Homology decompositions of the loops on 1–stunted Borel constructions of C_2–actions

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The Carlsson construction is a simplicial group whose geometric realization is the loop space of the 1–stunted reduced Borel construction. Our main results are: (i) given a pointed simplicial set acted upon by the discrete cyclic group C_2 of order 2, if the orbit projection has a section, then the loop space on the geometric realization of the Carlsson construction has a mod 2 homology decomposition; (ii) in addition, if the reduced diagonal map of the C_2–invariant set is homologous to zero, then the pinched sets in the above homology decomposition themselves have homology decompositions in terms of the C_2–invariant set and the orbit space. Result (i) generalizes a previous homology decomposition of the second author for trivial actions. To illustrate these two results, we compute the mod 2 Betti numbers of an example.

1 Introduction

A general problem in algebraic topology is to compute the homology of a loop space, or failing that, to give a homology decomposition of the loop space. We show in Theorem 1.1 that under some assumptions there is a mod 2 homology decomposition of a certain loop space $\Omega(X \rtimes_{C_2} W_\infty^1 C_2)$. This generalizes a previous homology decomposition of the second author for trivial actions (see [17] and Section 4 below).

The following notational conventions will be used throughout this paper. We reserve G to denote the discrete cyclic group C_2 of order 2, written multiplicatively with generator t. In particular $t^2 = 1$. Let X denote a pointed simplicial G–set. Denote by A the simplicial subset of X fixed under the G–action. Let $\mathbb{F}_2$ denote the finite field with two elements.

The 1–stunted reduced Borel construction $X \rtimes_{C_2} W_\infty^1 C_2$ is the homotopy cofiber of the inclusion X into its reduced Borel construction. Carlsson constructed a simplicial group $J^G[X]$ whose geometric realization is the loop space $\Omega(X \rtimes_{C_2} W_\infty^1 C_2)$. See Section 2 for details.

The *orbit projection* is the simplicial epimorphism $X \to X/G$ onto the orbit space. A *section of the orbit projection* is a simplicial map $j: X/G \to X$ such that the composite $X/G \xrightarrow{j} X \to X/G$ is the identity map on X/G. Simplicial G–sets whose orbit projection has a section is characterized in Proposition 4.1.

Theorem 1.1 *If the orbit projection has a section, then there is an isomorphism of $\mathbb{F}_2$–algebras:*

$$(1\text{-}1) \qquad \widetilde{H}_*(\Omega(X \rtimes_G W_\infty^1 G); \mathbb{F}_2) \cong \bigoplus_{s=1}^{\infty} \widetilde{H}_*((X/G)^{\wedge s}/\widetilde{\Delta}_s; \mathbb{F}_2),$$

where $\widetilde{\Delta}_0 = \widetilde{\Delta}_1 := *$ and

$$\widetilde{\Delta}_s := \{x_1 G \wedge \cdots \wedge x_s G \in (X/G)^{\wedge s} \mid \exists i = 1, \ldots, s-1 \ (x_i = x_{i+1} \in A)\} \text{ for } s \geq 2.$$

To compute the direct summands in (1-1), we consider the long exact sequence associated to the cofiber sequence $\widetilde{\Delta}_s \to (X/G)^{\wedge s} \to (X/G)^{\wedge s}/\widetilde{\Delta}_s$:

$$(1\text{-}2) \quad \cdots \to \widetilde{H}_*(\widetilde{\Delta}_s) \to \widetilde{H}_*((X/G)^{\wedge s}) \to \widetilde{H}_*((X/G)^{\wedge s}/\widetilde{\Delta}_s) \to \widetilde{H}_{*-1}(\widetilde{\Delta}_s) \to \cdots$$

Our next result gives a sufficient condition for the existence of a homology decomposition of the pinched set $\widetilde{\Delta}_s$.

The reduced diagonal map of A is the simplicial map $A \to A \wedge A$ given by $a \mapsto a \wedge a$ for all $a \in A_n$. A pointed simplicial map $f: Y \to Z$ is *mod 2 homologous to zero* if the induced map $f_*: \widetilde{H}_*(Y; \mathbb{F}_2) \to \widetilde{H}_*(Z; \mathbb{F}_2)$ is the zero map. We show that if the reduced diagonal map of A is mod 2 homologous to zero, then the mod 2 homology of $\widetilde{\Delta}_s$ is completely determined by the mod 2 homology of the fixed set A and the orbit space X/G.

We use the following multi-index notation. Let $\widetilde{b}_t(Y; \mathbb{F}_2) := \dim \widetilde{H}_t(Y; \mathbb{F}_2)$ denote the tth *reduced mod 2 Betti number* of Y. A *multi-index* $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a (possibly empty) sequence of positive integers. The *length* of this multi-index is $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and its *dimension* is $\dim \alpha = d$. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, we write for short the following product:

$$\widetilde{b}_\alpha(Y; \mathbb{F}_2) := \widetilde{b}_{\alpha_1}(Y; \mathbb{F}_2) \widetilde{b}_{\alpha_2}(Y; \mathbb{F}_2) \cdots \widetilde{b}_{\alpha_d}(Y; \mathbb{F}_2).$$

Theorem 1.2 *If the reduced diagonal map of A is mod 2 homologous to zero, then the reduced mod 2 Betti numbers of $\widetilde{\Delta}_s$ are given by*

$$\widetilde{b}_t(\widetilde{\Delta}_s; \mathbb{F}_2) \cong \sum_{\substack{|\lambda|+|\mu|=t-s+\dim\lambda+\dim\mu+1 \\ 2 \leq \dim\lambda+\dim\mu+1 \leq s}} c_{\lambda,\mu} \widetilde{b}_\lambda(X/G; \mathbb{F}_2) \widetilde{b}_\mu(A; \mathbb{F}_2),$$

where $c_{\lambda,\mu} = \binom{\dim\lambda+\dim\mu}{\dim\mu}\binom{s-\dim\lambda-\dim\mu-1}{\dim\mu-1}$.

The condition that the reduced diagonal map of A is mod 2 homologous to zero is quite general. For example, this condition is satisfied if A is the reduced suspension on some space (see Example 5.5).

The homology decompositions of Theorems 1.1 and 1.2 can be applied to compute the mod 2 Betti numbers of $\Omega(X \rtimes_G W_\infty^1 G)$ for certain pointed simplicial G–sets X. These homology decompositions are particularly effective when the orbit space X/G has many trivial homology groups. As an illustration, we compute the mod 2 Betti numbers in the following example.

Proposition 1.3 *Consider the G–space $S^2 \cup_{S^1} S^2$ formed by two 2–spheres S^2 with the antipodal action under which their equatorial circles are identified. The reduced mod 2 Betti numbers of the loop space of its 1–truncated Borel construction are*

$$\widetilde{b}_n(\Omega([S^2 \cup_{S^1} S^2] \rtimes_G E_\infty^1 G))$$

$$= \begin{cases} 1 + \displaystyle\sum_{r=k+1}^{2k} \sum_{J=1}^{2r-3} \binom{2k-r+J}{J}\binom{2r-2k-J-1}{J-1} & n = 2k,\ k \geq 1, \\ \displaystyle\sum_{r=k+2}^{2k+1} \sum_{J=1}^{r-k-1} \binom{2k-r+J+1}{J}\binom{2r-2k-J-2}{J-1} & n = 2k + 1,\ k \geq 0. \end{cases}$$

The outline of this paper is as follows. Carlsson's simplicial group construction $J^G[X]$ and the reduced 1–stunted Borel construction are introduced in Section 2. In Section 3, the augmentation ideal filtration of the group ring $\mathbb{F}_2(J^G[X])$ is considered. We construct simplicial algebras which are isomorphic to the graded algebra associated to this filtration. Theorem 1.1 is proved in Section 4. Theorem 1.2 is proved in Section 5 using the Mayer–Vietoris spectral sequence. Section 6 is devoted to the example $X = S^2 \cup_{S^1} S^2$ and the proof of Proposition 1.3.

This paper is based on the results in the first author's PhD thesis [6].

2 Preliminaries

To begin, we explain the concepts of the reduced Borel construction and its 1–stuntation.

Denote by WG any contractible simplicial set with a free G–action. Any two such simplicial sets are equivariantly homotopy equivalent. In our case where $G = C_2$, we take WG for the ∞–sphere S^∞ with the antipodal action. Let $EG := |WG|$ denote the geometric realization of WG.

The simplicial set WG is filtered by simplicial G–subsets:

(2-1) $$G \simeq W_0 G \subset W_1 G \subset \cdots \subset W_p G \subset \cdots \subset W_\infty G =: WG,$$

where $W_p G$ is the pth skeleton of WG. In fact $W_p G$ can be taken to be the $(p+1)^{\text{th}}$ fold join of G. In our case where $G = C_2$, it is standard to give $W_p G$ as the p–sphere S^p with the antipodal action.

The *bar construction of G* is the orbit space $\overline{W}G := WG/G$. In our case $\overline{W}G$ is homotopy equivalent to the infinite-dimensional real projective space:

$$\overline{W}G \simeq \mathbb{R}P^\infty.$$

The *classifying space of G* is the geometric realization $BG := |\overline{W}G|$. Since G is discrete, its classifying space BG is the Eilenberg–Mac Lane space $K(G, 1)$.

Consider an action of G on a simplicial set X. The free simplicial G–set associated to X is $X \times WG$ with the diagonal action. The *Borel construction of X* is the orbit space $X \times_G WG := (X \times WG)/G$. For example, the Borel construction of the G–action on the standard 0–simplex $\Delta[0] = *$ is the bar construction of G:

$$* \times_G WG \simeq \overline{W}G.$$

Suppose the G–action is pointed, that is to say, the simplicial set X has a basepoint which is fixed under the G–action. The *reduced Borel construction* of this pointed action, written $X \rtimes_G WG$, is the homotopy cofiber of $* \times_G WG \to X \times_G WG$.

More generally, let $X \times_G W_p G$ denote the orbit space $(X \times W_p G)/G$ of the diagonal action. For pointed actions, let $X \rtimes_G W_p G$ denote the homotopy cofiber of $* \times_G W_p G \to X \times_G W_p G$. For $q \geq p$, define the (p, q)–*stunted reduced Borel construction* $X \rtimes_G W_p^q G$ as the homotopy cofiber of $X \rtimes_G W_{q-1} G \to X \rtimes_G W_p G$. In particular, when $p = \infty$, we call $X \rtimes_G W_\infty^q G$ the q–*stunted reduced Borel construction* of the G–action on X.

