$\mathfrak{sl}_3$–foam homology calculations

LUKAS LEWARK

We exhibit a certain infinite family of three-stranded quasi-alternating pretzel knots, which are counterexamples to Lobb's conjecture that the $\mathfrak{sl}_3$–knot concordance invariant s_3 (suitably normalised) should be equal to the Rasmussen invariant s_2. For this family, $|s_3| < |s_2|$. However, we also find other knots for which $|s_3| > |s_2|$. The main tool is an implementation of Morrison and Nieh's algorithm to calculate Khovanov's $\mathfrak{sl}_3$–foam link homology. Our C++ program is fast enough to calculate the integral homology of, eg, the $(6, 5)$–torus knot in six minutes. Furthermore, we propose a potential improvement of the algorithm by gluing sub-tangles in a more flexible way.

57M25; 81R50

1 Introduction

During the last decade, $\mathfrak{sl}_n$–polynomials were categorified one after the other, beginning, of course, with Khovanov's categorification of the Jones polynomial [18] using cobordisms, followed by his categorification of the $\mathfrak{sl}_3$–polynomial [21] using foams, and finally, Khovanov and Rozansky's categorification of the $\mathfrak{sl}_N$–polynomials for arbitrary N [23] and of the Homflypt–polynomial itself [24] (see also Khovanov [22] and Webster [47]) using matrix factorisations.

All these homologies are completely combinatorial in nature – unlike eg the original knot Floer homology – meaning that their definition is in itself a description how to compute them. By hand, this direct way of computation is hardly practicable for any but the smallest knots, and using skein long exact sequences and criteria for thinness is much more efficient (see eg a computation of Mackaay and Vaz [33]). On a computer, however, even the straight-forward method, as implemented with some tweaks in the program KhoHo [42] by Shumakovitch, could already compute the $\mathfrak{sl}_2$–homology of knots with up to circa 20 crossings. Matrix factorisations, on the other hand, are more resistant to efficient treatment by a computer. Nevertheless, Carqueville and Murfet [5] wrote a program that is able to compute $\mathfrak{sl}_N$–homology of links with up to six crossings and $N \approx 18$. Webster's program [46] computes Homflypt-homology of short braids, but is largely limited to 3–stranded ones.

Yet it is desirable to be able to compute the Khovanov–Rozansky homologies of much larger knots, since some phenomena require a certain complexity of the knot to occur; see eg Hedden and Ording's 15–crossing knot, which demonstrates that the Rasmussen invariant s_2 and twice the Floer concordance invariant τ differ [12]. Or to mention an extreme example, see Freedman, Gompf, Morrison and Walker [8] for a link with 222 crossings whose Rasmussen invariant was worthwhile to compute at the time, because it could have provided a counterexample to the four-dimensional smooth Poincaré conjecture. Though this approach was later rebutted by Akbulut [1], the $\mathfrak{sl}_3$–analogue of the Rasmussen invariant, the s_3–invariant, might succeed where s_2 fails.
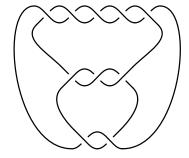
Bar-Natan's extension of $\mathfrak{sl}_2$–homology from link diagrams to tangles [2] (see also Khovanov [19]) led subsequently to his divide-and-conquer algorithm to compute $\mathfrak{sl}_2$–homology [3]. The speed of this algorithm depends primarily on the *girth* of the link diagram: this is the maximal number of intersection points of a horizontal line with the diagram (see eg [7] and Section 3.1 for details). An implementation by Green and Morrison called JavaKh [10] is able to compute the homology of knots of girth up to 14, eg the $(8, 7)$–torus knot. Mackaay and Vaz [32] and Morrison and Nieh [35] then extended $\mathfrak{sl}_3$–homology to tangles, and the latter describe in detail the ensuing algorithm. In this text, we present an implementation of this algorithm as a C++ program called FoamHo [26].

The algorithm can be improved by gluing sub-tangles in a more flexible way, along a *sub-tangle tree* instead of one after the other. This leads to the notion of the *recursive girth* of a link, which replaces the girth as main factor limiting calculation speed. Even though this improvement is not implemented in FoamHo yet, the program is still fast enough to calculate the integral homology, reduced or unreduced, of links with girth up to 10, such as the $(6, 5)$–torus knot. The $\mathfrak{sl}_3$–concordance invariant s_3, as defined by Lobb [31] (see also Wu [48] and Lobb [29]), may (in most cases) be extracted from the $\mathfrak{sl}_3$–homology by means of the spectral sequence converging to the filtered version of homology. This method was used for $\mathfrak{sl}_2$–homology by Freedman, Gompf, Morrison and Walker [8]. It does not depend on the conjectured convergence of the spectral sequence on the second page.

The most striking calculatory result obtained with FoamHo concerns this s_3–invariant, which shares many properties with s_2: eg, they agree on homogeneous and quasi-positive knots, and until now it was not known whether they were actually equal or not (see Lobb [31, Conjectures 1.5, 1.6]).

Conjecture Let s_3 be the $\mathfrak{sl}_3$–concordance invariant, normalised to take the same value as the Rasmussen invariant on the trefoil. If $a > b \geq 3$, $c \geq 2$ and

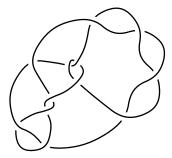$$a + 1 \equiv b + 1 \equiv c \equiv 0 \pmod 2,$$

Figure 1: On the left, the $(5, -3, 2)$–pretzel knot (aka 10_{125}), the smallest knot with $s_3 \neq s_2$; on the right (drawn with knotscape [13]), $12n_{340}$, the smallest knot with $|s_3| > |s_2|$

then the $(a, -b, c)$–pretzel knot $P(a, -b, c)$ has s_3–invariant

$$s_3(P(a, -b, c)) = a - b + \delta_c,$$

where $\delta_2 = -1$ and $\delta_c = -2$ for $c > 2$.

This statement is called a "conjecture" since it is established by `FoamHo`-calculations only for small values of a, b and c. In addition, however, we have a proof for the case $c > 2$ that does not rely on computer calculations [27]. See Section 3.5 for further details.

If one appends $b > c$ to the hypotheses of the conjecture, then the $(a, -b, c)$–pretzel knot is quasi-alternating (see Champanerkar and Kofman [6] and Greene [11]), and hence its s_2–invariant equals its classical knot signature (see Manolescu and Ozsváth [34]): $s_2(P(a, -b, c)) = \sigma(P(a, -b, c)) = a - b$. This value of the signature can be easily computed using Göritz matrices and the formula of Gordon and Litherland [9]. So for this infinite family of pretzel knots, the s_2– and s_3–invariant differ, and the latter gives a weaker bound for the slice genus than the former. However, there are knots for which this is different, eg $s_3(12n_{340}) = 1$ and $s_2(12n_{340}) = 0$.

The rest of the paper is organised as follows: Section 2 provides a brief, but essentially self-contained definition of the $\mathfrak{sl}_3$–foam homology as defined by Khovanov [21], Mackaay and Vaz [32] and Morrison and Nieh [35]. We also define reduced homology in the framework of foams. In Section 3.1, we give an account of the divide-and-conquer algorithm, including the computation of integral homology and the acceleration of the algorithm using a sub-tangle tree. In Section 3.2, we discuss how to extract s_3 from the $\mathfrak{sl}_3$–homology. Section 3.3 addresses some particular implementation issues and their resolution in `FoamHo`, and Section 3.4 presents the usage and characteristics of

the program itself. Finally, Section 3.5 states some results of `FoamHo` calculations, and compares them against previously known results.

