

## Abels's groups revisited

STEFAN WITZEL

We generalize a class of groups introduced by Herbert Abels to produce examples of virtually torsion free groups that have Bredon-finiteness length  $m - 1$  and classical finiteness length  $n - 1$  for all  $0 < m \leq n$ .

The proof illustrates how Bredon-finiteness properties can be verified using geometric methods and a version of Brown's criterion due to Martin Fluch and the author.

20J05, 22E40; 51E24, 57M07

Let  $\mathbb{G}_n$  be the algebraic group of invertible upper triangular  $(n + 1) \times (n + 1)$  matrices whose top left and lower right entry are 1. The groups  $\mathbb{G}_n(\mathbb{Z}[1/p])$ , where  $p$  is a prime, were introduced by Abels because they have interesting finiteness properties. Namely, it was shown by Abels, Brown and Strebel [1; 23; 2; 8] that  $\mathbb{G}_n(\mathbb{Z}[1/p])$  is of type  $F_{n-1}$  but not of type  $F_n$ .

Recall that a group  $\Gamma$  is of type  $F_n$  if it admits a classifying space  $X$  whose  $n$ -skeleton  $X^{(n)}$  is compact modulo the action of  $\Gamma$ . A *classifying space* is a contractible CW-complex on which  $\Gamma$  acts freely. Closely related to these topological finiteness properties are the homological finiteness properties of being of type  $FP_n$ : by definition  $\Gamma$  is of type  $FP_n$  if the trivial  $\mathbb{Z}\Gamma$ -module  $\mathbb{Z}$  admits a projective resolution  $(P_i)_{i \in \mathbb{N}}$  with  $P_i$  finitely generated for  $i \leq n$ . It is not hard to see that  $F_n \implies FP_n$ . The converse can only fail to hold at the second level, that is,  $F_2 + FP_n \implies F_n$ . Examples of groups that are of type  $FP_2$  but not of type  $F_2$  were given by Bestvina and Brady [4].

For a group  $\Gamma$  that has torsion it is sometimes more natural to consider a *classifying space for proper actions* for  $\Gamma$ . This is a CW-complex on which  $\Gamma$  acts rigidly in such a way that the fixed point set of every finite subgroup is (nonempty and) contractible and the fixed point set of every infinite subgroup is empty. We say that  $\Gamma$  is of type  $\underline{F}_n$  if it admits a classifying space for proper actions whose  $n$ -skeleton is compact modulo the action of  $\Gamma$ . There is a homology theory developed by Bredon [5] and generalized by Lück [17] that describes the homological aspects of proper actions just as usual homology does for free actions. In particular, we get a notion of Bredon-finiteness properties  $\underline{FP}_n$  and again  $\underline{F}_n \implies \underline{FP}_n$  and  $\underline{F}_2 + \underline{FP}_2 \implies \underline{F}_n$ . For the definition we refer the reader to Fluch and Witzel [11].

The lower finiteness properties have more concrete interpretations: a group is of type  $F_1$  if and only if it is finitely generated, it is of type  $F_2$  if and only if it is finitely presented, and it is of type  $\underline{FP}_0$  if and only if it has finitely many conjugacy classes of finite subgroups (Kropholler, Martínez-Pérez and Nucinkis [15, Lemma 3.1]).

We define the *classical finiteness length* of  $\Gamma$  to be the supremum over all those  $n$  for which  $\Gamma$  is of type  $F_n$ . The *Bredon-finiteness length* is defined analogously. We can now state a version of our Main Theorem.

**Main Theorem** *For  $0 < m \leq n$  there is a solvable algebraic group  $\mathbb{G}$  such that for every odd prime  $p$  the group  $\mathbb{G}(\mathbb{Z}[1/p])$  has classical finiteness length  $n - 1$  and has Bredon-finiteness length  $m - 1$ .*

Related results concerning solvable groups were obtained via more algebraic means by Kochloukova, Martínez-Pérez and Nucinkis [14]. Other results exploring the gap between classical and Bredon-finiteness properties were known before. Platonov [19] constructed a finitely generated solvable linear group with infinitely many conjugacy classes of finite subgroups. Bridson [6], inspired by an article of Grunewald and Platonov [13], constructed linear groups of type  $F_n$  but not of type  $\underline{FP}_0$ . The first separating examples involving higher Bredon-finiteness properties are due to Leary and Nucinkis [16, Examples 3,4].

The precise version of the Main Theorem involves some combinatorial conditions which are introduced in Section 1. In Section 2 we will establish some basic facts about Bruhat–Tits buildings and  $\text{CAT}(0)$ –spaces that will be needed in the proof. The classical finiteness length of the groups is proved in Section 3. The homological Bredon-finiteness length is verified in Section 4. The property  $\underline{F}_2$  is verified independently in Section 5. The appendix describes the natural simplicial model for the extended Bruhat–Tits building of  $\text{GL}_n(K)$ . This should be well known but the author could not find a good reference.

## Acknowledgments

I would like to thank Herbert Abels, Kai-Uwe Bux, Martin Fluch, Giovanni Gandini, Linus Kramer, Ian Leary, Marco Schwandt, Daniel Skodlerack and Matthew Zaremsky for fruitful conversations and helpful comments on the manuscript for this article. The referee made some very useful remarks for which I would like to thank her or him, in particular for the question that led to Section 5. I also gratefully acknowledge support through the SFBs 878 in Münster and 701 in Bielefeld.

### 1 Precise statement of Main Theorem

Fix  $n \in \mathbb{N}$  and consider two nonzero vectors with integer entries

$$v^1 = (v_1^1, \dots, v_{n+1}^1) \quad \text{and} \quad v^2 = (v_1^2, \dots, v_{n+1}^2)$$

which satisfy the following conditions:

- (i) The sequences  $(v_i^1)_i$  and  $(v_i^2)_i$  are monotonically decreasing.
- (ii)  $\sum_i v_i^1 > 0$  and  $\sum_i v_i^2 \leq 0$ .

Denote by  $\mathbb{G}_{v^1, v^2}$  the algebraic group that on a ring  $A$  evaluates to

$$\mathbb{G}_{v^1, v^2}(A) = \left\{ \begin{pmatrix} d_1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & d_{n+1} \end{pmatrix} \in \text{GL}_{n+1}(A) \mid \prod_i d_i^{v_i^1} = 1 = \prod_i d_i^{v_i^2} \right\}.$$

We define a new vector by

$$v := v^2 - \frac{\sum_i v_i^2}{\sum_i v_i^1} v^1.$$

Note that  $v$  satisfies  $\sum_i v_i = 0$ .

By a *partition* of  $I := \{1, \dots, n + 1\}$  we mean a set  $\mathcal{I} \subseteq \mathfrak{P}_{\neq \emptyset}(I)$  of nonempty subsets, its *blocks*, that disjointly cover  $I$ . A partition  $\{J^+, J^-\}$  is called *elementary admissible* (relative to  $v^1$  and  $v^2$ ) if  $\sum_{i \in J^+} v_i^1$  and  $\sum_{i \in J^-} v_i^2$  are even. The trivial partition  $\{I\}$  is also considered elementary admissible. A partition is called *admissible* if it is the (coarsest) common refinement of elementary admissible partitions. We say that  $\mathcal{I}$  is a *partition of  $v$*  if  $\sum_{i \in J} v_i = 0$  for every block  $J$  of  $\mathcal{I}$ . The *essential blocks* of a partition of  $v$  are the blocks  $J$  on which  $v$  is not constant zero, that is, for which there is an  $i \in J$  such that  $v_i \neq 0$ . The *essential dimension* of a partition of  $v$  is

$$\text{ed}(\mathcal{I}) = \sum_J (|J| - 1),$$

where the sum runs over the essential blocks of  $\mathcal{I}$ . We can now define

$$m = m(v^1, v^2) := \min\{\text{ed}(\mathcal{I}) \mid \mathcal{I} \text{ is an admissible partition of } v\}$$

and state:

**Main Theorem** (precise version) *Let  $n, v^1, v^2$  and  $m$  be as above. For every odd prime  $p$  the group  $\mathbb{G}_{v^1, v^2}(\mathbb{Z}[1/p])$  is of type  $F_{n-1}$  but not of type  $FP_n$  and is of type  $\underline{F}_{m-1}$  but not of type  $\underline{FP}_m$ .*

- Remarks**
- (i) Since the trivial partition is admissible, we have  $m \leq n$ . Since every essential block of a partition of  $v$  must have size at least 2, we have  $m \geq 1$ .
  - (ii) Admissibility of a partition is not a strong restriction. In fact, if all entries of  $v^1$  and  $v^2$  are even, then every partition is admissible.
  - (iii) The criterion from [11] can only be used to show that the group is of type  $\underline{FP}_{m-1}$  instead of  $\underline{F}_{m-1}$  because it does not cover type  $\underline{F}_2$ . The reason is that [7] does not directly translate to the context of proper actions. That the group is also of type  $\underline{F}_2$  if  $m > 2$  is shown in Section 5 using a criterion by Lück and Meintrup [18].
  - (iv) The restriction to odd primes is due to the fact that involutions on the building associated to  $\mathrm{GL}_{n+1}(\mathbb{Q}_2)$  have larger fixed point set than they should, cf Proposition 4.4. In the case  $p = 2$  Lemmas 4.1 and 4.2 imply that the group is of type  $\underline{FP}_0$ . It is not clear to the author what the higher Bredon-finiteness properties are in that case.