In this paper, we are interested in the 1–stunted Borel construction $X \rtimes_G W_\infty^1 G$. Since $X \rtimes_G W_0 G \simeq X \rtimes_G G \simeq (X \times_G G)/(* \times_G G) \simeq X$, the 1–stunted Borel construction is just the homotopy cofiber of the inclusion $X \hookrightarrow X \rtimes_G WG$. Denote by $|X| \rtimes_G E_\infty^1 G$ the geometric realization of $X \rtimes_G W_\infty^1 G$.

Carlsson [4] constructed a simplicial group $J^G[X]$ whose geometric realization is the loop space of the 1–stunted reduced Borel construction:

(2-2) $$|J^G[X]| \simeq \Omega(|X| \rtimes_G E_\infty^1 G).$$

Carlsson's construction is given in dimension n by

$$(2\text{-}3) \qquad J^G[X]_n := \frac{F[X_n \wedge G_n]}{\langle \forall x \in X_n \forall g, h \in G_n \ (x \wedge g) \cdot (xg \wedge h) \sim (x \wedge gh) \rangle},$$

where $F[S] = \mathrm{coker}(F(*) \to F(S))$ is the reduced free group on a pointed set S, where $F(\bullet)$ denotes the (unreduced) free group. The functor $F[\bullet] \colon \mathsf{PtSet} \to \mathsf{Grp}$ is the left adjoint to the inclusion functor $\mathsf{Grp} \hookrightarrow \mathsf{PtSet}$ that sends a group to its underlying set with the identity element as basepoint.

Carlsson's construction is the reduced universal simplicial group on the pointed simplicial action groupoid $X /\!\!/ G$:

$$J^G[X] \cong U[X /\!\!/ G].$$

For a pointed small groupoid H, its *reduced universal monoid* $U[H]$ is defined by the following cokernel:

$$U[H] := \mathrm{coker}(U(\mathrm{Aut}_H(*)) \to U(H)).$$

Here $\mathrm{Aut}_H(*)$ denotes the full subcategory of H whose only object is the basepoint and $H \colon \mathsf{Grpd} \to \mathsf{Grp}$ is the left adjoint of the inclusion functor $\mathsf{Grp} \hookrightarrow \mathsf{Grpd}$ that sends a group to the corresponding small groupoid with one object. The *reduced universal simplicial group* $U[G]$ of a small simplicial groupoid G is defined dimension-wise. Further details can be found in the first author's thesis [6]. This categorial viewpoint led to a unification of Carlsson's construction and a simplicial monoid construction of the second author [17], which contains the classical constructions of Milnor [12] and James [10] as special cases. An upcoming paper will further elaborate on this viewpoint.

3   Augmentation quotients as free simplicial modules

In this section, we construct two simplicial algebras each of which is isomorphic to the associated graded algebra of the augmentation ideal filtration of the group ring $\mathbb{F}_2(J^G[X])$ (see Proposition 3.5). In every dimension, each of these simplicial algebras is a quotient of a tensor algebra by a homogeneous ideal. Therefore each augmentation quotient is the reduced simplicial $\mathbb{F}_2$–module of a pointed simplicial set (see Corollary 3.6).

In our case where $G = C_2$, there is a natural isomorphism of pointed simplicial G–sets:

$$(3\text{-}1) \qquad\qquad J^G[X] \cong \frac{F[X]}{\langle \forall x \in X \ (x \cdot xt \sim 1) \rangle}.$$

Recall from Section 2 that $F[X]$ is the reduced free group on X. Via this natural isomorphism, we identify $J^G[X]$ with the right hand side of (3-1).

Let K be a field and H be a group. The elements of the group ring $K(H)$ are finite sums of the form $\sum_{\lambda \in K, h \in H} \lambda_h h$. The augmentation map $K(H) \to K$ is generated by $h \mapsto 1$ for $h \in H$. The kernel of this map is the augmentation ideal. Reserve I to denote the augmentation ideal of the group ring $\mathbb{F}_2(J^G[X])$. The augmentation ideal I is generated by $\bar{h} := h - 1$ where $h \in J^G[X]$. The powers of I filter the group ring

$$(3\text{-}2) \qquad \cdots \subseteq I^{s+1} \subseteq I^s \subseteq \cdots \subseteq I^1 \subseteq I^0 = \mathbb{F}_2(J^G[X])$$

(Quillen [14] calls this the I–filtration). We denote the spectral sequence associated to this filtration by $\{E^r\}$. The E^0 term is just the graded algebra $\bigoplus_{s=0}^{\infty} I^s / I^{s+1}$.

One of the simplicial algebras we construct is $A^G[X]$. For an $\mathbb{F}_2$–module M, let $T(M) = \bigoplus_{s=0}^{\infty} \underbrace{M \otimes_{\mathbb{F}_2} \cdots \otimes_{\mathbb{F}_2} M}_{s}$ denote the tensor $\mathbb{F}_2$–algebra on M.

Definition 3.1 Let X be a pointed simplicial G–set. Define the simplicial graded $\mathbb{F}_2$–algebra $A^G[X]$ dimension-wise by

$$(A^G[X])_n := \frac{T(\mathbb{F}_2[X_n] \otimes_{\mathbb{F}_2(G)} \mathbb{F}_2)}{\langle \forall a \in A_n \, (a \otimes_{\mathbb{F}_2(G)} 1)^2 \rangle}.$$

Here the G–action on X_n allows us to view $\mathbb{F}_2[X_n]$ as a right $\mathbb{F}_2(G)$–module, while $\mathbb{F}_2$ is viewed as an $\mathbb{F}_2(G)$–module where G acts trivially on the left. The tensor product $\mathbb{F}_2[X_n] \otimes_{\mathbb{F}_2(G)} \mathbb{F}_2$ is viewed as an $\mathbb{F}_2$–module.

Proposition 3.2 The augmentation quotient I/I^2 is generated by $\{\bar{x} + I^2 \mid x \neq *\}$.

Proof The identity $\bar{x}\bar{y} = \overline{xy} - \bar{x} - \bar{y}$ and the fact that $\bar{x}\bar{y} \in I^2$ implies that $(\bar{x} + I^2) + (\bar{y} + I^2) = \overline{xy} + I^2$.

The set $S := \{\bar{x} + I^2 \mid x \neq *\}$ thus generates

$$(3\text{-}3) \qquad \{\overline{x_1 x_2 \cdots x_n} + I^2 \mid x_1 \neq *, \ldots, x_n \neq *\}.$$

By (3-1), each nonidentity element of $J^G[X]$ is (the equivalence class of) a reduced word of the form $x_1 \cdots x_n$, hence I/I^2 is generated by (3-3) and thus also by S. This completes the proof. $\qquad\square$

Proposition 3.3 Let B be a graded algebra. Let $\hat{B}$ be the completion of B with respect to the filtration by degree,

$$\cdots \subset B_{\geq r} \subset \cdots \subset B_{\geq 1} \subset B_{\geq 0} = B,$$

where $B_{\geq r} := \bigoplus_{i=r}^{\infty} B_i$. The above filtration induces a filtration on $\widehat{B}$:

$$\cdots \subset \widehat{B}_{\geq r} \subset \cdots \subset \widehat{B}_{\geq 1} \subset \widehat{B}_{\geq 0} = \widehat{B}.$$

Then the map $\Theta\colon E^0(\widehat{B}) \to B$ whose r^{th} grade is given by $\Theta_r(f + \widehat{B}_{\geq r+1}) = f_r$, where f_r is the rth degree component of f, is an isomorphism of graded algebras.

Proof An element of $\widehat{B}$ is a formal power series of the form $f = f_0 + f_1 + \cdots f_i + \cdots$, where f_i is of degree i in B. An element of $\widehat{B}_{\geq r}$ is a formal power series of the form $f = f_r + f_{r+1} + \cdots$ whose lowest degree is at least r. The map Θ_r is well-defined since if $f \in \widehat{B}_{\geq r+1}$, then $f_r = 0$.

Let $\Lambda\colon B \to E^0(\widehat{B})$ be the map whose r^{th} grade is $\Lambda_r(f) = f + \widehat{B}_{\geq r+1}$. It is easy to check that Θ and Λ are inverse to one another. □

Lemma 3.4 *There is an isomorphism of graded algebras natural in X:*

$$\Phi\colon A^G[X] \to E^0,$$

$$x \otimes_{\mathbb{F}_2(G)} 1 \mapsto \bar{x} + I^2.$$

Proof Write $\otimes := \otimes_{\mathbb{F}_2(G)}$ for short.

We first verify that Φ is well-defined, that is to say, that the map $(x, 1) \mapsto \bar{x} + I^2$ is indeed $\mathbb{F}_2(G)$–linear. On the one hand, $(x \cdot t, 1) \mapsto \overline{xt} + I^2$, and on the other hand $(x, t \cdot 1) = (x, 1) \mapsto \bar{x} + I^2$. Since $x \cdot xt = 1$,

$$\bar{x} \cdot \overline{xt} + \bar{x} + \overline{xt} = (x-1)(xt-1) + (x-1) + (xt-1) = x \cdot xt - 1 = 0.$$

This implies $\bar{x} + \overline{xt} = -\bar{x} \cdot \overline{xt} \in I^2$ so that $\bar{x} + I^2 = -\overline{xt} + I^2 = \overline{xt} + I^2$ as the ground field is $\mathbb{F}_2$. Therefore both $(x \cdot t, 1)$ and $(x, t \cdot 1)$ are sent to the same thing which verifies the $\mathbb{F}_2(G)$–linearity of the map $(x, 1) \mapsto \bar{x} + I^2$.

Our definition $\Phi(x \otimes 1) = \bar{x} + I^2$ is given for $x \in X$, then it can be extended to a map $T(\mathbb{F}_2[X_n] \otimes_{\mathbb{F}_2(G)} \mathbb{F}_2) \to E^0$. This is because $\Phi(* \otimes 1) = \bar{*} + I^2 = 1 - 1 + I^2 = 0$ and the tensor algebra $T(\mathbb{F}_2[X_n] \otimes_{\mathbb{F}_2(G)} \mathbb{F}_2)$ is generated by elements of the form $x \otimes 1$. We check that this map factors through the defining equivalence relation of $A^G[X]$. Given $a \in A$, we have $\Phi((a \otimes 1)^2) = (\bar{a} + I^2)^2 = \bar{a}^2 + I^3$. And $\bar{a}^2 = (a-1)^2 = a^2 - 1 = 0$ since $a \in A$ implies $a^2 = a \cdot at = 1$. Thus $\Phi((a \otimes 1)^2) = \bar{a}^2 + I^3 = 0$, so we have a well-defined map $\Phi\colon A^G[X] \to E^0$.

