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Weak asymptotic hereditary asphericity for free product and HNN extension of groups

JOANNA JAKUS

Asymptotic hereditary asphericity (AHA) is a coarse property of metric spaces and groups, introduced by T Januszkiewicz and J Świątkowski in [3]. Conjecturally, this property is closed under amalgamated free products and HNN extensions over finite subgroups. We prove this conjecture for a slightly weaker property, weak asymptotic hereditary asphericity (AHA(−)), which is still strong enough for the purposes which AHA was used for in [3].

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1 Introduction

Asymptotic hereditary asphericity (AHA) is a property of metric spaces introduced by T Januszkiewicz and J Świątkowski [3]. Roughly, it says that all spheres (of dimension greater than 1) in a Rips complex may be filled by balls in a Rips complex with a larger constant. AHA is a quasiisometry invariant [3, Corollary 3.3], so it is also a well defined property of discrete groups. This property is used in [3] to show the existence of high-dimensional hyperbolic groups which contain no fundamental groups of closed nonpositively curved Riemannian manifolds of dimension greater than two as their subgroups. The examples of such groups are systolic groups, that is groups which act by simplicial automorphisms, properly discontinuously and cocompactly on systolic complexes (some simplicial analogues of CAT(0) spaces). Januszkiewicz and Świątkowski used the fact that every systolic group is AHA (and hence every finitely generated subgroup of a systolic group is AHA), but the fundamental group of a nonpositively curved manifold of dimension greater than two is never AHA.

Apart from the facts mentioned above, not much is known about AHA. It is known that metric spaces which uniformly embed in AHA spaces are also AHA [3, Proposition 3.2]. Another result, [3, Corollary 8.7], tells us that the Cartesian product of more than two infinite groups is never AHA. In [5] the author showed that groups of asymptotic dimension 1 are all AHA. Osajda and Świątkowski [4] prove that the boundary of any AHA group (hyperbolic or CAT(0)) contains no 2–disk.

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There is a natural question, *is a free product of two* AHA *groups also* AHA? It would seem that the answer is positive. In fact however, an attempt to prove this conjecture presents severe difficulties. The reason is that we do not have asymptotic analogues of many (even basic) concepts and facts of algebraic topology.

In this paper we introduce weak asymptotic hereditary asphericity $\mathrm{AHA}(-)$, which is a slightly weaker property than AHA. Roughly, instead of filling the spheres in Rips complex by balls, we require that they are filled by some simply connected manifolds. For this new property we prove the following two theorems:

(1) Let G and H be two weakly asymptotically hereditarily aspherical ($\mathrm{AHA}(-)$) groups, and let K be their common finite subgroup. Then $G *_K H$ (the free product with amalgamation over K) also has $\mathrm{AHA}(-)$.

(2) Let G be $\mathrm{AHA}(-)$ group and let K_1 and K_2 be isomorphic finite subgroups of G. Then the HNN extension of G, $G*_{(K_1=K_2)}$, also has $\mathrm{AHA}(-)$.

We also mention that for the purposes set out by Januszkiewicz and Świątkowski [3], for which AHA was invented, $\mathrm{AHA}(-)$ is sufficient.

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The author has previously published under the name Joanna Zubik.

2 Definition of asymptotic hereditary asphericity

Recall the definition of asymptotic hereditary asphericity in a form which was introduced by Januszkiewicz and Świątkowski [3]:

Definition 2.1 A metric space X is asymptotically hereditarily aspherical (shortly AHA), if for every $r>0$ there is $R \geqslant r$ such that for every $A \subset X$, any simplicial map $f\colon S \to \mathrm{Rips}_r(A)$ (S is any triangulation of the sphere S^k, $k \geqslant 2$) has a simplicial extension $F\colon B \to \mathrm{Rips}_R(A)$ for some triangulation B of the ball B^{k+1} such that $\partial B = S$.

Clearly, the constant R depends only on r. We will denote by $R(X,r)$ the value of R in a specific case.

AHA is a quasiisometry invariant property [3, Corollary 3.3]. Thus, it makes sense to speak about AHA for finitely generated groups; we say that a group is AHA if some its Cayley graph is AHA. Below we provide another definition which will turn out to be equivalent for geodesic spaces.

Definition 2.2 A geodesic space X has the property of AHA* if for every $r > 0$ there is $R^* \geqslant r$ such that for every path-connected set $A \subset X$, any simplicial map $f\colon S \to \mathrm{Rips}_r(A)$ (S is any triangulation of the sphere S^k, $k \geqslant 2$) has a simplicial extension $F\colon B \to \mathrm{Rips}_{R^*}(A)$ for some triangulation B of the ball B^{k+1} such that $\partial B = S$.

In this case the constant R^* also depends only on r and we will denote it by $R^*(X,r)$.

Lemma 2.3 *Let X be a geodesic space. Then X has* AHA *if and only if it has* AHA*.

