Fractional Dehn twists in
knot theory and contact topology

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Fractional Dehn twists give a measure of the difference between the relative isotopy class of a homeomorphism of a bordered surface and the Thurston representative of its free isotopy class. We show how to estimate and compute these invariants. We discuss the relationship of our work to stabilization problems in classical knot theory, general open book decompositions and contact topology. We include an elementary characterization of overtwistedness for contact structures described by open book decompositions.

1 Introduction

Given an automorphism h of a bordered surface S that is the identity map on ∂S, the fractional Dehn twist coefficient of h with respect to a component C of ∂S is a rational number. It measures the amount of rotation that takes place about C when isotoping h to its Thurston representative. See Section 2 for precise definitions of $c(h)$.

This concept was first studied by Gabai and Oertel [10], who called it the degeneracy slope, and was applied to problems related to Dehn fillings of manifolds containing essential laminations. It is also used, with different coordinates, by Roberts [19; 20] to describe surgeries on fibred knots in which she can construct taut foliations. Most recently it was used by Honda, Kazez and Matić [14; 15] to quantify the concept of right-veering that arises in contact topology.

Section 2 describes techniques for estimating and computing $c(h)$ and gives some examples. We prove a general result, Theorem 2.24, that $c(h) \in [0, \frac{1}{2}]$ if h is a stabilization, and begin to develop properties of the important case when $c(h) = \frac{1}{2}$.

In Section 3 we construct a class of examples, motivated by an example of Gabai [8], for which $c(h) = \frac{1}{2}$. The idea is to replace a fibred knot with its $(2, 1)$–cable. This creates a fibred knot with reducible monodromy and fractional Dehn twist coefficient $\frac{1}{2}$. Following this with surgery on an unknotted curve in the fiber of the $(2, 1)$–cable

can produce examples of fibred knots with pseudo-Anosov monodromy that also have fractional Dehn twist coefficient $\frac{1}{2}$.

Our interest in producing examples of fibred knots in S^3 with fractional Dehn twist coefficient $\frac{1}{2}$ arises from Conjecture 4.7. This conjecture states that a knot in S^3 with fractional Dehn twist coefficient $\frac{1}{2}$ can not be destabilized. Section 4 includes a brief history of this conjecture.

The relevance of Conjecture 4.7 in contact topology is discussed in Section 5. Denote by ξ the contact structure compatible with a pair (S, h). If true, this conjecture would provide the simplest known counterexamples to a conjecture of Honda, Kazez and Matić [14] by producing fibred knots in S^3 that are not stabilizations, are right-veering, yet are still overtwisted. The section includes some context for the conjecture of [14], references to earlier counterexamples for links bounding planar surfaces by Lekili [17], Lisca [18] and Ito and Kawamuro [16], and some remarks on work of Colin and Honda [5]. The technique we use for recognizing that a contact structure is overtwisted is completely elementary: we exhibit an overtwisted disk by finding an unknotted, untwisted curve on the Seifert surface of the knot.

2 Computing fractional Dehn twist coefficients

We recall Thurston's classification of surface automorphisms. Precise statements of this theorem can be found in Thurston [26, page 425] and in Fathi, Laudenbach and Poénaru [6, Section 4, Exposé 11]. A slightly weaker version can be found in Casson and Bleiler [4].

Theorem 2.1 [26; 4; 6] *Let S be an oriented hyperbolic surface with geodesic boundary, and let $h \in \mathrm{Homeo}(S, \partial S)$. Then h is freely isotopic to either*

(1) *a pseudo-Anosov homeomorphism ϕ,*

(2) *a periodic homeomorphism ϕ, in which case there is a hyperbolic metric for which S has geodesic boundary and such that ϕ is an isometry of S, or*

(3) *a reducible homeomorphism h' that fixes, setwise, a maximal collection of disjoint simple closed geodesic curves $\{C_j\}$ in S.*

To avoid overlap in the cases, we refer to a map as reducible only if it is not periodic. Given a reducible map, splitting S along $\bigcup_j C_j$ gives a collection of surfaces $S_1, \ldots, S_n \subset S$ with geodesic boundary that are permuted by h'. Choose an integer n_i so that $(h')^{n_i}$ maps S_i to itself. Maximality of $\{C_j\}$ implies that applying Thurston's classification theorem to $(h')^{n_i} \in \mathrm{Homeo}(S_i, \partial S_i)$ produces either a pseudo-Anosov or periodic representative.

Our primary focus will be in describing homeomorphisms from the point of view of a single boundary component $C \subset \partial S$ that is fixed pointwise by h. If such a map is reducible, let S_0 be the subsurface S_i (as above) of S which contains C. Then $h'(S_0) = S_0$, and we let ϕ_0 be the pseudo-Anosov or periodic representative of $h'|S_0$. With this notation, we make the following definition.

Definition 2.2 A map $\phi \in \mathrm{Homeo}(S, \partial S)$, freely isotopic to h, is called a *Thurston representative* of h if it is pseudo-Anosov, periodic or reducible with $\phi|_{S_0} = \phi_0$, where in the reducible case we require that a preferred component C of ∂S has been specified.

Let S be a compact surface with nonempty boundary. A pair (S, h), where h is a homeomorphism that restricts to the identity map on ∂S, determines a closed 3–manifold $M = S \times [0,1]/\approx$, where the equivalence relation $\approx$ identifies $(x, 1) \approx (h(x), 0)$ for all $x \in S$ and $(x, s) \approx (x, t)$ for all $x \in \partial S$ and $s, t \in [0, 1]$. The singular fibration with pages $S \times \{t\}$ is called the *open book determined by the data* (S, h).

Notice that if S is a disk, then necessarily h is isotopic rel boundary to the identity map. If S is an annulus, then h is isotopic rel boundary to some power of the Dehn twist about the core of S. Otherwise, S is hyperbolic.

We now restrict attention to the case that S is hyperbolic, and hence h has Thurston representative ϕ. Let l be the link in M given by $\partial S \times [0,1]/\approx$ and let $N(l)$ be a regular neighborhood of l. Notice that the link complement $M \setminus \mathrm{int}\, N(l)$ is homeomorphic to the mapping torus $S \times [0,1]/(x, 1) \sim (h(x), 0)$ of h and also to the mapping torus $S \times [0,1]/(x, 1) \sim (\phi(x), 0)$ of ϕ. In the discussions which follow, we will typically identify $M \setminus \mathrm{int}\, N(l)$ with the mapping torus of ϕ. We will also take advantage of the flow Φ, called the *suspension flow of* ϕ, obtained by integrating the vector field $\partial/\partial t$, where points of $M \setminus \mathrm{int}\, N(l)$ are given by $[(x, t)], x \in S, t \in [0, 1]$. Notice that the suspension flow Φ is pseudo-Anosov (respectively, periodic) when ϕ is pseudo-Anosov (respectively, periodic). In particular, when ϕ is periodic, all orbits of Φ are closed. When ϕ is pseudo-Anosov, there are an even number of closed orbits of $\Phi|\partial N(l)$, half attracting and half repelling.

Therefore, fixing a component C of ∂S and restricting the flow Φ to the component of $\partial N(l)$ corresponding to C, Φ necessarily has periodic orbits. Let γ be one such, and write

$$\gamma = p\lambda + q\nu,$$

where $\lambda = C$, ν is the meridian, and p, q are relatively prime integers. The *fractional Dehn twist coefficient* of h with respect to a component C of ∂S [14] is given by

$$c(h) = p/q.$$

In the context of surgery on knots in S^3, the reciprocal quantity, q/p, is called the *degeneracy slope* by Gabai [8]. See also [10] for motivating work describing the location of cusps from the point of view of a toroidal boundary component in a 3–manifold carrying an essential lamination. Fractional Dehn twists, though with different coordinates, also play an important role in Roberts' constructions of taut foliations [19; 20]. See [15] for a description of the change in coordinates.

An alternate description of the fractional Dehn twist coefficient, which is meant to emphasize that h looks like a fraction of a Dehn twist has been applied to the boundary of S is given as follows. First extend h by the identity map to a homeomorphism of $F = S \cup (\partial S \times I)$ where the annuli added to the boundary of S are glued along $\partial S \times \{1\}$. Next, freely isotope h to its Thurston representative, ϕ, on S, and extend this representative across $\partial S \times I$ so that the resulting homeomorphism of F is isotopic to $\mathrm{id} \cup h$ relative to ∂F. The extension can be chosen to be a shearing map, that is, to be a straight-line homotopy when lifted to the universal cover of $\partial S \times I$ thought of as a subset of the plane.

The periodic orbit γ determines a collection of points $\{x_0, \ldots, x_{q-1}\} \subset C$ labelled cyclically and ordered compatibly with the induced orientation on C. The shearing homeomorphism that extends ϕ across $C \times I$ maps $x_0 \times I$ to an arc that may spiral more than once around C in either direction. Lifting to the universal cover, $\widetilde{S}$, and indexing with integers rather than working mod q, it must end at some $x_p \times \{1\} \in \partial \widetilde{S}$. It then follows that $c(h) = p/q$.

We now describe how to compute $c(h)$ with respect to C by computing the action of h on some arcs. Let α be an essential oriented, properly embedded arc in S starting on C. Define $i_h(\alpha)$ to measure the number of intersections of α and $h(\alpha)$ which occur after their common initial point but in an annular neighborhood of C.

Topologically, this means first isotoping the arcs α and $h(\alpha)$, relative to their boundaries, so they intersect minimally. Then $i_h(\alpha)$ is defined to be a signed count of the number of points, x, in the intersection of the interiors of $h(\alpha)$ and α with the property that the union of the initial segments of these arcs, up to x, is contained in an annular neighborhood of C.

It is also useful to define $i_h(\alpha)$ geometrically by putting the arcs in a canonical position. We can think of arcs as living in $F = S \cup (\partial S \times I)$, a hyperbolic surface with geodesic boundary together with annuli extending the boundary components. Denote by $\overline{\alpha}$ the unique geodesic arc in the free isotopy class of α that is orthogonal to the initial and terminal boundary components of S. Now let α_0 and α_1 be monotonically spiraling arcs in $\partial S \times I$ such that $\alpha_0 * \overline{\alpha} * \alpha_1$ is an arc isotopic, relative to its endpoints, to α. Write $h(\alpha)_0 * \overline{h(\alpha)} * h(\alpha)_1$ for the canonical form of $h(\alpha)$.
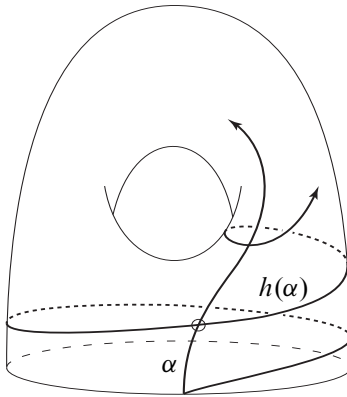
Figure 1: In this example, $i_h(\alpha) = 1$

The geometric description of $i_h(\alpha)$ depends on properties of its Thurston representative, ϕ. In all cases $\overline{\phi(\alpha)} = \overline{h(\alpha)}$.

If ϕ is pseudo-Anosov, it fixes no compact geodesic, and hence $\overline{h(\alpha)}$ and $\overline{\alpha}$ intersect minimally. It follows that $i_h(\alpha)$ is the number of intersections of the interiors of $h(\alpha)_0$ and α_0 counted with sign; namely,

$$i_h(\alpha) = \langle \operatorname{int} h(\alpha)_0, \operatorname{int} \alpha_0 \rangle.$$

The same formula for $i_h(\alpha)$ applies when ϕ is periodic, provided that $\overline{h(\alpha)} \neq \overline{\alpha}$. In the case that $\overline{h(\alpha)} = \overline{\alpha}$, ϕ is an orientation preserving isometry that agrees with the identity map at a point of ∂S, and hence is the identity map. It follows that h is the composition of Dehn twists along the boundary components of S.