Acknowledgements

2   The $\mathfrak{sl}_3$–foam homology of tangle diagrams

This section gives a definition of the $\mathfrak{sl}_3$–polynomial and its categorification, the $\mathfrak{sl}_3$–foam homology. Except for the definition of reduced homology using foams (Section 2.5), this section contains nothing essentially new. We just review the parts of [21; 35] that are relevant to the purpose of this section, which is to provide a self-contained definition of $\mathfrak{sl}_3$–homology with the objective of calculation in mind. Instead of choking tori we use dots, like Khovanov [21], and Mackaay and Vaz [32]. The origin of webs and the $\mathfrak{sl}_N$–polynomials lies in representation theory (Reshetikhin and Turaev [41] and Kuperberg [25]), an aspect we will not dwell on.

2.1   The $\mathfrak{sl}_3$–polynomial, naively

The $\mathfrak{sl}_3$–polynomial can be defined by a single skein relation involving only link diagrams. We will instead use the two skein relations (Sk$^+$) and (Sk$^-$), see below, because this allows a categorification using foams. These skein relations involve webs: a *closed web* is a plane oriented trivalent graph whose every vertex is either a source or a sink, and that may have vertex-less circles as additional edges. Note that the orientation of a single edge of a connected closed web already determines the orientations of all its edges. Thus, in drawings of webs, only the orientations of one edge per component will be specified.

A *tangle diagram* is the generic intersection of a link diagram with a disc; generic means that the disc's boundary intersects the diagram's strands transversally, and does not pass through a crossing. Let us define a map V from the set of smooth isotopy classes of link diagrams to the free $\mathbb{Z}[q^{\pm 1}]$–module on the set of smooth isotopy classes of closed webs. The map V is uniquely determined by the following two local relations,

which are interpreted naively for now (ie, apply these relations to all crossings of the link at once, then expand):

(Sk⁻)
$$V\left(\text{⬡}\right) = q^2 \cdot V\left(\text{⬡}\right) - q^3 \cdot V\left(\text{⬡}\right)$$

(Sk⁺)
$$V\left(\text{⬡}\right) = q^{-2} \cdot V\left(\text{⬡}\right) - q^{-3} \cdot V\left(\text{⬡}\right)$$

Next, we define an evaluation $\langle \cdot \rangle$ of closed webs, called the *Kuperberg bracket* (Kuperberg [25], Jaeger [14]), which associates to a closed web a Laurent polynomial in q. This evaluation is given by the three relations:

(C)
$$\left\langle \text{◯} \right\rangle = \left\langle \text{◯} \right\rangle = (q^{-2} + 1 + q^2) \cdot \left\langle \text{◯} \right\rangle$$

(D)
$$\left\langle \text{◌} \right\rangle = (q^{-1} + q) \cdot \left\langle \text{◌} \right\rangle$$

(S)
$$\left\langle \text{⬡} \right\rangle = \left\langle \text{◌◌} \right\rangle + \left\langle \text{⌣} \right\rangle$$

The $\mathfrak{sl}_3$–polynomial, which associates a Laurent polynomial in q to a link diagram, can now be obtained by composing V with $\langle \cdot \rangle$, and identifying the empty web with 1. Categorifying this construction is going to yield the $\mathfrak{sl}_3$–homology. However, it is advantageous to formalise the *localness* of the relations (Sk$^\pm$, C, D, S) before proceeding.

2.2 Planar algebras

While 2–categories do give a framework for webs and foams, they make sense only if one aims at interpreting webs as maps between oriented 0–manifolds; this aspect is not essential to the calculation of $\mathfrak{sl}_3$–homology, and so we use planar algebras instead. Planar algebras were introduced by Jones [16] to identify subfactors. They were subsequently used to describe locally defined knot invariants such as the Jones polynomial. Bar-Natan [2] introduced a categorified version of planar algebras called *canopolis* to describe Khovanov homology, a method adaptable to $\mathfrak{sl}_3$–homology [35]. We will use a slightly generalised version of planar algebras, working over arbitrary monoidal categories instead of over the category of vector spaces over a fixed field. In this way, a canopolis is a planar algebra as well.

Let $B_0 \subset \mathbb{R}^2$ be a closed disc, and $B_1, \ldots B_n \subset B_0^\circ$ be n pairwisely disjoint smaller closed discs. Punching out these discs yields a disc with holes, $H = B_0 \setminus \bigcup_{i=1}^n B_i^\circ$. Let M be a compact oriented one-dimensional smooth submanifold of H with

$M \cap \partial H = \partial M$; in other words, M is a collection of circles and of intervals whose endpoints lie on the boundary of the big discs or one of the smaller discs. An *input diagram* consists of M, H, and on each boundary component of H a base point that is not in ∂M. We consider input diagrams up to smooth isotopy, in the course of which boundary points of M may not cross the base points.

For every $i \in \{0, \ldots n\}$, the intersection $M \cap B_i$ is a finite set; at each of its points, the corresponding interval of M is either oriented towards the boundary ($+$ for $i > 0$, $-$ for $i = 0$), or away from it ($-$ for $i > 0$, $+$ for $i = 0$). Moreover, these points have a canonical order, given by starting from the base point and walking once around the circle in the counterclockwise direction. Thus the isotopy type of $M \cap B_i$ may be written as a sign-word ε_i, ie, a word over the alphabet $\{+, -\}$. The *boundary labelling* of H is the tuple $(\varepsilon_0, \ldots \varepsilon_n)$.
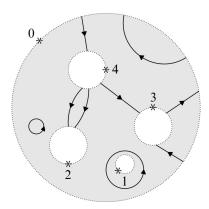

Figure 2: Example of an input diagram (also called spaghetti-and-meatballs diagram) with boundary labelling $(+-+-+, \varnothing, ++, ++-, +---)$

Now suppose (M, H) and (M', H') are two input diagrams, such that $\varepsilon'_0 = \varepsilon_k$ for a fixed $k \in \{1, \ldots n\}$. Then (M', H') may be shrunk and glued into B_k, base point on base point and boundary points on boundary points, resulting in a new input diagram with $n + n' - 1$ holes.

Let I be a subset of the set of all sign-words. Let C be a monoidal category, in the easiest case just the category Set of sets, and in the classical case the category of vector spaces over a fixed field. Then a *planar algebra* $\mathcal{P}$ over I and C consists of the following data:

- For each $\varepsilon \in I$, an object $\mathcal{P}_\varepsilon \in \mathsf{C}$.

- For each input diagram H with boundary labelling $(\varepsilon_0, \ldots \varepsilon_n)$ such that $\varepsilon_i \in I$ for each i, a C–morphism

$$\mathcal{P}_H \colon \bigotimes_{i=1}^{n} \mathcal{P}_{\varepsilon_i} \to \mathcal{P}_{\varepsilon_0}.$$

This data is required to satisfy the following axioms:

- Suppose H is an input diagram with $n = 1$ and $\varepsilon_0 = \varepsilon_1$ that consists only of appropriately oriented radial strands. Then $\mathcal{P}_H \colon \mathcal{P}_{\varepsilon_0} \to \mathcal{P}_{\varepsilon_1}$ is the identity morphism.