Next we show that Φ is an epimorphism. It suffices to show that, when Φ is restricted to the first grade, the map $\Phi_1\colon A_1^G[X] \to I/I^2$ is an epimorphism, since I/I^2 generates E^0. Using the isomorphism in (3-1), Proposition 3.2 implies that the augmentation

ideal is generated by $\bar{x}$ where $* \neq x \in X$. Since, for each $x \in X$, the element $\bar{x} + I^2$ is the image of $\Phi(x \otimes 1)$, this proposition implies that each element in I/I^2 has a preimage under Φ. Therefore Φ is an epimorphism.

To show that Φ is a monomorphism, choose a subset $B \subset X$ of elements not fixed by the G–action that decomposes X into the disjoint union $A \sqcup B \sqcup Bt$. Then the map $f : J^G[X] \to J^G[A] * F(B)$ that sends $a \mapsto a$ for $a \in A$ and $b \mapsto b, bt \mapsto b^{-1}$ for $b \in B$ is a group isomorphism. The map

$$e_1 : J^G[A] \to \widehat{A^G[X]}$$

generated by $a \mapsto a \otimes 1 + 1 \otimes 1$ is well-defined, because $e_1(a \cdot a) = (a \otimes 1 + 1 \otimes 1) \cdot (a \otimes 1 + 1 \otimes 1) = (a \otimes 1)(a \otimes 1) + (1 \otimes 1)(1 \otimes 1) = (1 \otimes 1)$ agrees with $e_1(1) = 1 \otimes 1$. Define

$$e_2 : F(B) \to \widehat{A^G[X]}, \quad b \mapsto b \otimes 1 + 1 \otimes 1.$$

In particular, $e_2(b^{-1}) = 1/(b \otimes 1 + 1 \otimes 1) = \sum_{i=0}^{\infty}(-1)^i (b \otimes 1)^i = \sum_{i=0}^{\infty}(b \otimes 1)^i$ as the ground field is $\mathbb{F}_2$. The universal property of the free product gives a map $e_1 * e_2 : J^G[A] * F(B) \to \widehat{A^G[X]}$. The universal property of the group ring then induces a map $\widehat{e_1 * e_2} : \mathbb{F}_2(J^G[X]) \to \widehat{A^G[X]}$. This induces a map between the associated graded algebras, $E^0(\widehat{e_1 * e_2}) : E^0(\mathbb{F}_2(J^G[A] * F(B))) \to E^0(\widehat{A^G[X]})$. Consider the composite

$$A^G[X] \xrightarrow{\Phi} E^0(\mathbb{F}_2(J^G[X])) \xrightarrow{E^0(\mathbb{F}_2(f))} E^0(\mathbb{F}_2(J^G[A] * F(B)))$$

$$\xrightarrow{E^0(\widehat{e_1 * e_2})} E^0(\widehat{A^G[X]}) \xrightarrow{\Theta} A^G[X].$$

Here Θ is the map given in Proposition 3.3. It is easy to check that this composite is the identity map on $A^G[X]$ and hence the first map Φ is a monomorphism, as required.

Finally, it is straightforward to check the naturality. This completes the proof. $\qquad\square$

Proposition 3.5 *There are isomorphisms of simplicial graded $\mathbb{F}_2$–algebras:*

$$\bigoplus_{s=0}^{\infty} I^s/I^{s+1} \cong A^G[X] \cong T(\mathbb{F}_2[X/G])/\langle \forall a \in A \, (aG)^2 \rangle,$$

where $T(\mathbb{F}_2[X/G])/\langle \forall a \in A \, (aG)^2 \rangle$ is the simplicial graded $\mathbb{F}_2$–algebra whose n^{th} dimension is $T(\mathbb{F}_2[X_n/G])/\langle \forall a \in A_n \, (aG)^2 \rangle$.

Proof Define $\Phi : A^G[X] \to E^0(\mathbb{F}_2(J^G[X]))$ dimension-wise using Lemma 3.4. In each dimension n, the map Φ_n is an isomorphism of graded algebras. The naturality

part of the same lemma implies that the map Φ commutes with faces and degeneracies and hence it is a simplicial map. Therefore Φ is an isomorphism of simplicial algebras:

$$(3\text{-}4) \qquad\qquad \Phi\colon A^G[X] \to E^0(\mathbb{F}_2(J^G[X])).$$

Denote the algebra $T(\mathbb{F}_2[X/G])/\langle\forall a \in A\,(aG)^2\rangle$ by T. Let $\phi\colon \mathbb{F}_2[X]\times\mathbb{F}_2 \to T$ send $(x,1) \mapsto xG$. Since $\phi(x\cdot t,1) = xtG = xG$ agrees with $\phi(x,t\cdot 1) = \phi(x,1) = xG$, this map factors to a map $\mathbb{F}_2[X]\otimes_{\mathbb{F}_2(G)}\mathbb{F}_2 \to T$ from the tensor product. The universal property of the tensor algebra defines a map $T(\mathbb{F}_2[X]\otimes_{\mathbb{F}_2(G)}\mathbb{F}_2) \to T$. We check that this map factors through the defining equivalence relations of $A^G[X]$. Given $a \in A$, indeed $(a\otimes 1)^2$ is sent to $(aG)^2$, which is in the quotient ideal of T. Thus we have a map $\tilde{\phi}\colon A^G[X] \to T$.

Let $\psi\colon X/G \to A^G[X]$ send $xG \mapsto x\otimes_{\mathbb{F}_2(G)}1$. This map ψ is well-defined since $xt\otimes_{\mathbb{F}_2(G)}1 = x\otimes_{\mathbb{F}_2(G)}t\cdot 1 = x\otimes_{\mathbb{F}_2(G)}1$. The universal property of the tensor algebra defines a map $\psi\colon T(\mathbb{F}_2[X/G]) \mapsto A^G[X]$. We check that this map factors through the defining equivalence relations of T. Given $a \in A$, indeed $(aG)^2$ is sent to $(a\otimes_{\mathbb{F}_2(G)}1)^2$, which is in the quotient ideal of $A^G[X]$. Thus we have a map $\tilde{\psi}\colon T \to A^G[X]$.

It is easy to check that $\tilde{\phi}$ and $\tilde{\psi}$ are inverses to each other and give an isomorphism

$$(3\text{-}5) \qquad\qquad A^G[X] \cong T = T(\mathbb{F}_2[X/G])/\langle\forall a \in A\,(aG)^2\rangle.$$

Combine the isomorphisms (3-4) and (3-5) to complete the proof. $\qquad\qquad\square$

Recall from the introduction that the pointed simplicial subset $\tilde{\Delta}_s$ of $(X/G)^{\wedge s}$ is defined as follows. Set $\tilde{\Delta}_0 = \tilde{\Delta}_1 := *$ and

$$\tilde{\Delta}_s := \{x_1 G \wedge \cdots \wedge x_s G \in (X/G)^{\wedge s} \mid \exists i = 1, \ldots, s-1\ (x_i = x_{i+1} \in A)\} \text{ for } s \geq 2.$$

Corollary 3.6 *For $s \geq 1$, there is an isomorphism of simplicial $\mathbb{F}_2$–modules*

$$I^s/I^{s+1} \overset{\cong}{\to} \mathbb{F}_2\big[(X/G)^{\wedge s}/\tilde{\Delta}_s\big],$$

$$\overline{x}_1\cdots\overline{x}_s + I^{s+1} \mapsto x_1 G \wedge \cdots \wedge x_s G.$$

Proof The proof of Proposition 3.5 gives an isomorphism of simplicial graded algebras

$$\tilde{\phi}\circ\Phi^{-1}\colon \bigoplus_{s=0}^{\infty} I^s/I^{s+1} \overset{\cong}{\to} T(\mathbb{F}_2[X/G])/\langle\forall a \in A\,(aG)^2\rangle,$$

$$\overline{x} + I \mapsto xG.$$

The sth grade of this isomorphism is

$$I^s/I^{s+1} \xrightarrow{\cong} \left(T(\mathbb{F}_2[X/G])/\langle \forall a \in A\,(aG)^2 \rangle\right)_s,$$

$$\overline{x}_1 \cdots \overline{x}_s + I^{s+1} \mapsto x_1 G \cdots x_s G.$$

The sth grade of the tensor algebra $T_s(\mathbb{F}_2[X/G])$ can be identified with $\mathbb{F}_2[(X/G)^{\wedge s}]$. Via this identification, the terms of degree s in the ideal $\langle \forall a \in A\,(aG)^2 \rangle$ are linear combinations of smash products $x_1 G \wedge \cdots \wedge x_s G$ such that, for some $i = 1, \ldots, s-1$, the elements x_i and x_{i+1} are equal and belong to A. The result follows by the definition of $\widetilde{\Delta}_s$. $\qquad\square$

4 Proof of Theorem 1.1

In this section, we show that the existence of a section of the orbit projection leads to a mod 2 homology decomposition of $J^G[X]$. There are two proof ingredients. First, we show that the powers of the augmentation ideal of $\mathbb{F}_2(J^G[X])$ have trivial intersection. Second, we show that the exact sequences $I^{s+1} \to I^s \to I^s/I^{s+1}$ are split. These imply that $\mathbb{F}_2(J^G[X])$ is isomorphic to E^0 and that the long exact sequence associated to $I^{s+1} \to I^s \to I^s/I^{s+1}$ splits into short exact sequences. Therefore the spectral sequence associated to the augmentation ideal filtration collapses at the E^1 term and converges to $H_*(J^G[X]; \mathbb{F}_2)$.

We begin with a characterization of the G–sets whose orbit projection has a section.

Proposition 4.1 *The orbit projection $X \to X/G$ has a section if and only if, there exist simplicial sets A and Y with A as a simplicial subset of Y, such that X is a pushout $Y \cup_A Yt$ with the action of swapping Y with Yt.*

Proof If j is a section of the orbit projection, then $X = \operatorname{im} j \cup_A (\operatorname{im} j)t$ where $A \subset X$ is the set fixed under the action.

Conversely, the orbit space of a pushout $Y \cup_A Yt$ is isomorphic to Y. Thus the map $Y \hookrightarrow Y \cup_A Yt$ that is the inclusion to the left copy of Y gives the required section. $\square$

For the G–set $Y \cup_A Yt$, its orbit space is isomorphic to Y and the set fixed under the action is A. There are two sections of the orbit projection. One section maps the orbits space to Y, the other section maps the orbit space to Yt.