We give some examples that in particular allow us to recover the previous formulation of the Main Theorem. Denote the standard basis of  $\mathbb{Z}^{n+1}$  by  $a_1, \dots, a_{n+1}$ .

**Example 1.1** If  $v^1 = a_1$  and  $v^2 = -a_{n+1}$ , then  $\Gamma = \mathbb{G}_{v^1, v^2}(\mathbb{Z}[1/p])$  is just Abels's group  $\mathbb{G}_n(\mathbb{Z}[1/p])$ . In this case  $v = a_1 - a_{n+1}$  and the elementary admissible partition into  $J^+ = \{1, n+1\}$  and  $J^- = \{2, \dots, n\}$  shows that  $m = 1$ . Therefore, the Main Theorem states that  $\Gamma$  is of type  $F_{n-1}$  but not of type  $FP_n$  and is of type  $\underline{FP}_0$  but not of type  $\underline{FP}_1$ . The classical finiteness length was known by Abels and Brown [2, Theorem A], and Brown [8, Theorem 6.1]. To prove the first part of the theorem we use metric versions of the methods used there. Part of the translation is done in the appendix.

**Example 1.2** For  $0 < m \leq n$ , we may take  $v^1 = 2 \sum_i a_i$  and  $v^2 = -ma_{n+1} + \sum_{i=1}^m a_i$ . Then  $v = v^2$  and every partition of  $v$  must contain  $\{1, \dots, m, n+1\}$  in one block and therefore have essential dimension at least  $m$ . The partition into  $J^+ = \{1, \dots, m, n+1\}$  and  $J^- = \{m+1, \dots, n\}$  is elementary admissible and has essential dimension  $m$ . Thus we get groups of Bredon-finiteness length  $m-1$  and classical finiteness length  $n-1$  and recover the original formulation of the Main Theorem.

**Example 1.3** As an example of how admissibility comes into play let  $n = 2k$  be even and consider the vectors  $v^1 = a_1 + \dots + a_{k+1}$  and  $v^2 = -a_{k+1} - \dots - a_{n+1}$ . Then  $v = a_1 + \dots + a_k - a_{k+2} - \dots - a_{n+1}$ . A partition of  $v$  with the minimal essential dimension of  $k$  is one into  $\{1, n+1\}, \dots, \{k, k+2\}$ . However, this partition

is not admissible. If  $k$  is even, a partition of  $v$  of the minimal admissible essential dimension  $3/2 \cdot k$  is one into  $\{1, 2, n, n + 1\}, \dots, \{k - 1, k, k + 1, k + 2\}$ . If  $k$  is odd, the minimal admissible essential dimension is  $3/2 \cdot (k - 1) + 2$  and realized by the partition  $\{1, 2, n, n + 1\}, \dots, \{k - 2, k - 1, k + 3, k + 4\}, \{k, k + 1, k + 2\}$ . So if we set  $\Gamma = \mathbb{G}_{v^1, v^2}(\mathbb{Z}[1/p])$  and  $\Gamma' = \mathbb{G}_{2v^1, 2v^2}(\mathbb{Z}[1/p])$ , we get:  $\Gamma$  is a subgroup of finite index in  $\Gamma'$ . The Bredon-finiteness length of  $\Gamma'$  is  $k$  while the Bredon-finiteness length of  $\Gamma$  is  $3/2 \cdot k$ , respectively  $3/2 \cdot (k - 1) + 2$ . Of course, all groups considered here are virtually torsion free and hence virtually of type  $FP_{n-1}$  (because classical and Bredon-finiteness conditions coincide for torsion free groups).

The plan to prove the Main Theorem is as follows. The group  $\Gamma := \mathbb{G}(\mathbb{Z}[1/p])$  acts on the extended Bruhat–Tits building  $X^1$  associated to  $GL_{n+1}(\mathbb{Q}_p)$ . Cell stabilizers are arithmetic and thus of type  $F_\infty$ . The vectors  $v^1$  and  $v^2$  define horospheres  $H^1$  and  $H^2$  that are invariant under the action of  $\Gamma$ . Moreover, the action of  $\Gamma$  on  $H^1 \cap H^2$  is cocompact. The horosphere  $H^1$  can be identified with the (non-extended) Bruhat–Tits building  $X$  in such a way that  $H^1 \cap H^2$  is identified with a horosphere in  $X$ . It is known that horospheres in  $X$  are  $(n - 2)$ -connected. More precisely, let  $\beta$  be the Busemann function whose 0-level is the horosphere. Then the maps  $\beta^{-1}([0, s]) \hookrightarrow \beta^{-1}([0, s + 1])$  induce isomorphisms in  $\pi_k$  for  $k < n - 1$  and epimorphisms that are infinitely often non-injective in  $\pi_{n-1}$ .

With these ingredients, the classical finiteness length follows from Brown's criterion, which we state below. But first we have to recall some definitions. Recall that a space  $X$  is  $n$ -connected if  $\pi_k(X) = 1$  for  $k \leq n$  and is  $n$ -acyclic if  $\tilde{H}_k(X) = 0$  for  $k \leq n$ . The action of a group  $\Gamma$  on a CW-complex  $Z$  is called *rigid* if the stabilizer of every cell fixes that cell pointwise. A system of groups  $(A_s \rightarrow A_{s+1})_{s \in \mathbb{N}}$  is called *essentially trivial* if for every  $s$  there is a  $t \geq s$  such that the map  $A_s \rightarrow A_t$  is trivial.

**Theorem 1.4** (Brown's criterion [8, Theorems 2.2, 3.2]) *Let  $\Gamma$  act rigidly on a CW-complex  $Z$ . Assume that  $Z$  is  $(n - 1)$ -connected. Assume also that the stabilizer of each  $k$ -cell is of type  $F_{n-k}$ . Let  $(Z_s)_{s \in \mathbb{N}}$  be a filtration of  $Z$  by  $\Gamma$ -invariant and  $\Gamma$ -cocompact subspaces. Then  $\Gamma$  is of type  $F_n$  if and only if the system*

$$\pi_k(Z_s) \rightarrow \pi_k(Z_{s+1})$$

*is essentially trivial for  $k < n$ . The same statement holds with “ $(n - 1)$ -connected” replaced by “ $(n - 1)$ -acyclic”, “ $\pi_k$ ” replaced by “ $\tilde{H}_k$ ”, and “ $F_n$ ” replaced by “ $FP_n$ ”.*

To determine the Bredon-finiteness length, we have to take torsion into account. The only torsion elements that  $\Gamma$  contains are of order 2. Moreover, every finite subgroup is conjugate to a group of diagonal matrices. The fixed point set of such a group

is a product of extended Bruhat–Tits buildings. More precisely, it is the extended Bruhat–Tits building of the centralizer of the finite group. The products that can arise are described by admissible partitions. The horosphere in the fixed point set is a product of a horosphere in the essential factors, those corresponding to essential blocks, and of the remaining factors. Its connectivity is two less than the essential dimension. From this, the Bredon-finiteness length can be deduced using the following version of Brown’s criterion from [11]:

**Theorem 1.5** *Let  $\Gamma$  act rigidly on a CW–complex  $Z$ . Assume that for every finite subgroup  $F < \Gamma$  the fixed point set  $Z^F$  is  $(n - 1)$ –acyclic. Assume also that the stabilizer of each  $k$ –cell is of type  $\underline{FP}_{n-k}$ . Let  $(Z_s)_{s \in \mathbb{N}}$  be a filtration of  $Z$  by  $\Gamma$ –invariant and  $\Gamma$ –cocompact subspaces. Then  $\Gamma$  is of type  $\underline{FP}_n$  if and only if for  $k < n$  the following holds: for every  $s$  there is a  $t \geq s$  such that the maps*

$$\tilde{H}_k(Z_s^F) \rightarrow \tilde{H}_k(Z_t^F)$$

*are trivial for all finite subgroups  $F$ .*

## 2 Buildings

From now on we fix  $n$ ,  $v^1$ ,  $v^2$  and  $p$  and write  $\mathbb{G}$  for  $\mathbb{G}_{v^1, v^2}$  and put  $\Gamma := \mathbb{G}(\mathbb{Z}[1/p])$ . To prove the theorem we have to exhibit a space  $Z_0$  on which  $\Gamma$  acts cocompactly with good stabilizers. The finiteness properties of  $\Gamma$  will then correspond to the connectivity of  $Z_0$ .