- Let H and H' be two input diagrams with boundaries $(\varepsilon_0, \ldots \varepsilon_n)$ and $(\varepsilon'_0, \ldots \varepsilon'_n)$, respectively. Suppose that for a fixed $k \in \{1, \ldots n\}$, $\varepsilon'_0 = \varepsilon_k$. Let H'' be the input diagram obtained from gluing H' into the kth hole of H. Then the morphism $\mathcal{P}_{H''}$ is equal to the composition of the morphisms $\mathcal{P}_H$ and $\mathcal{P}_{H'}$, ie,

$$\mathcal{P}_{H''} = \mathcal{P}_H \circ \left(\mathrm{id}_{\bigotimes_{i=1}^{k-1} \mathcal{P}_{\varepsilon_i}} \otimes \mathcal{P}_{H'} \otimes \mathrm{id}_{\bigotimes_{i=k+1}^{n} \mathcal{P}_{\varepsilon_i}} \right).$$

If $F \colon \mathsf{C} \to \mathsf{C}'$ is a monoidal functor, one may define the planar algebra $F(\mathcal{P})$ over I and C' by defining $F(\mathcal{P})_\varepsilon = F(\mathcal{P}_\varepsilon)$ and

$$F(\mathcal{P})_H \colon \bigotimes_{i=1}^{n} F(\mathcal{P})_{\varepsilon_i} \to F(\mathcal{P})_{\varepsilon_0}$$

to be the composition $F(\mathcal{P}_H) \circ \gamma$, where γ is the natural transformation

$$\bigotimes_{i=1}^{n} F(\mathcal{P}_{\varepsilon_i}) \to F\left(\bigotimes_{i=1}^{n} \mathcal{P}_{\varepsilon_i} \right)$$

which comes with the functor F because it is monoidal. Examples of this construction include, for a planar algebra $\mathcal{P}$ over Set, replacing for all $\varepsilon \in I$ the set $\mathcal{P}_\varepsilon$ by the free R–modules for some ring R by means of applying the left-adjoint of the forgetful functor from the category of R–modules to Set, or the quotient $\mathcal{P}$ by an equivalence relation on $\mathcal{P}$, by which we mean a collection of equivalence relations on all the $\mathcal{P}_\varepsilon$ that respect the planar algebra structure.

Suppose $\mathcal{P}$ and $\mathcal{P}'$ are planar algebras over I, I' and C, C′, respectively, such that $I \subset I'$. A *planar algebra morphism* from $\mathcal{P}$ to $\mathcal{P}'$ consists of a functor $F \colon \mathsf{C}' \to \mathsf{C}$ and an I–indexed collection of C–morphisms $\mathcal{P}_\varepsilon \to F(\mathcal{P}'_\varepsilon)$ that respect the planar algebra structure, ie, commute with the maps $\mathcal{P}_H$ and $\mathcal{P}'_H$. The functor F will typically be a forgetful functor.

Tangle diagrams with a base point on the boundary, considered – as input diagrams – up to smooth isotopy, form a planar algebra $\mathcal{T}$ over Set and the set I_0 of sign-words $\varepsilon_1 \cdots \varepsilon_m$ with $\sum_{j=1}^{m} \varepsilon_j = 0$. Let us elaborate on this example: the planar algebra $\mathcal{T}$ associates to a sign-word ε with an equal number of both signs the (countably infinite) set $\mathcal{T}_\varepsilon$ of all tangles diagrams with boundary ε, modulo smooth isotopy, and to an input diagram H with n holes (see eg Figure 2) a function that maps a tuple of n tangle diagrams with appropriate boundaries to a new, bigger tangle diagram, by gluing each of the n tangle diagrams into the corresponding hole of H. One easily verifies that the planar algebra axioms are satisfied.

2.3 The $\mathfrak{sl}_3$–polynomial in the context of planar algebras

Suppose the unit circle intersects a closed web generically; as in the definition of tangle diagrams, this means that the circle intersects the edges of the closed web transversally, and does not pass through a vertex. Then the intersection of the closed web with the unit disc is called a *web*. As for input diagrams, we fix a base point on the boundary of a web, and encode the isotopy type of the boundary by a sign-word, in which $+$ stands for a strand oriented away from the boundary, and $-$ for a strand oriented towards it. Note that a sign-word $\varepsilon_1 \cdots \varepsilon_m$ is the boundary of some web if and only if $\sum_{j=1}^{m} \varepsilon_j \equiv 0 \pmod 3$. Denote by I_3 the set of all such sign-words. Webs, up to smooth isotopy, form a planar algebra $\mathcal{W}$ over I_3 and Set.

Let $\mathcal{W}_\varepsilon^q$ be the free $\mathbb{Z}[q^{\pm 1}]$–module on $\mathcal{W}_\varepsilon$. Then $\mathcal{W}^q$ forms a planar algebra over I_3 and the category of $\mathbb{Z}[q^{\pm 1}]$–modules. In $\mathcal{W}^q$, we may interpret the relations (C), (D) and (S) as relations on $\mathcal{W}_\varnothing^q$, $\mathcal{W}_{-+}^q$ or $\mathcal{W}_{+-}^q$ and $\mathcal{W}_{-+-+}^q$ or $\mathcal{W}_{+-+-}^q$, respectively. Denote by $\mathcal{W}^{qr}$ the quotient by the generated equivalence relation.

Let W, W' be two webs and $\varphi\colon W \to W'$ a diffeomorphism; just of the webs themselves, not taking into account the ambient discs. We call φ a *web diffeomorphism* if it preserves the order of the boundary points, and the cyclic ordering of edges around vertices. Note that in the quotient $\mathcal{W}^{qr}$, the equivalence class of a web is already determined by its *web diffeomorphism* type. In $\mathcal{W}$, this distinction is slightly coarser than the isotopy type, since eg web diffeomorphisms do not take the orientation and relative position of closed components into account.

The two skein relations (Sk$^\pm$) determine a unique morphism $V\colon \mathcal{T} \to \mathcal{W}^q$ of planar algebras, $\mathcal{T}$ being the planar algebra of tangle diagrams. A link diagram L may be seen as element of $\mathcal{T}_\varnothing$. The equivalence class $[V(L)] \in \mathcal{W}^{qr}$ has a unique member that is a $\mathbb{Z}[q^{\pm 1}]$–multiple of the empty web. The coefficient equals the $\mathfrak{sl}_3$–polynomial of the link diagram. Reidemeister invariance may be shown by proving that the tangle diagrams with two, four and six boundary points corresponding to the Reidemeister moves I, II and III have in each case the same image under V.

2.4 The $\mathfrak{sl}_3$–homology in the context of a special kind of planar algebras: canopolis

To categorify the $\mathfrak{sl}_3$–polynomial, one needs to understand foams, the cobordisms of webs. They should locally look like a web times an interval; thus a foam could vaguely be defined as a topological space whose points have neighbourhoods that look either like $\mathbb{R}^2$, or like the letter Y times an interval. The precise construction goes as follows: suppose that for all $i \in \{1, 2, 3\}$, Σ^i are compact oriented smooth (generally not connected) 2–manifolds that each have m boundary components $S_1^i, \ldots, S_m^i$. Let $\varphi_j \colon S_j^1 \to S_j^2$ and $\psi_j \colon S_j^1 \to S_j^3$ be orientation preserving diffeomorphisms. Consider the quotient of $\Sigma^1 \sqcup \Sigma^2 \sqcup \Sigma^3$ by the equivalence relation generated by $[x] = [\varphi_j(x)] = [\psi_j(x)]$ for all j. The images of the S_j^1 in the quotient are called *singular circles*, and the images of connected components of the Σ^i are called *facets*. There are three facets adjacent to each singular circle. Associate a non-negative integer to each facet. Such an integer d will graphically be represented by drawing d dots on the facet, which may roam the facet freely, but may not cross a singular circle. Such a quotient, together with the dots and with a choice of cyclic ordering of the three facets around each singular circle is called a *prefoam*.