In the case that ϕ is reducible let h', ϕ_0 and S_0 be as in Definition 2.2. If $\alpha \subset S_0$, then the formulas given above can be applied to compute $i_{h'}(\alpha) = i_h(\alpha)$. Otherwise, the geometric description breaks into cases. Although an analysis of these cases is straightforward, it seems to yield little insight, and so we do not include this here.

Definition 2.3 [14] Let α and β be properly embedded oriented arcs in S with a common initial point. Suppose, without changing notation, that they have been isotoped, while fixing endpoints, to intersect transversely in the minimum possible number of points. We say α is *to the right of* β if the frame consisting of the tangent vector to α followed by the tangent vector to β agrees with the orientation on S. Similarly, we say α is *to the left of* β if the frame consisting of the tangent vector to β followed by the tangent vector to α gives the orientation on S. This definition implies that every arc is both to the left and right of itself.

See [14] for several equivalent definitions.

Definition 2.4 [14] A homeomorphism h that restricts to the identity map on ∂S is called *right-veering at C* if for every embedded oriented arc α with initial endpoint on C, $h(\alpha)$ is to the right of α. A map is *right-veering* if it is right-veering at every boundary component of S. Similarly, a homeomorphism h that restricts to the identity map on ∂S is called *left-veering at C* if for every embedded oriented arc α with initial endpoint on C, $h(\alpha)$ is to the left of α. A map is *left-veering* if it is left-veering at all components of the boundary of S.

The first step in using i_h to compute $c(h)$ is the following proposition. We now return our focus to a component C of ∂S. In what follows, $c(h)$ denotes the fractional Dehn twist coefficient with respect to C and properly embedded oriented arcs have initial point on C as necessary.

Proposition 2.5 [14] *Let $h \in \mathrm{Homeo}(S, \partial S)$. Then h is right-veering at C if and only if $c(h) > 0$, and similarly h is left-veering at C if and only if $c(h) < 0$.*

Corollary 2.6 *If α is a properly embedded essential oriented arc in S, then $h(\alpha)$ is to the right of α implies $c(h) \geq 0$, and $h(\alpha)$ is to the left of α implies $c(h) \leq 0$.* □

This gives a quick way to show that $c(h) = 0$ by producing two essential arcs, one moved to the right by h and the other moved to the left by h. See Example 2.8. If $c(h) = 0$ it is always possible to find such a pair of arcs. Note that if ϕ is periodic or ϕ is reducible with periodic ϕ_0, then we can choose a single arc α fixed by h (and hence moved both to the left and to the right by h).

For h with $c(h) = p/q$ with $p \neq 0$, the next proposition shows how to produce a homeomorphism g for which computing $c(g) = 0$ is equivalent to showing $c(h) = p/q$.

Proposition 2.7 *If $g = \tau_C^{-p}(h^q)$, then $c(g) = qc(h) - p$.* □

Propositions 2.7, 2.9 and 2.20 make it possible to estimate $c(h)$ experimentally.

Example 2.8 The knot 8_{20} is fibred, with fiber as shown in Figure 2. This can be deduced using the product disk techniques of Gabai [7]; in fact, the fiber shown can be constructed by inductively summing together four Hopf bands. The knot 8_{20} is known to be hyperbolic (See, for example, Adams [1], Adams, Brock and Bugbee et al [2] and [8].) Denote its pseudo-Anosov monodromy by h. The simple closed curve A (respectively B) in Figure 2 is the core of a left (respectively, right) Hopf band summand. The properly embedded arc α (respectively, β) is the dual to A (B respectively) and hence is mapped to the right (respectively, left) by the monodromy h. By Corollary 2.6, $c(h) = 0$.
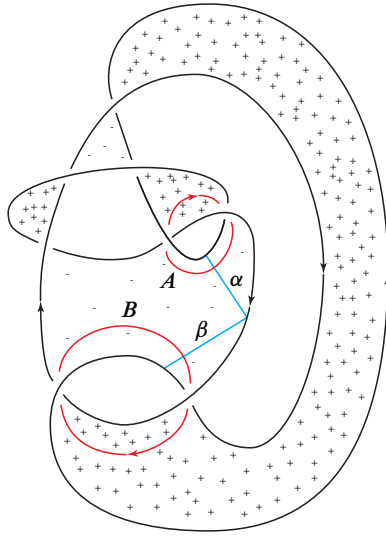
Figure 2: The Seifert surface for 8_{20}

Proposition 2.9 *Suppose h is right-veering at C and α is a properly embedded essential arc with initial point on C. Then either*

- $c(h) \notin \mathbb{Z}$ *and* $i_h(\alpha) = \lfloor c(h) \rfloor$ *or*

- $c(h) \in \mathbb{Z}$ *and* $i_h(\alpha) \in \{c(h) - 1, c(h)\}$.

Proof Let ϕ be the Thurston representative of h and let ϕ_C denote the restriction of ϕ to C. Let $\{x_0, x_1, \ldots, x_{q-1}\}$ be any periodic orbit of ϕ_C. We choose $\{x_0, x_1, \ldots, x_{q-1}\}$ to be ordered cyclically about C, with order compatible with the induced orientation on C.

Now consider the shearing homeomorphism (as described in the alternate description of $c(h)$ found earlier in this section) that extends ϕ across $C \times I$ to yield h, and, in particular, maps the arc $x_0 \times I$ to an arc connecting $x_0 \times \{0\}$ to $x_{p \bmod q} \times \{1\}$. Denote the lift of this homeomorphism to $\tilde{C} \times I$ by f; so, in particular, using the earlier notation, f maps the arc $x_0 \times I$ to the straight line joining $x_0 \times \{0\}$ to $x_p \times \{1\}$. Define $g \colon \tilde{C} \to \tilde{C}$ by $g(z) = z'$, where f maps the arc $z \times I$ to an arc connecting $z \times \{0\}$ to $z' \times \{1\}$. (So g is a lift of ϕ_C.)

Set $n = \lfloor c(h) \rfloor$ and notice that

$$\phi_C([x_0, x_1)) = [x_r, x_{(r+1) \bmod q}) \quad \text{and} \quad g([x_0, x_1)) = [x_p, x_{(p+1)}),$$

where $r = p - nq$, and therefore satisfies $0 \le r \le q - 1$. Also,

$$r = 0 \iff c(h) \in \mathbb{Z}.$$

Now recall that $i_h(\alpha) = \langle \text{int } h(\alpha)_0, \text{int } \alpha_0 \rangle$, where α has canonical form $\alpha_0 * \overline{\alpha} * \alpha_1$ and $h(\alpha)$ has the canonical form $h(\alpha)_0 * \overline{h(\alpha)} * h(\alpha)_1$. Let $z = \overline{\alpha}(0) \in C$. We may assume that $z \in [x_0, x_1)$, since we may relabel the indices of the x_i as necessary while preserving their cyclic order. Therefore, by Lemma 2.10, $\overline{h(\alpha)}(0) \in [x_r, x_{(r+1) \bmod q})$. If $r \ne 0$, then no lift of z lies in $[x_{r+nq}, x_{r+1+nq})$ and it follows that $i_h(\alpha) = \lfloor c(h) \rfloor$. If $r = 0$, then both $\phi_C(z)$ and z have unique lifts in $[x_{nq}, x_{nq+1})$. Label these lifts $\widetilde{\phi_C(z)}$ and $\widetilde{z}$ respectively. If $\widetilde{\phi_C(z)}$ lies to the right of $\widetilde{z}$ in $[x_{nq}, x_{nq+1})$, then $i_h(\alpha) = c(h)$. Otherwise, $i_h(\alpha) = c(h) - 1$. This is illustrated in Figure 3.
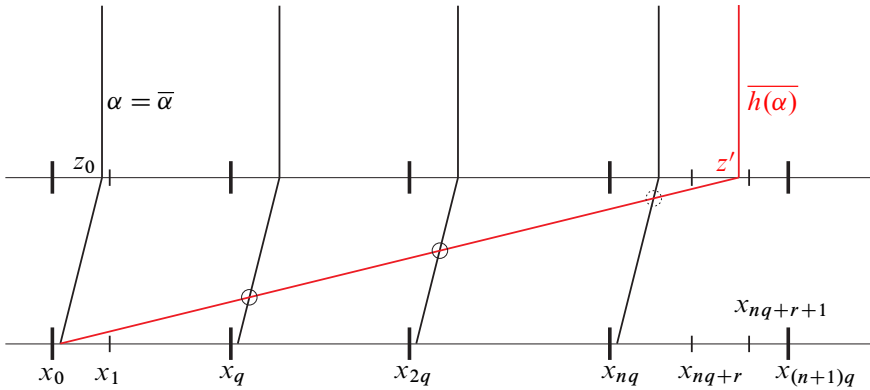

Figure 3: The existence of the third circled intersection depends on r and the position of z'

To conclude, therefore, it suffices to prove the following lemma.

Lemma 2.10 *Suppose β is a properly embedded geodesic arc orthogonal to the boundary (thus $\overline{\beta} = \beta$) with $\beta(0) \in [x_0, x_1)$. Then $\overline{\phi(\beta)}(0) \in [x_r, x_{(r+1) \bmod q})$.*

Proof Notice that if ϕ is periodic or reducible with ϕ_0 periodic, then ϕ, respectively ϕ_0, is an isometry. Hence, in either case, $\overline{\phi(\beta)}(0) = \phi(\beta)(0)$. So we may assume that ϕ is pseudo-Anosov or reducible with ϕ_0 pseudo-Anosov.

Let λ^s and λ^u denote the stable and unstable geodesic laminations of ϕ, if ϕ is pseudo-Anosov, or of ϕ_0, if ϕ is reducible with ϕ_0 pseudo-Anosov. Let $L_1^s, \dots, L_n^s$, $n \ge 1$, denote the leaves of λ^s bounding the complementary region of λ^s containing C, ordered cyclically about C. Similarly, let $L_1^u, \dots, L_n^u$ denote the leaves of λ^u bounding the complementary region of λ^u containing C, ordered cyclically about C.

We shall call the geodesics perpendicular to C and extending between each consecutive pair L_i^u, L_{i+1}^u the *unstable prongs* and the geodesics perpendicular to C and extending between each consecutive pair L_i^s, L_{i+1}^s the *stable prongs*. See Figure 4. Notice that ϕ fixes the unstable and stable prongs since it fixes the stable and unstable laminations. Notice also that the endpoints of the unstable (respectively, stable) prongs coincide exactly with the repelling (respectively, attracting) periodic points of ϕ_C. In particular, the periodic orbit $\{x_0, x_1, \ldots, x_{q-1}\}$ consists exactly of the endpoints of some subset of prongs. Now consider $\phi(\beta)$. Since the geodesic β intersects the stable
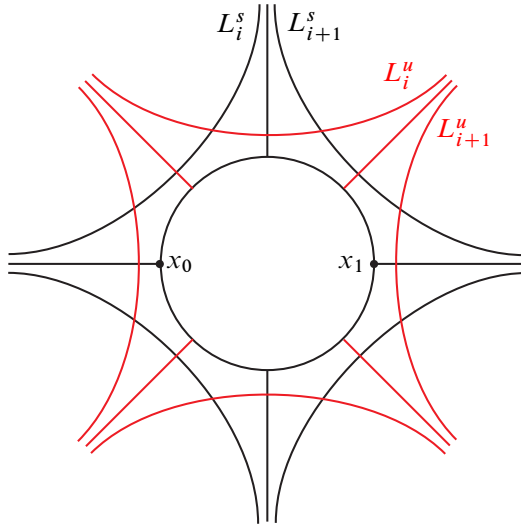

Figure 4: The case $q = 2$ is shown with x_0 and x_1 as endpoints of two of the stable prongs

and unstable laminations and prongs transversally and minimally, so does $\phi(\beta)$. Thus $\phi(\beta)(0) \in [x_r, x_{(r+1) \bmod q})$ implies $\overline{\phi(\beta)}(0) \in [x_r, x_{(r+1) \bmod q})$. $\square$

This completes the proof of Proposition 2.9. $\square$

The proof of Proposition 2.9 also yields the following.