Proof Clearly AHA implies AHA*. Suppose now X has AHA*. We need to check that it has also AHA. Consider any r, S, A and $f\colon S \to \mathrm{Rips}_r(A)$, as in Definition 2.1. Since f is a continuous function on a connected set, its image is also connected. Therefore $\mathrm{Im}(f)$ is included in one of connected components of $\mathrm{Rips}_r(A)$. Let $A_1 \subset A$ be the set of vertices of this component. Then $\mathrm{Im}(f) \subset \mathrm{Rips}_r(A_1)$. Consider the r–neighborhood $N_r(A_1) = \{x \in X : (\exists a \in A_1)\,(d(a,x) \leqslant r)\}$ of A_1. Such a set is path-connected. To show this, let us notice two things. First, by the definition of $N_r(A_1)$, we can connect any point of $N_r(A_1)$ with a point of A_1 by a geodesic which is not longer than r and consequently is included in $N_r(A_1)$. Second, notice that any two points x, y from A_1 can be connected by a geodesic in $N_r(A_1)$. Indeed, since $\mathrm{Rips}_r(A_1)$ is connected, there exists a path along edges which connects the vertices x and y. Since X is a geodesic space, for every two adjacent vertices of this path there exists a geodesic which connects them and which is not longer than r (since adjacent vertices in $\mathrm{Rips}_r(X)$ are at distance at most r in X) and consequently is included in $N_r(A_1)$. It shows path-connectedness of $N_r(A_1)$.

Now, view f as a simplicial map from S to $\mathrm{Rips}_r(N_r(A_1))$. By the assumption that X has AHA*, there exists an extension $F^*\colon B \to \mathrm{Rips}_{R^*}(N_r(A_1))$ of the map f, for a constant $R^* = R^*(X,r)$ and for B such that $\partial B = S$.

For each $x \in N_r(A_1)$ choose an element a_x from the set A_1 such that $d(x,a_x) \leqslant r$. For $x \in A_1$ we put $a_x = x$.

Consider the map $F\colon V(B) \to A_1$ (with $V(\cdot)$ denoting the vertex set), given by $F(v) = a_{F^*(v)}$. Now check the distance between the images of vertices of any edge $[v,w]$ in B. We have:

$$\begin{aligned} d(F(v),F(w)) &= d(a_{F^*(v)}, a_{F^*(w)}) \\ &\leqslant d(a_{F^*(v)}, F^*(v)) + d(F^*(v),F^*(w)) + d(F^*(w), a_{F^*(w)}) \\ &\leqslant r + R^* + r = R^* + 2r. \end{aligned}$$

Hence the map F can be extended to a simplicial map $B \to \mathrm{Rips}_R(A_1) \subset \mathrm{Rips}_R(A)$, with $R = R^* + 2r$. Therefore X has AHA with the constant $R(X,r) = R$. $\square$

3 Weak asymptotic hereditary asphericity

Definition 3.1 A metric space X is weakly asymptotically hereditarily aspherical (AHA($-$)) if for every $r > 0$ there is $R \geqslant r$ such that for every $A \subset X$, any simplicial map $f\colon S \to \mathrm{Rips}_r(A)$ (S is any triangulation of the sphere S^k, $k \geqslant 2$) has a simplicial extension $F\colon E \to \mathrm{Rips}_R(A)$ for some triangulation E of a simply connected manifold of dimension $k+1$, such that $\partial E = S$.

AHA($-$) is a quasiisometry invariant. A proof of this is analogous to the proof of the fact that AHA is a quasiisometry invariant [3, Corollary 3.3]. As before, for geodesic spaces we have an equivalent definition of AHA($-$):

Definition 3.2 A geodesic space X has AHA($-$) if for every $r > 0$ there is $R^* \geqslant r$ such that for every path-connected set $A \subset X$, any simplicial map $f\colon S \to \mathrm{Rips}_r(A)$ (S is any triangulation of the sphere S^k, $k \geqslant 2$) has a simplicial extension $F\colon E \to \mathrm{Rips}_{R^*}(A)$ for some triangulation E of a simply connected manifold of dimension $k+1$, such that $\partial E = S$.

A proof of equivalence of the two above definitions is analogous to the proof of Lemma 2.3. The second definition will be more useful for us. We will use the notation $R^*(X,r)$ for the value of R^* in a specific case. Clearly such $R^*(X,r)$ is not unique.

Remark As we mentioned in Section 1, AHA($-$) is sufficient for the purposes [3] for which AHA was invented. Indeed, Januszkiewicz and Świątkowski needed only the fact that systolic groups are S^2FRC, which means that for every $r > 0$ there is $R \geqslant r$ such that any 2–spherical cycle $f\colon S \to \mathrm{Rips}_r(X)$ which is nullhomologous in $\mathrm{Rips}_r(X)$ has a filling D in $\mathrm{Rips}_R(X)$ (ie a 3–chain such that ∂D is the image under f of the fundamental cycle in S) with the vertex set of D contained in the image under f of the vertex set of S; see [3, Section 6]. As we can easily see, S^2FRC is implied by AHA as well as by AHA($-$).

4 One property of maps from spheres to the wedge sum of two spaces

Let (X, x_0) and (Y, y_0) be based geodesic spaces. Consider the space $X \vee Y$, that is the wedge sum of X and Y (the disjoint union of spaces X and Y after identification of x_0 with y_0). Denote by p the point $x_0 = y_0$ in $X \vee Y$. If d_X is the metric on X and d_Y is the metric on Y, we define a metric d on $X \vee Y$ in the following way:

$$d(x,y) = \begin{cases} d_X(x,y) & \text{if } x,y \in X, \\ d_Y(x,y) & \text{if } x,y \in Y, \\ d_X(x,x_0) + d_Y(y_0,y) & \text{if } x \in X,\ y \in Y. \end{cases}$$

Let S be a triangulation of the sphere S^k and let $f\colon S \to \mathrm{Rips}_r(X \vee Y)$ be a simplicial map. Denote by f' the map $f|_{V(S)}$. We can identify the vertex set of $\mathrm{Rips}_r(X \vee Y)$ with the set $X \vee Y$, hence we can treat f' as a map from $V(S)$ to $X \vee Y$.