Now consider a smooth embedding of a prefoam into $\mathbb{R}^3$, ie, an embedding that is smooth on the facets and on the singular circles. Such an embedding induces cyclic orderings of the facets around each singular circle by the left-hand rule (see Figure 3). If these cyclic orderings agree with those given by the prefoam, the image of the embedding is called a *closed foam*.
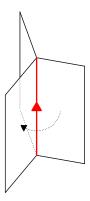

Figure 3: Cyclic ordering of facets around a singular circle of a closed foam. Singular circles are drawn as a thick red line.

Instead of directly defining foams with a boundary, we construct them as the intersection of a closed foam with a *can*, ie, the cylinder

$$B \times [0, 1] = \{(x, y, z) \mid x^2 + y^2 \le 1 \text{ and } 0 \le z \le 1\}.$$

This construction is sufficiently general, since we do not need to consider foams with a boundary as abstract, but merely as embedded objects. Taking the intersection with a cylinder instead of with a sphere separates the boundary into two parts (the intersection with the cylinder's side does not carry information), and the foam is understood as a map from the lower to the upper part. Let us state the conditions under which the intersection of a closed foam with the cylinder B is called a *foam*:

- The boundary of the cylinder intersects the facets and singular circles of the closed foam transversally.

- The side $(\partial B) \times [0, 1]$ of the cylinder intersects the closed foam in finitely many vertical lines, and is disjoint from all singular circles.

- The intersections with the top and bottom of the cylinder are webs.

- The base point of the top and the base point of the bottom web have the same $x-$ and $y-$coordinates.
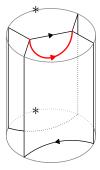
See Figure 4 for an example.

Figure 4: A singular saddle is the intersection of a closed theta foam (not drawn) with a can (drawn grey). It has three facets, one singular edge and degree 1. Orientations and base-points of the domain and codomain are specified.

We consider foams up to isotopies that, on the side of the cylinder, do not depend on the $z-$coordinate. A connected component of the intersection of a singular circle with the cylinder is called a *singular edge*. The web on the bottom of the cylinder is called *domain* of the foam, and the *codomain* of the foam is defined as the web on the top of the cylinder, with the orientation of each edge reversed. As usual, let χ denote the

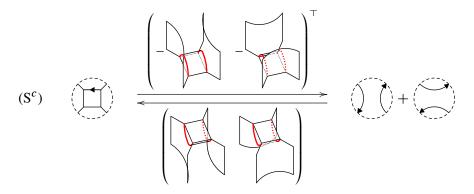Euler characteristic. Then the *degree* of a foam f is defined by

$$\deg f =$$
$$\chi(\text{domain of } f) + \chi(\text{codomain of } f) + 2(\text{total number of dots on } f) - 2\chi(f).$$

Foams can be glued in two ways: if the domain of one foam agrees with the codomain of another, they can be glued by stacking them on top of each other. Or, they can be glued into the cylindrical holes of a thickened input diagram. The degree is additive with respect to both of these operations.

Webs with a fixed boundary and the foams between them thus constitute a *graded category*, ie, a category whose morphisms have an integral rank that is additive under composition. Let us define a planar algebra $\mathcal{W}^c$ over I_3 [1] and the category GCat of small graded categories: to $\varepsilon \in I_3$, associate the category whose set of objects is $\mathcal{W}_\varepsilon$, and whose morphisms $W \to W'$ between two webs $W, W' \in \mathcal{W}_\varepsilon$ are the foams with domain W and codomain W'. If H is a planar input diagram, then $\mathcal{W}_H^c$ is the functor that acts as $\mathcal{W}_H$ on the objects, and glues foams into a thickened version of H.

Next, $\mathcal{W}^{cq}$ may be constructed by applying a functor from GCat to ACat, the category of small additive categories: replace webs by $\mathbb{Z}[q^{\pm 1}]$–linear combinations of webs, and foams by matrices of $\mathbb{Z}$–linear combination of foams, where morphisms from $q^\alpha \cdot W$ to $q^\beta \cdot W'$ are the foams with degree $\alpha - \beta$. So the categories $\mathcal{W}_\varepsilon^{cq}$ are not graded, but have instead a shift operator for their objects. In this planar algebra, consider the following morphisms:

(S^c)

Demanding that the first morphism be zero, and the following three pairs of morphisms be mutually inverse, for any placement of the base point, generates an equivalence relation on the planar algebra $\mathcal{W}^{cq}$. Let $\mathcal{W}^{cqr}$ be the quotient of $\mathcal{W}^{cq}$ by this equivalence relation.

Notice that the relations (T^c), (C^c), (D^c), (S^c) are not those of Khovanov's original definition [21]; rather, Khovanov builds on the TQFT given by the Frobenius algebra $\mathbb{Z}[X]/(X^3)$, and defines the evaluation of foams by fixing the evaluation of the closed theta foam, and then using the universal BHMV–construction (Blanchet, Habegger, Masbaum and Vogel [4]). From this, the above relations follow. Since this text's aim is to give a self-contained definition of $\mathfrak{sl}_3$–homology focusing on calculations, these relations are used as the definition, not as a corollary of the definition. See [35, Lemma 3.5] for a proof that they actually are sufficient.

Let us consider some examples of $\mathcal{W}^{cqr}_{\varepsilon}$. For $\varepsilon = \varnothing$, the isomorphism classes of objects of this category are in correspondence with the elements of the free $\mathbb{Z}[q^{\pm 1}]$–module generated by the empty web $\varnothing$. Non-zero morphisms $q^\alpha \cdot \varnothing \to q^\beta \cdot \varnothing$ exist only if $\alpha = \beta$, and in this case the morphism $\mathbb{Z}$–module is just $\mathbb{Z}$. The $\mathbb{Z}[q^{\pm 1}]$–module Khovanov [21] associates to a closed web W can be recovered as

$$\bigoplus_{\alpha \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{W}^{cqr}_\varnothing}(q^0 \cdot \varnothing, q^\alpha \cdot W).$$

Similarly, the isomorphism classes of objects of $\mathcal{W}^{cqr}_{+-}$ are in correspondence with the elements of the free $\mathbb{Z}[q^{\pm 1}]$–module generated by the web

which is just an interval. The morphisms are more complicated, however, since there are non-zero foams of three different degrees: rectangles with zero, one or two dots.

So the morphism $\mathbb{Z}$–module from

$$q^{\alpha} \cdot \bigcirc\!\!\!*\!\!\!\longrightarrow q^{\beta} \cdot \bigcirc\!\!\!*$$

is $\mathbb{Z}$ if $\alpha - \beta \in \{0, 2, 4\}$, and trivial otherwise.

Let a diffeomorphism between two foams be called a *foam diffeomorphism* if its restriction to the top (bottom) of the cylinder constitutes a web diffeomorphism between the domains (codomains) of the two foams. In $\mathcal{W}^{cqr}$, the equivalence class of a foam is already determined by its foam diffeomorphism type; this is not the case in $\mathcal{W}^{cq}$, where, eg, a cylindrical foam tied into a knot is not the identity of the circular web.