In the case where the coefficient ring is a field, there is a characterization of group rings for which the powers of the augmentation ideal to have trivial intersection. We recall below the characterization if the coefficient ring is a field of prime characteristic.

We use the following terminology from group theory. A group has *bounded exponent* if there exists an integer $n \geq 0$ such that every element of the group has order at most n. We say $\mathcal{P}$ is a *property of groups* if (i) the trivial group has the property $\mathcal{P}$ and (ii) given isomorphic groups G and H, the group G has property $\mathcal{P}$ if and only if the group H has property $\mathcal{P}$. A group G is residually $\mathcal{P}$ if, for each nonidentity element $x \in G$, there exists a group epimorphism $\phi \colon G \to H$ where H is a $\mathcal{P}$–group such that $\phi(x) \neq 1$.

Proposition 4.2 (Passi [13, Theorem 2.26]) *Let J be the augmentation ideal of a group ring $K(H)$ where K is a field of characteristic prime p. Then $\bigcap_n J^n = 0$ if and only if H is residually nilpotent p–group of bounded exponent.*

We will need the following result of Gruenberg [8].

Lemma 4.3 *The free product of finitely many residually finite p–groups is a residually finite p–group.*

Let C_∞ denote the infinite cyclic group and C_p denote the cyclic group of order p.

Proposition 4.4 *A free product of arbitrarily many copies of C_∞ and C_p is a residually finite p–group.*

Proof Let a group G which is a free product of copies of C_∞ and C_p. We write $G = *_{i \in I} H_i$, where I is an index set and H_i is isomorphic to a copy of either C_∞ or C_p. For each $i \in I$, fix a generator t_i of H_i.

Let a word $w = t_{i_1}^{n_{i_1}} \cdots t_{i_k}^{n_{i_k}}$ be given. Let $H = H_{i_1} * \cdots * H_{i_k}$. Let $\psi \colon G \to H$ be the group homomorphism given by

$$\psi(t_j) = \begin{cases} t_j & \text{if } j = i_1, \ldots, i_k, \\ 1_H & \text{otherwise.} \end{cases}$$

Then $\psi(w)$ is a nonidentity element of H.

It is easy to show that C_p and C_∞ are both residually finite p–groups. Thus Lemma 4.3 implies that the group H is a residually finite p–group. Since $\psi(w)$ is a nonidentity element of H, there exists a group epimorphism $\phi \colon H \to K$ where K is a finite p–group such that $\phi(\psi(w)) \neq 1$. Since for w, there is an epimorphism $G \xrightarrow{\psi} H \xrightarrow{\phi} K$, the group G is a residually finite p–group. $\square$

The following proposition is straightforward and its proof is omitted.

Proposition 4.5 *Let X be a pointed G–set. If X is written as a disjoint union $A \sqcup B \sqcup Bt$, then there is a group isomorphism*

$$J^G[X] \to J^G[A] * F(B),$$
$$a \mapsto a,$$
$$b \mapsto b.$$

In particular $\phi(bt) = \phi(b)^{-1} = b^{-1}$.

Corollary 4.6 *Let X be a pointed G–set. The augmentation ideal I of $\mathbb{F}_2(J^G[X])$ satisfies $\bigcap_n I^n = 0$.*

Proof Write X as a disjoint union $A \sqcup B \sqcup Bt$, then Proposition 4.5 gives an isomorphism $J^G[X] \cong J^G[A] * F(B)$. The group $J^G[A]$ is a free product of copies of C_2 while the free group $F(B)$ is a free product of copies of C_∞. Proposition 4.4 applies to show that $J^G[X]$ is a residually finite 2–group. Since a finite 2–group is a nilpotent 2–group of bounded exponent, the group $J^G[X]$ is a residually nilpotent 2–group of bounded exponent. Then the result follows from Proposition 4.2. $\qquad\square$

This corollary implies that the spectral sequence $\{E^r\}$ is weakly convergent.

Proposition 4.7 *Let J be the augmentation ideal of a group ring $K(H)$ with coefficients in a field K. If $\bigcap_n J^n = 0$ and the short exact sequence $J^{s+1} \to J^s \to J^s/J^{s+1}$ is split for all s, then there is an isomorphism of K–modules:*

$$K(H) \cong \bigoplus_{s=0}^{\infty} J^s/J^{s+1}.$$

Proof Since the coefficients are taken in a field, the split short exact sequences imply that $J^s \cong J^{s+1} \oplus J^s/J^{s+1}$ for all s. An easy induction shows that $K(H) \cong J^n \oplus \left(\bigoplus_{s=0}^{n-1} J^s/J^{s+1}\right)$ for all n. Thus there is an isomorphism of K–modules for each n:

$$\bigoplus_{s=0}^{n-1} J^s/J^{s+1} \cong K(H)/J^n.$$

This allows us to identify the filtered system

$$K(H)/J^1 \to \bigoplus_{s=0}^{1} J^s/J^{s+1} \to \cdots \to \bigoplus_{s=0}^{n-1} J^s/J^{s+1} \to \cdots$$

with the filtered system

$$K(H)/J^1 \to K(H)/J^2 \to \cdots \to K(H)/J^n \to \cdots .$$

Therefore the colimits are isomorphic as K–modules:

$$\bigoplus_{s=0}^{\infty} J^s/J^{s+1} \cong \varinjlim_n \bigoplus_{s=0}^{n-1} J^s/J^{s+1}$$

$$\cong \varinjlim_n K(H)/J^n$$

$$\cong K(H)\Big/ \bigcap_n J^n$$

$$= K(H),$$

where we used the assumption that $\bigcap_n J^n$ is trivial in the last step. □

Proof of Theorem 1.1 First we show that the following short exact sequence is split for each s:

$$(4\text{-}1) \qquad\qquad\qquad I^{s+1} \to I^s \to I^s/I^{s+1}.$$

For $s = 0$, the short exact sequence (4-1) always splits for any group ring. For $s \geq 1$, Corollary 3.6 gives an isomorphism $I^s/I^{s+1} \to \mathbb{F}_2\big[(X/G)^{\wedge s}/\widetilde{\Delta}_s\big]$ defined by $\overline{x}_1 \cdots \overline{x}_s + I^{s+1} \mapsto x_1 G \wedge \cdots \wedge x_s G$. Via this isomorphism, it suffices to show that the following map has a section:

$$\alpha\colon I^s \to \mathbb{F}_2\big[(X/G)^{\wedge s}/\widetilde{\Delta}_s\big],$$
$$\overline{x}_1 \cdots \overline{x}_s \mapsto x_1 G \wedge \cdots \wedge x_s G.$$

By Proposition 4.1, the assumption that the orbit projection has a section allows us to write $X = Y \cup_A Yt$. Then every orbit is of the form yG for some $y \in Y$. Define $\beta\colon \mathbb{F}_2\big[(X/G)^{\wedge s}/\widetilde{\Delta}_s\big] \to I^s$ by $\beta(y_1 G \wedge \cdots y_s G) = \overline{y}_1 \cdots \overline{y}_s$ for $y_1, \dots, y_s \in Y$. The map β is well-defined since if there exists some $i = 1, \dots, s-1$ such that both y_i and y_{i+1} are equal to some $a \in A$, then $\overline{y}_i \overline{y}_{i+1} = (a-1)(a-1) = a^2 - 1 = 1 - 1 = 0$ as $a^2 = 1$ in $J^G[X]$ so that $\beta(y_1 G \wedge \cdots \wedge y_s G) = 0$. Then β is a section of α:

$$\alpha(\beta(y_1 G \wedge \cdots \wedge y_s G)) = \alpha(\overline{y}_1 \cdots \overline{y}_s) = y_1 G \wedge \cdots \wedge y_s G.$$

Thus we have shown that the exact sequences (4-1) are split for each s.

We have shown that $\bigcap_n I^n = 0$ in Corollary 4.6. Thus Proposition 4.7 implies

$$(4\text{-}2) \qquad\qquad \mathbb{F}_2(J^G[X]) \cong \bigoplus_{s=0}^{\infty} I^s/I^{s+1} \cong \mathbb{F}_2 \oplus \bigoplus_{s=1}^{\infty} I^s/I^{s+1}.$$

Using Corollary 3.6 and taking homotopy, we have

$$\pi_*(\mathbb{F}_2(J^G[X])) \cong \pi_*(\mathbb{F}_2) \oplus \bigoplus_{s=1}^{\infty} \pi_* \big(\mathbb{F}_2[(X/G)^{\wedge s}/\widetilde{\Delta}_s]\big).$$

Using the Dold–Thom Theorem (see the last section in [17]), this becomes

$$H_t(J^G[X]; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 \oplus \big(\bigoplus_{s=1}^{\infty} \widetilde{H}_0((X/G)^{\wedge s}/\widetilde{\Delta}_s; \mathbb{F}_2)\big) & \text{if } t = 0, \\ \bigoplus_{s=1}^{\infty} \widetilde{H}_t((X/G)^{\wedge s}/\widetilde{\Delta}_s; \mathbb{F}_2) & \text{otherwise.} \end{cases}$$

Thus the reduced homology of $J^G[X]$ is

$$\widetilde{H}_*(J^G[X]; \mathbb{F}_2) \cong \bigoplus_{s=1}^{\infty} \widetilde{H}_*((X/G)^{\wedge s}/\widetilde{\Delta}_s; \mathbb{F}_2).$$

Now the homotopy equivalence (2-2) completes the proof. □

Note that the splitting of the short exact sequence (4-1) implies that the associated long exact sequence in homology splits into short exact sequences. Thus the spectral sequence $\{E^r\}$ collapses at the E^1 term. The isomorphism (4-2) between $\mathbb{F}_2(J^G[X])$ and E^0 implies that this spectral sequence converges to $H_*(J^G[X]; \mathbb{F}_2)$.

Theorem 1.1 should be compared with the following result of the second author.