The starting point for the construction of  $Z_0$  is the (reduced) Bruhat–Tits building  $X$  associated to  $\mathrm{GL}_{n+1}(\mathbb{Q}_p)$  (see Bruhat and Tits [9], and Tits [24]). Its construction is based on the fact that  $\mathbb{Q}_p$  is a local field with valuation  $v$ . Recall that  $X$  is a thick, irreducible, Euclidean building of type  $\tilde{A}_n$  and in particular is a CAT(0)–space (Abramenko and Brown [3, Theorem 11.16]). We denote by  $X^1$  the extended building  $X \times L$ , where  $L$  is a Euclidean line. The action of  $\mathrm{GL}_{n+1}(\mathbb{Q}_p)$  on  $X^1$  is given by

$$(2-1) \quad g \cdot (x, r) = \left( g \cdot x, r - \frac{1}{n+1} v(\det g) \right);$$

see [9, Paragraphe 2]. We write  $\mathrm{pr}_1: X^1 \rightarrow X$  for the projection onto the first factor.

We will consider the following subgroups of  $\mathrm{GL}_{n+1}$ : the group  $\mathbb{B}$  of upper triangular matrices, the group  $\mathbb{T}$  of diagonal matrices, and the group  $\mathbb{U}$  of strict upper triangular matrices. Plain letters will denote the corresponding groups of  $\mathbb{Q}_p$  points, that is  $G = \mathbb{G}(\mathbb{Q}_p)$ ,  $B = \mathbb{B}(\mathbb{Q}_p)$  and so on.

There is a unique apartment  $\Sigma^1$  of  $X^1$  that is invariant under the action of  $T$ . We can identify  $\Sigma^1$  with  $\mathbb{R}^{n+1}$  in such a way that the action of  $T$  is given by

$$(2-2) \quad \text{diag}(d_1, \dots, d_{n+1}).(x_1, \dots, x_{n+1}) = (x_1 + v(d_1), \dots, x_{n+1} + v(d_{n+1})).$$

With this identification the apartment  $\Sigma := \Sigma^1 \cap X$  of  $X$  is the hyperplane  $\ell^\perp$ , where  $\ell = (1, \dots, 1)$ . The boundary  $\partial\Sigma^1$  is an apartment of the spherical building  $\partial X^1$  that is fixed by  $T$ . The group  $B$  fixes a chamber  $C^1$  of  $\partial\Sigma^1$ . Making use of the above identification, we can describe the chamber  $C^1$  as follows. The standard root system  $\alpha_i = a_i - a_{i+1}$ ,  $1 \leq i \leq n$ , of type  $A_n$  defines a cone

$$\bar{C}^1 = \{x \in \Sigma^1 \mid (\alpha_i \mid x) \geq 0\}$$

in  $\Sigma^1$  and  $C^1$  is the boundary of  $\bar{C}^1$ .

As a last ingredient from the theory of buildings consider the morphism  $\eta: B \rightarrow T$  that takes each matrix to its diagonal. Its kernel is  $U$ . There is a corresponding map  $\rho: X^1 \rightarrow \Sigma^1$ , the retraction onto  $\Sigma^1$  centered at  $C^1$ . It can be described by the property that it takes a ray  $\gamma: [0, \infty) \rightarrow X^1$  whose endpoint lies in  $C^1$  to a ray  $\rho \circ \gamma: [0, \infty) \rightarrow \Sigma^1$  that coincides with the original ray on an infinite interval. Both maps are linked by the relation

$$(2-3) \quad \rho(b.x) = \eta(b).\rho(x)$$

for  $b \in B$  and  $x \in X^1$ . The image under  $\eta$  of  $G$  is just  $G \cap T$  (and similarly for  $\Gamma$ ).

As a consequence of (2-3) we observe that  $U$  not only fixes  $C^1$  but for every point  $\xi \in C^1$  leaves invariant every Busemann function centered at  $\xi$ .

### 3 Classical finiteness properties

It is time to shed some light on the seemingly mysterious notions of Section 1. Our group  $\Gamma$  is a subgroup of  $\text{GL}_{n+1}(\mathbb{Q}_p)$  and therefore acts on the extended building  $X^1$ . From (2-2) we see:

**Observation 3.1** Let  $w$  be a vector in  $\Sigma^1$ . An element  $g = \text{diag}(d_i) \in T$  leaves  $w^\perp$  invariant if and only if  $\sum_i w_i v(d_i) = 0$ .

Assume in addition that  $w \in \mathbb{Z}^{n+1}$  and write  $d_i = a_i \cdot p^{v(d_i)}$  with  $a_i \in \mathbb{Z}_p^\times$ . Then  $\prod_i d_i^{w_i} = p^{\sum_i w_i v(d_i)} \cdot \prod_i a_i^{w_i}$  where the second factor lies in  $\mathbb{Z}_p^\times$ . Therefore  $\prod_i d_i^{w_i} = 1$  is sufficient for  $g$  to leave  $w^\perp$  invariant. Regarding  $v^1$  and  $v^2$  as vectors in  $\Sigma^1$  we obtain with (2-3):

**Observation 3.2** The group  $G \cap T$  leaves  $(v^1)^\perp$  and  $(v^2)^\perp$  invariant. Consequently  $G$  leaves  $H^1 := \rho^{-1}((v^1)^\perp)$  and  $H^2 := \rho^{-1}((v^2)^\perp)$  invariant.

This discussion suggests that  $H^1 \cap H^2$  is the right space for  $\Gamma$  to act on. We can now explain the geometric meaning of the conditions on  $v^1$  and  $v^2$  in Section 1. Condition (i) means that  $v^1$  and  $v^2$  point into  $\bar{C}^1$ . This in turn implies that  $H^1$  and  $H^2$  are in fact horospheres. Indeed, let  $\xi^j$  be the endpoint of the geodesic ray spanned by  $v^j$  and let  $\beta^j$  be the Busemann function corresponding to  $[0, \xi^j]$ . Then  $\beta^j((v^j)^\perp) = 0$  and the fact that  $\xi^j \in C^1$  implies that  $\beta^j = \beta^j \circ \rho$ .

Condition (ii) implies that  $v^1$  does not lie in  $\Sigma$  but the geodesic segment  $[v^1, v^2]$  meets  $\Sigma$ . In fact, the intersection point is just  $v$ . As before, let  $\xi$  be the endpoint of the geodesic ray spanned by  $v$  and let  $\beta$  be the corresponding Busemann function.