Proposition 2.11 Let $\phi \colon S \to S$ be a Thurston representative, C a component of ∂S, and β a properly embedded geodesic arc orthogonal to ∂S with initial point in C. Let $\mathcal{P}$ be any finite union of ϕ–orbits in C and, and let I be a component of the complement in C of $\mathcal{P}$. Then $\phi(\beta)(0) \in I$ if and only if $\overline{\phi(\beta)}(0) \in I$.

We also have the following related results.

Lemma 2.12 [20, Lemma 5.1] *Let $\phi\colon S \to S$ be a Thurston representative which is pseudo-Anosov or reducible with ϕ_0 pseudo-Anosov, and C a component of ∂S. Let $\mathcal{P}$ be the set of periodic points of $\phi|_C$ in C, and let I be a component of the complement in C of $\mathcal{P}$. Then there is a properly embedded geodesic arc, β, orthogonal to ∂S with initial point in I. If ϕ is reducible, with ϕ_0 pseudo-Anosov, then β can be chosen to lie in S_0.*
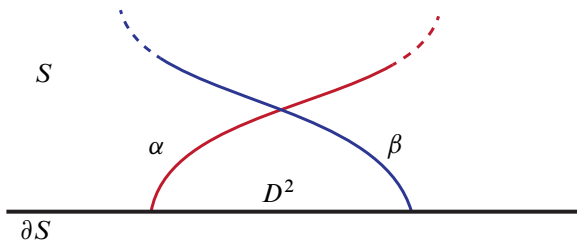

Figure 5: A boundary trivial intersection

Definition 2.13 Let α and β be two properly embedded arcs in a compact surface S. We call a point of intersection $p \in \alpha \cap \beta$ a *boundary trivial intersection* if there is a disk in S bounded by three arcs: subarcs α_1 of α, β_1 of β, and γ_1 of ∂S, with $\alpha_1 \cap \beta_1 = \{p\}$. This is illustrated in Figure 5.

Lemma 2.14 [20, Lemma 2.2] *Let $\phi\colon S \to S$ be a Thurston representative which is pseudo-Anosov. Then if β is a properly embedded geodesic arc orthogonal to ∂S, β and $\phi(\beta)$ have no boundary trivial intersections.*

Now let $\mathcal{P} = \{x_0, y_0, \ldots, x_{k-1}, y_{k-1}\}$ be the set of periodic points of pseudo-Anosov ϕ_C, ordered cyclically about C, with order compatible with the induced orientation on C. Suppose the x_i, $0 \le i \le k-1$, are the attracting periodic points and the y_i, $0 \le i \le k-1$, are the repelling periodic points. Consider the oriented intervals (x_i, y_i) and $(y_i, x_{(i+1) \bmod k})$, $0 \le i < k$, with orientation induced by the orientation of C.

Definition 2.15 Each interval of the form (x_i, y_i) is called ϕ–*slow* and each interval of the form $(y_i, x_{(i+1) \bmod k})$ is called ϕ–*fast*.

Now suppose that m is any positive integer such that ϕ^m fixes $\mathcal{P}$ pointwise. Notice that for points z in the ϕ–fast intervals, $\phi^m(z) > z$ whereas for points z in the ϕ–slow intervals, $\phi^m(z) < z$. See Figure 6. We therefore have the following.
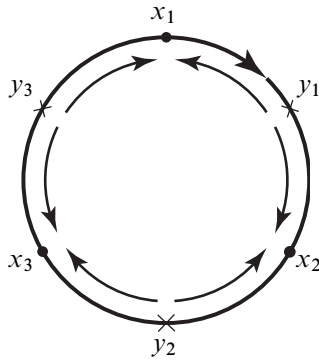
Figure 6: The pseudo-Anosov action of ϕ^m, when $k = 3$

Proposition 2.16 *Let* $\phi\colon S \to S$ *be a Thurston representative which is pseudo-Anosov or reducible with* ϕ_0 *pseudo-Anosov, and* C *a component of* ∂S. *Let* m *be any positive integer such that* ϕ^m *fixes the periodic points of* $\phi|_C$ *pointwise.*

If (x_i, y_i) *is a* ϕ*–slow interval, and* β *is a properly embedded geodesic arc* β *orthogonal to* ∂S *with initial point in* (x_i, y_i), *then* $\overline{\phi^m(\beta)}(0) < \beta(0)$, *where the ordering is given by the ordering on* (x_i, y_i).

And if $(y_i, x_{(i+1) \bmod k})$ *is a* ϕ*–fast interval, and* β *is a properly embedded geodesic arc* β *orthogonal to* ∂S *with initial point in* $(y_i, x_{(i+1) \bmod k})$, *then* $\overline{\phi^m(\beta)}(0) > \beta(0)$, *where the ordering is given by the ordering on* $(y_i, x_{(i+1) \bmod k})$.

Proof By symmetry, it suffices to consider the case that $\beta(0)$ lies in a ϕ–slow interval. Since $\beta(0)$ lies in a ϕ–slow interval, $\phi^m(\beta)(0) < \beta(0)$, where the ordering $<$ is given by the ordering on (x_i, y_i). Since the cyclic ordering in C of endpoints of arcs in S agrees with the cyclic ordering of their geodesic representatives, provided the arcs have no boundary trivial intersections, it therefore follows from [Lemma 2.14](#) that $\overline{\phi^m(\beta)}(0) < \beta(0)$. $\qquad\square$

Corollary 2.17 *Let* $\phi\colon S \to S$ *be a Thurston representative which is pseudo-Anosov or reducible with* ϕ_0 *pseudo-Anosov, and* C *a component of* ∂S. *Let* m *be any positive integer such that* ϕ^m *fixes the periodic points of* $\phi|_C$ *pointwise.*

Then there is a properly embedded geodesic arc β *orthogonal to* ∂S *with initial point in a* ϕ*–slow interval of* C *such that* $\overline{\phi^m(\beta)}(0) < \beta(0)$. *Symmetrically, there is a properly embedded geodesic arc* β *orthogonal to* ∂S *with initial point in a* ϕ*–fast interval of* C *such that* $\overline{\phi^m(\beta)}(0) > \beta(0)$.

Proof This follows immediately from [Lemma 2.12](#) and [Proposition 2.16](#). $\qquad\square$

Corollary 2.18 If h is right-veering at C and α is a properly embedded essential arc with initial point on C, then $i_h(\alpha) \leq c(h) \leq i_h(\alpha) + 1$.

Corollary 2.19 If h is right-veering at C and there exist essential arcs α and β with $i_h(\alpha) < i_h(\beta)$, then $c(h) = i_h(\beta) = i_h(\alpha) + 1$.

For completeness, we record the left-veering version of these results.

Proposition 2.20 Suppose h is left-veering at C and α is a properly embedded essential arc with initial point on C. Then either

- $c(h) \notin \mathbb{Z}$ and $i_h(\alpha) = \lceil c(h) \rceil$ or
- $c(h) \in \mathbb{Z}$ and $i_h(\alpha) \in \{c(h), c(h) + 1\}$.

Corollary 2.21 If h is left-veering at C and α is a properly embedded essential arc with initial point on C, $i_h(\alpha) - 1 \leq c(h) \leq i_h(\alpha)$.   □

Proposition 2.22 If h is neither right nor left-veering at C, that is $c(h) = 0$, then for any α, $i_h(\alpha) = 0$.

Proof If $h(\alpha) = \alpha$, then $i_h(\alpha) = 0$. So we may assume $h(\alpha) \neq \alpha$.

Let $g = T_C h$ where T_C is a right Dehn twist about C. By Proposition 2.7,

$$c(g) = c(h) + 1 = 1,$$

thus g is right-veering and by Proposition 2.9, $i_g(\alpha) \leq c(g) = 1$. Since $h(\alpha) \neq \alpha$, it follows that $i_h(\alpha) = i_g(\alpha) - 1 \leq 0$.

Next let $g' = T_C^{-1} h$. Then $c(g') = c(h) - 1 = -1$, thus g' is left-veering and by Proposition 2.20, $-1 = c(g') \leq i_{g'}(\alpha)$. It follows that $i_h(\alpha) = i'_g(\alpha) + 1 \geq 0$.   □

Definition 2.23 Let $g \in \text{Homeo}(S', \partial S')$ and let a be a properly embedded arc in S'. Define S to be the union of S' and a band attached to $\partial S'$ along neighborhoods of the endpoints of a in S' so that the resulting surface is orientable. This presents S as the union of S' and an annulus that intersect in a regular neighborhood of a. Let T be a positive Dehn twist on the annulus. Let $\hat{g}$ and $\hat{T}$ be the extensions of g and T, respectively, to the portion of S where they are not already defined, by the identity map. The *positive stabilization of g along a* is the homeomorphism $h = \hat{T} \circ \hat{g} \colon S \to S$. Refer to a as the *plumbing arc* of the stabilization.

Theorem 2.24 If h is a positive stabilization, and S has connected boundary, then $0 \leq c(h) \leq \frac{1}{2}$.

Proof A regular neighborhood of a can be thought of as a square with edges e and w contained in the boundary of S' and edges n and s that are parallel to a in S'. Figure 7 shows this plumbing square. Consider the arcs $h(w)$ and $\beta = h^{-1}(w)$. To show that $c(h) \leq \frac{1}{2}$, it is enough, by Proposition 2.7, to show that $c(h^2) \leq 1$. This in turn will follow, by Proposition 2.9, if we show $i_{h^2}(\beta) = 0$.

To see this, we follow the arc β from its initial point, keep track of the edges labelled n, e, w and s that it crosses, and argue that if it eventually intersects $h^2(\beta)$, the intersection point will not contribute to $i_{h^2}(\beta)$. The first edge that β crosses is e. From this point on, it never crosses either w or e. Thus if it intersects $h^2(\beta)$, it must do so in the portion f of β in the square bounded by n, e, w and s. If after hitting e, β intersects f, the initial segments of β and f are not boundary parallel since their union misses n. In this case, this point of $\beta \cap f$ contributes nothing to $i_{h^2}(\beta)$, subsequent intersections can not contribute either, and we conclude $i_{h^2}(\beta) = 0$.

Now consider the case instead that β first hits e, then s, and then s again. This forces an intersection of $\beta \cap f$, and for the same reason, we conclude $i_{h^2}(\beta) = 0$.

This leaves the case that β intersects e, s and n in order. At this point, either the entire arc β misses f, in which case there is nothing to prove, or β intersects f. In the latter case, the union of the initial segments of β and f are a nonseparating curve dual to n, and hence not boundary parallel. $\square$
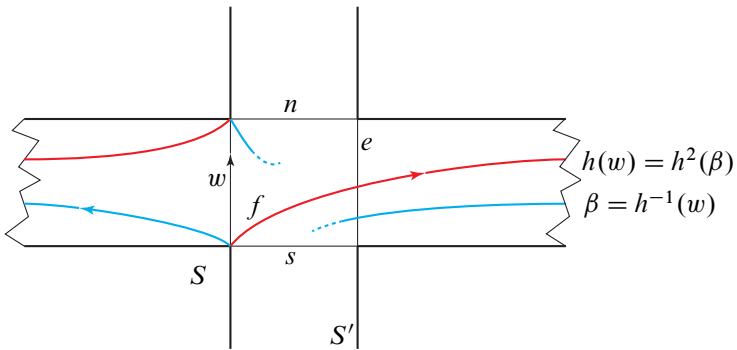

Figure 7: The surface S is obtained from S' by adding a Hopf band, shown horizontally, along the plumbing square bounded by n, e, w and s

Example 2.25 Let A be an annulus, and let T_A be a positive Dehn twist about the core of A. Let $g = T_A^k$ with $k \geq 5$. Next let $\sigma \subset A$ be an arc that cuts A into a disk, and let $h \colon S \to S$ be the positive stabilization of h' along σ. Since S is a punctured torus, a straightforward homology computation shows that h is pseudo-Anosov. Since h is a product of positive Dehn twists, it is right-veering, hence $c(h) > 0$.