Proposition 4.8 [17, Theorem 1.1] *Let $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and let X be a pointed space. Suppose that H_* is a multiplicative homology theory such that (1) both $\bar{H}_*(FP^\infty)$ and $\bar{H}_*(FP_2^\infty)$ are free $H_*(\mathrm{pt})$–modules; and (2) the inclusion of the bottom cell $S^d \to FP^\infty$ induces a monomorphism in the homology. Then there is a multiplicative filtration $\{F_r H_* \Omega(FP^\infty \wedge X) \mid r \geq 0\}$ of $H_* \Omega(FP^\infty \wedge X)$ such that $F_0 = H_*(\mathrm{pt})$ and*

$$F_s/F_{s-1} \cong \Sigma^{(d-1)s} \bar{H}_*(X^{\wedge s}/\hat{\Delta}_s),$$

where $d = \dim_{\mathbb{R}} F$, Σ is the suspension, $\hat{\Delta}_1 = *$ and $\hat{\Delta}_s = \{x_1 \wedge \cdots \wedge x_s \in X^{\wedge s} \mid x_i = x_{i+1}$ for some $i\}$ for $s > 1$. Furthermore, this filtration is natural with respect to X.

Take $F = \mathbb{R}$. In this case, the above result holds for the reduced mod 2 homology. Since $\mathbb{F}_2$ is a field, the multiplicative filtration yields the homology decomposition:

$$(4\text{-}3) \qquad \widetilde{H}_*(\Omega(\mathbb{R}P^\infty \wedge X); \mathbb{F}_2) = \bigoplus_{s=0}^{\infty} \widetilde{H}_*(X^{\wedge s}/\hat{\Delta}_s; \mathbb{F}_2).$$

If $G = C_2$ acts on X trivially, then X coincides with its orbit space X/G. This induces an isomorphism of simplicial sets for each r:

$$(X/G)^{\wedge s}/\widetilde{\Delta}_s \cong X^{\wedge s}/\widehat{\Delta}_s.$$

The 1–stunted reduced Borel construction has the following geometric realization for the trivial action:

$$|X \rtimes_G W^1_\infty G| \simeq \mathbb{R}P^\infty \wedge X.$$

Therefore our homology decomposition in Theorem 1.1 generalizes (4-3).

5   Proof of Theorem 1.2

We have shown in the previous section that, if the orbit projection has a section, then $\widetilde{H}_*(J^G[X]; \mathbb{F}_2) \cong \bigoplus \widetilde{H}_*((X/G)^{\wedge s}/\widetilde{\Delta}_s; \mathbb{F}_2)$. The pinched set $\widetilde{\Delta}_s$ can be written as the following union (see Corollary 5.2):

$$(5\text{-}1) \quad \big(\overline{\Delta}(A) \wedge (X/G)^{\wedge s-2}\big) \cup \big((X/G) \wedge \overline{\Delta}(A) \wedge (X/G)^{\wedge s-3}\big) \cup \cdots$$
$$\cup \big((X/G)^{\wedge s-2} \wedge \overline{\Delta}(A)\big),$$

where $\overline{\Delta}(A) := \{aG \wedge aG \mid a \in A\} \subset (X/G)^{\wedge 2}$.

Given a pointed simplicial set Y written as a union $Y_1 \cup \cdots \cup Y_N$ of pointed simplicial subsets, the Mayer–Vietoris spectral sequence allows one to approximate the homology of Y in terms of the homology of the intersections of the Y_i. Expression (5-1) suggests using the Mayer–Vietoris spectral sequence to study the homology of $\widetilde{\Delta}_s$. This can be combined with Theorem 1.1 to obtain further information about the mod 2 homology of $J^G[X]$. We illustrate this in Proposition 1.3.

We briefly review the Mayer–Vietoris spectral sequence. References for this spectral sequence are Cai [3], Chen, Lü and the second author [5] and Hatcher [9]. Suppose that $Y = Y_1 \cup \cdots \cup Y_N$ is a pointed simplicial set with each Y_i a pointed simplicial subset of Y. Associated with Y is an abstract simplicial complex K with vertices $1, 2, \ldots, N$ and $\{i_1, \ldots, i_p\} \in K$ for $Y_{i_1} \cap \cdots \cap Y_{i_p}$. For each $\mathcal{I} = \{i_1, \ldots, i_p\} \in K$, define $Y_{\mathcal{I}} = Y_{i_1} \cap \cdots \cap Y_{i_p}$. In particular $Y_\varnothing = Y$.

For any simplicial set W, let $\mathbb{Z}W$ denote the free simplicial abelian group on W. One has a chain complex $(\mathbb{Z}W, \partial)$:

$$\mathbb{Z}W_0 \xleftarrow{\partial} \mathbb{Z}W_1 \xleftarrow{\partial} \mathbb{Z}W_2 \xleftarrow{\partial} \cdots,$$

where $\partial = \sum_{i=0}^n (-1)^i d_i$ and d_i is the ith face of the simplicial abelian group $\mathbb{Z}W$. The homology of this chain complex is the integral homology of W (see Goerss and

Jardine [7, page 5]):

$$H_*(W; \mathbb{Z}) \cong H_*(\mathbb{Z}W, \partial).$$

If W is pointed, its mod 2 reduced homology of W is given by:

$$\widetilde{H}_*(W; \mathbb{F}_2) = \operatorname{coker}\left(H_*(\mathbb{Z}* \otimes \mathbb{F}_2, \partial) \to H_*(\mathbb{Z}W \otimes \mathbb{F}_2, \partial)\right).$$

Let $E_{p,q} = \bigoplus_{\#\mathcal{I}=p}(\mathbb{Z}Y_{\mathcal{I}} \otimes \mathbb{F}_2)_q$ where $\#\mathcal{I}$ is the number of elements in the set $\mathcal{I}$. Then $E = \bigoplus_{p,q} E_{pq}$ is a double complex. For $\alpha_q^{\mathcal{I}} \in (\mathbb{Z}Y_{\mathcal{I}} \otimes \mathbb{F}_2)_q$, we have that the vertical differential is $\partial^v(\alpha_q^{\mathcal{I}}) := \partial\alpha_q^{\mathcal{I}}$, which is the above differential of the chain complex $\mathbb{Z}Y_{\mathcal{I}} \otimes \mathbb{F}_2$. For $\alpha_q^{\mathcal{I}} \in (\mathbb{Z}Y_{\mathcal{I}} \otimes \mathbb{F}_2)_q$ where $\mathcal{I} = \{i_1, \ldots, i_p\}$, the horizontal differential is then

$$\partial^h(\alpha_q^{\mathcal{I}}) := \alpha_q^{\partial\mathcal{I}} := \sum_{j=1}^{p}(-1)^j \alpha_q^{\partial_j\mathcal{I}},$$

where $\partial_j\mathcal{I} := (i_1, \ldots, \hat{i}_j, \ldots, i_p)$ has $p-1$ elements by omitting the jth term. Here $\alpha_q^{\partial_j\mathcal{I}}$ is an element of $(\mathbb{Z}Y_{\partial_j\mathcal{I}} \otimes \mathbb{F}_2)_q$ via the inclusion $Y_{\mathcal{I}} \hookrightarrow Y_{\partial_j\mathcal{I}}$.

Write $E_p = \bigoplus_q E_{p,q}$. The homology of E_0 is the mod 2 homology of Y (see [9]):

$$\widetilde{H}_*(Y; \mathbb{F}_2) \cong H_*(E_0).$$

There is an exact sequence (see Bott and Tu [2, page 94]):

$$0 \to E_N \xrightarrow{\partial_N^h} \cdots \xrightarrow{\partial_1^h} E_0 \to 0.$$

Denote $F_0 = \operatorname{im}\partial_1^h, \ldots, F_{N-2} = \operatorname{im}\partial_{N-1}^h, F_{N-1} = \operatorname{im}\partial_N^h$. Then we have the short exact sequences

$$0 \to E_N \to F_{N-1} \to 0,$$
$$0 \to F_{N-1} \to E_{N-1} \to F_{N-2} \to 0,$$
$$0 \to F_{N-2} \to E_{N-2} \to F_{N-3} \to 0,$$
$$\vdots$$
$$0 \to F_1 \to E_1 \to F_0 \to 0.$$

With respect to the differential $\partial^v \colon E_{p,q} \to E_{p,q-1}$, we obtain long exact sequences

$$\cdots \to H_q(F_{N-2}) \xrightarrow{i} H_q(E_{N-2}) \xrightarrow{j} H_q(F_{N-3}) \xrightarrow{\zeta} H_{q-1}(F_{N-2}) \to \cdots$$

$$\vdots$$

$$\cdots \to H_q(F_1) \xrightarrow{i} H_q(E_1) \xrightarrow{j} H_q(F_0) \xrightarrow{\zeta} H_{q-1}(F_1) \to \cdots$$

This long exact sequence can be written as an exact couple where i has bidegree $(0, 0)$, j has bidegree $(0, -1)$ and ζ has bidegree $(-1, 1)$:

$$
\begin{array}{ccc}
H_*(F_*) & \xrightarrow{\quad (0,0) \quad} & H_*(E_*) \\
 & \overset{i}{\underset{(-1,1)}{\nwarrow}} \quad \overset{(0,-1)}{\underset{\zeta}{\swarrow}} \quad \overset{}{\underset{j}{\searrow}} & \\
 & H_*(F_*). &
\end{array}
$$

The resulting spectral sequence is the Mayer–Vietoris spectral sequence

$$
\{E^r_{p,q}(X_1 \cup \cdots \cup X_N), d^r\} \Rightarrow H_{p+q-1}(E_0) = \widetilde{H}_{p+q-1}(X; \mathbb{F}_2),
$$

where the rth differential $d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ is induced by $i \circ \zeta^{-r+1} \circ j$ for $r \geq 1$. Note that $H_t(E_0) = \bigoplus_{p+q-1=t} E^\infty_{p,q}$. Let t be the dimension of the homology of E_0, the E^1 term of this spectral sequence is

$$
(5\text{-}2) \qquad\qquad E^1 = \bigoplus_{p+q-1=t} \bigoplus_{\substack{X_\mathcal{I} \neq \varnothing \\ \#\mathcal{I}=p\geq 1}} \widetilde{H}_q(X_\mathcal{I}; \mathbb{F}_2),
$$

where p and q range over nonnegative integers in the direct sum.

For the rest of this paper, we write $\widetilde{H}(\bullet)$ as $\widetilde{H}(\bullet; \mathbb{F}_2)$ for short. Recall from the introduction that the pointed simplicial subset $\widetilde{\Delta}_s$ of $(X/G)^{\wedge s}$ is defined as follows. Set $\widetilde{\Delta}_0 = \widetilde{\Delta}_1 := *$ and

$$
\widetilde{\Delta}_s := \{x_1 G \wedge \cdots \wedge x_s G \in (X/G)^{\wedge s} \mid \exists i = 1, \ldots, s-1 \, (x_i = x_{i+1} \in A)\} \text{ for } s \geq 2.
$$

These simplicial sets $\widetilde{\Delta}_s$ have the following alternative inductive definition.