Consider the vertices of the preimage of the set $X \setminus \{p\}$ (analogically, of the set $Y \setminus \{p\}$) under f'. The full subcomplex of S which is spanned on these vertices can consist of several connected components. Such components for $X \setminus \{p\}$ and $Y \setminus \{p\}$ form a family $\mathcal{W}_f$ of connected and pairwise disjoint subcomplexes of S. The following observation will be crucial for us: there exists a complex $W \in \mathcal{W}_f$ that we can enlarge by some vertices from the preimage of $\{p\}$ under f' in such a way that its regular neighborhood has connected boundary. In other words, the complement of that neighborhood is connected in S.

Before we will show it, we introduce some useful notations and definitions. For a simplicial complex Z we denote by $\mathrm{Bar}(Z)$ its barycentric subdivision. For a subcomplex Z_0 of a complex Z, we denote by $N_{\mathrm{Bar}}(Z_0, Z)$ the subcomplex of $\mathrm{Bar}(Z)$ which is the union of all simplices intersecting Z_0.

Definition 4.1 A peripheral subcomplex of a simplicial sphere S for a map f is such a subcomplex W of S that

- either $f'(V(W)) \subset X$ and $f'(v) \in Y$ for each v adjacent to W, or $f'(V(W)) \subset Y$ and $f'(v) \in X$ for each v adjacent to W;
- the complement of $N_{\mathrm{Bar}}(W, S)$ in S is connected.

Fact 4.2 *There exists a peripheral subcomplex for f in S.*

Proof Let us consider three cases.

Case 1 There is $W \in \mathcal{W}_f$ such that $N_{\mathrm{Bar}}(W, S)$ does not disconnect S. Then W is a desired peripheral subcomplex for f in S.

Case 2 For every $W \in \mathcal{W}_f$, $N_{\mathrm{Bar}}(W, S)$ disconnects S, but there is $W \in \mathcal{W}_f$ such that vertices of all subcomplexes from $\mathcal{W}_f$ except W lie in one of the connected components of the complement of $N_{\mathrm{Bar}}(W, S)$. It indicates that f maps vertices of other components to p. Then the full subcomplex spanned on these vertices and vertices of W is peripheral.

Case 3 Previous cases do not occur. Then for every $W \in \mathcal{W}_f$, the vertices of subcomplexes from $\mathcal{W}_f$ except W lie in at least two connected components of the complement of $N_{\mathrm{Bar}}(W, S)$.

Choose a sequence W_n of elements from $\mathcal{W}_f$ in the following way. Let W_1, W_2 be any elements from $\mathcal{W}_f$. For $i > 2$ we choose W_i recursively. Assume W_{i-1} is chosen. Vertices of subcomplexes from the family $\mathcal{W}_f$ (except for W_{i-1}) lie in at least two connected components of the complement of $N_{\mathrm{Bar}}(W_{i-1}, S)$. The subcomplex W_{i-2} is contained in one of them. Choose any element of $\mathcal{W}_f$ which is contained in any other component and denote it by W_i. Notice, that W_i is different than $W_1, \dots, W_{i-1}$, because $W_1, \dots, W_{i-3}$ are contained in the same connected component of the complement of $N_{\mathrm{Bar}}(W_{i-1}, S)$ as W_{i-2}.

We obtain an infinite sequence W_n of disjoint subcomplexes of S, which is impossible since S is the triangulation of a sphere.

Hence, only the first two cases can occur and the proof is complete. □

5 AHA(−) for a wedge sum

Theorem 5.1 *If geodesic spaces X and Y have* AHA(−), *then also $X \vee Y$ has this property. Moreover, for any fixed r the wedge sum $X \vee Y$ satisfies the condition of* AHA(−) *with the constant $R^*(X \vee Y, r)$ equal to the maximum of constants $R^*(X, r)$ and $R^*(Y, r)$.*

Proof Denote by p the basepoint of $X \vee Y$. Fix any r and consider any A and S (a sphere of dimension k) as in Definition 3.2.

Define $R = \max\{R_x, R_y\}$, where $R_x := R^*(X, r)$ and $R_y := R^*(Y, r)$.

We will consider a map $f\colon S \to \mathrm{Rips}_r(A)$ together with the map f' ($f|_{V(S)}$ viewed as a map $V(S) \to A$) and the family $\mathcal{W}_f$ (of connected disjoint subcomplexes of S). We will show the existence of an appropriate extension of f by induction on $|\mathcal{W}_f|$.

If f is such that $|\mathcal{W}_f| \leqslant 1$, then $f'(V(S)) \subset X$ or $f'(V(S)) \subset Y$ and the existence of an appropriate extension of f is an immediate consequence of the fact that X and Y are AHA(−).