**Lemma 3.3** *The restriction  $\text{pr}_1|_{H^1}$  is a homeomorphism that takes horoballs centered at  $\xi^2$  to horoballs centered at  $\xi$ .*

We prove a more general statement:

**Proposition 3.4** *Let  $X^1$  be a CAT(0)-space that decomposes as  $X^1 = X \times L$ , where  $L$  is a Euclidean line. Let  $\text{pr}_1: X^1 \rightarrow X$  be the projection onto the first factor. Let  $\xi^1 \in \partial X^1 \setminus \partial X$  and let  $H^1$  be a horosphere centered at  $\xi^1$ . The restriction  $\text{pr}_1|_{H^1}$  is a homeomorphism.*

*Moreover, if  $\xi^2 \in \partial X^1$  is such that  $\angle(\xi^1, \xi^2) \neq \pi$  and that the unique geodesic  $[\xi^1, \xi^2]$  meets  $\partial X$  in a point  $\xi$ , then  $\text{pr}_1|_{H^1}$  takes horoballs around  $\xi^2$  to horoballs around  $\xi$ .*

**Proof** We identify  $L$  with  $\mathbb{R}$  and write elements of  $X^1$  as pairs  $(x, r)$  with  $x \in X$  and  $r \in \mathbb{R}$ . We also let  $\infty$  and  $-\infty$  denote the endpoints of  $L$ . Let  $\beta^1$  be the Busemann function centered at  $\xi^1$  so that  $H^1 = (\beta^1)^{-1}(0)$ . For  $x \in X$  we may consider the Euclidean half-plane spanned by the geodesic ray  $\text{pr}_1([x, \xi^1])$  and the line  $L$ . In that half-plane it is easy to verify that

$$(3-1) \quad \beta^1(x, r) - \beta^1(x, s) = \cos \theta_1 (r - s),$$

where  $\theta_1 = \angle(\xi^1, \infty)$  (see Figure 1). From this it follows that  $\text{pr}_1|_{H^1}$  is a homeomorphism with inverse:

$$x \mapsto \left(x, -\frac{1}{\cos \theta_1} \beta^1(x, 0)\right)$$

For the second statement set  $\theta_2 = \angle(\xi^2, \infty)$  and observe that (3-1) holds analogously. We define  $\beta := \beta^2 - (\cos \theta_2)/(\cos \theta_1) \beta^1$ . Note that this is a positive combination of



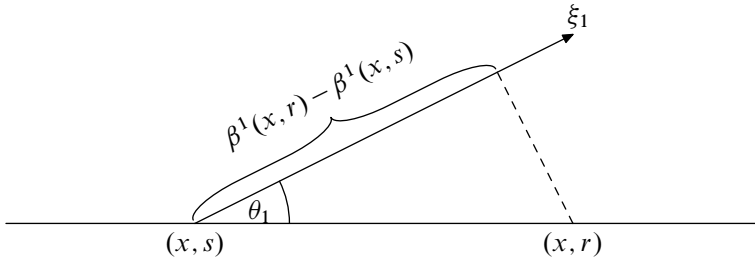


Figure 1:  $\beta^1(x, r) - \beta^1(x, s) = \cos \theta_1(r - s)$

$\beta^1$  and  $\beta^2$  by the assumption that  $[\xi_1, \xi_2] \cap \partial X \neq \emptyset$ . Therefore it is up to scaling a Busemann function centered at a point in  $[\xi^1, \xi^2]$ . Moreover,

$$\begin{aligned} \beta(x, r) - \beta(x, s) &= \beta^2(x, r) - \beta^2(x, s) - \frac{\cos \theta_2}{\cos \theta_1}(\beta^1(x, r) - \beta^1(x, s)) \\ &= \cos \theta_2(r - s) - \frac{\cos \theta_2}{\cos \theta_1} \cos \theta_1(r - s) \\ &= 0 \end{aligned}$$

hence  $\beta$  is centered at  $\xi$  and we may in particular regard it as a reparametrized Busemann function on  $X$ . For  $(x, r) \in X^1$  with  $\beta^1(x, r) = 0$  we clearly have  $\beta^2(x, r) = \beta(x, r)$ , which shows the second claim.  $\square$

Our next goal is to show that the action of  $\Gamma$  on  $H^1 \cap H^2$  is cocompact. The first step is the following consequence of the cocompactness result of [2].

**Proposition 3.5** *The building  $X^1$  is covered by translates of  $\Sigma^1$  under  $\mathbb{B}(\mathbb{Z}[1/p])$ . In short,  $\mathbb{B}(\mathbb{Z}[1/p]).\Sigma^1 = X^1$ .*

**Proof** By [2, Proposition 2.1(b)],  $\mathbb{B}(\mathbb{Z}[1/p])$  acts transitively on the lattices in  $\mathbb{Q}_p^{n+1}$ , which by the appendix correspond to the vertices of  $X^1$ . Let  $c$  be a chamber of  $X^1$  and let  $\gamma$  be a geodesic ray that starts in a vertex, ends in  $C^1$  and meets the interior of  $c$  at a time  $t$ . Let  $\gamma' = \rho \circ \gamma$ , which is also a geodesic ray because  $\gamma$  ends in  $C^1$  and  $\rho$  is centered at  $C^1$ . Let  $d \subseteq \Sigma^1$  be the chamber that contains  $\gamma'(t)$ . Let  $g \in \mathbb{B}(\mathbb{Z}[1/p])$  be such that  $g.\gamma'(0) = \gamma(0)$ . Since  $B$  fixes  $C^1$  it follows that  $g \circ \gamma' = \gamma$  and in particular  $g.d = c$ .  $\square$

Now cocompactness of  $\Gamma$  follows using that  $v^1$  and  $v^2$  lie in  $\mathbb{Z}^n$ .

**Lemma 3.6** (i)  $\Gamma \cap T$  acts cocompactly on  $(v^1)^\perp \cap (v^2)^\perp$ .  
 (ii)  $\Gamma$  acts cocompactly on  $H^1 \cap H^2$ .

**Proof** For the first part note that  $\mathbb{T}(\mathbb{Z}[1/p])$  acts on  $\Sigma^1$  through  $\mathbb{Z}^n$  and the intersection  $\Gamma \cap T$  acts as the stabilizer in  $\mathbb{Z}^n$  of  $(v^1)^\perp \cap (v^2)^\perp$ . Since  $v^1$  and  $v^2$  lie in  $\mathbb{Z}^n$ , the  $\mathbb{Z}$ -module  $(v^1)^\perp \cap (v^2)^\perp \cap \mathbb{Z}^n$  has rank  $n - 2$ , so the stabilizer acts cocompactly.

Now let  $K \subseteq \Sigma^1$  be compact such that its translates under  $\Gamma$  cover  $(v^1)^\perp \cap (v^2)^\perp$  and let  $x \in H^1 \cap H^2$  be arbitrary. By Proposition 3.5 there is a  $b \in \mathbb{B}(\mathbb{Z}[1/p])$  such that  $b.x \in \Sigma^1$ . Clearly there is an  $s \in \mathbb{T}(\mathbb{Z}[1/p])$  such that  $sb \in \mathbb{U}(\mathbb{Z}[1/p])$ . But then we necessarily have  $sb.x \in H^1 \cap H^2 \cap \Sigma^1 = (v^1)^\perp \cap (v^2)^\perp$ . Therefore, by the first part, there is a  $t \in \Gamma \cap T$  such that  $tsb.x \in K$ . Since  $tsb \in \Gamma$  this closes the proof.  $\square$

Since  $X^1$  is locally compact we get immediately:

**Corollary 3.7** *For every  $s > 0$  the action of  $\Gamma$  on  $(\beta^2)^{-1}([0, s]) \cap H^1$  is cocompact.*

The connectivity of horospheres in Euclidean buildings has been established by Bux and Wortman [10]:

**Theorem 3.8** *Let  $X$  be a thick Euclidean building and  $\zeta \in \partial X$ . Let  $\beta$  be a Busemann function centered at  $\zeta$ . Let  $X_0$  be the least factor of  $X$  such that  $\zeta \in \partial X_0$  and let  $m$  be its dimension. Then for  $r \leq s$  the set  $\beta^{-1}([r, s])$  is  $(m - 2)$ -connected. Moreover there is a  $t \geq s$  such that the map*

$$\pi_{m-1}(\beta^{-1}([r, s])) \rightarrow \pi_{m-1}(\beta^{-1}([r, t]))$$

*is not injective. In particular  $\beta^{-1}([r, s])$  is not  $(m - 1)$ -connected.*

Since this is slightly stronger than [10, Theorem 7.7], we briefly sketch how their machinery gives our statement.

**Proof sketch** Let  $\beta$  be a Busemann function centered at  $\zeta$ . In general, if  $X = X_0 \times X_1$  with  $\zeta \in \partial X_0$ , then  $\beta$  is constant on  $\{x\} \times X_1$  for every  $x \in X_0$ . That is,  $\beta^{-1}([r, s]) = \beta|_{X_0}^{-1}([r, s]) \times X_1$ . Since  $X_1$  is contractible this shows in particular that  $\beta^{-1}([r, s])$  and  $\beta|_{X_0}^{-1}([r, s])$  are homotopy equivalent. So we may assume that  $X_0 = X$ .

Now the statement that  $\beta^{-1}([r, s])$  is  $(m - 2)$ -connected is almost [10, Theorem 7.7], except that there it is stated for  $\beta^{-1}((-\infty, s])$ . But  $\beta$  is a concave function, so horoballs are convex and  $\beta^{-1}([r, s])$  is a deformation retract of  $\beta^{-1}((-\infty, s])$ .