Finally, let $\mathcal{W}^{cqrt}$ be the planar algebra obtained from $\mathcal{W}^{cqr}$ by setting $\mathcal{W}^{cqrt}_{\varepsilon}$ to be the category of bounded chain complexes (up to chain homotopy equivalence) over $\mathcal{W}^{cqr}_{\varepsilon}$. More formally, this means applying a monoidal functor $K\colon \mathsf{ACat} \to \mathsf{ACat}$. The natural transformation $K(C_1) \otimes K(C_2) \to K(C_1 \otimes C_2)$ is given by

$$(*) \quad (P_i, g_i)_i \otimes (Q_j, h_j)_j \mapsto \Big(\bigoplus_{i+j=k} P_i \otimes Q_j, \sum_{i+j=k} g_i \otimes \mathrm{id}_{Q_j} + (-1)^i \, \mathrm{id}_{P_i} \otimes h_j \Big)_k.$$

In the notation of complexes, the module at t–degree 0 is underlined. We may now define the $\mathfrak{sl}_3$–chain complex as the planar algebra morphism $V^c\colon \mathcal{T} \to \mathcal{W}^{cqrt}$ uniquely determined by the skein relations:

$$(\mathrm{Sk}^{-c}) \qquad V^c\left(\bigotimes\right) = \underline{q^2 \cdot \bigcirc} \xrightarrow{} q^3 \cdot \bigcirc$$

$$(\mathrm{Sk}^{+c}) \qquad V^c\left(\bigotimes\right) = q^{-3} \cdot \bigcirc \xrightarrow{} \underline{q^{-2} \cdot \bigcirc}$$

Note that these relations each just give the complex of one tangle diagram, and no cone or flattening is involved. The minus-sign that must intervene to arrange anti-commutativity of differentials along squares is hidden in $(*)$. Although the placement of this sign is somewhat arbitrary, V^c is still unique, since chain complexes are considered only up to homotopy equivalence.

Reidemeister invariance may, just as for the $\mathfrak{sl}_3$–polynomial, be shown by inspecting the small tangle diagrams corresponding to Reidemeister moves [35, section 4.2]. So the $\mathfrak{sl}_3$–chain complex is, up to chain homotopy equivalence, an invariant of tangles.

A link diagram lives in $\mathcal{T}_\varnothing$, and is mapped to a chain complex in the category of $\mathbb{Z}[q^{\pm 1}]$–linear combination of closed webs and matrices of $\mathbb{Z}$–linear combination of foams between them. Due to the relations, there is a homotopy to a chain complex which contains only empty webs and closed foams. The latter are in turn just integral multiples of the empty foam. Setting the empty foam to 1, a chain complex in the category of free graded abelian groups is obtained. Its homology is the $\mathfrak{sl}_3$–*homology* of the link.

2.5 Reduced $\mathfrak{sl}_3$–homology

A *reduced* version of $\mathfrak{sl}_N$–homology has been introduced by Khovanov and Rozansky [23] in the context of matrix factorisations. This section contains a definition in the context of foams.
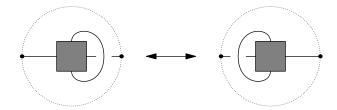

Figure 5: The effect on the corresponding $+--$tangle diagram of sliding the base-point through a crossing

Let D be a diagram of a link L with a base-point. Cutting D open at the base-point, one obtains a tangle diagram D' with boundary $+-$. Let us consider how D' changes if the base-point is slid along the component of the link on which it lies. As long as the base-point does not pass a crossing, the isotopy type of D' clearly does not change. On passing a crossing, D' changes in the way depicted in Figure 5: the strand that does not contain the base-point is pulled from one side of D' to the other, over the rest of the link. This pulling-over of the strand can of course be realised as a sequence of Reidemeister moves. Therefore, we may equivalently consider links with a marked component instead of a base-point. Two tangle diagrams with boundary $+-$ represent the same link with a marked component if and only if they are connected by a finite sequence of Reidemeister moves: because, if a Reidemeister move of a link diagram happens away from the base-point, it directly translates to a Reidemeister move of the corresponding $+--$tangle. Otherwise, the base-point may be slid away prior to the Reidemeister move.

So, the homotopy type of the $\mathfrak{sl}_3$–chain complex C of D' is an invariant of links with a marked component; in particular, it is a knot invariant. As noted in Section 2.4, if

C is fully simplified, it contains only one kind of web – an interval – and three kinds of foams: the identity foam of the interval with zero, one or two dots. Now map the identity foam to one, and foams of higher degree to zero. Thus one obtains a chain complex of free graded abelian groups. Its homology is the reduced $\mathfrak{sl}_3$–homology of L, an invariant of links with a marked component. As simple examples show, its value may indeed depend on the choice of marked component, and neither determines the unreduced version, nor is it determined by it (see Section 3.5).

3 Calculations

3.1 The algorithm

The above definition of $\mathfrak{sl}_3$–homology gives a straightforward way of practical calculation: for an n–crossing diagram, take n copies of the chain complex of $\mathrm{Sk}^{\pm c}$, and take the tensor product ($*$) of these complexes as determined by the diagram. This is the same as forming the *cube of resolutions*. Transform this chain complex of closed webs and foams between them into a homotopy equivalent chain complex of empty webs and closed foams using the three relations (C^c), (D^c), (S^c); finally, evaluate the closed foams, and calculate the homology of the emerging integral chain complex.

In the algorithm described by Morrison and Nieh [35], it is just the order in which these steps are taken that is changed: we do not apply $(\mathrm{Sk}^{\pm c})$ to all crossings at once, but to one after the other, and try to simplify the chain complex at each step as much as possible. Examples for the manual application of this algorithm can be found in [35].

At each step, one manipulates a chain complex of tangles; if these tangles have less boundary points, then there are fewer different tangles and cobordisms between them and thus the calculations demand less memory and will, heuristically, go faster. Therefore, one glues the crossings in such an order that the cardinal of the boundary of the intermediate tangles is minimised (or at least, as low as one sees possible). This minimum is precisely the *girth* of the link diagram, which is thus the main factor limiting the speed of the algorithm.

In this text, we propose a variation of this algorithm based on sub-tangle trees, which is potentially faster, because intermediate tangles will have smaller boundary.

Definition 3.1 Let D be a non-split link diagram with crossings enumerated from 1 to n. Decompose D into small tangle diagrams D_i containing the ith crossing, respectively. A *sub-tangle tree* of D is a full binary tree with a tangle diagram at each node, such that:

- The leaves are decorated by the D_i.
- The root is decorated by D.
- Every node that is not a leaf has two children decorated with adjacent tangle diagrams, and is decorated itself with the union of those two tangle diagrams.

Let the girth of a sub-tangle tree be the maximum number of boundary points of the tangle diagrams at all nodes. The *recursive girth* of D is the minimum of the girths of all its sub-tangle trees.
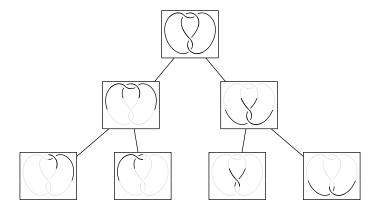

Figure 6: Example of a sub-tangle tree of the figure-eight-knot with girth 4

In general, the recursive girth of a diagram is smaller than its girth. For example, pretzel links have recursive girth 4 and girth 6, and the 222–crossing link considered in [8] has girth ≈ 24 and recursive girth at most 16.

Let us give a description of the algorithm. Given a link diagram D, find a sub-tangle tree of D with small girth. The algorithm consists of simplifying the $\mathfrak{sl}_3$–chain complex of the tangle diagram at each node, one after the other. In the final step, the chain complex of the diagram at the root, which is D itself, is handled. After simplifications, this will be a chain complex of vectors of q–shifted empty webs and matrices whose entries are integer multiples of the empty foam; identifying the empty foam with 1, this becomes a complex of free graded abelian groups, and its homology may be calculated – separately for each q–degree – using the Smith normal form.

In the beginning, only the complexes at the leaves are known, which are given by the relations (Sk$^{\pm c}$). During each step, fix a node whose two children have complexes C_1 and C_2 that are already known and simplified. Now, decorate this node with the tensor product $C_1 \otimes C_2$ (see $(*)$) and simplify this chain complex as described below. After as many steps as D has crossings, the process reaches the root and is finished.