By Theorem 2.24, $c(h) \leq \frac{1}{2}$. Since $\chi(S) = -1$ the stable lamination can have only 2 prongs, or equivalently cusps, about ∂S. From this it follows that $c(h)$ is a multiple of $\frac{1}{2}$. It now follows that $c(h) = \frac{1}{2}$.

Proposition 2.26 *Let h be a homeomorphism of a surface S with connected boundary. Suppose that h is a positive stabilization of a homeomorphism g of S' along a plumbing arc a. Suppose $c(h) = \frac{1}{2}$. If S' is an annulus, then $g = T_{S'}^k$, where $T_{S'}^k$ is a right Dehn twist about the core of S' and $k \geq 5$. Otherwise, $i_g(a) \geq 1$ and $i_g(a^{-1}) \geq 1$, and so $c(g) \geq 1$ on each boundary component of S'.*

Proof Suppose that $c(h) = \frac{1}{2}$. Recall from Definition 2.23 that $h = \widehat{T} \circ \widehat{g} \colon S \to S$.

Let n, e, w and s be the properly embedded arcs introduced in the proof of Theorem 2.24 and let $\sigma = h(w)$. Notice that $\sigma = \widehat{T}(w)$ is the arc shown in the first image of Figure 9. To simplify notation, we will isotope n, e, w, s and σ so that $n = \overline{n}$, $w = e = \overline{e} = \overline{w}$, $s = \overline{s}$ and $\sigma = \overline{\sigma}$ in S. Notice that the cyclic ordering in C of endpoints of arcs in S agrees with the cyclic ordering of their geodesic representatives, provided the arcs have no boundary trivial intersections. In particular, therefore, the cyclic ordering of the endpoints of n, s, w and σ is as shown in Figure 9.

Notice that $n = s$ if and only if S' is an annulus and in this case, $c(h) = \frac{1}{2}$ if and only if $g = T_{S'}^k$, where $T_{S'}^k$ is a right Dehn twist about the core of S' and $k \geq 5$. So we may restrict attention to the case that S' is not an annulus and $n \cap s = \varnothing$.

Step 1: Characterize twisting greater than 1 By symmetry, it suffices to show that $i_g(a) \geq 1$. Focus attention on $n \cap h(s)$. If this is nonempty, follow the arc $h(s)$ from its initial point $s(0)$ to its first intersection $h(s)(t_0) = n(t_1)$ with n. Computing $i_g(a)$ requires consideration of geodesic representatives, $\overline{a}$, of arcs, like a, in S'. Notice that even though $n \neq s$, we have $\overline{\overline{n}} = \overline{\overline{s}} = \overline{\overline{a}}$. It follows that $i_g(a) \geq 1$ if and only if $n \cap h(s) \neq \varnothing$ with $h(s)[0, t_0] * n[0, t_1]$ boundary parallel in S'. See Figure 8.

Step 2: Introduce reference points on the boundary C of S We next introduce reference points $x_1, y_1, \ldots, x_{2N}, y_{2N}$ about the boundary $C = \partial S$. These are defined below and shown in Figure 9. As usual, let ϕ denote the Thurston representative of h.

First consider the case that ϕ is periodic, or ϕ is reducible with ϕ_0 periodic. Since $c(h) = \frac{1}{2}$, there are distinct periodic points x_1 and y_1 in C with the following properties. First $\{x_1, x_2 = \phi(x_1)\}, \{y_1, y_2 = \phi(y_1)\}$ are disjoint order two orbits of ϕ. Next $w(0) \in (x_1, y_1)$, and finally $\{x_1, y_1, x_2, y_2\}$ is ordered cyclically and compatibly with the induced orientation of C. This is shown in Figure 9 by setting $N = 1$.
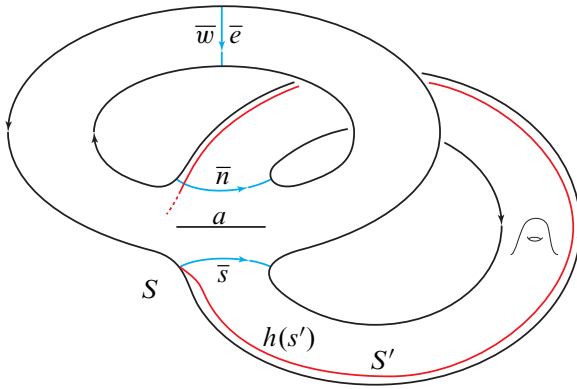
Figure 8

Next consider the case that ϕ is pseudo-Anosov, or ϕ is reducible with ϕ_0 pseudo-Anosov. Since $c(h) = \frac{1}{2}$, there are an even number, $2N$ say, of attracting periodic points and an even number, again $2N$, of repelling critical points in C with respect to ϕ or ϕ_0 respectively. Let $\{x_1, y_1, \ldots, x_{2N}, y_{2N}\}$ be a listing of these, labelled cyclically and ordered compatibly with the induced orientation of C. Cut C open along these periodic points to obtain $4N$ pairwise disjoint intervals

$$(x_1, y_1), (y_1, x_2), \ldots, (x_{2N}, y_{2N}), (y_{2N}, x_1).$$

We may further assume that the periodic points are labelled so that $w(0) \in (x_1, y_1)$. Since $c(h) = \frac{1}{2}$ and $\overline{h(w)} = \sigma$, it follows from Proposition 2.11 that

$$\sigma(0) \in (x_{N+1}, y_{N+1}).$$

Next we show that necessarily $w(1) \in (x_{N+1}, y_{N+1})$ and $\sigma(1) \in (x_1, y_1)$. First notice that $w(0), \sigma(0), w(1), \sigma(1)$ are cyclically ordered about C, as shown in Figure 9. Since $c(h) = \frac{1}{2}$, it therefore follows from Proposition 2.11 that $w(1) \in (x_{N+1}, y_{N+1})$ and $\sigma(1) \in (x_1, y_1)$. This is illustrated in the second image of Figure 9.

Step 3: Choose a reference arc Using the fact that $c(h) = \frac{1}{2}$, a reference arc γ satisfying $i_{h^2}(\gamma) = 1$, $\gamma = \overline{\gamma}$, and $\gamma(0) \in (w(0), \sigma(0))$ can be chosen as follows.

If ϕ is periodic, then let γ be any properly embedded geodesic arc orthogonal to ∂S and with initial point in $(w(0), \sigma(0))$. Since w and σ admit no boundary trivial intersections, such arcs exist. If ϕ is reducible with ϕ_0 periodic, then let γ be any properly embedded geodesic arc in S_0 orthogonal to ∂S and with initial point in $(w(0), \sigma(0))$. Again, since w and σ admit no boundary trivial intersections, such arcs exist. In either case, $\phi^2(\gamma) = \gamma$ and so $h^2(\gamma) = T_C(\gamma)$.
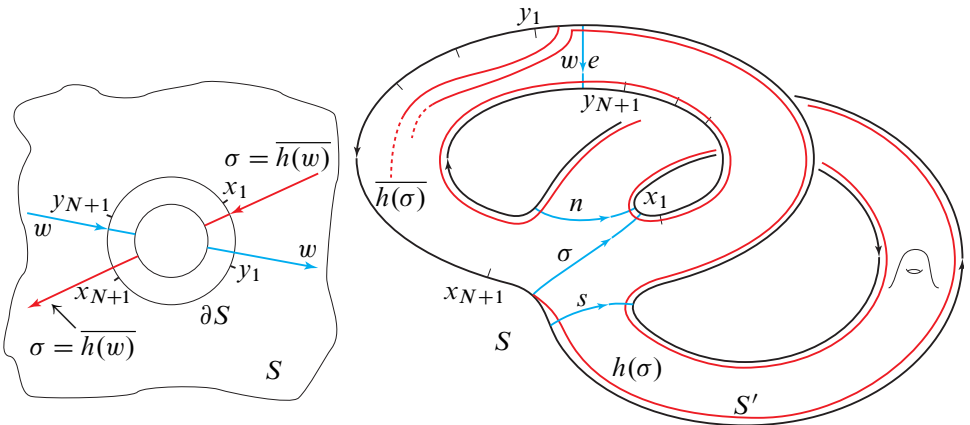
Figure 9: Two views of ∂S

Consider next the case that ϕ is pseudo-Anosov or reducible with ϕ_0 pseudo-Anosov. If the intervals (x_1, y_1) and (x_2, y_2) are ϕ–slow, necessarily $(w(0), \sigma(0))$ contains a ϕ–fast interval I, and by Corollary 2.17, there is a properly embedded geodesic arc γ orthogonal to ∂S with initial point in I. Otherwise, the intervals (x_1, y_1) and (x_2, y_2) are ϕ–fast, and hence $I = (x_2, \sigma(0))$ lies in a ϕ–fast interval. In this case, the fact that σ is in efficient position with respect to the unstable prong with initial point x_2 means that there is a properly embedded geodesic arc γ orthogonal to ∂S with initial point in I. In each case, the interval I is ϕ–fast and hence, by Proposition 2.16, $\overline{\phi^2(\gamma)}(0) > \gamma(0)$. It follows immediately that $i_{h^2}(\gamma) = 1$. This is illustrated in Figure 10.
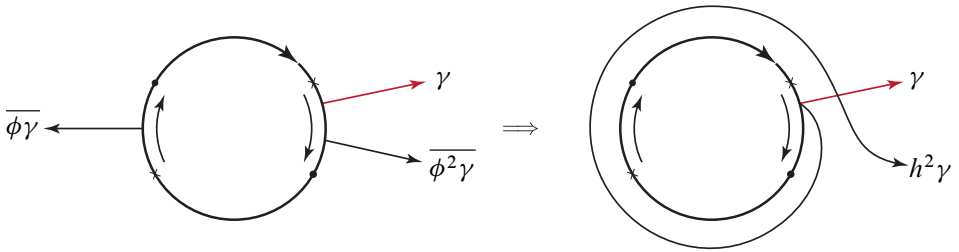

Figure 10: The initial point of γ is in a ϕ–fast interval and $i_{h^2}(\gamma) = 1$

Step 4: Action on reference arc We now derive a contradiction to the assumption that $i_g(a) < 1$ by computing $\overline{\phi^2(\gamma)}(0)$. Recall that $h = \hat{T} \circ \hat{g} \colon S \to S$ and notice that the points in

$$\{w(0), \gamma(0), \sigma(0), s(0), n(0), w(1), \sigma(1), w(0)\}$$

are listed in cyclic order about C.

Since $\gamma(0) \in (w(0), \sigma(0))$ and $\sigma = \overline{\phi(w)}$,

$$\overline{\phi(\gamma)}(0) \in \left(\overline{\phi(w)}(0), \overline{\phi(\sigma)}(0)\right) = \left(\sigma(0), \overline{\phi(\sigma)}(0)\right)$$

and hence

$$\overline{\phi^2(\gamma)}(0) \in \left(\overline{\phi(\sigma)}(0), \overline{\phi^2(\sigma)}(0)\right).$$

Now we notice that $i_g(a) < 1$ implies that $\overline{\phi(s)}(0) \in (s(0), n(0))$ and so

$$\overline{\phi(\sigma)}(0) \in (\sigma(0), n(0)) \subset (\sigma(0), w(1)).$$

But this means that

$$\overline{\phi^2(\gamma)}(0) \in \left(\overline{\phi(\sigma)}(0), \overline{\phi(w)}(1)\right) = \left(\overline{\phi(\sigma)}(0), \sigma(1)\right) \subset (\sigma(0), \sigma(1)).$$

So $\overline{\phi^2(\gamma)}(0) \in (\sigma(0), \sigma(1))$ and hence $i_{h^2}(\gamma) = 0$, a contradiction. This is illustrated in Figure 11. $\qquad \square$
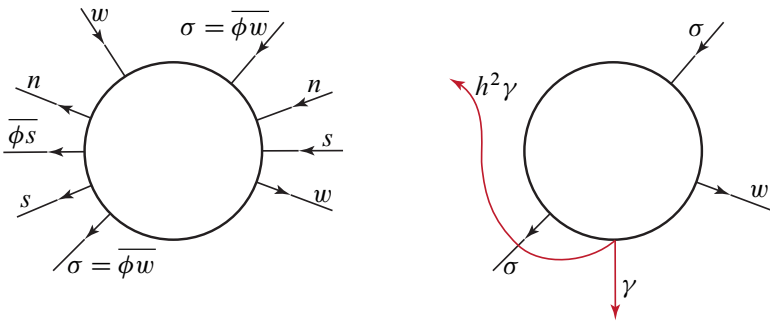

Figure 11: Endpoint ordering with the assumption $i_g(a) < 1$

3 Stallings' twists applied to $(2, 1)$–cables.

Let S be a surface properly embedded in a 3–manifold M, and let C be an unknot which is properly embedded in S. We say that C is *untwisted relative to S* if its regular neighborhood in S is an untwisted annulus. This means, if D is an embedded disk in M bounded by C, then C is untwisted relative to S if and only if D can be isotoped relative to its boundary so that D and S are transverse along C. A *Stallings' twist* [25] is a surgery along any such unknotted untwisted C.