It remains to verify that  $\pi_{k-1}(\beta^{-1}([r, s])) \rightarrow \pi_{k-1}(\beta^{-1}([r, t]))$  is not injective for sufficiently large  $t$ . In the language of [10] this requires showing that there is a barycenter  $(\overset{\circ}{\tau}_1, \dots, \overset{\circ}{\tau}_n)$  of  $\beta$ -height greater than  $r$  so that  $\text{Lk}^\downarrow(\overset{\circ}{\tau}_1, \dots, \overset{\circ}{\tau}_n)$  is not  $(m - 1)$ -connected. But we can take each  $\tau_i$  to be a special vertex of the corresponding

factor. Then  $\text{Lk}^\downarrow(\tau_i^\circ)$  is an open hemisphere complex in an irreducible, thick spherical building. These are not contractible by [22, Theorem B]; cf the proof of [10, Lemma 6.6]. Therefore  $\text{Lk}^\downarrow(\tau_1^\circ, \dots, \tau_n^\circ)$  is not  $(m-1)$ -connected, which gives the desired statement.  $\square$

With these preparations in place it is a routine matter to prove the first part of the Main Theorem:

**Theorem 3.9** *The group  $\Gamma = \mathbb{G}(\mathbb{Z}[1/p])$  is of type  $F_{n-1}$  but not of type  $FP_n$ .*

**Proof** By Lemma 3.3 the set  $Z = (\beta^2)^{-1}([0, \infty)) \cap H^1$  is homeomorphic to a horoball in  $X$ , which is contractible, being a convex subset of a CAT(0) space.

We want to apply Brown's criterion, Theorem 1.4. The filtration we consider is:

$$Z_i := (\beta^2)^{-1}([0, i]), \quad i \in \mathbb{N}.$$

The action on each of these spaces is cocompact by Corollary 3.7. The stabilizers are of type  $F_\infty$  by [2, Theorem B(b)]. By Lemma 3.3 the terms of the filtration are homeomorphic to the intersection of a horoball and a horoball complement in  $X$ . Since  $X$  is irreducible, Theorem 3.8 implies that they are  $(n-2)$ -connected, so in particular the system  $(\pi_k(Z_i))_i$  is essentially trivial for  $k < n-1$ . The theorem also implies that the system  $(\tilde{H}_{n-1}(Z_i))_i$  is not essentially trivial.  $\square$

## 4 Bredon-finiteness properties

To determine the Bredon-finiteness properties of  $\Gamma$  we have to understand the torsion and its fixed point sets.

**Lemma 4.1** *Every torsion element of  $\Gamma$  has order 2. In fact the same is true of every torsion element of  $\mathbb{B}(\mathbb{Z}[1/p])$ .*

**Proof** Consider the homomorphism  $\eta|_{\mathbb{B}(\mathbb{Z}[1/p])}: \mathbb{B}(\mathbb{Z}[1/p]) \rightarrow \mathbb{T}(\mathbb{Z}[1/p])$ . Its kernel is  $\mathbb{U}(\mathbb{Z}[1/p])$ , which is torsion-free. The image  $\mathbb{T}(\mathbb{Z}[1/p]) \cong (\mathbb{Z}[1/p]^\times)^{n+1}$  is isomorphic to  $(\{\pm 1\} \times \mathbb{Z})^{n+1}$  and therefore contains only torsion of order 2. Thus if  $g \in \Gamma$  has finite order, then  $\eta(g^2) = 1$  and hence  $g^2 = 1$ .  $\square$

**Lemma 4.2** *Let  $R$  be an integral domain in which 2 is a unit. Every element of order 2 in  $\mathbb{B}(R)$  is conjugate via an element of  $\mathbb{U}(R)$  to a diagonal matrix.*

**Proof** If  $g^2 = \text{id}$ , then the diagonal entries  $d_i$  of  $g$  satisfy  $d_i^2 = 1$ . Since  $R$  is an integral domain the only solutions of this equation are  $\pm 1$ . Hence every column of  $g$  gives rise to a column of precisely one of

$$\frac{1}{2}(\text{id} + g) \quad \text{and} \quad \frac{1}{2}(\text{id} - g)$$

whose diagonal entry is 1. Collecting these columns gives a matrix in  $\mathbb{U}(R)$  whose columns are eigenvectors of  $g$ .  $\square$

Putting both lemmas together, we obtain:

**Proposition 4.3** *Every finite subgroup of  $\Gamma$  is conjugate to a subgroup of  $\Gamma \cap T$ .*

**Proof** Lemma 4.1 implies that every finite subgroup of  $\Gamma$  is abelian. So if  $g$  and  $h$  span a finite group, then  $h$  leaves the eigenspaces of  $g$  invariant. Applying Lemma 4.2 inductively gives a basis of  $(\mathbb{Z}[1/p])^{n+1}$  of simultaneous eigenvectors of any finite subgroup.  $\square$

We can now explain the significance of the remaining notions from Section 1. To understand torsion in  $\Gamma$  it suffices by Proposition 4.3 to understand torsion in  $T \cap \Gamma$ . Every diagonal involution can be described by a partition into the indices with entry  $+1$  and  $-1$  respectively. This partition is elementary admissible if and only if the involution is an element of  $\Gamma$ .

The fixed point set of such an element will turn out to be the extended Bruhat–Tits building of its centralizer and in particular decomposes into a direct product of the extended buildings corresponding to the  $+1$  and  $-1$  eigenspace respectively. Clearly, the fixed point set  $Y$  of a finite group is the intersection of the fixed point sets of its generators and therefore decomposes as a product of buildings that correspond to blocks of an admissible partition.

Each of these extended buildings is a direct product of a line and a building. If the least factor of  $Y$  that contains  $\xi$  in its boundary has a line as a direct factor, then  $Y \cap H$  is contractible. If on the other hand  $\xi$  is contained in the boundary of the building factors, which happens if and only if the partition is a partition of  $v$ , then Theorem 3.8 implies that the connectivity of  $Y \cap H$  is determined by the least factor that contains it. A building factor contributes if and only if its block is essential and therefore the Bredon-finiteness length of  $\Gamma$  is controlled by the essential dimension.

Fixed point sets of finite order automorphisms of  $X^1$  are generally well-studied; see for example Prasad and Yu [20]. Inner involutions, that is, automorphisms that come from involutions in  $\text{GL}_{n+1}(K)$ , are particularly easy to understand. Their fixed point sets can be described as follows.

**Proposition 4.4** *Let  $K$  be a local field of residue characteristic  $\neq 2$ . Let  $\sigma \in \text{GL}_{n+1}(K)$  be an involution. Let  $V^+$  and  $V^-$  be the eigenspaces of  $\sigma$  associated to the eigenvalues  $+1$  and  $-1$  respectively. The fixed point set of  $\sigma$  on  $X^1$  is equivariantly isometric to the extended building associated to the group  $\text{GL}(V^+) \times \text{GL}(V^-)$  (which is the centralizer of  $\sigma$  in  $\text{GL}_{n+1}(K)$ ).*

We give two proofs that are essentially the same but refer to different models of  $X^1$ . For the first recall that  $X^1$  is a simplicial complex whose vertices are  $\mathcal{O}$ -lattices in  $K^{n+1}$ ; see the appendix. Here  $\mathcal{O}$  is the valuation ring of  $K$ .

**Proof 1** Let  $\Lambda$  be a lattice in  $V = K^{n+1}$ . Let  $\Lambda^+ = \Lambda \cap V^+$  and  $\Lambda^- = \Lambda \cap V^-$ . We have to show that  $\Lambda$  is fixed by  $\sigma$  if and only if  $\Lambda = \Lambda^+ + \Lambda^-$  because the lattices meeting the second condition are the ones lying in the building associated to  $\text{GL}(V^+) \times \text{GL}(V^-)$ .

Clearly if  $\Lambda = \Lambda^+ + \Lambda^-$ , then  $\sigma\Lambda = \Lambda^+ - \Lambda^- = \Lambda$ , so  $\Lambda$  is  $\sigma$ -invariant. For the converse note that  $2 \in \mathcal{O}^\times$  by assumption. Assume that  $\Lambda$  is  $\sigma$ -invariant and let  $f \in \Lambda$ . Then

$$f = \frac{1}{2}(f + \sigma f) + \frac{1}{2}(f - \sigma f),$$

where  $\frac{1}{2}(f \pm \sigma f) \in \Lambda^\pm$ . This closes the proof. □

For the second proof recall that the points of  $X^1$  correspond to splittable norms on  $K^{n+1}$ ; see [9, Théorème 2.11]. Here norms are understood additively as in [9, 1.1]: A norm on a  $K$ -vector space  $V$  is a map  $K \rightarrow \mathbb{R} \cup \{\infty\}$  such that for  $f, f' \in V$  and  $k \in K$  the following hold:

- (i)  $\alpha(kf) = v(k) + \alpha(f)$ .
- (ii)  $\alpha(f + f') \geq \inf\{\alpha(f), \alpha(f')\}$ .
- (iii)  $\alpha(f) = \infty$  if and only if  $f = 0$ .