There are two ways to simplify a chain complex: firstly, apply the circle, digon or square relation wherever possible. Secondly, apply Gaußian elimination to all matrix entries that are plus or minus an invertible foam:

Lemma 3.2 (Gaußian elimination [3]) *Over an additive category with an isomorphism h, the chain complex*

$$\cdots \longrightarrow P \xrightarrow{\begin{pmatrix} * \\ g \end{pmatrix}} Q \oplus R \xrightarrow{\begin{pmatrix} h & i \\ j & k \end{pmatrix}} S \oplus T \xrightarrow{\begin{pmatrix} * & \ell \end{pmatrix}} U \longrightarrow \cdots$$

is homotopy equivalent to

$$\cdots \longrightarrow P \xrightarrow{g} R \xrightarrow{k - j \circ h^{-1} \circ i} T \xrightarrow{\ell} U \longrightarrow \cdots.$$

The additive category in question is $\mathcal{W}_\varepsilon^{cqr}$, whose objects are $R[q^{\pm 1}]$–linear combinations of webs with boundary ε, and whose morphisms are matrices of R–linear combinations of foams. The base ring R is usually $\mathbb{Z}$, but one may also just calculate homology over some field. Gaußian elimination may then be applied more often, namely to all non-zero multiples of invertible foams instead of just plus or minus an invertible foam. This may increase the algorithm's speed.

As a rule, use the circle/digon/square-relations only when it is not possible to apply Gaußian elimination. This is to prevent those relations from altering a foam that is plus or minus the identity and could thus be removed.

3.2 Extracting the s_3–concordance invariant from homology

There is a spectral sequence $E_\bullet$ from the graded $\mathfrak{sl}_N$–homology converging to a the filtered version. For $N = 2$, Freedman, Gompf, Morrison and Walker [8, Section 5.2] show how this allows the extraction of the Rasmussen invariant from Khovanov homology. We strengthen their method and adapt it to $N = 3$.

The filtered $\mathfrak{sl}_3$–homology has Poincaré polynomial $q^{-2s_3}(q^{-2} + 1 + q^2)$. The differential on the kth page of the spectral sequence has (t, q)–degree $(1, -6k)$: this is because the differential of the filtered $\mathfrak{sl}_3$–complex preserves the degree modulo 6. Let KR_3 be the Poincaré polynomial of the graded $\mathfrak{sl}_3$–homology of a fixed knot. Then $E_\bullet$ gives rise to a decomposition of the form

$$KR_3(t, q) = q^{-2s_3}(q^{-2} + 1 + q^2) + \sum_{k=1}^{\infty} \zeta_k(t, q) \cdot (1 + tq^{6k}),$$

for some Laurent polynomials $\zeta_k(t,q)$ with non-negative coefficients. Reversely, given the value of KR_3, one may easily determine all possible such decompositions. The conjecture that $E_\bullet$ converges on the second page translates as $\zeta_k(t,q) = 0$ for all $k \geq 2$. Assuming the conjecture is true, the above decomposition is unique; otherwise, however, it is generally not. But even if there are several possible decompositions, it may happen they all share the same value for s_3. This need not be the case, either, as the example $KR_3(t,q) = 1 + q^2 + q^4 + q^6 + tq^{12}$ demonstrates: either $s_3 = -1$, $\zeta_1(t,q) = q^6$, $\zeta_{\neq 1}(t,q) = 0$ or $s_3 = -2$, $\zeta_2(t,q) = 1$, $\zeta_{\neq 2}(t,q) = 0$. The first knots for which s_3 cannot be uniquely determined from $\mathfrak{sl}_3$–homology are $12n_{118}$, $12n_{210}$, $12n_{214}$ and $12n_{318}$ (see Section 3.5 for further examples). At any rate, one obtains a list of possible values for s_3, and for most small knots, only one value is possible.

3.3 Implementation issues

An implementation of the algorithm of Section 3.1 is possible because webs and foams need only be considered up to web and foam diffeomorphisms (see Sections 2.3 and 2.4), and their diffeomorphism type contains only finitely much information. A web is determined by the following data:

- Its boundary, given as a sign-word.
- The number of edges that are circles.
- For each vertex, the triple of the vertices or boundary points it is connected to, listed in counterclockwise order.

Likewise, a foam is completely encoded by the following information:

- Its domain and codomain web.
- The start and end point of every singular edge that is an interval.
- For each singular edge, the three facets adjacent to it, in the order specified by the left-hand rule (see Figure 3).
- The genus of and number of dots on each facet.
- Each boundary component of each facet, given as an ordered tuple of edges; in this tuple, edges of the domain and codomain web, and singular edges alternate.

In order to reduce the complexity of foams, `FoamHo` applies the well-known relations shown in Figure 7 whenever possible. In particular, these relations are sufficient to evaluate closed foams.

Gaußian elimination is in fact only applied to plus or minus an identity foam; there are other isomorphisms than those, but they are difficult to detect for a computer program, and so rare that looking for them does not seem worthwhile.
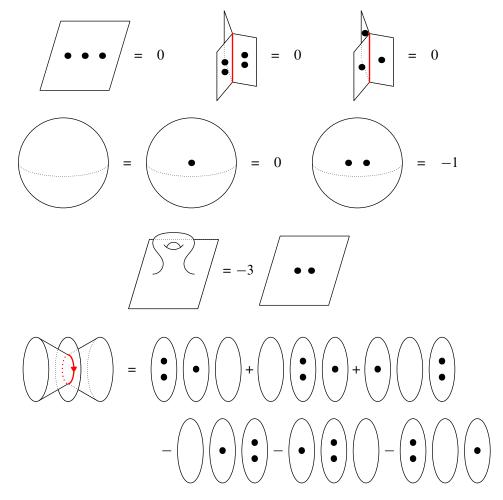
Figure 7: Foam relations

3.4 `FoamHo`, a $\mathfrak{sl}_3$–calculator

The algorithm described in the previous sections was implemented by the author as a C++ program.[2] It computes the integral or rational (homology over finite fields is not yet implemented), reduced or unreduced homology of knot or link diagrams given in braid, planar diagram or Dowker–Thistlethwaite notation; it also attempts to find the value of the s_3–concordance invariant, using the spectral sequence from $\mathfrak{sl}_3$–homology to filtered $\mathfrak{sl}_3$–homology. If it is impossible to extract the s_3–invariant, a list of possible

---

[2]The PARI/GP-library [38] was used to calculate the Smith normal form, and the MPIR library [36] was used for arbitrary precision integers and rationals.

values is printed, and the value which corresponds to the convergence of the spectral sequence on the second page is highlighted.

The current version does not make use of the sub-tangle trees yet, and glues instead one crossing after the other. Apart from that, there is surely some room for rendering the program faster and less memory-hungry by optimising the code, without changing the algorithm; eg, webs and foams are encoded in a redundant way, and tightening this would reduce the memory consumption.

The program was baptised `FoamHo` in recognition of Shumakovitch's `KhoHo` [42]. It has been released under the GPL,[3] and its source code as well as compiled versions for Linux and Windows may be downloaded from [26]. There are also tables of the homology and the s_3–concordance invariant of small knots and links available.

Let us give a rough idea of the capabilities of the program for large knots. On an AMD Opteron (2.2 GHz), the (6,5)–torus knot's homology can be calculated in six minutes, using 80 MB of RAM, the (8,5)–torus knot takes 50 minutes and 275 MB of RAM, but the (7,6)–torus knot is out of the reach of 5 GB of RAM. In other words, links of girth 10 can be calculated until circa 40 crossings, while links of girth 12 would demand a well-equipped computer.