Assume that appropriate extensions exist for maps f such that $|\mathcal{W}_f| \leqslant n$. We will show that we can extend in an appropriate way also f with $|\mathcal{W}_f| = n+1$. If $f'(V(S)) \subset X$

or $f'(V(S)) \subset Y$, then the existence of an appropriate extension of f is again a consequence of the fact that X and Y are AHA$(-)$. Notice that in the opposite case the basepoint p belongs to A. Indeed, if $f'(V(S)) \cap X \neq \varnothing$ and $f'(V(S)) \cap Y \neq \varnothing$ then let $x \in X$ and $y \in Y$ be points from these intersections respectively. Then $x, y \in A$ and it follows from path-connectedness of A that there exists a path from x to y. Every such a path has to pass through the point p, so $p \in A$.

Choose a peripheral subcomplex of S for f and denote it by S_1. Let S_2 be the subcomplex of S spanned on the rest of vertices. Without loss of generality we can assume that the image of the vertex set of S_1 by f' is included in X.

Now consider $\mathrm{Bar}(S)$. Let

$$S_0 := N_{\mathrm{Bar}}(S_1, S) \cap N_{\mathrm{Bar}}(S_2, S).$$

S_0 is equal to $\partial N_{\mathrm{Bar}}(S_i, S)$ (for $i = 1, 2$), so it is a closed manifold of dimension $k-1$. Notice that it may be not a sphere S^{k-1}, because S_1 may be not a disk D^k.

Define the map $f^*\colon V(\mathrm{Bar}(S)) \to \mathrm{Rips}_r(A)$, such that

$$f^*(v) = \begin{cases} f(v) & \text{for } v \in V(S), \\ p & \text{for } v \in V(S_0), \\ f(w) & \text{for } v \in V(\mathrm{Bar}(S)) \backslash (V(S) \cup V(S_0)), \end{cases}$$

where w is any of the vertices from $V(S)$ which are closest to v. It is easy to check that the distance in A of the images of vertices of any edge from $\mathrm{Bar}(S)$ is not bigger than r. For that reason f^* can be extended in a simplicial way to $f^*\colon \mathrm{Bar}(S) \to \mathrm{Rips}_r(A)$.

Let us construct a new complex, starting from $\mathrm{Bar}(S)$. Along $S_0 \subset \mathrm{Bar}(S)$ we glue S_1^o (a copy of $N_{\mathrm{Bar}}(S_1, S)$), and S_2^o (a copy of $N_{\mathrm{Bar}}(S_2, S)$). Consequently we obtain three spheres: $N_{\mathrm{Bar}}(S_1, S) \cup S_2^o$, $S_2^o \cup S_1^o$, $S_1^o \cup N_{\mathrm{Bar}}(S_2, S)$, whose intersection is S_0, and the first sphere is glued with the second along S_2^o and the second sphere is glued with the third along S_1^o.

With the use of f^*, we will construct auxiliary simplicial maps defined on spheres described above. By the inductive assumption we will be able to extend these maps to simply connected manifolds such that boundaries of these manifolds will be those spheres. With the use of the obtained extensions we will construct an extension of f^* which we will use to construct an extension of f.

Let $f_1^*\colon N_{\mathrm{Bar}}(S_1, S) \cup S_2^o \to \mathrm{Rips}_r(A)$ be an auxiliary simplicial map defined on vertices as follows:

$$f_1^*(v) = \begin{cases} f^*(v) & \text{for } v \in V(N_{\mathrm{Bar}}(S_1, S)), \\ p & \text{for } v \in S_2^o. \end{cases}$$

It is clear that f_1^* is well defined. After identifying $V(\mathrm{Rips}_r(A))$ with A we observe that $f_1^*[V(N_{\mathrm{Bar}}(S_1,S))] \subset X \cap A$. Since $f_1^*[V(S_2^o)] = \{p\}$, the whole image of $f_1^*|_{V(N_{\mathrm{Bar}}(S_1,S) \cup S_2^o)}$ is contained in $X \cap A$. Notice that $X \cap A$ is path-connected. Indeed, if $x_1, x_2 \in X \cap A$, then it follows from path-connectedness of A that there exists a path in A from x_1 to x_2. If this path is not contained in X, then its segments are contained in Y. However, every time when such a path exits X or comes back to X, it passes also through the point p. For that reason every segment, which is contained in Y may be deleted, to obtain a path that is completely contained in X.

The space X has AHA($-$), so there exists an extension $F_1\colon E_1 \to \mathrm{Rips}_{R_x}(A \cap X)$ of f_1^*, for some triangulation E_1 of a simply connected manifold E_1^{k+1} such that $\partial E_1 = N_{\mathrm{Bar}}(S_1,S) \cup S_2^o$.

Define the next auxiliary map as follows. Let $f_2^*\colon N_{\mathrm{Bar}}(S_2,S) \cup S_1^o \to \mathrm{Rips}_r(A)$ be a simplicial extension of the map defined on the vertex set in the following way:

$$f_2^*(v) = \begin{cases} f^*(v) & \text{for } v \in V(N_{\mathrm{Bar}}(S_2,S)), \\ p & \text{for } v \in S_1^o. \end{cases}$$

It is clear that f_2^* is well defined. Notice that $|\mathcal{W}_{f_2^*}| = n$, so by the inductive assumption there exists an extension $F_2\colon E_2 \to \mathrm{Rips}_R(A)$ of f_2^*, for some triangulation E_2 of a simply connected manifold E_2^{k+1} such that $\partial E_2 = N_{\mathrm{Bar}}(S_2,S) \cup S_1^o$.