Stallings' twists will be applied to $(2,1)$–cables and exploited in Section 5. As a first step we show in this section that the monodromies associated to the construction are pseudo-Anosov and have fractional Dehn twist coefficient $\frac{1}{2}$.

Consider an open book $S^3 = (F, h)$, with connected binding k, such that h is isotopic to a pseudo-Anosov map ψ and $c(h) = 0$. (Recall that 8_{20} is the binding of one such open book. There are infinitely many such. Take as binding, for example, any hyperbolic fibred knot which contains as summand the fiber of the figure-8 knot.) Let N denote a regular neighborhood of k. Abuse notation and let F also denote the compact fiber of the surface bundle $S^3 \setminus \text{int } N$, and consider two copies of this compact fiber: $F_0 = F \times \{0\}$ and $F_{1/2} = F \times \{\frac{1}{2}\}$. Choose a hyperbolic metric with geodesic boundary for F, and let F_0 and $F_{1/2}$ inherit this metric.

Let $N_0 \subset \text{int } N$ be a second smaller regular neighborhood of k and let K denote the $(2,1)$–cable of k, embedded in ∂N_0 (see Rolfsen [21, page 112]). Letting $\iota\colon S^1 \times D^2 \to S^3$ be an embedding with image N_0, the image under ι of the $(2,1)$ torus knot (shown in Figure 12) is what is standardly known as the $(2,1)$–cable of k.
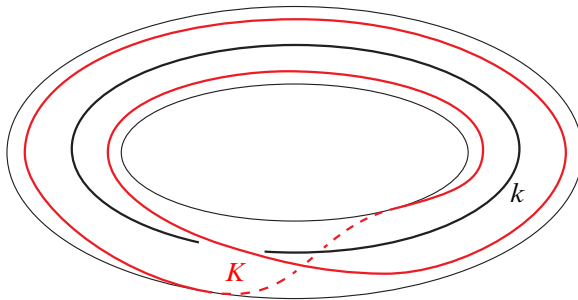

Figure 12: The $(2,1)$–cable of k

Let P denote a pair of pants properly embedded in $N \setminus \text{int } N_1$, with boundary components $\partial F_0 \cup \partial F_{1/2}$ in ∂N and a third boundary component lying on ∂N_1, where N_1 is a small regular neighborhood of K lying in the interior of N. See Figure 13.

Set $S = F_0 \cup P \cup F_{1/2}$. Extend the hyperbolic metric on $F_0 \cup F_{1/2}$ over P to obtain a hyperbolic metric on S with geodesic boundary. Notice that S can be thought of as the union of two copies of F connected by a 1–handle. Let $\alpha = \overline{\alpha}$ denote the geodesic representative orthogonal to ∂S of a cocore of this 1–handle. We may choose this hyperbolic metric on S so that there is an orientation preserving isometry, $r\colon P \to P$ say, which interchanges the two components of $P \setminus \alpha$.

Lemma 3.1 gives two slightly different formulas for the monodromy map S. Depending on the context, it is enough to describe a representative of the free isotopy class, in
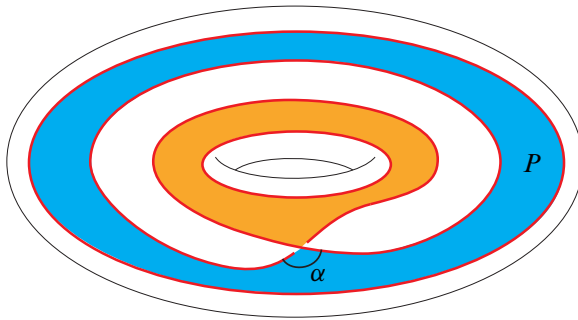
Figure 13

which case we use H_0 and r. To compute fractional Dehn twist coefficients we need to describe a representative of the relative isotopy class. This is done by adjusting r so that it fixes ∂S pointwise. To accomplish this, add a collar neighborhood $\partial S \times I$ to S, and let R be the homeomorphism of $P \cup (\partial S \times I)$ that extends r and is a shear to the right by half a rotation on $\partial S \times I$. For simplicity of notation, we incorporate the collar into S and think of R as a homeomorphism of S.

Lemma 3.1 *The surface S is a fiber of K in the fibred 3–manifold $S^3 \setminus N_1$. Representatives of the free and relative isotopy classes of the monodromy classes are H_0 and $H_0'\colon S \to S$, respectively, which are given by*

$$H_0(z) = \begin{cases} (x, \tfrac{1}{2}) & \text{if } z = (x, 0) \text{ lies in } F_0, \\ (h(x), 0) & \text{if } z = (x, \tfrac{1}{2}) \text{ lies in } F_{1/2}, \\ (T_{\partial F_0} T_{\partial F_{1/2}})^{-1} r(z) & \text{if } z \text{ lies in } P, \end{cases}$$

$$H_0'(z) = \begin{cases} H_0(z) & \text{if } z \notin P, \\ (T_{\partial F_0} T_{\partial F_{1/2}})^{-1} R(z) & \text{if } z \in P. \end{cases}$$

Proof As $(S^3 \setminus \mathrm{int}\, N) \setminus S$ inherits a product structure from the one on $(S^3 \setminus \mathrm{int}\, N) \setminus F$, it suffices to understand a compatible product structure on $(N \setminus \mathrm{int}\, N_1) \setminus P$.

A meridional flow line for $S^3 \setminus \mathrm{int}\, N$ is shown on the left of Figure 14. It splits into two subarcs that are flow lines for the $(2,1)$–cable of k. In the center of the figure several, of an interval's worth, of these upward flowing subarcs are shown. The initial points of this collection of arcs form an interval that is mapped by the monodromy to the terminal points. On the right an analogous collection of downward flowing subarcs is shown.

Figure 15 shows P flattened out, and it records the positions of the arcs constructed above under the monodromy map. The monodromy, up to isotopy fixing ∂S, is determined by the action on these arcs. Direct computation shows that $(T_{\partial F_0} T_{\partial F_{1/2}})^{-1} R(z)$ agrees with the monodromy map on these arcs.   □
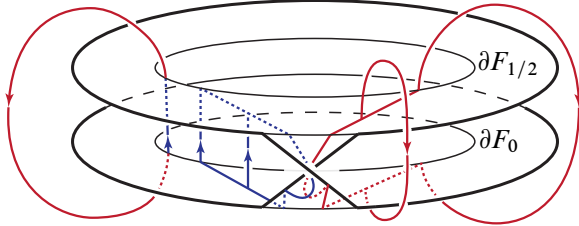

Figure 14: Each curve indicated with an arrow is a flow line of $H_0': S \to S$. The monodromy is computed by comparing the locations of initial and terminal points of an interval's worth of flowlines.
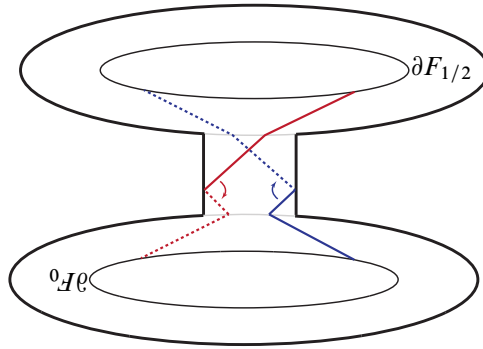

Figure 15: Before and after images of two arcs in P under the flow

Choose a simple closed curve C in S with two properties. First, it intersects $F_0 \cup F_{1/2}$ in two essential arcs, $C_0 = C \cap F_0$ and $C_{1/2} = C \cap F_{1/2}$. Second, C_i is nonseparating in F_i for each $i = 0, \frac{1}{2}$. Now set $H = T_C \circ H_0$.

Theorem 3.2 *Suppose that h has pseudo-Anosov Thurston representative and satisfies $c(h) = 0$. Let H be the map obtained from the above construction applied to h. Then the Thurston representative Ψ of H is pseudo-Anosov.*

Proof To simplify the exposition, it helps to pass to a closed hyperbolic surface $\hat{S}$ naturally associated to S. Topologically, $\hat{S}$ is defined by setting

$$\hat{S} = S \cup_\partial D^2,$$

the surface obtained from S by capping off ∂S with a disk. Notice that

$$\widehat{S} = F_0 \cup \widehat{P} \cup F_{1/2},$$

where $\widehat{P} = P \cup_\partial D^2$ is an annulus. The homeomorphism H_0 extends to a homeomorphism $\widehat{H}_0 \colon \widehat{S} \to \widehat{S}$ given by

$$\widehat{H}_0(z) = \begin{cases} (x, \frac{1}{2}) & \text{if } z = (x, 0) \text{ lies in } F_0, \\ (h(x), 0) & \text{if } z = (x, \frac{1}{2}) \text{ lies in } F_{1/2}, \\ (T_{\partial F_0} T_{\partial F_{1/2}})^{-1} \widehat{r}(z) & \text{if } z \text{ lies in } \widehat{P}, \end{cases}$$

where $\widehat{r} \colon \widehat{P} \to \widehat{P}$ is a homeomorphism, uniquely determined up to isotopy, extending $r \colon P \to P$.

Now consider the closed hyperbolic surface $F_0 \cup_f F_{1/2}$, obtained by gluing the hyperbolic surfaces F_0 and $F_{1/2}$ by the isometry $f \colon \partial F_0 \to \partial F_1$ given by $f(x, 0) = (x, \frac{1}{2})$. Choose a diffeomorphism from $\widehat{S}$ to $F_0 \cup_f F_{1/2}$, and use this diffeomorphism to pull back the hyperbolic metric on $F_0 \cup_f F_{1/2}$ to a hyperbolic metric on $\widehat{S}$. Let A denote the geodesic simple closed curve which is the pull back to $\widehat{S}$ of $\partial F_0 = \partial F_{1/2} \subset F_0 \cup_f F_{1/2}$. Notice that

$$\widehat{H}_0(z) = \begin{cases} (x, \frac{1}{2}) & \text{if } z = (x, 0) \text{ lies in } F_0, \\ (h(x), 0) & \text{if } z = (x, \frac{1}{2}) \text{ lies in } F_{1/2}, \\ T_A^{-2} \, \widehat{r}(z) & \text{if } z \text{ lies in } \widehat{P}. \end{cases}$$

Set

$$\widehat{H} = T_C \circ \widehat{H}_0.$$

The following characterization of periodic and reducible maps follows immediately from Thurston's classification of automorphisms of surfaces, Theorem 2.1.