Detailed usage instructions for `FoamHo` can be found in the README file that is distributed along with the program's source code and binaries. The following appetising example session demonstrates the computation of the $(4, 3)$–torus knot's homology. User input begins with a dollar sign and is printed bold:

```
$ ./foamho -h

foamho, a sl3-homology calculator, version 1.1.

Usage: foamho [OPTIONS] braid | pd | dt NOTATION

For example, the following three commands all compute the figure-8-knot's
integral homology:

                foamho braid aBaB
                foamho pd "[[4,2,5,1],[8,6,1,5],[6,3,7,4],[2,7,3,8]]
                foamho dt "[4,6,8,2]"

Options:
-q              Compute rational homology instead of integral.
-r              Compute reduced homology instead of unreduced. You
may give
                a number right after -r to indicate the marked strand
```

---

[3]See http://www.gnu.org/licenses/gpl.html.

```
                    (useful for links).

-g                  Do not attempt to optimise the girth.
-v                  Display some progress information.
-vv                 Display more detailed progress information.
-t                  Display time and memory consumption.
-h                  Display this help message and exit.


Written in 2012/2013 by Lukas Lewark, lewark@math.jussieu.fr.
All feedback is welcome.

$ ./foamho -t braid abababab

Girth-optimised link diagram (modified pd notation) [[2,4,3,1],[5,7,6,4],
    [6,9,8,3],[7,11,10,9],[10,13,12,8],[11,15,14,13],[14,16,1,12],[15,5,2,16]].
Girth: 6.
Calculating...
Done. Result:
Rational homology: (q^-14 + q^-12 + q^-10) + t^2(q^-16 + q^-14) +
    t^3(q^-22 + q^-20) + t^4(q^-20 + 2q^-18 + q^-16) +
    t^5(q^-26 + 2q^-24 + q^-22) + t^6(q^-22) + t^7(q^-28 + q^-26)
    + t^8(q^-32)
Total rank: 19
Rational homology is not self-dual => the link is chiral.
3-torsion: t^3(q^-18) + t^5(q^-22 + q^-20) + t^7(q^-26 + q^-24) +
    t^8(q^-30 + q^-28)
The 3-torsion part of homology is not self-dual => the link is chiral.
s_3-concordance invariant: -12
Run time in seconds: 2
Memory consumption in megabytes: 9.5
```

3.5 Results of calculations

Pretzel knots Since pretzel knots have girth only 6, their homology can be computed particularly quickly. FoamHo calculations agree with the conjecture (see Section 1) for $b + 2 = a \leq 69$, $c = 4$ and $b < a \leq 55$, $c = 2$.

Building on these results, one may easily use the sharper slice Bennequin inequality (Lobb [30], Kawamura [17]) and Livingston's crossing-switch inequality [28, Corollary 3], to confirm the conjecture for the infinite family of a, b, c satisfying $b < a \leq 69$ and $c \geq 4$.

Furthermore, we are able to prove the conjecture for $c > 2$, and show $\delta_2 \in \{-1, -2\}$, without relying on computer calculations. That proof makes use of the spectral sequences developed by Rasmussen [39] and involves bounds on the $\mathfrak{sl}_N$–concordance invariants of a knot which are induced by its reduced triply graded Homflypt-homology.

For details, see the upcoming paper [27].

s_3 and s_2 for small knots   Of the $59\,937$ prime knots with 14 crossings or less, there are 361 for which the s_3 and s_2 concordance invariants differ, and for 63 the absolute value of s_3 is greater, ie, the lower bound to the slice genus coming from $\mathfrak{sl}_3$ is better than the one coming from $\mathfrak{sl}_2$. For all these 361 knots, the difference between s_2 and s_3 is ± 1. A 16–crossing prime knot example for a greater difference is given by the conjecture (see introduction): the $(7, -5, 4)$–pretzel knot, with $s_3 = 0$ and $s_2 = 2$.

The s_3–concordance invariant cannot be determined for certain knots with 14 crossings or less, so the statistics in the preceding paragraph are based on the assumption that the spectral sequence to the filtered version of homology converges on the second page.

s_3 and the Floer-concordance invariant τ   Let τ be the knot concordance invariant coming from Floer homology, as defined by Ozsváth and Szabó [37]. Hedden and Ording [12] found examples of knots K for which $2 = s_2(K) \neq 2\tau(K) = 0$. `FoamHo` calculations yield the following results for these knots:

$$s_3(D_+(T_{2,3}, 2)) = 2 \qquad s_3(D_+(T_{2,7}, 8)) \in \{2, 3, 4\}$$
$$s_3(D_+(T_{2,5}, 5)) \in \{2, 3\} \qquad s_3(D_+(T_{2,7}, 7)) \in \{2, 3, 4, 5\}$$
$$s_3(D_+(T_{2,5}, 4)) \in \{2, 3, 4\} \qquad s_3(D_+(T_{2,7}, 6)) \in \{2, 3, 4, 5, 6\}$$

All but the first knot are examples of how the method of Section 3.2 may fail to completely determine the value of s_3. However, since all of the above knots have slice genus equal to 1, they all have an s_3–invariant equal to 2. So these knots do not give examples for $s_2 \neq s_3$, but they demonstrate that $s_3 \neq 2\tau$.

Thinness   Khovanov [20] called a knot *H–thin* if its unreduced Khovanov homology is supported in only two diagonals, or, equivalently, if its reduced Khovanov homology is supported in only one. Thinness was generalised to $\mathfrak{sl}_N$–homologies by Rasmussen [40]. Alternating knots are H–thin, but not necessarily N–*thin* for $N > 2$. We find the knots $11a_{263}, 12a_{36}, 12a_{694}, 12a_{804}, 12a_{811}, 12a_{817}, 12a_{829}, 12a_{832}$ to be the smallest examples of alternating knots that are not 3–thin. None of these knots is two-bridge, which agrees with Rasmussen's theorem [39] that two-bridge knots are N–thin for all N.

Mutation   Integral reduced $\mathfrak{sl}_3$–homology has been proven to be invariant under mutation of knots by Jaeger [15]; for unreduced homology, invariance appears to be an open question. Our calculations confirm that all mutant families up to 13 crossings have the same reduced and unreduced integral $\mathfrak{sl}_3$–homology. We use the lists provided by Stoimenow [45].

Torsion   All prime knots and links for which the homology was computed, including knots with up to 12 and links with up to 11 crossings, have 3–torsion, with the exception of the Hopf links, whose homology is torsion-free. This is reminiscent of the omnipresence of 2–torsion in Khovanov homology remarked by Shumakovitch [43]. Exemplary calculations also show the existence of 2–, 4–, 5– and 8–torsion, which are rather scarce. Small knots have torsion-free reduced homology, but large enough knots like the $(8, 5)$–torus knot have not.

Reduced and unreduced homology   Reduced $\mathfrak{sl}_2$–homology was conjectured by Khovanov [20] to be determined by its unreduced counterpart; more precisely, that the rank of reduced homology were one less than the rank of unreduced homology. This is true for small knots, but Shumakovitch produced 15–crossing counterexamples [44]. For $\mathfrak{sl}_3$–homology, even considering only knots with crossing number six is enough to see that there is no linear relationship between the ranks of reduced and unreduced homology.

Previous computations   `FoamHo` calculations are in agreement with the results of Carqueville and Murfet [5], who compute reduced and unreduced rational $\mathfrak{sl}_3$–homology of all prime knots and links with up to six crossings.