Let B_o be the simplicial cone over the sphere $S_1^o \cup S_2^o$. We will show that $E_0 := E_1 \cup B_o \cup E_2$ is a simply connected manifold of dimension $k+1$. To do this, we will use Van Kampen's Theorem; see Hatcher [2, Theorem 1.20]. E_1, B_o and E_2 are path-connected and simply connected. Also, the intersection of any pair as well as the intersection of all three of these complexes are path-connected. By Van Kampen's Theorem, $\pi_1(E_0)$ is trivial, since it is isomorphic to the free product of fundamental groups of complexes E_1, B_o and E_2, which are trivial.

Now we will construct an auxiliary map which will be a simplicial extension of f^*, defined on E_0. Since $R = \max\{R_x, R_y\}$, we have $\mathrm{Rips}_{R_x}(A \cap X) \subset \mathrm{Rips}_{R_x}(A) \subset \mathrm{Rips}_R(A)$, so we can view F_1 as a function to $\mathrm{Rips}_R(A)$. Define $F_0\colon E_0 \to \mathrm{Rips}_R(A)$ on the vertex set of E_0:

$$F_0(v) = \begin{cases} F_1(v) & \text{for } v \in V(E_1), \\ F_2(v) & \text{for } v \in V(E_2), \\ p & \text{for } v \in V(B_o). \end{cases}$$

It is easy to see that for edges of E_0 the distance in A between the images of their vertices is not bigger than R. For this reason we can extend F_0 to a simplicial map.

In this way we have constructed E_0 — a triangulation of the manifold E_0^{k+1} such that the triangulation of the boundary of E_0 is equal to the barycentric subdivision $\mathrm{Bar}(S)$ of the simplicial sphere S — and we have defined the simplicial map F_0 on E_0, which is an extension of f^*. We will join E_0 with the product $S^k \times [0,1]$ to obtain a manifold with the simplicial sphere S as its boundary.

We triangulate $S^k \times \{0\}$ and $S^k \times \{1\}$ as S; for $v \in V(S)$ let v^0 (respectively v^1) define the corresponding vertex in $S^k \times \{0\}$ (respectively in $S^k \times \{1\}$). For a simplex $[v_0, \dots, v_i]$ (for $i \leqslant k$) $[v_0^0, \dots, v_i^0, v_0^1, \dots, v_i^1]$ defines a prism in $S^k \times [0,1]$. We triangulate each prism in such a way that on $S^k \times \{1\}$ we have the initial triangulation, on $S^k \times \{0\}$, the barycentric subdivision of S, and that we do not have any new additional vertices except for these in $S^k \times \{0\}$ (for a precise description of such a triangulation, see Hatcher [2, pages 121–122]). We join $S^k \times [0,1]$, triangulated in this way, with E_0 (by identifying $S^k \times \{0\}$ with ∂E_0) and as a result we obtain a manifold with S as a triangulation of its boundary. Denote it by E.

Since $\partial E = S$, f is defined on ∂E. We define a simplicial extension $F\colon E \to \mathrm{Rips}_R(A)$ of f, on the vertex set of E:

$$F(v) = \begin{cases} f(v) & \text{for } v \in V(S), \\ F_0(v) & \text{for } v \in V(E_0). \end{cases}$$

To see that it is well defined it is sufficient to check what happens for edges $[v, w]$, where $v \in V(S)$ and $w \in V(\partial E_0)$. According to our notation, $v = v^1$ and

$$w \in V\big(\mathrm{Bar}[v^0, v_1^0, \dots, v_k^0]\big)$$

for some simplex $[v, v_1, \dots, v_k]$ in S. Then the distance in A between the images of v and w under F is

$$\begin{aligned} d(F(v), F(w)) = d(f(v), F_0(w)) &\leqslant \max\big\{d(f(v), f(x)) : x \in \{v, v_1, \dots, v_k\}\big\} \\ &\leqslant r, \end{aligned}$$

since f is a simplicial map, so $F(v)$ and $F(w)$ span an edge in $\mathrm{Rips}_R(A)$.

Consequently, F is a required extension of f, which completes the inductive step. This proves that $X \vee Y$ satisfies the condition of AHA(−) with the constant $R^*(X \vee Y, r)$ equal to $R = \max\{R_x, R_y\}$. □

Now consider an infinite iterated wedge sum of spaces X_n $(n = 1, 2, \dots)$, that is the space

$$X = \bigcup X_n',$$

where $X'_1 := X_1$ and $X'_{n+1} := X'_n \vee X_{n+1}$ for $n \geqslant 1$ (in this wedge sum we identify any distinguished points of X'_n and X_{n+1}). The metric on X'_{n+1}, restricted to X'_n, is equal to the metric on X'_n, so we can define a metric on X in the following way: for $x, y \in X$ there exists n such that $x, y \in X'_n$ and we put $d_X(x, y) := d_{X'_n}(x, y)$.

