_2^{s'-2}a_0^{p-r+1}h_{2,1}h_{1,1}h_{0}h_{1,1} + \cdots \neq 0, \\
d_1(a_2^{s'-1}a_0^{p-r-1}h_{1,2}h_{1,1}h_{1,1}) &= (s' - 1)a_2^{s'-1}a_0^{p-r}h_{2,0}h_{1,2}h_{1,1} + \cdots \neq 0, \\
d_1(a_3a_2^{s'-2}a_0^{p-r}h_{2,0}h_{1,1}h_{1,1}) &= a_2^{s'-2}a_0^{p-r+1}h_{3,0}h_{2,0}h_{1,1} + \cdots \neq 0, \\
d_1(a_3a_2^{s'-1}a_0^{p-r-1}b_{1,0}h_{1,1}) &= a_2^{s'-1}a_0^{p-r}h_{3,0}b_{1,0}h_{1,1} + \cdots \neq 0, \\
d_1(a_3a_2^{s'-1}a_0^{p-r-1}h_{1,1}b_{1,1}h_{1,1}) &= a_2^{s'-1}a_0^{p-r}h_{3,0}h_{1,1}b_{1,1} + \cdots \neq 0, \\
d_1(a_3a_2^{s'-1}a_0^{p-r}h_{1,1}h_{1,1}) &= (s' - 1)a_2^{s'-1}a_0^{p-r+1}h_{2,0}h_{1,2}h_{1,1} + \cdots \neq 0, \\
d_1(a_2^{s'-1}a_0^{p-r}h_{2,1}h_{2,0}h_{1,1}) &= a_2^{s'-1}a_0^{p-r}h_{2,0}h_{1,2}h_{1,1} + \cdots \neq 0, \\
d_1(a_2^{s'-1}a_0^{p-r}h_{2,1}h_{1,1}h_{1,1}) &= a_2^{s'-1}a_0^{p-r+1}h_{2,1}h_{1,1}h_{0}h_{1,1} + \cdots \neq 0, \\
d_1(a_2^{s'-1}a_0^{p-r-1}h_{2,1}b_{1,1}h_{1,1}) &= a_2^{s'-1}a_0^{p-r}h_{2,2}h_{1,1}b_{1,1} + \cdots \neq 0, \\
d_1(a_3a_2^{s'-1}a_0^{p-r-2}h_{1,1}b_{1,1}h_{1,1}) &= a_2^{s'-1}a_0^{p-r}h_{3,0}h_{1,1}b_{1,1} + \cdots \neq 0, \\
d_1(a_4^{p-2}a_2^{s'-1}h_{4,0}h_{3,0}h_{2,0}) &= a_4^{p-2}a_2^{s'-1}h_{3,1}h_{3,0}h_{2,0}h_{1,0} + \cdots \neq 0.
\end{align*}

Obviously, the first May differential of every generator contains a term which is not contained in the first May differential of the other generators. This implies that all the first May differentials of the generator are linearly independent.

**Case 2:** $s' = p - 2$  In this case, we have that the carry sequence $k = (0, 0, 1, \ldots, 1)$ and $c = (s', s' + 1, p + 1, p - 1, \ldots, p - 1, 0)$. Similar to Subcase 1.3, we get the five generators in the $E_{1}^{s'-r+2,t,*}$ as follows:

\begin{align*}
a_4^{s'}a_0^{p-r-1}h_{3,1}h_{1,2}h_{1,2} &= 0, \quad a_4^{s'}a_0^{p-r-1}h_{2,2}h_{2,1}h_{1,2}, \\
a_4^{s'-1}a_3a_0^{p-r-1}h_{2,2}h_{2,2}h_{2,1} &= 0, \quad a_4^{s'-1}a_3a_0^{p-r-1}h_{3,1}h_{2,2}h_{2,1}, \\
a_4^{s'-2}a_3^2a_0^{p-r-1}h_{3,1}h_{2,2}h_{2,2} &= 0.
\end{align*}

Consider the first May differential,

\begin{align*}
d_1(a_4^{s'}a_0^{p-r-1}h_{2,2}h_{2,1}h_{1,2}) &= s'a_4^{s'-1}a_0^{p-r}h_{4,0}h_{2,2}h_{2,1}h_{1,2} + \cdots \neq 0, \\
d_1(a_4^{s'-1}a_3a_0^{p-r-1}h_{3,1}h_{2,2}h_{1,2}) &= -a_4^{s'-1}a_0^{p-r}h_{3,1}h_{3,0}h_{2,2}h_{1,2} + \cdots \neq 0.
\end{align*}

Similarly, the first May differentials of the generator are linearly independent.
From the above results, it is easy to see that $E_2^{s+2-r',t'-r+1,*} = 0$ for $r \geq 2$. It follows that $\text{Ext}_A^{s+2-r',t'-r+1,*}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$.

\section*{4 Proof of the main theorem}

In this section, we give the proof of the main theorem.

\textbf{Proof of Theorem 1.1} From [2], $(i_1)_*(h_0 h_n) \in \text{Ext}_A^{2,pnq+q}(H^* M, \mathbb{Z}_p)$ is a permanent cycle in the ASS and converges to a nontrivial element $\xi_n \in \pi_{p^n q+q-2} M$. Consider the composite

$$
\Sigma p^n q+q-2 S \xrightarrow{\xi_n} M \xrightarrow{i_2} V(1) \xrightarrow{\beta^s} \Sigma -s(p+1)q V(1) \xrightarrow{j_1 j_2} \Sigma -s(p+1)q+q+2 S.
$$

Since $\xi_n$ is represented by $(i_1)_*(h_0 h_n) \in \text{Ext}_A^{2,pnq+q}(H^* M, \mathbb{Z}_p)$ in the ASS, then $f$ is represented by

$$
\tilde{c} = (j_1 j_2)_*(\beta^s)_*(i_2)_*(i_1)_*(h_0 h_n) = (j_1 j_2 \beta^s i_2 i_1)_*(h_0 h_n)
$$
in the ASS.

By using the Yoneda products, we know that the composite

$$
\text{Ext}_A^{0,0}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{(i_2 i_1)_*} \text{Ext}_A^{0,0}(H^* M, \mathbb{Z}_p) \xrightarrow{(j_1 j_2)_*(\beta^s)_*} \text{Ext}_A^{s,s p q+(s-1)q+s-2}(\mathbb{Z}_p, \mathbb{Z}_p)
$$
is a multiplication by

$$
\tilde{\beta}_s \in \text{Ext}_A^{s,s p q+(s-1)q+s-2}(\mathbb{Z}_p, \mathbb{Z}_p).
$$

Hence $f$ is represented by

$$
\tilde{\beta}_s h_0 h_n \in \text{Ext}_A^{s+2,p^n q+s p q+s q+s-2}(\mathbb{Z}_p, \mathbb{Z}_p)
$$
in the ASS.

From Theorem 3.1, we see that $\tilde{\beta}_s h_0 h_n \neq 0$. Moreover, from Theorem 3.2 it follows that $\tilde{\beta}_s h_0 h_n$ can not be hit by any differential in the ASS. Thus the $\tilde{\beta}_s h_0 h_n$ survives nontrivially to a homotopy element of $\pi_\ast S$.

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