Lemma 3.3 *A homeomorphism h of a surface S has a periodic or reducible representative if and only if there exists a possibly immersed multicurve $\gamma = \rho_1 \cup \cdots \cup \rho_n$ where each ρ_i is an essential, embedded, curve in S, and, up to isotopy, γ is fixed by h. In the periodic case, each ρ_i may be assumed to be nonseparating, and in the reducible case, γ may be chosen to be embedded.* $\square$

Corollary 3.4 *If the Thurston representative $\widehat{\Psi}$ of $\widehat{H}$ is pseudo-Anosov, then the Thurston representative Ψ of H is also pseudo-Anosov.*

Proof By contradiction, suppose that H is either periodic or reducible, and let γ be the multicurve guaranteed by Lemma 3.3. Since the curves γ, $H_0(\gamma)$, and $T_C^{-1}(\gamma)$ and the image of the isotopy between $H_0(\gamma)$ and $T_C^{-1}(\gamma)$ all lie in S and therefore in

$\hat{S}$, it follows that $H(\gamma)$ is isotopic to γ. Furthermore, no component of γ is boundary parallel in S. Thus applying Lemma 3.3 to $\gamma \subset \hat{S}$ shows that $\hat{H}$ is either periodic or reducible. □

Theorem 3.2 therefore follows from the following theorem. □

Theorem 3.5 *The Thurston representative $\hat{\Psi}$ of $\hat{H}$ is pseudo-Anosov.*

Proof Suppose instead that $\hat{\Psi}$ is periodic or reducible. Choose γ as guaranteed by Lemma 3.3 so that it is either an embedded curve or is a union of nonseparating curves. Necessarily $\hat{H}_0(\gamma)$ is isotopic to $T_C^{-1}(\gamma)$. We next establish notation for a sequence of lemmas that will be used to derive a contradiction in Corollary 3.11.

Let λ^s and λ^u denote the stable and unstable geodesic laminations of ψ. Let $L_1^s, \ldots, L_n^s$, $n \geq 1$, denote the leaves of λ^s bounding the complementary region of λ^s containing ∂F, ordered cyclically about ∂F. Similarly, let $L_1^u, \ldots, L_n^u$ denote the leaves of λ^u bounding the complementary region of λ^u containing ∂F, ordered cyclically about ∂F.

The unstable prongs, ie, the geodesics perpendicular to ∂F found between each consecutive pair L_i^u, L_{i+1}^u, canonically cut each L_i^s into two open intervals L_{i-}^s and L_{i+}^s. Similarly, the stable prongs canonically cut each L_i^u into two open intervals L_{i-}^u and L_{i+}^u. Since $c(h) = 0$, we may assume that both ψ and h fix the leaves $L_i^s, L_i^u, L_{i\pm}^s$, and $L_{i\pm}^u$, for $1 \leq i \leq n$.

To simplify the exposition and the notation, we now assume that A, C, and γ have been isotoped so that they are each geodesic in $\hat{S}$. Let η denote the geodesic representative of $\hat{H}_0(\gamma)$, and choose an isotopy representative of T_C^{-1} so that $T_C^{-1}(\gamma)$ is a geodesic.

Let W denote the closed complementary region of $\lambda^s \times \{0, \frac{1}{2}\}$ which contains A. Since γ meets $\lambda^s \times \{0, \frac{1}{2}\}$ efficiently, so does $\hat{H}_0(\gamma)$. It follows that we can isotope $\hat{H}_0(\gamma)$ to be geodesic while preserving this efficient intersection with $\lambda^s \times \{0, \frac{1}{2}\}$. The following therefore holds.

Lemma 3.6 *If a component of $\hat{H}_0(\gamma) \cap W$ is a path connecting $L_i^s \times \{0\}$ and $L_j^s \times \{\frac{1}{2}\}$ for some i, j, $1 \leq i, j \leq n$, then the corresponding component of $\eta \cap W$ is also a path connecting $L_i^s \times \{\frac{1}{2}\}$ and $L_j^s \times \{0\}$.* □

Lemma 3.7 *Let O be a geodesic simple closed curve in $\hat{S}$, and for each integer l, let O_l denote the geodesic representative of $T_A^l(O)$. If a component of $O \cap W$ is a path connecting $L_i^s \times \{0\}$ and $L_j^s \times \{\frac{1}{2}\}$ for some i, j, $1 \leq i, j \leq n$, then the corresponding component of $O_l \cap W$ is also a path connecting $L_i^s \times \{\frac{1}{2}\}$ and $L_j^s \times \{0\}$.*

Proof The leaves $L_i^s \times \{0\}$ which intersect O are uniquely determined by the constraint that the intersection $O \cap (L_i^s \times \{0\})$ is efficient. This efficiency of intersection is unaffected by composition with T_A^l. □

Symmetric statements hold relative to the unstable lamination $\lambda^u \times \{0, \frac{1}{2}\}$.

Lemma 3.8 *Both $\gamma \cap C \neq \varnothing$ and $\gamma \cap A \neq \varnothing$.*

Proof If $\gamma \cap C = \varnothing$, then γ is fixed by T_C^{-1} and hence, up to isotopy, also by $\widehat{H}_0$. Since ψ is pseudo-Anosov, $\widehat{H}_0$ fixes only A, up to isotopy, and it follows that $\gamma \cap C \neq \varnothing$.

It follows that if $\gamma \cap A = \varnothing$, then necessarily $T_C^{-1}(\gamma) \cap A \neq \varnothing$. But we know $\widehat{H}_0(\gamma) = T_C^{-1}(\gamma)$ and hence

$$|T_C^{-1}(\gamma) \cap A| = |\widehat{H}_0(\gamma) \cap A| = |\widehat{H}_0(\gamma) \cap \widehat{H}_0(A)| = |\gamma \cap A| = 0.$$

So $\gamma \cap A \neq \varnothing$ (and $T_C^{-1}(\gamma) \cap A \neq \varnothing$). □

We now use C to get a measure of the amount of twisting of $\widehat{H}_0(\gamma)$ and $T_C^{-1}(\gamma)$ respectively about A. We do this as follows. First consider the components of $\widehat{S} \setminus (\gamma \cup A \cup C)$. Call such a complementary region a *triangular disk region* if it is a disk with piecewise geodesic boundary consisting of exactly three geodesic subarcs, one contained in each of γ, A and C respectively. Choose a regular neighborhood X of A so that

(1) all such triangular disk regions are contained in the interior of X,

(2) all intersections of ∂X with C, γ, $\widehat{H}_0(\gamma)$, and $T_C^{-1}(\gamma)$ are transverse,

(3) $C \cap X$ consists of two essential embedded arcs: σ_1 and σ_2,

(4) $\gamma \cap X$ consists of a essential embedded arcs: $\gamma_1, \ldots, \gamma_a$, for some $a \geq 2$,

(5) $\eta \cap X$ consists of a essential embedded arcs: $\eta_1, \ldots, \eta_a$,

(6) $T_C^{-1}(\gamma) \cap X$ consists of a essential embedded arcs: $\tau_1, \ldots, \tau_a$, and

(7) for $1 \leq i \leq 2$ and $1 \leq j \leq a$, each of the arcs σ_i, γ_j, η_j, and τ_j has nonempty (necessarily minimal) intersection with each of $\bigcup L_i^s \times \{0\}$, $\bigcup L_i^s \times \{\frac{1}{2}\}$, $\bigcup L_i^u \times \{0\}$, and $\bigcup L_i^u \times \{\frac{1}{2}\}$.

In (4)–(6) we are using the fact that $\widehat{H}_0(A) = A$, thus

$$|T_C^{-1}(\gamma) \cap A| = |\widehat{H}_0(\gamma) \cap A| = |\gamma \cap A| = a.$$

Choose the indices j, $1 \leq j \leq a$, so that η_j is the component corresponding to $\widehat{H}_0(\gamma_j)$.

Orient the arcs σ_1, σ_2, γ_i, η_i, and τ_i, $1 \leq i \leq a$, so that they run from $\partial X \cap (F \times \{0\})$ to $\partial X \cap (F \times \{\frac{1}{2}\})$.

Lemma 3.9 *For all i and j, $1 \le i \le a$ and $1 \le j \le 2$, $|\langle \tau_i, \sigma_j \rangle| \le 1$.*

Proof Since $\tau_i \subset T_C^{-1}(\gamma)$ and $\sigma_j \subset C$ each component τ_i can meet each of σ_1 and σ_2 at most once.   □

Lemma 3.10 *For some i_0 and j_0, $1 \le i_0 \le a$ and $1 \le j_0 \le 2$, $\langle \gamma_{i_0}, \sigma_{j_0} \rangle < 0$.*

Proof Points of intersection $\gamma \cap C$ lie either in X or outside X. The condition that all triangular disk regions lie in X guarantees that any point of intersection $\gamma \cap C$ lying outside X results in increased geometric intersection of $T_C^{-1}(\gamma)$ with A. And any point of intersection $\gamma \cap C$ lying inside X and with positive intersection number $\langle \gamma_i, \sigma_j \rangle$ results in increased geometric intersection of $T_C^{-1}(\gamma)$ with A. Since $\gamma \cap C$ is nonempty, it follows that if all intersection numbers $\langle \gamma_i, \sigma_j \rangle$ are nonnegative, then necessarily $|T_C^{-1}(\gamma) \cap A| > |\gamma \cap A|$. This is impossible.   □

Rechoose indices as necessary so that

$$\langle \gamma_1, \sigma_1 \rangle < 0$$

and η_1 is the component of η corresponding to $\hat{H}_0(\gamma_1)$.

Corollary 3.11 *The geodesic representative of $\hat{H}_0(\gamma)$, η, and $T_C^{-1}(\gamma)$ are not equal, and therefore $\hat{H}(\gamma)$ is not isotopic to γ.*

Proof Focus on γ_1 and $T_A^2(\eta_1)$, the latter arc isotoped rel endpoints to be geodesic and oriented so that it runs from $\partial X \cap (F \times \{0\})$ to $\partial X \cap (F \times \{\frac{1}{2}\})$. The arc $\gamma_1 \cap W$ is a path from $L_i^s \times \{0\}$ to $L_j^s \times \{\frac{1}{2}\}$ for some i, j, $1 \le i, j \le n$. Since $\hat{H}_0$ switches $F \times \{0\}$ and $F \times \{\frac{1}{2}\}$, and since $c(h) = 0$, it follows from Lemma 3.7 that both $\eta_1 \cap W$ and $T_A^2(\eta_1) \cap W$ are paths from $L_j^s \times \{0\}$ to $L_i^s \times \{\frac{1}{2}\}$. There are two possibilities: either $T_A^2(\eta_1) \cap \gamma_1 = \varnothing$ or $T_A^2(\eta_1) \cap \gamma_1 \ne \varnothing$.

Consider first the case that $T_A^2(\eta_1) \cap \gamma_1 = \varnothing$. In this case,

$$\langle T_A^2(\eta_1), \sigma_1 \rangle \le 0.$$

Since η_1 intersects A in a point, applying T_A changes the intersection number by 1, and we have

$$\langle \eta_1, \sigma_1 \rangle \le -2.$$

It follows from Lemma 3.9 that $\eta_1 \ne \tau_m$ for all m, $1 \le m \le a$.

Next consider the case that $T_A^2(\eta_1) \cap \gamma_1 \ne \varnothing$. Recall that η_1 is the portion of the geodesic, corresponding to γ_1, that is formed by applying $\hat{H}_0$ to γ. The action of $\hat{H}_0$

on W is T_A^{-2}, thus $T_A^2(\eta_1)$ corresponds to acting on γ_1 by the map $\hat{r}$ used in the definition of $\hat{H}_0$. If $i \neq j$, $\hat{r}$ would force $T_A^2(\eta_1) \cap \gamma_1 = \varnothing$. It follows that $i = j$. Moreover, $\langle \gamma_1, \sigma_1 \rangle < 0$ implies $|\langle T_A^2(\eta_1), \sigma_1 \rangle| \leq 1$.