Theorem 5.2 *Let X be an infinite iterated wedge sum of geodesic spaces X_n ($n \in \mathbb{N}$),*

$$X = \bigcup X'_n.$$

Assume that each space X_n has AHA(−) *and for fixed r the sequence of constants $R^*(X_n, r)$ is bounded above by R_0. Then X has* AHA(−)*. Moreover, for any fixed r the space X satisfies the condition of* AHA(−) *with the constant $R^*(X, r)$ equal to R_0.*

Proof Let us fix any positive real number r.

First we observe that for each $n \geqslant 1$, X'_n has AHA(−). Indeed, by Theorem 5.1, X'_2 has AHA(−) and the constant $R^*(X'_2, r)$ is equal to $\max\{R^*(X_1, r), R^*(X_2, r)\} \leqslant R_0$. Assume that for $n \leqslant k$ the space X'_n has AHA(−) and the constant $R^*(X'_n, r)$ is equal to R_0. Then, since X'_{k+1} is the wedge sum $X'_k \vee X_{k+1}$, it also has AHA(−) (by Theorem 5.1) and $R^*(X'_{k+1}, r) = \max\{R_0, R_{k+1}\} = R_0$.

Consider any simplicial map $f\colon S \to \mathrm{Rips}_r(A)$ for a path-connected set $A \subset X$ and for S; a triangulation of S^m, $m \geqslant 2$. The triangulation S has always finitely many vertices, so $f[V(S)]$ is contained in X'_k for some k. Notice that the set $A \cap X'_k$ is path-connected. Indeed, for $x, y \in A \cap X'_k$ it follows from the path-connectedness of A that there exists a path in A which connects these points. If this path is not contained in X'_k, then notice that every time this path exits X'_k, it has to pass through the basepoint of the wedge sum $X'_k \vee X_i$ for some $i > k$ and afterwards it has to come back also through this point. Consequently, every segment which is not contained in X'_k can be deleted, to obtain a path which is completely contained in X'_k.

Since X'_k has AHA(−), there exists $F\colon E \to \mathrm{Rips}_{R_0}(A \cap X'_k) \subset \mathrm{Rips}_{R_0}(A)$ for E; a triangulation of E^{m+1} such that $\partial E = S$. □

6 AHA(−) for a free product

Analogously as for AHA, we say that a group is AHA(−) if some its Cayley graph is AHA(−). Since AHA(−) is quasiisometry invariant it is a well defined property for groups.

Theorem 6.1 *Let G and H be groups with* AHA$(-)$*. Then the free product $G * H$ also has this property.*

Proof Fix any generating sets B_G, B_H for G, H respectively. As a generating set of $G * H$ we take $B_G \cup B_H$. Consider the Cayley graph of the product $G * H$ with respect to this generating set (we denote it by $\mathrm{Cay}(G * H, B_G \cup B_H)$).

Notice that $\mathrm{Cay}(G*H, B_G \cup B_H)$ is an infinite iterated wedge sum of spaces which are copies of $\mathrm{Cay}(G, B_G)$ and $\mathrm{Cay}(H, B_H)$ (one can see the construction of the Cayley graph of $G * H$ in Cannon [1, page 280], taking trivial K in the amalgamated free product $G *_K H$ which is considered there). By Theorem 5.2, $\mathrm{Cay}(G * H, B_G \cup B_H)$ has AHA$(-)$, where for fixed r we have

$$R^*(\mathrm{Cay}(G * H, B_G \cup B_H), r) = \max \left\{ R^*(\mathrm{Cay}(G, B_G), r), R^*(\mathrm{Cay}(H, B_H), r) \right\}.$$

□

We want to prove the analogous theorem for a free product with amalgamation over nontrivial finite subgroup, that is for $G *_K H$ with a nontrivial finite K.

First, we describe the Cayley graph of $G *_K H$. As a generating set for K we take the whole group. Then we extend K to a generating set B_G of the group G and to a generating set B_H of the group H. We take $B_G \cup B_H$ as a generating set of $G *_K H$. Consider the Cayley graph $\mathrm{Cay}(G *_K H, B_G \cup B_H)$. It is constructed from copies of Cayley graphs of groups G and H. We can reconstruct it by taking $\mathrm{Cay}(G, B_G)$ and next gluing along each subgraph corresponding to some left coset of the subgroup K in G the subgraph corresponding to the left coset eK in the graph $\mathrm{Cay}(H, B_H)$ together with the whole new copy of $\mathrm{Cay}(H, B_H)$. In the second step, along each subgraph corresponding to some (other than eK) left coset of the subgroup K in the glued copy of $\mathrm{Cay}(H, B_H)$, we glue the subgraph corresponding to the left coset eK together with a new copy of $\mathrm{Cay}(G, B_G)$. We continue the construction in this fashion. More details of the construction of a Cayley graph of a free amalgamated product can be seen in, eg, Cannon [1, page 280].

Let us modify the graph described above. Since AHA$(-)$ is a quasiisometry invariant, it suffices to show AHA$(-)$ for a graph quasiisometric to $\mathrm{Cay}(G *_K H, B_G \cup B_H)$.

First, modify the graph $\mathrm{Cay}(G, B_G)$. To each subgraph which is contained in this graph and corresponds to some left coset of the subgroup K in G let us add one new vertex and join it by edges with the rest of vertices of the subgraph. Assume that all edges in the graph have length equal to 1. Denote the resulting graph by C_G.

Fact 6.2 *The graph C_G is quasiisometric to the graph* $\mathrm{Cay}(G, B_G)$.