A norm  $\alpha$  is said to *split* over a decomposition  $V = V_1 \oplus V_2$  if  $\alpha(f_1 + f_2) = \inf\{\alpha(f_1), \alpha(f_2)\}$  for  $f_i \in V_i$ , [9, 1.4]. This clearly gives a notion of when a norm splits over a decomposition into more than two summands and a norm on  $V$  is said to be *splittable* if there is a decomposition of  $V$  into one-dimensional subspaces over which it splits.

**Proof 2** It suffices to show that a norm  $\alpha$  is  $\sigma$ -invariant if and only if splits over  $V^+ \oplus V^-$ .

Again it is clear that  $\alpha$  is  $\sigma$ -invariant if it splits over  $V^+ \oplus V^-$ . So suppose that  $\alpha$  is  $\sigma$ -invariant and let  $f \in K^{n+1}$ . Write

$$f = \frac{1}{2}(f + \sigma f) + \frac{1}{2}(f - \sigma f),$$

where  $\frac{1}{2}(f \pm \sigma f) \in V^\pm$ . Then  $\alpha(f) \geq \inf\{\frac{1}{2}(f + \sigma f), \frac{1}{2}(f - \sigma f)\}$  but also

$$\alpha(\frac{1}{2}(f \pm \sigma f)) \geq \inf\{\alpha(\frac{1}{2}f), \alpha(\frac{1}{2}\sigma f)\} = \alpha(f)$$

because  $\alpha$  is  $\sigma$  invariant and  $2 \in \mathcal{O}^\times$ . This shows that  $\alpha(f) = \inf\{\frac{1}{2}(f + \sigma f), \frac{1}{2}(f - \sigma f)\}$  as desired. □

For diagonal matrices we get the following more explicit statement. For brevity we write  $\pm$  to mean either  $+$  or  $-$  consistently in each expression.

**Corollary 4.5** *Assume that the residue characteristic of  $K$  is not 2. Let  $\sigma = \text{diag}(d_i)$  be of order 2 in  $T$ . Let  $J^\pm = \{i \in \{1, \dots, n + 1\} \mid d_i = \pm 1\}$ .*

*The fixed point set  $Y$  of  $\sigma$  decomposes as a direct product  $Y = X^+ \times L^+ \times X^- \times L^-$  where  $X^\pm$  are buildings of type  $\tilde{A}_{|J^\pm|-1}$  and  $L^\pm$  are Euclidean lines. More precisely,  $\rho(X^\pm) = \langle a_i - a_j \mid i, j \in I^\pm \rangle$  and  $L^\pm = \langle \sum_{i \in J^\pm} a_i \rangle$ .*

**Proof** Let  $e_1, \dots, e_{n+1}$  be the standard basis for  $K^{n+1}$ . The eigenspaces of  $\sigma$  are  $V^\pm = \langle e_i \mid i \in J^\pm \rangle$ . Let  $X^{\pm 1}$  be the extended building of  $\text{GL}(V^\pm)$ . Since the inclusion  $X^{+1} \times X^{-1} \rightarrow X^1$  is  $\text{GL}(V^+) \times \text{GL}(V^-)$ -equivariant we can determine the factors by looking at the invariant subspaces. Moreover, since everything commutes with  $\rho$ , it suffices to look at the action of  $T$  on  $\Sigma^1$ . We see that  $X^{\pm 1} \cap \Sigma^1$  is just the span of the  $a_i, i \in J^\pm$  and that  $X^{\pm 1}$  decomposes further as  $X^\pm \times L^\pm$ . □

As a consequence we get:

**Proposition 4.6** *Let  $F$  be a finite subgroup of  $\Gamma \cap T$ . Then there is an admissible partition of  $I$  into blocks  $J_1, \dots, J_k$  such that the fixed point set  $Y$  of  $F$  decomposes as a direct product*

$$Y = X_1 \times L_1 \times \dots \times X_k \times L_k$$

*with  $L_\ell$  a Euclidean line and  $X_\ell$  a  $(|J_\ell|-1)$ -dimensional building. The factors satisfy  $\rho(X_\ell) = \langle a_i - a_j \mid i \in J_\ell \rangle$  and  $L_\ell = \langle \sum_{i \in I_\ell} a_i \rangle$ .*

*Conversely, if  $\mathcal{I}$  is the common refinement of elementary admissible partitions  $\mathcal{I}_1, \dots, \mathcal{I}_r$ , and  $\sigma_1, \dots, \sigma_r \in \Gamma$  are the corresponding involutions, then the partition arising above for  $F = \langle \sigma_1, \dots, \sigma_r \rangle$  is  $\mathcal{I}$ .*

We are now ready to prove the second part of the Main Theorem.

**Theorem 4.7**  $\Gamma$  is of type  $\underline{FP}_{m-1}$  but not of type  $\underline{FP}_m$ .

**Proof** We consider the same setup as in the proof of Theorem 3.9 but this time apply Theorem 1.5 instead of Brown's classical criterion.

The stabilizers are of type  $\underline{FP}_\infty$  by [2, Proof of Theorem B(b); 15, Theorem 1.1].

We will verify the following statements, which are stronger than the needed connectivity hypotheses:

- (i)  $Z_i^F$  is  $(m - 2)$ -connected for all  $i \in \mathbb{N}$  and every finite  $F \leq \Gamma$ .
- (ii) There is a finite  $F \leq \Gamma$  such that the maps of the system  $(\tilde{H}_{m-1}(Z_i^F))_{i \in \mathbb{N}}$  are infinitely often not injective.

From these assertions the result follows.

The case  $F = \{1\}$  of (i) has already been verified in the proof of Theorem 3.9. So now we look at a nontrivial finite subgroup  $F$ . By Proposition 4.3  $F$  is conjugate to a group of diagonal matrices and since conjugation does not change the homotopy type of the fixed point set, we may as well assume that  $F$  is diagonal.

The fixed point set  $Y = (X^1)^F$  is described by Proposition 4.6 and decomposes according to an admissible partition  $\mathcal{I} = \{J_1, \dots, J_k\}$  as a product of Euclidean buildings  $X_\ell, 1 \leq \ell \leq k$  and a Euclidean space  $L_1 \times \dots \times L_k$ . Proposition 3.4 implies that the map  $\text{pr}_1$  identifies the intersection  $Y \cap H^1$  with  $Y \cap X$  in such a way that horoballs around  $\xi^2$  are mapped to horoballs around  $\xi$ .

If  $\xi$  is not contained in the boundary of the product of the  $X_\ell$ , then it includes an acute angle with the endpoint of a direct factor of  $Y$  that is a Euclidean line. In that case Proposition 3.4 implies that  $Y_i$  is contractible. That  $\xi$  is contained in the boundary of the product of the  $X_\ell$  is equivalent to the condition that  $v$  is perpendicular to  $L_\ell$  for all  $\ell$ , which is to say that  $\mathcal{I}$  is a partition of  $v$ . If this is the case, then the minimal factor of  $Y$  that contains  $\xi$  in its boundary is the product of those  $X_\ell$  for which  $J_\ell$  is essential. Therefore Theorem 3.8 implies that  $Y \cap Z = Z^F$  is  $(\text{ed}(\mathcal{I}) - 2)$ -connected and in particular  $(m - 2)$ -connected.

Finally we verify (ii). If  $m = n$ , the statement has been verified in the proof of Theorem 3.9 for  $F$  the trivial group. So assume  $m < n$ . Let  $\mathcal{I}$  be an admissible partition of  $v$  with  $\text{ed}(\mathcal{I}) = m$ . Then  $\mathcal{I}$  is the coarsest common refinement of essentially admissible partitions  $\mathcal{I}_1, \dots, \mathcal{I}_r$  that correspond to diagonal involutions  $\sigma_1, \dots, \sigma_r \in \Gamma$ . Let  $F = \langle \sigma_1, \dots, \sigma_r \rangle$  and let  $Y := (X^1)^F$  be its fixed point set. Proposition 4.6

describes the structure of  $Y$ . In particular it implies that  $\xi$  lies in the boundary of a factor of  $Y$  that is a building of dimension  $m$ . Therefore Theorem 3.8 shows that the maps in the directed system  $\tilde{H}_{m-1}(Z_i^F)$  are infinitely often not injective.  $\square$

## 5 The property $\underline{F}_2$

To complete the proof of the Main Theorem it remains to show that  $\Gamma$  is of type  $\underline{F}_2$  provided  $m > 2$ . This is because of the following statement, which is a consequence of the proof of [18, Theorem 0.1 (b)].