Proposition 5.1 *The simplicial sets $\widetilde{\Delta}_s$ can be defined inductively by:*

$$
\begin{aligned}
\widetilde{\Delta}_0 &= \widetilde{\Delta}_1 = *, \\
\widetilde{\Delta}_2 &= \overline{\Delta}(A), \\
\widetilde{\Delta}_s &= \left(\widetilde{\Delta}_{s-1} \wedge (X/G)\right) \cup \left((X/G)^{\wedge s-2} \wedge \overline{\Delta}(A)\right) \quad \text{for } s \geq 3.
\end{aligned}
$$

Proof We have $\widetilde{\Delta}_0 = \widetilde{\Delta}_1 = *$ by definition. It is easy to check that $\widetilde{\Delta}_2 = \overline{\Delta}(A)$. We will show that

$$
\widetilde{\Delta}_s = \left(\widetilde{\Delta}_{s-1} \wedge (X/G)\right) \cup \left((X/G)^{\wedge s-1} \wedge \overline{\Delta}(A)\right).
$$

Let an element $x_1 G \wedge \cdots \wedge x_s G$ of $\widetilde{\Delta}_s$ be given. There are two cases: either $x_{s-1} = x_s \in A$ or $x_i = x_{i+1} \in A$ for some $1 \leq i < s - 1$. In the former case $x_1 G \wedge \cdots \wedge x_s G$ belongs to $(X/G)^{\wedge s-1} \wedge \overline{\Delta}(A)$. In the latter case $x_1 G \wedge \cdots \wedge x_s G$

belongs to $\widetilde{\Delta}_{s-1} \wedge (X/G)$. Hence in either case $x_1 G \wedge \cdots \wedge x_s G$ belongs to the union $\widetilde{\Delta}_{s-1} \wedge (X/G) \cup (X/G)^{\wedge s-1} \wedge \overline{\Delta}(A)$. This proves one inclusion.

The proof of the reverse inclusion is similar.   □

Corollary 5.2 *For $s \geq 2$, the simplicial set $\widetilde{\Delta}_s$ decomposes into the following union:*

$$\left(\overline{\Delta}(A) \wedge (X/G)^{\wedge s-2}\right) \cup \left((X/G) \wedge \overline{\Delta}(A) \wedge (X/G)^{\wedge s-3}\right) \cup \cdots \cup \left((X/G)^{\wedge s-2} \wedge \overline{\Delta}(A)\right).$$

Before we prove this corollary, we introduce multi-index notation to abbreviate the expressions. Recall from the introduction that a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a (possibly empty) sequence of positive integers. The *length* of this multi-index is $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and its *dimension* is $\dim \alpha = d$.

Definition 5.3 For $k \geq 2$, let $\overline{\Delta}^k(A)$ denote the pointed simplicial subset of $(X/G)^{\wedge k}$ whose elements are $aG \wedge \cdots \wedge aG$ for some $a \in A$. We set $\overline{\Delta}^1(A) := X/G$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, denote by $\overline{\Delta}^\alpha$ the pointed simplicial set $\overline{\Delta}^{\alpha_1}(A) \wedge \cdots \wedge \overline{\Delta}^{\alpha_d}(A)$.

The pointed simplicial set $\overline{\Delta}^\alpha$ is a subset of

$$(X/G)^{\wedge \alpha_1} \wedge \cdots \wedge (X/G)^{\wedge \alpha_d} = (X/G)^{\wedge \alpha_1 + \cdots + \alpha_d} = (X/G)^{\wedge |\alpha|}.$$

Proof of Corollary 5.2 We proceed by induction. For $s = 2$, the right hand side reduces to $\overline{\Delta}(A)$. Indeed Proposition 5.1 says that $\widetilde{\Delta}_2 = \overline{\Delta}(A)$.

Suppose the above decomposition holds for some $s \geq 2$. In the shorthand notation, the induction hypothesis becomes

$$\widetilde{\Delta}_s = \overline{\Delta}^{(2,1,\ldots,1,1)} \cup \overline{\Delta}^{(1,2,1,\ldots,1)} \cup \cdots \cup \overline{\Delta}^{(1,1\ldots,1,2)},$$

where all the multi-indices are of length s. Thus

$$\begin{aligned}
\widetilde{\Delta}_{s+1} &= \left(\widetilde{\Delta}_s \wedge (X/G)\right) \cup \left((X/G)^{\wedge s} \wedge \overline{\Delta}(A)\right) \\
&= \left[\left(\overline{\Delta}^{(2,1,\ldots,1,1)} \cup \overline{\Delta}^{(1,2,1,\ldots,1)} \cup \cdots \cup \overline{\Delta}^{(1,1\ldots,1,2)}\right) \wedge (X/G)\right] \cup \overline{\Delta}^{(1,1,\ldots,1,2)} \\
&= \overline{\Delta}^{(2,1,\ldots,1,1)} \cup \overline{\Delta}^{(1,2,1,\ldots,1)} \cup \cdots \cup \overline{\Delta}^{(1,\ldots,1,2,1)} \cup \overline{\Delta}^{(1,1,\ldots,1,2)}.
\end{aligned}$$

Then we can use $\overline{\Delta}^\beta \wedge (X/G) = \overline{\Delta}^{(\beta,1)}$ in the last line. This proves the induction step.   □

Recall from the introduction that a pointed simplicial map $f\colon Y \to Z$ is *mod 2 homologous to zero* if the induced map on homology $\widetilde{H}_*(Y;\mathbb{F}_2) \to \widetilde{H}_*(Z;\mathbb{F}_2)$ is the zero map. Since we are using homology only with coefficients in $\mathbb{F}_2$ throughout, we skip the reference to "mod 2". Let $f_1,\ldots,f_k$ be pointed simplicial maps. If f_i is homologous to zero for some $i = 1,\ldots,k$, then the smash product $f_1 \wedge \cdots \wedge f_k$ is homologous to zero. This is because the induced map $(f_1 \wedge \cdots \wedge f_k)_*$ on homology is just the tensor product $(f_1)_* \otimes \cdots \otimes (f_k)_*$.

Proposition 5.4 *Let α and β be multi-indices of length s. Suppose that the reduced diagonal map of A is homologous to zero. If $\overline{\Delta}^\alpha$ is a proper subset of $\overline{\Delta}^\beta$, then the inclusion $\overline{\Delta}^\alpha \hookrightarrow \overline{\Delta}^\beta$ is homologous to zero.*

Proof The higher reduced diagonal map $d_k\colon A \to A^{\wedge k}$ is given by $a \mapsto a \wedge \cdots \wedge a$ for $a \in A_n$. We first show that for $k \geq 2$, the higher reduced diagonal map $d_k\colon A \to A^{\wedge k}$ is homologous to zero. This map is a monomorphism with image $\overline{\Delta}^k(A) \cong A$. We can write d_k as a composite:

$$d_k\colon A \to A \wedge A \xrightarrow{1_A \wedge d_{k-1}} A \wedge \overline{\Delta}^{k-1}(A) \hookrightarrow A^{\wedge k}.$$

Since the first map is homologous to zero by assumption, d_k is homologous to zero.

Now we return to the proposition. First consider the case where $\overline{\Delta}^\alpha$ is just $\overline{\Delta}^s(A)$, that is the case where $\dim \alpha = 1$. Since $\overline{\Delta}^\alpha$ is a proper subset of $\overline{\Delta}^\beta$ by assumption, $e := \dim \beta \geq 2$. There is a commutative diagram

$$
\begin{array}{ccc}
\overline{\Delta}^s(A) & \rightarrowtail & \overline{\Delta}^\beta \\
{\scriptstyle d_s}\big\uparrow {\scriptstyle \cong} & & \big\uparrow {\scriptstyle d_{\beta_1} \wedge \cdots \wedge d_{\beta_e}} \\
A & \xrightarrow{\ d_e\ } & \underbrace{A \wedge \cdots \wedge A}_{e}.
\end{array}
$$

Since $e \geq 2$, the reduced diagonal map d_e is homologous to zero from what we have shown above. Since $A \xrightarrow{d_s} \overline{\Delta}^s(A)$ is an isomorphism, the inclusion $\overline{\Delta}^s(A) \hookrightarrow \overline{\Delta}^\beta$ is homologous to zero.

Finally we prove the general case where $\dim \alpha = d > 1$. Since $\overline{\Delta}^\alpha$ is a proper subset of $\overline{\Delta}^\beta$ by assumption, we can decompose the multi-index β into $\beta = (\gamma^{(1)},\ldots,\gamma^{(d)})$ such that $\alpha_1 = |\gamma^{(1)}|,\ldots,\alpha_d = |\gamma^{(d)}|$. Thus the inclusion map $\overline{\Delta}^\alpha \hookrightarrow \overline{\Delta}^\beta$ decomposes into a smash product of the inclusions $\overline{\Delta}^{\alpha_j}(A) \hookrightarrow \overline{\Delta}^{\gamma^{(j)}}$:

$$
\begin{array}{ccc}
\overline{\Delta}^\alpha & \rightarrowtail & \overline{\Delta}^\beta \\
\big\| & & \big\| \\
\overline{\Delta}^{\alpha_1}(A) \wedge \cdots \wedge \overline{\Delta}^{\alpha_d}(A) & \longrightarrow & \overline{\Delta}^{\gamma^{(1)}} \wedge \cdots \wedge \overline{\Delta}^{\gamma^{(d)}}.
\end{array}
$$

Each inclusion $\bar{\Delta}^{\alpha_j}(A) \hookrightarrow \bar{\Delta}^{\gamma^{(j)}}$ reduces to the case above. Hence it is homologous to zero. Thus after taking the smash product, the inclusion $\bar{\Delta}^{\alpha} \hookrightarrow \bar{\Delta}^{\beta}$ is homologous to zero. (Actually one j is enough.) □

Example 5.5 There are spaces whose reduced diagonal map is homologous to zero. For example, if $A = \Sigma Y$, then the reduced diagonal map of A is null-homotopic and thus homologous to zero. The *weak category* of a space A is the least k such that the k–fold reduced diagonal $A \to A^{\wedge k}$ is null-homotopic (see Berstein and Hilton [1, Definition 2.2]). For example, a noncontractible suspension space has weak category 2.