We skip the proof since it is easy to see that the embedding $\mathrm{Cay}(G, B_G) \to C_G$ is a quasiisometry.

We modify the graph $\mathrm{Cay}(H, B_H)$ in the same way as we did with $\mathrm{Cay}(G, B_G)$, obtaining the graph C_H.

Now, instead of the graph $\mathrm{Cay}(G *_K H, B_G \cup B_H)$ consisting of copies of $\mathrm{Cay}(G, B_G)$ and copies of $\mathrm{Cay}(H, B_H)$ joined in the appropriate way along copies of $\mathrm{Cay}(K, K)$, consider a graph which consists of copies of C_G and C_H joined in single vertices. That is, for each copy of $\mathrm{Cay}(G, B_G)$ and $\mathrm{Cay}(H, B_H)$ in $\mathrm{Cay}(G *_K H, B_G \cup B_H)$ we consider a copy of C_G and C_H respectively, and we make the following identifications: If a copy of $\mathrm{Cay}(G, B_G)$ and a copy of $\mathrm{Cay}(H, B_H)$ in $\mathrm{Cay}(G *_K H, B_G \cup B_H)$ have nonempty intersection (equivalently, their intersection is a copy of $\mathrm{Cay}(K, K)$), then their corresponding copies of C_G and C_H are adjacent in the new graph, meaning that they are joined by identifying appropriate vertices added to copies of $\mathrm{Cay}(K, K)$ in C_G and C_H (which correspond to a copy of $\mathrm{Cay}(K, K)$ mentioned above). Denote this new graph by C_{GH}.

Fact 6.3 *The graph C_{GH} is quasiisometric to the graph* $\mathrm{Cay}(G *_K H, B_G \cup B_H)$.

Proof Since a graph with the standard geodesic metric is quasiisometric to its vertex set (with the restricted metric), it suffices to find a quasiisometry between vertex sets of $\mathrm{Cay}(G *_K H, B_G \cup B_H)$ and C_{GH} (with restricted metrics). Define

$$f\colon V(\mathrm{Cay}(G *_K H, B_G \cup B_H)) \to V(C_{GH})$$

in the following way. All copies of $V(\mathrm{Cay}(G, B_G))$ and $V(\mathrm{Cay}(H, B_H))$, included in $V(\mathrm{Cay}(G *_K H, B_G \cup B_H))$, are mapped by identity to $V(C_{GH})$ (onto their corresponding copies). Since each vertex of $V(\mathrm{Cay}(G *_K H, B_G \cup B_H))$ is either a vertex of some copy of $\mathrm{Cay}(G, B_G)$ or of some copy of $\mathrm{Cay}(H, B_H)$, the map f is defined in this way on all vertices.

Let us check that f is a quasiisometric embedding. Take any

$$x, y \in V(\mathrm{Cay}(G *_K H, B_G \cup B_H)).$$

It is clear that $d(x, y) \leqslant d(f(x), f(y))$ by the construction of C_{GH} and the definition of f. Consider $g(x, y)$, a geodesic with endpoints x and y in $\mathrm{Cay}(G *_K H, B_G \cup B_H)$. A geodesic $g(f(x), f(y))$ in C_{GH} is not longer than $g(x, y)$ lengthened by two edges in each vertex. The reason is that in all vertices where $g(x, y)$ goes from a copy of $\mathrm{Cay}(G, B_G)$ to a copy of $\mathrm{Cay}(H, B_H)$ (or the opposite) the corresponding path in C_{GH} is two edges longer. These two edges connect a vertex in a copy of $\mathrm{Cay}(G, B_G)$

with a vertex in a copy of $\mathrm{Cay}(H, B_H)$ in C_{GH}. Thus we have $d(f(x), f(y)) \leqslant |g(x, y)| + 2|V(g(x, y))| = 3d(x, y) + 2$.

Consequently we have $\frac{1}{3}d(x, y) - 2 \leqslant d(x, y) \leqslant d(f(x), f(y)) \leqslant 3d(x, y) + 2$, which shows that f is a quasiisometric embedding.

It remains to check that every vertex in C_{GH} is within a constant distance from the image of f. Notice that each vertex of C_{GH} is either in a copy of $V(\mathrm{Cay}(G, B_G))$ or $V(\mathrm{Cay}(H, B_H))$ (so in the image of f), or it is one of additional vertices (so its distance from the image of f is 1). Hence we see that every vertex in C_{GH} is within the distance 1 from the image of f. It completes the proof that f is a quasiisometry. □

Theorem 6.4 *Assume that groups G and H have* AHA(−) *and that a group K is their common finite subgroup. Then $G *_K H$ has also* AHA(−)*.*

Proof It suffices to show that the graph C_{GH} has AHA(−).

Notice that C_{GH} is an infinite iterated wedge sum of spaces which are copies of C_G and C_H. Graphs C_G and C_H have AHA(−) and they are geodesic spaces (since they are connected graphs). By Theorem 5.2 the space C_{GH} has AHA(−) and for a fixed r the constant $R^*(C_{GH}, r)$ is equal to $\max\{R^*(C_G, r), R^*(C_H, r)\}$. □

7 AHA(−) for HNN extension

To study the behavior of AHA(−) under HNN extensions over finite subgroups let us look at first at a Cayley graph of HNN extension.