**Proposition 5.1** *If  $\Gamma$  is of type  $\underline{F}_2$  then for every  $n$  it is of type  $\underline{F}_n$  if and only if it is of type  $\underline{FP}_n$ .*

As mentioned in the introduction we have to treat this case separately because the criterion in [11] does not cover the property  $\underline{F}_2$ . Instead we use the following criterion due to Lück and Meintrup [18, Theorem 0.1(d)].

**Theorem 5.2** *A group  $\Gamma$  is of type  $\underline{F}_2$  if and only if it is of type  $\underline{FP}_0$  and for every finite subgroup  $F$  of  $\Gamma$  the “Weyl group”  $N_\Gamma(F)/F$  is finitely presented.*

Note that if any of the groups  $N_\Gamma(F)/F$ ,  $N_\Gamma(F)$ , or  $Z_\Gamma(F)$  is finitely presented, then they all are because  $F$  is finite (hence finitely presented) and  $Z_\Gamma(F)$  has finite index in  $N_\Gamma(F)$ . The analogous statement to Theorem 5.2 for higher Bredon-homological finiteness properties is also true; see [15, Lemma 3.1].

**Theorem 5.3** *If  $m > 2$  then  $\Gamma$  is of type  $\underline{F}_2$ .*

**Proof** Let  $F$  be a finite subgroup of  $\Gamma$ . Using Theorem 5.2 it suffices to show that  $N_\Gamma(F)$  is finitely presented. By Proposition 4.3 we may again assume that  $F$  is a subgroup of  $\Gamma \cap T$ . Then the normalizer of  $F$  in  $\text{GL}_{n+1}(\mathbb{Q}_p)$  is just a product of copies of  $\text{GL}_{m_i}(\mathbb{Q}_p)$  for some  $m_i, 1 \leq i \leq k$ , acting on its associated building  $Y = (X^1)^F = X_1^1 \times \cdots \times X_k^1$ .

We can therefore apply Proposition 3.5 to each of the factors to see that the translates of  $\Sigma^1$  under the normalizer of  $F$  in  $\mathbb{B}(\mathbb{Z}[1/p])$  cover  $Y$ . Since all of  $\Gamma \cap T$  normalizes  $F$ , Lemma 3.6(i) implies that  $N_\Gamma(F) \cap T$  acts cocompactly on  $(v^1)^\perp \cap (v^2)^\perp$ . As in the proof of Lemma 3.6(ii) we can conclude that  $N_\Gamma(F)$  acts cocompactly on  $Y \cap H^1 \cap H^2$ .

The stabilizer in  $\Gamma$  of the fundamental lattice  $[\mathbb{Z}_p e_1 + \cdots + \mathbb{Z}_p e_{n+1}]$  is virtually a product of unipotent upper triangular groups with entries in  $\mathbb{Z}$ , which is of type  $F_\infty$



and in particular finitely presented. All other stabilizers are commensurable to this one and thus also finitely presented.

Applying Theorem 1.4 we see that  $N_\Gamma(F)$  is finitely presented if and only if  $Z^F$  is 1-connected. But that  $Z^F$  is 1-connected was shown as part of the proof of Theorem 4.7. □

### Appendix: The extended building of $GL_n(K)$ as a simplicial complex

Let  $K$  be a field equipped with a discrete valuation, let  $\mathcal{O}$  be its valuation ring, and let  $\pi$  be a uniformizing element. Let  $V = K^n$ . By an  $\mathcal{O}$ -lattice in  $V$  (or just a lattice) we mean an  $\mathcal{O}$ -submodule  $\Lambda$  of  $V$  such that the map  $K \otimes_{\mathcal{O}} \Lambda \rightarrow V$  is an isomorphism. We denote by  $\Delta^1$  the simplicial complex whose vertices are the  $\mathcal{O}$ -lattices in  $V$  and whose simplices are flags

$$\Lambda_0 < \dots < \Lambda_k$$

of lattices such that  $\pi\Lambda_k < \Lambda_0$ .

Clearly  $GL(V)$  acts on  $\Delta^1$ . Taking the quotient modulo the action of  $K^\times$  gives a projection

$$\delta: \Delta^1 \rightarrow \Delta$$

where  $\Delta$  has as vertices homothety classes  $[\Lambda]$  of lattices  $\Lambda$ . It is clear from the definition that  $\Delta$  is just the affine building associated to  $SL_n(V)$ ; see for example Ronan [21, Chapter 9.2]. In particular,  $X$  can be regarded as the geometric realization of  $\Delta$ .

We want to see that similarly  $X^1$  can be regarded as the geometric realization of  $\Delta^1$ . We have to be a little careful because even though the projection  $X^1 \rightarrow X$  of metric spaces as well as the projection  $\Delta^1 \rightarrow \Delta$  of simplicial complexes admit splittings (by isometric respectively simplicial embeddings) these splittings do not coincide under the identification we want to make (see Figure 2).

One way to deal with this would be to construct appropriate subdivisions of  $\Delta$  and  $\Delta^1$  under which the metric splitting becomes simplicial. Instead, we exhibit an equivariant homeomorphism  $|\Delta^1| \rightarrow |\Delta| \times \mathbb{R}$  that is not induced by a simplicial map.

To do so we use the following definitions from Grayson [12]. The *length*  $\text{length}(M)$  of a  $\mathcal{O}$ -module  $M$  is the length of a maximal chain of proper submodules of  $M$ . For two arbitrary lattices  $\Lambda$  and  $\Lambda_0$  the *index*  $\text{ind}(\Lambda, \Lambda_0)$  is  $\text{length}(\Lambda/\Lambda_1) - \text{length}(\Lambda_0/\Lambda_1)$

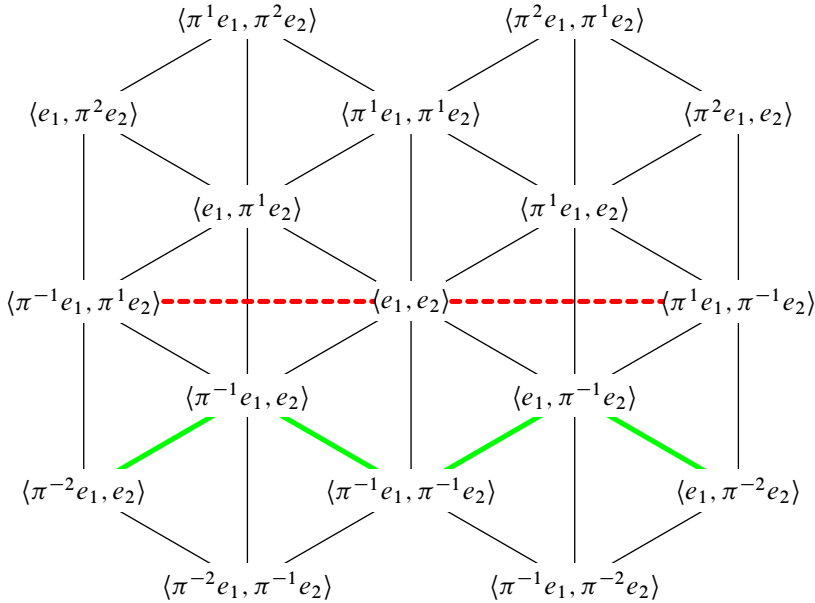


Figure 2: Part of the fundamental apartment of the extended building  $\Delta^1$  for  $SL_2(K)$ . A simplicial subcomplex that is isomorphic to the fundamental apartment of  $\Delta$  is drawn in bold green. A subspace that is isometric to a fundamental apartment of  $X$  is dashed in bold red.

for any lattice  $\Lambda_1$  contained in  $\Lambda$  and  $\Lambda_0$ . Finally we fix a lattice  $\Lambda_0$  and define the map  $\varepsilon: |\Delta^1| \rightarrow \mathbb{R}$  by

$$\varepsilon(\Lambda) = \frac{1}{n} \text{ind}(\Lambda, \Lambda_0)$$

on vertices and extending affinely. This last map is defined differently by Grayson, who wants the map to be simplicial.