Given a subset $\mathcal{I}$ of $\{(2, 1, \ldots, 1, 1), \ldots, (1, 1 \ldots, 1, 2)\}$, define $\bar{\Delta}_{\mathcal{I}} = \bigcap_{\alpha \in \mathcal{I}} \bar{\Delta}^{\alpha}$.

Proposition 5.6 *The collections*

$$\{\bar{\Delta}_{\mathcal{I}} \mid \mathcal{I} \subset \{(2, 1, \ldots, 1, 1), \ldots, (1, 1 \ldots, 1, 2)\}, \#\mathcal{I} = p\}$$

and $\{\bar{\Delta}^{\alpha} \mid \dim \alpha = s - p\}$ *are equal for* $p = 1, \ldots, s - 1$. *Here all the multi-indices are of length* s.

Proof We proceed by induction on p. The base step $p = 1$ is obvious.

Let $\mathcal{I} = \{\gamma^{(j_1)}, \ldots, \gamma^{(j_p)}\}$ where $j_1 < \cdots < j_p$ and $\gamma^{(j)}$ is the multi-index $(1, \ldots, 2, \ldots, 1)$ with 2 as the jth entry. By the inductive hypothesis, there exists some β of dimension $s - (p - 1)$ such that $\bar{\Delta}_{\mathcal{J}} = \bar{\Delta}^{\beta}$ where $\mathcal{J} = \{\gamma^{(j_1)}, \ldots, \gamma^{(j_{p-1})}\}$. Recall that $\bar{\Delta}_{\mathcal{J}} = \bar{\Delta}^{\gamma^{(j_1)}} \cap \cdots \cap \bar{\Delta}^{\gamma^{(j_{p-1})}}$. Since j_{p-1} is the largest term in this intersection, we can decompose β into $(\beta', 1, \ldots, 1)$ where there are $(s - j_{p-1} - 1)$ 1s. Since β has dimension $s - p + 1$, β' has dimension

$$(s - p + 1) - (s - j_{p-1} - 1) = j_{p-1} - p + 2.$$

There are two cases: either $j_p = j_{p-1} + 1$ or $j_p > j_{p-1} + 1$.

Consider the case where $j_p = j_{p-1} + 1$. Then $\bar{\Delta}_{\mathcal{I}} = \bar{\Delta}^{\delta}$ where, writing $e := \dim \beta'$,

$$\delta = (\beta'_1, \ldots, \beta'_{e-1}, \beta'_e + 1, \underbrace{1, \ldots, 1}_{s - j_{p-1} - 2}).$$

Then

$$\dim \delta = e + (s - j_{p-1} - 2) = (j_{p-1} - p + 2) + (s - j_{p-1} - 2) = s - p,$$

which proves the induction step for this case.

Next consider the case where $j_p > j_{p-1} + 1$. Recall $\beta = (\beta', \overbrace{1, \ldots, 1}^{s - j_{p-1} - 1})$. Then $\bar{\Delta}_{\mathcal{I}} = \bar{\Delta}^{\epsilon}$. Here ϵ is modified from β by contracting an adjacent pair of 1s at the j_p^{th} and $(j_p + 1)^{\text{th}}$ places into a 2. In any case $\dim \epsilon = \dim \beta - 1 = s - p$. This proves the induction step for this cases and completes the whole proof. □

Corollary 5.7 Let $\mathcal{I} \subset \{(2, 1, \ldots, 1, 1), \ldots, (1, 1 \ldots, 1, 2)\}$ *where the multi-indices are of length s. For each $j = 1, \ldots, \#\mathcal{I}$, the inclusion map $\bar{\Delta}_{\mathcal{I}} \hookrightarrow \bar{\Delta}_{\partial_j \mathcal{I}}$ is homologous to zero.*

Recall that if $\mathcal{I} = \{\gamma^{(1)}, \ldots, \gamma^{(k)}\}$, then $\bar{\Delta}_{\partial_j \mathcal{I}} = \bigcap_{i \neq j} \bar{\Delta}^{\gamma^{(i)}}$ is the intersection omitting the jth term.

Proof Proposition 5.6 shows that there exists the multi-indices α and β of length s such that $\bar{\Delta}_{\mathcal{I}} = \bar{\Delta}^{\alpha}$ and $\bar{\Delta}_{\partial_j \mathcal{I}} = \bar{\Delta}^{\beta}$. Since $\bar{\Delta}_{\mathcal{I}}$ is a proper subset of $\bar{\Delta}_{\partial_j \mathcal{I}}$, then Proposition 5.4 shows that the inclusion $\bar{\Delta}_{\mathcal{I}} \hookrightarrow \bar{\Delta}_{\partial_j \mathcal{I}}$ is homologous to zero, as required. $\qquad \square$

Lemma 5.8 *If the reduced diagonal map of A is homologous to zero, then the Mayer–Vietoris spectral sequence of $\widetilde{\Delta}_s = \bar{\Delta}^{(2,1,\ldots,1,1)} \cup \cdots \cup \bar{\Delta}^{(1,1\ldots,1,2)}$ collapses at the E^1 term so that*

$$\widetilde{H}_t(\widetilde{\Delta}_s) \cong \bigoplus_{\substack{\#\mathcal{I}+q-1=t \\ \#\mathcal{I} \geq 1}} \widetilde{H}_q(\bar{\Delta}_{\mathcal{I}}).$$

Here $\mathcal{I}$ ranges over the nonempty subsets of $\{(2, 1, \ldots, 1, 1), \ldots, (1, 1 \ldots, 1, 2)\}$.

Proof The differential of the E^1–term is given as the following composition:

$$d^1_{p,q} \colon E^1_{p,q} \xrightarrow{j} H_q(F_{p-1}) \xrightarrow{i} E^1_{p-1,q}.$$

The homology class of $\alpha_q^{\mathcal{I}}$ in $\widetilde{H}_q(\bar{\Delta}_{\mathcal{I}})$ is mapped to the homology class of

$$\sum_{j=1}^{\#\mathcal{I}} (-1)^j \alpha_q^{\partial_j \mathcal{I}}$$

in $\bigoplus_{j=1}^{\#\mathcal{I}} \widetilde{H}_q(\bar{\Delta}_{\partial_j \mathcal{I}})$. Since the reduced diagonal is homologous to zero, Corollary 5.7 tells us that each map $\widetilde{H}_q(\bar{\Delta}_{\mathcal{I}}) \to \widetilde{H}_q(\bar{\Delta}_{\partial_j \mathcal{I}})$ is zero. Therefore $\alpha_q^{\partial_j \mathcal{I}} = 0$ and $\sum_{j=1}^{\#\mathcal{I}} (-1)^j \alpha_q^{\partial_j \mathcal{I}} = 0$ so that the differential d^1 is the zero map. Therefore the Mayer–Vietoris spectral sequence collapses at the E^1 term.

Using the expression (5-2) for the E^1 term,

$$\widetilde{H}_t(\widetilde{\Delta}_s) \cong \bigoplus_{\substack{p+q-1=t \\ \bar{\Delta}_{\mathcal{I}} \neq \varnothing \\ \#\mathcal{I}=p \geq 1}} \bigoplus \widetilde{H}_q(\bar{\Delta}_{\mathcal{I}}) \cong \bigoplus_{\substack{\#\mathcal{I}+q-1=t \\ \#\mathcal{I} \geq 1}} \widetilde{H}_q(\bar{\Delta}_{\mathcal{I}}),$$

since no $\bar{\Delta}_{\mathcal{I}}$ is empty. $\qquad \square$

Proof of Theorem 1.2 The above lemma implies the following expression for the mod 2 Betti numbers:

$$\widetilde{b}_t(\widetilde{\Delta}_s) = \sum_{\substack{\#\mathcal{I}+q-1=t \\ \#\mathcal{I}\geq 1}} \widetilde{b}_q(\overline{\Delta}_\mathcal{I}).$$

To simplify this expression, recall Proposition 5.6 which states that for $p = 1, \ldots, s-1$, the collections $\{\overline{\Delta}_\mathcal{I} \mid \#\mathcal{I} = p\}$ and $\{\overline{\Delta}^\alpha \mid \dim\alpha = s - p\}$ are identical. Thus the above expression becomes:

$$(5\text{-}3) \qquad \widetilde{b}_t(\widetilde{\Delta}_s) = \sum_{\substack{|\alpha|=s \\ (s-\dim\alpha)+q-1=t \\ \dim\alpha\leq s-1}} \widetilde{b}_q(\overline{\Delta}^\alpha) = \sum_{\substack{|\alpha|=s \\ q-\dim\alpha=t-s+1 \\ \dim\alpha\leq s-1}} \widetilde{b}_q(\overline{\Delta}^\alpha).$$

Notice if $\dim\alpha = d$, then

$$\widetilde{b}_q(\overline{\Delta}^\alpha) = \sum_{|\nu|=q} \widetilde{b}_{\nu_1}(\overline{\Delta}^{\alpha_1}(A)) \cdots \widetilde{b}_{\nu_d}(\overline{\Delta}^{\alpha_d}(A)).$$

Since $\overline{\Delta}^1(A) = X/G$ by convention and $\overline{\Delta}^k(A)$ is isomorphic to A for $k \geq 2$, the homology of $\widetilde{\Delta}_s$ depends only on the homology of A and X/G. There must exists constants $c_{\lambda,\mu}$ depending on the multi-indices λ and μ such that

$$\widetilde{b}_t(\widetilde{\Delta}_s) = \sum_{\lambda,\mu} c_{\lambda,\mu} \widetilde{b}_\lambda(X/G) \widetilde{b}_\mu(A).$$

Let I denote $\dim\lambda$ and J denote $\dim\mu$. Thus $c_{\lambda,\mu}$ is the number of multi-indices α which are permutations of $(1, \ldots, 1, a_1, \ldots, a_J)$ (I 1s) for integers $a_1, \ldots, a_J \geq 2$ that satisfy $I + a_1 + \cdots + a_J = s$. After making the substitution $b_i = a_i - 2$, this condition is equivalent to $b_1 + \cdots + b_J = s - I - 2J$ where each b_i is a nonnegative integer. There are $\binom{(s-I-2J)+(J-1)}{J-1} = \binom{s-I-J-1}{J-1}$ nonnegative integer solutions $(b_1, \ldots, b_J)$ to this equation. Thus we have $c_{\lambda,\mu} = \binom{I+J}{J}\binom{s-I-J-1}{J-1}$.