References

[1]   **S Akbulut**, *Cappell–Shaneson homotopy spheres are standard*, Ann. of Math. 171 (2010) 2171–2175   MR2680408

[2]   **D Bar-Natan**, *Khovanov's homology for tangles and cobordisms*, Geom. Topol. 9 (2005) 1443–1499   MR2174270

[3]   **D Bar-Natan**, *Fast Khovanov homology computations*, J. Knot Theory Ramifications 16 (2007) 243–255   MR2320156

[4]   **C Blanchet**, **N Habegger**, **G Masbaum**, **P Vogel**, *Topological quantum field theories derived from the Kauffman bracket*, Topology 34 (1995) 883–927   MR1362791

[5]   **N Carqueville**, **D Murfet**, *Computing Khovanov–Rozansky homology and defect fusion* `arXiv:1108.1081`

[6]   **A Champanerkar**, **I Kofman**, *Twisting quasi-alternating links*, Proc. Amer. Math. Soc. 137 (2009) 2451–2458   MR2495282

[7]   **M Freedman**, *Complexity classes as mathematical axioms*, Ann. of Math. 170 (2009) 995–1002   MR2552117

[8]   **M Freedman**, **R Gompf**, **S Morrison**, **K Walker**, *Man and machine thinking about the smooth* 4–*dimensional Poincaré conjecture*, Quantum Topol. 1 (2010) 171–208   MR2657647

[9]   **C M Gordon**, **R A Litherland**, *On the signature of a link*, Invent. Math. 47 (1978) 53–69   MR0500905

[10] **J Green**, **S Morrison**, *JavaKh* (2005) Available at `katlas.math.toronto.edu`

[11] **J Greene**, *Homologically thin, non-quasi-alternating links*, Math. Res. Lett. 17 (2010) 39–49   MR2592726

[12] **M Hedden**, **P Ording**, *The Ozsváth–Szabó and Rasmussen concordance invariants are not equal*, Amer. J. Math. 130 (2008) 441–453   MR2405163

[13] **J Hoste**, **M Thistlethwaite**, *Knotscape* (1999) version 1.01 Available at `http://www.math.utk.edu/~morwen/knotscape.html`

[14] **F Jaeger**, *A new invariant of plane bipartite cubic graphs*, Discrete Math. 101 (1992) 149–164   MR1172374

[15] **T C Jaeger**, *Khovanov–Rozansky homology and Conway mutation* (2011) `arXiv:1101.3302`

[16] **V F R Jones**, *Planar algebras, I* (1998) `arXiv:math/9909027`

[17] **T Kawamura**, *The Rasmussen invariants and the sharper slice-Bennequin inequality on knots*, Topology 46 (2007) 29–38   MR2288725

[18] **M Khovanov**, *A categorification of the Jones polynomial*, Duke Math. J. 101 (2000) 359–426

[19] **M Khovanov**, *A functor-valued invariant of tangles*, Algebr. Geom. Topol. 2 (2002) 665–741   MR1928174

[20] **M Khovanov**, *Patterns in knot cohomology, I*, Experiment. Math. 12 (2003) 365–374   MR2034399

[21] **M Khovanov**, sl(3) *link homology*, Algebr. Geom. Topol. 4 (2004) 1045–1081

[22] **M Khovanov**, *Triply-graded link homology and Hochschild homology of Soergel bimodules*, Internat. J. Math. 18 (2007) 869–885   MR2339573

[23] **M Khovanov**, **L Rozansky**, *Matrix factorizations and link homology*, Fund. Math. 199 (2008) 1–91   MR2391017

[24] **M Khovanov**, **L Rozansky**, *Matrix factorizations and link homology, II*, Geom. Topol. 12 (2008) 1387–1425   MR2421131

[25] **G Kuperberg**, *Spiders for rank* 2 *Lie algebras*, Comm. Math. Phys. 180 (1996) 109–151   MR1403861

[26] **L Lewark**, *FoamHo* (2012) Available at `maths.dur.ac.uk/~vxhn54/foamho.html`

[27] **L Lewark**, *Rasmussen's spectral sequences and the $\mathfrak{sl}_n$–concordance invariants*, in preparation (2013)

[28] **C Livingston**, *Computations of the Ozsváth–Szabó knot concordance invariant*, Geom. Topol. 8 (2004) 735–742   MR2057779

[29] **A Lobb**, *A slice genus lower bound from* sl(n) *Khovanov–Rozansky homology*, Adv. Math. 222 (2009) 1220–1276   MR2554935

[30] **A Lobb**, *Computable bounds for Rasmussen's concordance invariant*, Compos. Math. 147 (2011) 661–668   MR2776617

[31] **A Lobb**, *A note on Gornik's perturbation of Khovanov–Rozansky homology*, Algebr. Geom. Topol. 12 (2012) 293–305   MR2916277

[32] **M Mackaay**, **P Vaz**, *The universal* sl$_3$–*link homology*, Algebr. Geom. Topol. 7 (2007) 1135–1169   MR2336253

[33] **M Mackaay**, **P Vaz**, *The reduced HOMFLY-PT homology for the Conway and the Kinoshita–Terasaka knots* (2008)   arXiv:0812.1957

[34] **C Manolescu**, **P Ozsváth**, *On the Khovanov and knot Floer homologies of quasi-alternating links*, from: "Proceedings of Gökova Geometry-Topology Conference 2007", (S Akbulut, T Önder, R J Stern, editors), GGT Conference, Gökova (2008) 60–81   MR2509750

[35] **S Morrison**, **A Nieh**, *On Khovanov's cobordism theory for* $\mathfrak{su}_3$ *knot homology*, J. Knot Theory Ramifications 17 (2008) 1121–1173   MR2457839

[36] *MPIR, version 11.0.0* (2012) Available at   http://www.mpir.org/

[37] **P Ozsváth**, **Z Szabó**, *Heegaard Floer homology and alternating knots*, Geom. Topol. 7 (2003) 225–254   MR1988285

[38] *PARI/GP, version 2.5.3* (2012) Available at   http://pari.math.u-bordeaux.fr/

[39] **J A Rasmussen**, *Some differentials on Khovanov–Rozansky homology* (2006)   arXiv:math/0607544

[40] **J A Rasmussen**, *Khovanov–Rozansky homology of two-bridge knots and links*, Duke Math. J. 136 (2007) 551–583   MR2309174

[41] **N Y Reshetikhin**, **V G Turaev**, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. 127 (1990) 1–26   MR1036112

[42] **A Shumakovitch**, *KhoHo* (2003) Available at   http://www.geometrie.ch/KhoHo/

[43] **A Shumakovitch**, *Torsion of the Khovanov homology* (2004)   arXiv:math/0405474

[44] **A Shumakovitch**, *Khovanov homology theories and their applications*, from: "Perspectives in analysis, geometry, and topology", (I Itenberg, B Jöricke, M Passare, editors), Progr. Math. 296, Springer, New York (2012) 403–430   MR2884045

[45] **A Stoimenow**, *Knot data tables* (2012) Available at   mathsci.kaist.ac.kr/~stoimeno/ptab

[46] **B Webster**, *Kr.m2* (2005) Available at   http://katlas.math.toronto.edu/wiki/User:Ben/KRhomology

[47] **B Webster**, *Khovanov–Rozansky homology via a canopolis formalism*, Algebr. Geom. Topol. 7 (2007) 673–699   MR2308960

[48] **H Wu**, *The Khovanov–Rozansky cohomology and Bennequin inequalities* (2007) arXiv:math/0703210

Institut de Mathématiques de Jussieu (IMJ) – Paris Rive Gauche
Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France

lewark@math.jussieu.fr

http://www.math.jussieu.fr/~lewark/