If $\langle T_A^2(\eta_1), \sigma_1 \rangle \leq 0$, the argument of the first case applies. So we may assume $\langle T_A^2(\eta_1), \sigma_1 \rangle = 1$. This is possible only if γ_1, $T_A^2\eta_1$ and σ_1 are all paths connecting $L_i^s \times \{0\}$ to $L_i^s \times \{\frac{1}{2}\}$, pairwise isotopic through paths connecting $L_i^s \times \{0\}$ to $L_i^s \times \{\frac{1}{2}\}$. In other words, up to composition by T_A^l for some integer l, they must lie as shown in Figure 16 (above). The arc η_1 is added in Figure 16 (below).
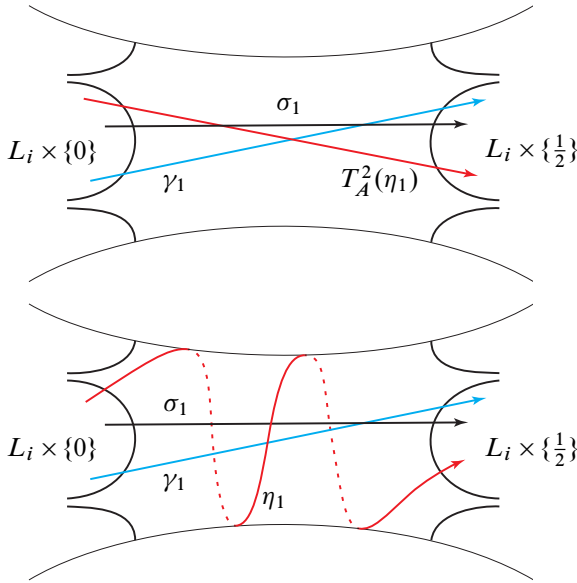

Figure 16

The next step is to show there is a geodesic arc, β, properly embedded and essential in W, such that β is disjoint from $\gamma_1 \cup T_A^2(\eta_1)$. When $n > 1$, choose β to be a geodesic path from $L_p^s \times \{0\}$ to $L_p^s \times \{\frac{1}{2}\}$ for some $p \neq i$. When $n = 1$, we argue similarly but work instead with the leaves $L_{1\epsilon}^s$, $\epsilon \in \{\pm\}$. The arcs γ_1, $T_A^2\eta_1$ and σ_1 are all paths connecting $L_{1\epsilon}^s \times \{0\}$ to $L_{1\epsilon}^s \times \{\frac{1}{2}\}$, pairwise isotopic through paths connecting $L_{1\epsilon}^s \times \{0\}$ to $L_{1\epsilon}^s \times \{\frac{1}{2}\}$. We then choose β to be a geodesic path from $L_{1\delta}^s \times \{0\}$ to $L_{1\delta}^s \times \{\frac{1}{2}\}$, where $\delta = -\epsilon$.

Thus in either case, β satisfies $|\beta \cap \eta_1| = 2$, whereas $|\beta \cap \tau_m| \leq 1$ for all $l, 1 \leq l \leq a$. So again we may conclude $\eta_1 \neq \tau_m$ for all $m, 1 \leq m \leq a$. $\qquad \square$

Theorem 3.5 now follows from Corollary 3.11 $\qquad \square$

Proposition 3.12 With H'_0 as defined in [Lemma 3.1](#), and $H' = T_C \circ H'_0$,

$$\tfrac{1}{2} \le c(H') \le 1.$$

Proof Let $\alpha \subset S$ be the arc fixed by the involution $r \colon P \to P$. To estimate $c(H')$, we first compute $i_{H'}(\alpha)$. Loosely speaking, it is enough to compute just R and T_C on the initial portion of α. More precisely, $H'(\alpha) = \beta_1 * \beta_2$, where β_1 is an arc ending transversely on $F \times \{0\}$, and β_2 is an arc which starts on $F \times \{0\}$ and has an essential first return to $F \times \{0\}$. [Figure 17](#) shows β_1 which is computed by applying R and T_C to the initial portion of α. Only the first intersection of β_1 and α can contribute to $i_{H'}(\alpha)$, but this does not yield a boundary parallel union of initial segments. It follows that $i_{H'}(\alpha) = 0$, and thus by [Proposition 2.9](#), $0 \le c(H') \le 1$.
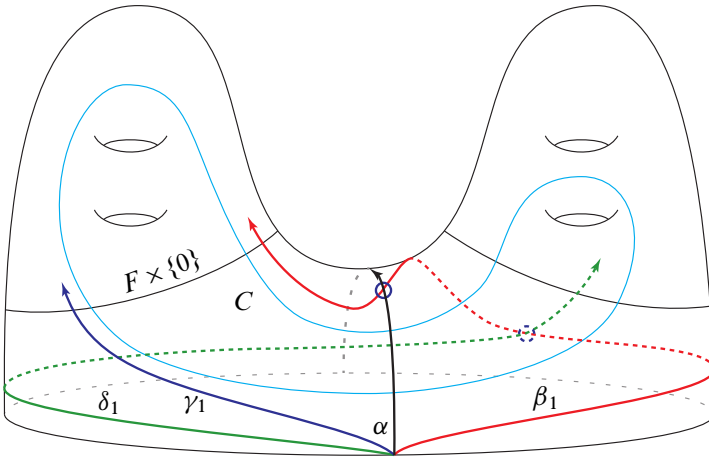

Figure 17

The next step is to estimate $c(H'^2)$ by computing $H'^{-1}(\alpha)$ and comparing it to $H'(\alpha)$. Towards this end, write $T_C^{-1}(\alpha) = \gamma_1 * \gamma_2$, where γ_1 is essentially truncated at $F \times \{0\}$ as above. To compute $H'^{-1}(\alpha) = \delta_1 * \delta_2$, where δ_1 is essentially truncated at $F \times \{\tfrac{1}{2}\}$ by applying the remaining portion of H'^{-1} to β_1. The single intersection of γ_1 and β_1 is enough to conclude that $i_{H'^2}(H'^{-1}(\alpha)) = 1$. Thus by [Proposition 2.9](#) we have $1 \le c(H'^2) \le 2$ and it follows by [Proposition 2.7](#) that $\tfrac{1}{2} \le c(H') \le 1$. $\qquad\square$

We show in the next section, [Corollary 4.6](#), that in S^3, $c(H')$ can only be $\tfrac{1}{2}$.

4 Stabilization in S^3

The first portion of this section is devoted to proving the result, Theorem 4.5, that the only values fractional Dehn twist coefficients take on in S^3 are 0 and $1/n$, where n is an integer satisfying $|n| \geq 2$.

This leads to the statement of Conjecture 4.7, or in light of Theorem 4.5, the equivalent statement, Conjecture 4.8, either of which imply that a fibred knot in S^3 with fractional Dehn twist coefficient $\frac{1}{2}$ can never be destabilized. This consequence, in the most important case when the monodromy is pseudo-Anosov, was first stated, without proof, by Gabai in [8].

Part of our interest in this result is that it appears to depend on deep facts about S^3, most likely Gabai's thin position strategy, in an essential way. Indeed, very simple examples, such as the examples of Proposition 4.9, show the result fails even in lens spaces.

We begin by considering more closely the case that the Thurston representative ϕ of the monodromy h is either periodic or reducible with ϕ_0 periodic.

Proposition 4.1 (Seifert [24]) *Let $S^3 = (S, h)$ be an open book decomposition which has connected binding k. If the Thurston representative of h is periodic, then k is the unknot or a (p, q)–torus knot for some relatively prime integers p and q satisfying $|p| > 1$.*

Proposition 4.2 *Let $S^3 = (S, h)$ be an open book decomposition which has connected binding k. If the Thurston representative, ϕ, of h is reducible with ϕ_0 periodic, then k is a (p, q)–cable knot, for some relatively prime integers p and q satisfying $|p| > 1$, and $c(h) = 1/(pq)$.*

Proof Suppose ϕ is reducible with $\phi_0 \colon S_0 \to S_0$ periodic. Denote the boundary components of S_0 by $C = k = \partial S, C_1, \ldots, C_n$. Let T_i denote the torus in S^3 that is the union of flow lines of ϕ_0 intersecting C_i. Since ϕ_0 may permute the C_i, the T_i are not necessarily distinct. By Alexander [3] and Schubert [22], each T_i bounds a solid torus V_i in S^3.

Each T_i is essential in $S^3 - k$. To see this, notice that an infinite cyclic cover of $S^3 - k$ can be created by gluing copies of $S^3 - k$ split along S. The inverse image of T_i in this cover is an infinite cylinder which is incompressible since C_i is essential in S.

It follows that every V_i contains k. By connectivity of S_0 there can be only one such solid torus, thus $V = V_1 = \cdots = V_n$ and $T = T_1 = \cdots = T_n$. Consider a closed orbit

γ of ϕ_0 that lies on the boundary of a regular neighborhood $N(k)$ of k. If this is a meridian, then $c(h) = 0$. Otherwise the orbits of ϕ_0, together with k, define a Seifert fiber space structure on V.

A Seifert fiber space structure on V can have at most one singular fiber (Hatcher [12, Theorem 2.3]). If k is a singular fiber, the leaf space of $V - N(k)$ is an annulus. Moreover, this annulus is covered by S_0. It follows that S_0 is itself an annulus, thereby contradicting the choice of S_0 as a reducing surface for ϕ.

It follows that k is a nonsingular fibre and hence a (p, q)–cable of the core of V. Let K_V denote the knot which is the core of V, and let X_V denote the complement of V in S^3. Coordinates (p, q) are chosen so that a $(1, 0)$ curve is a longitude for K_V, and a $(0, 1)$ curve is a meridian for K_V. A *cabling annulus* is an annulus in V such that one boundary component is a (p, q) curve on the boundary of V, the interior is embedded in the interior of V, and the other boundary component is a p to one cover of K_V. Notice that $c(h) = 1/(pq)$ is the slope of the cabling annulus.

Let S'' denote the subsurface $S \setminus \text{int } S_0$ of S. Notice that the orientation on S_0 induces a ϕ_0–invariant orientation on the intersection $T \cap S = C_1 \cup \cdots \cup C_n$. Equivalently, the oriented complementary surface S'' intersects ∂X_V in $n \geq 1$ parallel consistently oriented simple closed curves: $C_1, \ldots, C_n$. Necessarily these curves are $(1, 0)$ curves and S'' has n connected components. Moreover, the complement of S'' in X_V is an I–bundle. It follows that K_V is fibred and S'' consists of n copies of a fiber S_V of K_V. It follows also that $p = n$. Moreover, since S_0 cannot be an annulus, $|p| > 1$.

To compute $c(h)$, consider a periodic orbit γ near C, and express it in (longitude, meridian) coordinates with respect to C. The orbits of ϕ_0 define the Seifert fibration on V, and since C is a nonsingular fiber, γ is also a (p, q) curve with respect to K_V. It follows that γ intersects a meridian of C once. To compute the number of times γ intersects S, it is enough to compute the intersection between a (p, q) curve and the p $(1, 0)$ curves that are $S \cap T$. It follows that γ is a $(1, pq)$ curve with respect to C, thus $c(h) = 1/(pq)$. □

Corollary 4.3 *Let $S^3 = (S, h)$ be an open book decomposition which has connected binding k. Suppose that the Thurston representative, ϕ, of h is either periodic or reducible with ϕ_0 periodic. Then either k is the unknot and $c(h) = 0$ or $c(h) = 1/n$, where $|n| \geq 2$ is the slope of the cabling annulus.*

Proof Since k is either a (p, q)–torus knot or a (p, q)–cable knot, it follows that either k is the unknot, or the cabling annulus is incompressible and boundary incompressible in $S^3 - \text{int } N(k)$. Since the cabling annulus has slope pq, it follows that either k is the unknot and $c(h) = 0$ or k is not the unknot and $c(h) = 1/(pq)$. □

Theorem 4.4 [8] *Let $S^3 = (S, h)$ be an open book decomposition which has connected binding k. Suppose that ϕ is either pseudo-Anosov or reducible with ϕ_0 pseudo-Anosov. Then either $c(h) = 0$ or $c(h) = 1/r$, where $2 \le |r| \le 4(\text{genus}(k)) - 2$.*

Proof [8, Theorem 8.8] states that the degeneracy, $d(k)$, of the complement of k has one of the forms

(1) $r(1/0)$, with $1 \le r \le 4(\text{genus}(k)) - 2$,

(2) $r/1$, with $2 \le |r| \le 4(\text{genus}(k)) - 2$.