**Proposition A.1** *The map  $|\delta| \times \varepsilon: |\Delta^1| \rightarrow |\Delta| \times \mathbb{R}$  is a  $GL(V)$ -equivariant homeomorphism.*

Here, the action of  $GL(V)$  on  $|\Delta^1| \times \mathbb{R}$  is as in (2-1).

**Proof** That the map is continuous and  $GL(V)$ -equivariant is clear. Bijectivity follows from Lemma A.2 below: let  $x \in |\Delta|$  be the convex combination

$$x = \sum_{j=0}^{n-1} \alpha_j [\Lambda_j],$$

where we choose the representatives  $\Lambda_j$  so that  $\varepsilon(\Lambda_j) = j/n$ . Let further  $r \in \mathbb{R}$ . The lemma exhibits an  $i \in \mathbb{Z}$  and a  $\beta \in [0, 1)$  such that

$$\tilde{x} = \sum_{j=1}^{n-1} \alpha_{(i+j) \bmod n} (\pi^{(i+j) \operatorname{div} n} \Lambda_{(i+j) \bmod n}) + \beta \alpha_{i \bmod n} (\pi^{i \operatorname{div} n} \Lambda_{i \bmod n}) + (1 - \beta) \alpha_{(i+n) \bmod n} (\pi^{(i+n) \operatorname{div} n} \Lambda_{(i+n) \bmod n})$$

satisfies  $\varepsilon(\tilde{x}) = r$ ,  $|\delta|(\tilde{x}) = x$ , and  $\tilde{x}$  is unique with these properties (here  $i \operatorname{div} j = \lfloor i/j \rfloor$ ).

To see that the map is closed, it suffices to consider sets of the form  $C = |\sigma| \times [a, b]$  for  $\sigma$  a cell of  $|\Delta|$  and show that the restriction

$$(|\delta| \times \varepsilon)^{-1}(C) \rightarrow C$$

is closed. But sets of the form  $(|\delta| \times \varepsilon)^{-1}(C)$  are clearly compact. □

**Lemma A.2** *Let  $\alpha_j, 0 \leq j \leq n-1$ , be such that  $\alpha_j \geq 0$  and  $\sum \alpha_j = 1$ . For  $i \in \mathbb{Z}$ , set*

$$c_i = \sum_{j=0}^{n-1} \alpha_{(i+j) \bmod n} \frac{i+j}{n}.$$

*The intervals  $[c_i, c_i + \alpha_{i \bmod n})$  with  $i \in \mathbb{Z}$  and  $\alpha_{i \bmod n} > 0$  disjointly cover  $\mathbb{R}$ .*

*In other words, for every  $r \in \mathbb{R}$  there are  $i \in \mathbb{Z}$  and  $\beta \in [0, 1)$  so that*

$$r = \beta \alpha_{i \bmod n} \frac{i}{n} + \sum_1^{n-1} \alpha_{(i+j) \bmod n} \frac{i+j}{n} + (1 - \beta) \alpha_{(i+n) \bmod n} \frac{i+n}{n}$$

*and these are unique if we require  $\alpha_{i \bmod n} \neq 0$ .*

**Proof** This amounts to saying  $c_{i+1} - c_i = \alpha_{i \bmod n}$ , which is elementary. □

## References

- [1] **H Abels**, *An example of a finitely presented solvable group*, from: "Homological group theory", (CTC Wall, editor), London Math. Soc. Lecture Note Ser. 36, Cambridge Univ. Press (1979) 205–211 MR564423
- [2] **H Abels, KS Brown**, *Finiteness properties of solvable S–arithmetic groups: An example*, from: "Proceedings of the Northwestern conference on cohomology of groups", J. Pure Appl. Algebra 44 (1987) 77–83 MR885096
- [3] **P Abramenko, KS Brown**, *Buildings: Theory and applications*, Graduate Texts in Mathematics 248, Springer, New York (2008) MR2439729

- [4] **M Bestvina, N Brady**, *Morse theory and finiteness properties of groups*, *Invent. Math.* 129 (1997) 445–470 MR1465330
- [5] **G E Bredon**, *Equivariant cohomology theories*, *Lecture Notes in Mathematics* 34, Springer, Berlin (1967) MR0214062
- [6] **MR Bridson**, *Finiteness properties for subgroups of  $GL(n, \mathbf{Z})$* , *Math. Ann.* 317 (2000) 629–633 MR1777113
- [7] **K S Brown**, *Presentations for groups acting on simply-connected complexes*, *J. Pure Appl. Algebra* 32 (1984) 1–10 MR739633
- [8] **K S Brown**, *Finiteness properties of groups*, from: “Proceedings of the Northwestern conference on cohomology of groups”, *J. Pure Appl. Algebra* 44 (1987) 45–75 MR885095
- [9] **F Bruhat, J Tits**, *Schémas en groupes et immeubles des groupes classiques sur un corps local*, *Bull. Soc. Math. France* 112 (1984) 259–301 MR788969
- [10] **K-U Bux, K Wortman**, *Connectivity properties of horospheres in Euclidean buildings and applications to finiteness properties of discrete groups*, *Invent. Math.* 185 (2011) 395–419 MR2819164
- [11] **M Fluch, S Witzel**, *Brown’s criterion in Bredon homology*, to appear in *Homology, Homotopy Appl.* arXiv:1206.0962
- [12] **DR Grayson**, *Finite generation of  $K$ -groups of a curve over a finite field (after Daniel Quillen)*, from: “Algebraic  $K$ -theory, Part I”, (R K Dennis, editor), *Lecture Notes in Math.* 966, Springer, Berlin (1982) 69–90 MR689367
- [13] **F Grunewald, V P Platonov**, *On finite extensions of arithmetic groups*, *C. R. Acad. Sci. Paris Sér. I Math.* 325 (1997) 1153–1158 MR1490116
- [14] **DH Kochloukova, C Martínez-Pérez, B E A Nucinkis**, *Centralisers of finite subgroups in soluble groups of type  $FP_n$* , *Forum Math.* 23 (2011) 99–115 MR2769866
- [15] **P H Kropholler, C Martínez-Pérez, B E A Nucinkis**, *Cohomological finiteness conditions for elementary amenable groups*, *J. Reine Angew. Math.* 637 (2009) 49–62 MR2599081
- [16] **I J Leary, B E A Nucinkis**, *Some groups of type  $VF$* , *Invent. Math.* 151 (2003) 135–165 MR1943744
- [17] **W Lück**, *Transformation groups and algebraic  $K$ -theory*, *Lecture Notes in Mathematics* 1408, Springer, Berlin (1989) MR1027600
- [18] **W Lück, D Meintrup**, *On the universal space for group actions with compact isotropy*, from: “Geometry and topology: Aarhus”, (K Grove, I H Madsen, E K Pedersen, editors), *Contemp. Math.* 258, Amer. Math. Soc. (2000) 293–305 MR1778113
- [19] **V P Platonov**, *The theory of algebraic linear groups and periodic groups*, *Izv. Akad. Nauk SSSR Ser. Mat.* 30 (1966) 573–620 MR0199279 In Russian; translated in *Am. Math. Soc. Transl. II. Ser.* 69 (1968) 61–110

- [20] **G Prasad, J-K Yu**, *On finite group actions on reductive groups and buildings*, Invent. Math. 147 (2002) 545–560 MR1893005
- [21] **M Ronan**, *Lectures on buildings*, Perspectives in Mathematics 7, Academic Press, Boston, MA (1989) MR1005533
- [22] **B Schulz**, *Spherical subcomplexes of spherical buildings*, Geom. Topol. 17 (2013) 531–562 MR3039769
- [23] **R Strebel**, *Finitely presented soluble groups*, from: “Group theory”, (K W Gruenberg, J E Roseblade, editors), Academic Press, London (1984) 257–314 MR780572
- [24] **J Tits**, *Reductive groups over local fields*, from: “Automorphic forms, representations and  $L$ -functions, Part 1”, (A Borel, W Casselman, editors), Proc. Sympos. Pure Math. 33, Amer. Math. Soc. (1979) 29–69 MR546588

*Mathematisches Institut, Westfälische Wilhelms-Universität Münster  
Einsteinstraße 62, 48149 Münster, Germany*

s.witzel@uni-muenster.de

<http://www.math.uni-muenster.de/u/stefan.witzel/>

Received: 8 October 2012