Let K_1, K_2 be finite subgroups of a group G and let $\varphi\colon K_1 \to K_2$ be an isomorphism. Fix a generating set B_G for G, which contains $K_1 \cup K_2$. If G has a presentation $\langle B_G \mid R\rangle$, then HNN extension $G*_\varphi$ has the presentation

$$\langle B_G, t \mid R, t\varphi(k)t^{-1} = k : k \in K_1\rangle.$$

The reader is referred to Cannon [1, page 281] for the detailed description of the graph $\mathrm{Cay}(G*_\varphi, B_G \cup \{t\})$, we only outline the construction. Notice that the graph $\mathrm{Cay}(G*_\varphi, B_G \cup \{t\})$ consists of copies of graphs $\mathrm{Cay}(G, B_G)$ joined in an appropriate way. To reconstruct it take a copy of $\mathrm{Cay}(G, B_G)$. Consider a subgraph of $\mathrm{Cay}(G, B_G)$ corresponding to some left coset of the subgroup K_1 in G. We connect this subgraph with the subgraph corresponding to the left coset eK_2 in a new copy of $\mathrm{Cay}(G, B_G)$ by a family of edges corresponding to multiplying by t, in a manner corresponding to the isomorphism φ. We do the same for all subgraphs corresponding to left cosets of K_1 in the initial copy of $\mathrm{Cay}(G, B_G)$. We repeat the analogous process for all subgraphs corresponding to the left cosets of K_2 in the initial copy of $\mathrm{Cay}(G, B_G)$, connecting them with subgraphs corresponding to the left cosets eK_1 in subsequently added copies of $\mathrm{Cay}(G, B_G)$. We repeat the process for all new copies of $\mathrm{Cay}(G, B_G)$, considering all subgraphs corresponding to left cosets of K_1 and K_2 they contain. We continue the construction in this fashion and after infinite number of steps we obtain the graph $\mathrm{Cay}(G{*_\varphi}, B_G \cup \{t\})$.

Consider the graph with vertices corresponding to copies of $\mathrm{Cay}(G, B_G)$ contained in $\mathrm{Cay}(G{*_\varphi}, B_G \cup \{t\})$ and edges joining the vertices which correspond to copies of $\mathrm{Cay}(G, B_G)$ joined by a family of edges in $\mathrm{Cay}(G{*_\varphi}, B_G \cup \{t\})$. Such a graph is a tree (see Cannon [1, page 281]).

Instead of the graph $\mathrm{Cay}(G{*_\varphi}, B_G \cup \{t\})$ we want to consider another graph which is quasiisometric to it. To describe it, consider at first any copy of $\mathrm{Cay}(G, B_G)$ embedded in $\mathrm{Cay}(G{*_\varphi}, B_G \cup \{t\})$, together with edges which have one endpoint in this copy. As we mentioned, these edges correspond to multiplying by t or t^{-1}. We can group them into families in such a way that every family consists of all edges connecting the initial copy of $\mathrm{Cay}(G, B_G)$ with another copy. For each such family separately we take the vertex set, choose all vertices that are not in the initial copy of $\mathrm{Cay}(G, B_G)$ and identify them with each other. Denote the resulting graph by C_G.

Fact 7.1 *The graph C_G is quasiisometric to* $\mathrm{Cay}(G, B_G)$.

We skip the proof since it is easy to see that the embedding $\mathrm{Cay}(G, B_G) \to C_G$ is a quasiisometry.

Now, instead of the graph $\mathrm{Cay}(G{*_\varphi}, B_G \cup \{t\})$ consisting of copies of $\mathrm{Cay}(G, B_G)$ connected in the appropriate way by families of edges, consider the graph which consists of copies of C_G joined in single vertices. That is, for each copy of $\mathrm{Cay}(G, B_G)$ in $\mathrm{Cay}(G{*_\varphi}, B_G \cup \{t\})$ we consider a copy of C_G. For two adjacent copies in $\mathrm{Cay}(G{*_\varphi}, B_G \cup \{t\})$ of $\mathrm{Cay}(G, B_G)$ (meaning these copies are connected by a family of edges) we join their corresponding copies of C_G by identifying appropriate vertices (in which edges corresponding to edges from the family mentioned above are glued together). Denote the resulting graph by C.

Fact 7.2 *The graph C is quasiisometric to* $\mathrm{Cay}(G{*_\varphi}, B_G \cup \{t\})$.

We omit a proof of this fact, because it is a repetition of the proof of Fact 6.3.

Theorem 7.3 *Let K_1, K_2 be finite isomorphic subgroups of a group G and let φ be an isomorphism between K_1 and K_2. If G has* AHA(−), *then its* HNN *extension $G{*_\varphi}$ also has* AHA(−).

Proof It suffices to show that the graph C, which we described above, has $\mathrm{AHA}(-)$. Notice at first that C is an infinite iterated wedge sum of spaces which are copies of C_G. Since G has $\mathrm{AHA}(-)$, C_G also has this property (as a consequence of Fact 7.1). By Theorem 5.2 the graph C has $\mathrm{AHA}(-)$ and $R^*(C,r) = R^*(C_G,r)$. □

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Instytut Matematyczny, Uniwersytet Wrocławski
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

zubik@math.uni.wroc.pl

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