See [20] for a second proof of the lower bound in (2). As noted in Section 2, when $M = S^3$, $c(h)$ is the reciprocal of the slope of $d(k)$. It follows that $c(h)$ is either 0 or $1/r$ for some integer r, $2 \le |r| \le 4(\text{genus}(k)) - 2$. □

Combining Corollary 4.3 and Theorem 4.4 proves the following.

Theorem 4.5 *If (S, h) is an open book decomposition of S^3 with connected binding, then $c(h)$ equals 0 or $1/n$ for some integer n, $|n| \ge 2$.* □

Theorem 4.4 and Proposition 3.12 immediately imply the following.

Corollary 4.6 *Let H_0' be the monodromy of a (2,1)-cable of a knot in S^3 with pseudo-Anosov monodromy and fractional Dehn twist coefficient 0. Then $H' = T_C \circ H_0'$, as constructed above, satisfies $c(H') = \frac{1}{2}$.* □

Among all the possible values of the fractional Dehn twist coefficient that arise in S^3, it is $\frac{1}{2}$ that plays the most mysterious role in knot theory. The following was first stated by Gabai in [8] along with a possible proof strategy.

Conjecture 4.7 *Let $S^3 = (S, h)$ be an open book decomposition which is stabilized and has connected binding K. Then $c(h) \ne \frac{1}{2}$.*

By Theorem 4.5, this conjecture can also be stated as follows.

Conjecture 4.8 *Let $S^3 = (S, h)$ be an open book decomposition which is stabilized and has connected binding. Then either $c(h) = 0$ or $c(h) = 1/n$, where $n \ge 3$.*

An appealing aspect of these conjectures is that very simple examples show that they are false outside of S^3. Specifically, Example 2.25 in Section 2 proves the following.

Proposition 4.9 *Let $p \geq 5$ and $q = 1$. Then $L(p, q)$ has an open book decomposition with connected binding which is stabilized and satisfies $c(h) = \frac{1}{2}$.* □

5 Applications

In this section, we give an application of our results in contact topology by associating to each pair (S, h) a contact structure ξ via the Giroux correspondence [11]. This correspondence is many to one, and Giroux shows that the failure of injectivity is generated by positive stabilization (see Definition 2.23). A natural goal is to try to determine properties of ξ from a single pair (S, h).

Some results in this direction are as follows [14]. If $c(h) \leq 0$ then ξ is overtwisted. If $c(h) \geq 1$ then ξ is tight. Contrast this with the uncheckable theorem that ξ is tight if and only if for *every* (S, h) determining ξ, $c(h) > 0$. This leads to the conjecture by Honda, Kazez and Matić [14] that if a single representative is not destabilizable and has $c(h) > 0$ (or equivalently is right-veering) then ξ is tight.

Lekili [17] produced counterexamples to this conjecture with S a 4–times punctured sphere. Lisca [18] constructed an infinite number of counterexamples for the same surface. Ito and Kawamuro [16] have produced an even larger set of counterexamples on the 4 times punctured sphere.

This should be contrasted with the work of Colin and Honda [5] in which they show that if an open book has connected boundary, pseudo-Anosov monodromy, and fractional Dehn twist coefficient $k/n > 0$, then $k > 1$ implies the associated contact structure is not only tight, but universally tight, and the universal cover is $\mathbb{R}^3$.

Our examples show that it is often very easy to recognize an overtwisted contact structure using essentially no technology. It is enough, by Proposition 5.3, to find a nonseparating, untwisted, unknotted curve contained in S.

Our construction involves surgery on an unknotted curve in S, so the first step is to record some results on different framings of the curve before and after surgery.

Let $M = (S, h)$ be an open book decomposition and let C be a simple closed curve embedded in the interior of S which bounds an embedded disk in M. Let N be a regular neighborhood of C. Note that although there is a unique canonical meridian μ, there are two natural choices of longitude in this setting:

(1) λ_D, the slope of the single curve $D \cap \partial N$, and

(2) λ_S, the slope shared by the two curves $S \cap \partial N$.

Using the orientation convention that $\langle \lambda_S, \mu \rangle = \langle \lambda_D, \mu \rangle = 1$ with respect to the outward pointing normal on ∂N, define the integer $\mathrm{tw}_C(S)$, the *twisting of S along C*, by writing $\lambda_S = \lambda_D + \mathrm{tw}_C(S)\mu$.

If the orientation for λ_D is reversed, then μ and hence λ_S also have their orientations reversed. It follows that the sign of $\mathrm{tw}_C(S)$ does not depend on the choice of orientation of D. In particular, from the point of view of D, if the sign of $\mathrm{tw}_C(S)$ is negative, an annular neighborhood of C in S twists to the left, when traveling around ∂D. Thus $\mathrm{tw}_C(S)$ serves as a topological stand-in for the Thurston–Bennequin invariant.

Perform a $p\mu + q\lambda_D$ surgery along C, for some relatively prime integers p and q, and let Y denote the manifold thus obtained from M.

Proposition 5.1 *The manifold Y is homeomorphic to M if and only if $p/q = \pm 1/q$.*

Proof Let B denote a regular neighborhood of D in M. Note that $M = M \,\sharp\, S^3$ with summing sphere ∂B. So it suffices to consider the case that $M = S^3$. If $p = \pm 1$, simply cut and twist around D to realize the homeomorphism between M and Y. Computing $\pi_1(Y) = \langle \mu \mid p\mu \rangle = \mathbb{Z}_p$ shows the converse is also true. $\qquad \square$

Proposition 5.2 *If $\mathrm{tw}_C(S) = 0$ and $p = -1$, then Y has an open book decomposition given by $(S, T_C^q h)$. Moreover, the analogues of C, D and S exist in Y as embedded objects, and in Y it remains the case that $\mathrm{tw}_C(S) = 0$.*

Proof Let A be an annular neighborhood of C in S and let $N = A \times [\frac{1}{2}, 1]$. Let b be nonseparating embedded arc in A and let $R = b \times [\frac{1}{2}, 1]$ be a rectangular compressing disk for N.

To construct the desired homeomorphism from Y to the manifold built from $(S, T_C^q h)$, start with the identity map from $(S \times I) - N \to (S \times I) - N$. Next extend the homeomorphism to $A \times \{1\} \to A \times \{1\}$ as follows. Let $x, y \in A$. At the quotient level, we have $(x, 1) \sim (h(x), 0)$ and $(y, 1) \sim (T_C^q h(y), 0)$, thus we define $(x, 1) \mapsto (y, 1)$ if $h(x) = T_C^q h(y)$.

Next we compute the preimage of ∂R to compute the required filling of ∂N in the domain. To be explicit about orientations, pick an orientation for C. Choose the orientation on λ_S so that it is isotopic to C in $N(C)$. The orientation on λ_S determines the orientation on μ. See Figure 18. The only portion of ∂R not mapped by the identity is the arc $b \times \{1\}$. From the formula above, we see the preimage of $b \times \{1\}$ is an arc $a \times \{1\}$ in $A \times \{1\}$ such that $h(a) = T_C^q h(b)$, or equivalently, $a = h^{-1} T_C^q h(b)$. It follows that the preimage of ∂R in M is a $\mu - q\lambda_S = \mu - q\lambda_D$ curve, thus the homeomorphism on $(S \times I) - N$ extends to a homeomorphism defined on all of Y. $\square$
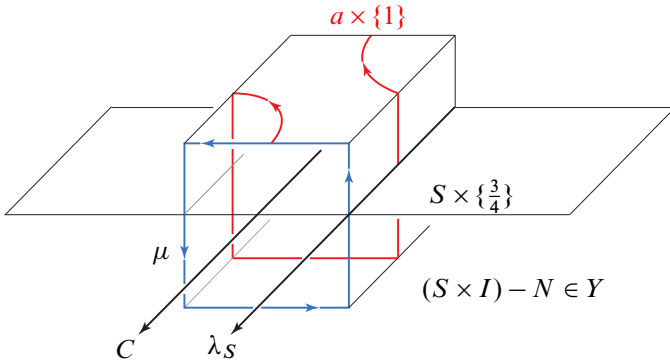
Figure 18

Proposition 5.3 If C bounds an embedded disk D in $M = (S, h)$, $\mathrm{tw}_C(S) \geq 0$, and C is nonseparating in S, then the contact structure determined by (S, h) is overtwisted.

Proof Since C is nonseparating, it is nonisolating. Thus by Honda's Legendrian realization principle [13], we may assume C is Legendrian. The Thurston–Bennequin invariant of $C = \partial D$ is equal to $\mathrm{tw}_C(S)$, thus if $\mathrm{tw}_C(S) = 0$, D is an overtwisted disk. In the remaining cases, C can be isotoped to decrease the Thurston–Bennequin invariant and thereby produce an overtwisted disk.   □

Corollary 5.4 If C bounds an embedded disk D in $M = (S, h)$, $\mathrm{tw}_C(S) = 0$, and C is nonseparating in S, then the contact structure determined by $(S, T_C^q h)$ is overtwisted for any integer q.

Proof The invariant $\mathrm{tw}_C(S)$ has two possible interpretations depending on whether it is computed in $M = (S, h)$ or in $Y = (S, T_C^q h)$.

In M we have $\lambda_S = \lambda_D + \mathrm{tw}_C(S)\mu$. By Proposition 5.2 we have in Y, $\mu' = \mu - q\lambda_D$ and consequently, $\lambda_S = \lambda_D + \mathrm{tw}_C(S)(\mu' + q\lambda_D) = (1 + \mathrm{tw}_C(S)q)\lambda_D + (\mu' \, \mathrm{tw}_C(S))$. When $\mathrm{tw}_C(S) = 0$ in M, $\lambda_S = \lambda_D$. This formula is satisfied on just the boundary of the regular neighborhood of C, thus it holds in Y as well.   □

Example 5.5 Let F be the Seifert surface for the knot 8_{20} as described in Example 2.8. Using the notation of Section 3, form the $(2, 1)$–cable of F, and denote its Seifert surface by $S = F_0 \cup P \cup F_{\frac{1}{2}}$. This gives an open book decomposition (S, H_0') of S^3. To produce the curve C used in the definition of $H = T_C \circ H_0$ consider the curves A and B that live on F as shown in Figure 2.

Pick an arc $a \subset F_0$ that runs from a point of $A \times \{0\}$ to the point where the twisted band is added in the formation of the $(2, 1)$–cable. Choose an arc $b \subset F_{\frac{1}{2}}$ similarly corresponding to $B \times \{\frac{1}{2}\}$. In addition the arcs should be chosen so that A, B, and the projections of a, b to F have no interior intersections. Define C to be the band sum, in S, of $A \times \{0\}$ and $B \times \{\frac{1}{2}\}$ along an arc consisting of a, b, and an arc that runs across the twisted band.

Since A and B are nonseparating in F, C does not separate when restricted to either F_0 or $F_{\frac{1}{2}}$ and Theorem 3.2 applies. Combining Theorem 3.2, Proposition 2.5, Corollary 4.6, and Proposition 5.3 shows that H is pseudo-Anosov, right-veering with fractional Dehn twist coefficient $\frac{1}{2}$, and the associated contact structure is overtwisted.

Theorem 5.6   *There exists an infinite collection of fibred knots in S^3 such that the monodromy is pseudo-Anosov, right-veering with fractional Dehn twist coefficient $\frac{1}{2}$, and the associated contact structure is overtwisted.*

Proof   It is enough, by the construction of Example 5.5, to produce knots with pseudo-Anosov monodromy whose fibers contain two disjoint Hopf bands of opposite signs. One such family of examples are the nontorus fibred 2–bridge knots of genus 2 and higher. The fibers of such knots correspond to words in L and R that use both letters and have even length; see [23] and Gabai and Kazez [9]. The letters correspond to positive and negative Hopf bands plumbed together in a vertical stack. As long as the surface has genus at least 2, the word will have length at least 4. The desired Hopf bands correspond to a nonadjacent pair of letters L and R which necessarily exist. ☐

If Conjecture 4.7 is true, then none of the examples of Theorem 5.6 and Example 5.5 can be destabilized, and thus they would all provide additional counterexamples to the conjecture of [14].

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