Since $q = |\nu| = |\lambda| + |\mu|$ and $\dim\nu = \dim\lambda + \dim\mu$, so the condition $q - \dim\alpha = t - s + 1$ in (5-3) is equivalent to $|\lambda| + |\mu| = t - s + \dim\lambda + \dim\mu + 1$. Similarly, since $\dim\alpha = \dim\nu$, the condition $\dim\alpha \leq s - 1$ in (5-3) is equivalent to $s \geq \dim\lambda + \dim\mu + 1$. Thus we obtain

$$(5\text{-}4) \qquad \widetilde{b}_t(\widetilde{\Delta}_s; \mathbb{F}_2) \cong \sum_{\substack{|\lambda|+|\mu|=t-s+\dim\lambda+\dim\mu+1 \\ 2\leq\dim\lambda+\dim\mu+1\leq s}} c_{\lambda,\mu} \widetilde{b}_\lambda(X/G; \mathbb{F}_2) \widetilde{b}_\mu(A; \mathbb{F}_2)$$

as required. $\qquad\square$

Note that (5-4) is in fact a finite sum, since the condition $\dim \lambda + \dim \mu + 1 \le s$ implies that $|\lambda| + |\mu| = t - s + (\dim \lambda + \dim \mu + 1) \le t - s + s = t$. As the length is bounded above, there can only be finitely many λ and μ that satisfy $\dim \lambda + \dim \mu + 1 \le s$.

6 Proof of Proposition 1.3

We illustrate the efficacy of the homology decompositions in Theorem 1.1 and Theorem 1.2 by computing all the mod 2 Betti numbers of $\Omega(X \rtimes_G E_\infty^1 G)$ for an example where $X = S^2 \cup_{S^1} S^2$. The discrete group $G = C_2$ acts on the 2–sphere S^2 antipodally with the equatorial circle S^1 as the fixed set. The G–space X is formed by taking two 2–spheres S^2 with the antipodal action and identifying their equatorial circles.

This pointed G–space is equivariantly homotopy equivalent to the following. Take two pairs of discs (that is, four discs in total), and identify all the boundary circles. Let G act on this union $D^2 \cup D^2 \cup D^2 \cup D^2$ by switching the discs in each pair.

Proof of Proposition 1.3 Put a simplicial G–structure on the G–space. Write the simplicial G–set as $X = S_1^2 \cup_{S^1} S_2^2$. The subscripts serve to distinguish each of the two S^2s. For $i = 1, 2$, let D_i^+ denote the upper hemisphere of S_i^2 and D_i^- the lower hemisphere. The antipodal G–action sends each upper hemisphere to the lower hemisphere, so $D_i^- = D_i^+ t$. Then

$$
\begin{aligned}
X &= (D_1^+ \cup D_2^+) \cup_{S^1} (D_1^- \cup D_2^-) \\
&= (D_1^+ \cup D_2^+) \cup_{S^1} (D_1^+ t \cup D_2^+ t) \\
&= (D_1^+ \cup D_2^+) \cup_{S^1} (D_1^+ \cup D_2^+) t.
\end{aligned}
$$

By Proposition 4.1, the orbit projection of X has a section. Thus Theorem 1.1 applies (here and below we suppress the coefficient $\mathbb{F}_2$ in the notation):

$$
(6\text{-}1) \qquad \tilde{H}_n(\Omega(X \rtimes_G W_\infty^1 G)) = \bigoplus_{s=1}^\infty \tilde{H}_n\big((S^2)^{\wedge s}/\tilde{\Delta}_s\big),
$$

Since $S^1 \to S^1 \wedge S^1 \cong S^2$ is mod 2 homologous to zero, Theorem 1.2 also applies:

$$
\tilde{b}_n(\tilde{\Delta}_s) = \sum_{J \ge 1} \sum_{I = n-s+1} \binom{I+J}{J}\binom{t-2I-J}{J-1} \underbrace{\tilde{b}_2(S^2) \cdots \tilde{b}_2(S^2)}_{I} \otimes \underbrace{\tilde{b}_1(S^1) \cdots \tilde{b}_1(S^1)}_{J}.
$$

Since $\tilde{b}_2(S^2) = \tilde{b}_1(S^1) = 1$, the Betti number is

$$
\begin{aligned}
(6\text{-}2) \qquad \tilde{b}_n(\tilde{\Delta}_s) &= \sum_{J \geq 1} \sum_{I = n-s+1} \binom{I+J}{J}\binom{n-2I-J}{J-1} \\
&= \sum_{J \geq 1} \binom{n-s+1+J}{J}\binom{n-2(n-s+1)+J}{J-1} \\
&= \sum_{J \geq 1} \binom{n-s+1+J}{J}\binom{2s-n-J-2}{J-1} \\
&= \sum_{J=1}^{2s-3} \binom{n-s+1+J}{J}\binom{2s-n-J-2}{J-1}.
\end{aligned}
$$

Note that if the binomial coefficient $\binom{2s-n-J-2}{J-1}$ is nonzero, then $2s-n-J-2 \geq J-1$. That is, $n \leq 2s - 2J - 1 \leq 2s - 3$ since $J \geq 1$. Thus $\tilde{H}_n(\tilde{\Delta}_s) = 0$ if $n > 2s - 3$. Combining this observation with the fact that the only nontrivial homology group of $((S^2 \cup_{S^1} S^2 t)/G)^{\wedge s} = (S^2)^{\wedge s} = S^{2s}$ is in the $2s^{\text{th}}$ dimension, the short exact sequence $\tilde{\Delta}_s \to S^{2s} \to S^{2s}/\tilde{\Delta}_s$ induces the following long exact sequence in homology:

$$
\begin{aligned}
\cdots &\longrightarrow 0 \longrightarrow \tilde{H}_{2s+2}(S^{2s}/\tilde{\Delta}_s) \\
&\longrightarrow 0 \longrightarrow 0 \longrightarrow \tilde{H}_{2s+1}(S^{2s}/\tilde{\Delta}_s) \\
&\longrightarrow 0 \longrightarrow \tilde{H}_{2s}(S^{2s}) = \mathbb{F}_2 \longrightarrow \tilde{H}_{2s}(S^{2s}/\tilde{\Delta}_s) \\
&\longrightarrow 0 \longrightarrow 0 \longrightarrow \tilde{H}_{2s-1}(S^{2s}/\tilde{\Delta}_s) \\
&\longrightarrow 0 \longrightarrow 0 \longrightarrow \tilde{H}_{2s-2}(S^{2s}/\tilde{\Delta}_s) \\
&\longrightarrow \tilde{H}_{2s-3}(\tilde{\Delta}_s) \longrightarrow 0 \longrightarrow \tilde{H}_{2s-3}(S^{2s}/\tilde{\Delta}_s) \\
&\qquad\qquad\qquad \vdots \\
&\longrightarrow \tilde{H}_1(\tilde{\Delta}_s) \longrightarrow 0 \longrightarrow \tilde{H}_1(S^{2s}/\tilde{\Delta}_s) \\
&\longrightarrow 0
\end{aligned}
$$

Thus

$$
\tilde{H}_n(S^{2s}/\tilde{\Delta}_s) = \begin{cases} 0 & n \geq 2s+1, \\ \mathbb{F}_2 & n = 2s, \\ 0 & n = 2s-1, \\ \tilde{H}_{n-1}(\tilde{\Delta}_s) & n \leq 2s-2. \end{cases}
$$

For $k \geq 1$, applying this result to (6-1) gives

$$\tilde{H}_{2k}(\Omega(X \rtimes_G W_\infty^1 G)) \cong \tilde{H}_{2k}(S^{2k}/\tilde{\Delta}_k) \oplus \bigoplus_{r=k+1}^{\infty} \tilde{H}_{2k}(S^{2r}/\tilde{\Delta}_r)$$

$$\cong \mathbb{F}_2 \oplus \bigoplus_{r=k+1}^{\infty} \tilde{H}_{2k-1}(\tilde{\Delta}_r).$$

By (6-2), the even Betti numbers are:

$$\tilde{b}_{2k}(\Omega(X \rtimes_G W_\infty^1 G)) = 1 + \sum_{r=k+1}^{\infty} \sum_{J=1}^{2r-3} \binom{2k-r+J}{J}\binom{2r-2k-J-1}{J-1}$$

$$= 1 + \sum_{r=k+1}^{2k} \sum_{J=1}^{2r-3} \binom{2k-r+J}{J}\binom{2r-2k-J-1}{J-1}.$$

Here the upper bound $r \leq 2k$ is obtained by observing that $\binom{2k-r+J}{J}$ is nonzero only if $2k - r + J \geq J$ or $r \leq 2k$. Similarly we can compute the odd Betti numbers:

$$\tilde{b}_{2k+1}(\Omega(X \rtimes_G W_\infty^1 G)) = \sum_{r=k+2}^{2k+1} \sum_{J=1}^{r-k-1} \binom{2k-r+J+1}{J}\binom{2r-2k-J-2}{J-1}$$

for $k \geq 0$.

Take geometric realization to obtain the required result. □

Using these formulas, we compute by hand the Betti numbers in the dimension 1 to 12 to be

$$\{\tilde{b}_n(\Omega(X \rtimes_G E_\infty^1 G); \mathbb{F}_2)\}_{n=1,\dots,12} = \{0, 2, 1, 5, 5, 14, 19, 42, 66, 131, 221, 417\}.$$

A search with the Online encyclopedia of integer sequences [15] gives the sequence A052547. For $n \geq 0$, set a_n to be the coefficient of x^n in the power series expansion of $(1-x)/(x^3 - 2x^2 - x + 1)$. The encyclopedia informs us that, for $1 \leq n \leq 12$:

$$a_n = \tilde{b}_n(\Omega(X \rtimes_G E_\infty^1 G); \mathbb{F}_2).$$

Note that for $n = 0$, the initial term $a_0 = 1$ of sequence A052547 differs from $\tilde{b}_0(\Omega(X \rtimes_G E_\infty^1 G); \mathbb{F}_2) = 0$; this is because we are using the reduced homology. This leads us to conjecture the following.

Conjecture 6.1 *The reduced mod 2 Poincaré series of $\Omega(X \rtimes_G E^1_\infty G)$ is*

$$\sum_{n=0}^{\infty} \widetilde{b}_n(\Omega(X \rtimes_G E^1_\infty G); \mathbb{F}_2)\, x^n = \frac{1-x}{x^3 - 2x^2 - x + 1} - 1.$$

The sequence a_n has a geometric interpretation in terms of diagonals lengths in the regular heptagon with unit side length (see Steinbach [16] and Lang [11]). These diagonal lengths are related to the Chebyshev polynomials, which are important in approximation theory.

Acknowledgments

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