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Gravitational anomaly cancellation and modular invariance

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In this paper, by combining modular forms and characteristic forms, we obtain general anomaly cancellation formulas of any dimension. For $(4k+2)$–dimensional manifolds, our results include the gravitational anomaly cancellation formulas of Alvarez-Gaumé and Witten in dimensions 2, 6 and 10 [1] as special cases. In dimension $4k+1$, we derive anomaly cancellation formulas for index gerbes. In dimension $4k+3$, we obtain certain results about eta invariants, which are interesting in spectral geometry.

53C27, 53C80

1 Introduction

In [1], it is shown that in certain parity-violating gravity theories in $4k+2$ dimensions, when Weyl fermions of spin $\frac{1}{2}$ or spin $\frac{3}{2}$ or self-dual antisymmetric tensor fields are coupled to gravity, perturbative anomalies occur. Alvarez-Gaumé and Witten calculate the anomalies and show that there are cancellation formulas for these anomalies in dimensions 2, 6 and 10. Let $\widehat{I}_{1/2}$, $\widehat{I}_{3/2}$ and $\widehat{I}_A$ be the spin $\frac{1}{2}$, spin $\frac{3}{2}$ and antisymmetric tensor anomalies, respectively. By direct computations, Alvarez-Gaumé and Witten find anomaly cancellation formulas in dimensions 2, 6 and 10, respectively,

$$-\widehat{I}_{1/2}+\widehat{I}_A=0, \tag{1-1}$$

$$21\widehat{I}_{1/2}-\widehat{I}_{3/2}+8\widehat{I}_A=0, \tag{1-2}$$

$$-\widehat{I}_{1/2}+\widehat{I}_{3/2}+\widehat{I}_A=0. \tag{1-3}$$

These anomaly cancellation formulas can tell us how many fermions of different types should be coupled to the gravity to make the theory anomaly free. Atiyah and Singer [7] and Alvarez, Singer and Zumino [2] relate the above types of anomalies to the family index theorem.

When perturbative anomalies cancel, this means that the effective action is invariant under gauge and coordinate transformations that can be reached continuously from the

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identity. In [24], Witten introduces the global anomaly by asking whether the effective action is invariant under gauge and coordinate transformations that are not continuously connected to the identity. Witten's work suggests that the global anomaly should be related to the holonomy of a natural connection on the determinant line bundle of the family of Dirac operators.

From the topological point of view, anomaly measures the nontriviality of the determinant line bundle of a family of Dirac operators. The perturbative anomaly detects the real first Chern class of the determinant line bundle while the global anomaly detects the integral first Chern class beyond the real information (cf Freed [15]).

For a family of Dirac operators on an even-dimensional closed manifold, the determinant line bundle over the parametrizing space carries the Quillen metric as well as the Bismut–Freed connection compatible with the Quillen metric such that the curvature of the Bismut–Freed connection is the two-form component of the integration of the $\widehat{A}$–form of the vertical tangent bundle along the fiber in the Atiyah–Singer family index theorem [10; 11]. The curvature of the Bismut–Freed connection (after multiplied by $\sqrt{-1}/(2\pi)$) is the representative of the real first Chern class of the determinant line bundle. In this paper, by studying modular invariance of certain characteristic forms, we derive cancellation formulas for the curvatures of the determinant line bundles of family signature operators and family tangent twisted Dirac operators on $(4k+2)$–dimensional manifolds (Theorems 2.2.1 and 2.2.2). When $k = 0, 1, 2$, ie in dimensions 2, 6, and 10, our cancellation formulas give the Alvarez-Gaumé and Witten anomaly cancellation formulas (1-1)–(1-3) (see Corollaries 2.2.1–2.2.3 and Theorem 2.2.5 as well as its proof).

For the global anomaly, in [11], Bismut and Freed prove the holonomy theorem suggested by Witten. Later, to detect the integral information of the first Chern class of the determinant line bundle, Freed uses Sullivan's $\mathbb{Z}/k$ manifolds [15]. In this paper, we also give cancellation formulas for the holonomies of the Bismut–Freed connections on the determinant line bundles of family signature operators and family tangent twisted Dirac operators on $(4k+2)$–dimensional manifolds over torsion loops which appear in the data of $\mathbb{Z}/k$ surfaces (Theorems 2.2.3 and 2.2.4 and Corollaries 2.2.1–2.2.3).

The general anomaly cancellation formulas in dimension $4k$ have been studied by the second author in [19]. One can naturally ask if there are similar results in odd dimensions.

For a family of Dirac operators on an odd-dimensional manifold, Lott [21] constructs an abelian gerbe-with-connection whose curvature is the three-form component of the Atiyah–Singer families index theorem. This gerbe is called the index gerbe, which is a higher analogue of the determinant line bundle. As Lott remarks in his paper, the curvature of such gerbes are certain nonabelian gauge anomalies from a Hamiltonian point of view. In this paper, we derive anomaly cancellation formulas for the curvatures of the index gerbes of family odd signature operators and family tangent twisted Dirac operators on $(4k+1)$–dimensional manifolds (Theorems 2.3.1 and 2.3.2 and Corollaries 2.3.1–2.3.3). Moreover, as the K^1 index of the fiberwise signature operators vanishes, we can also derive anomaly cancellation formulas on the de Rham cohomology level (but not on the form level), which do not involve the signature operators (Theorems 2.3.3 and 2.3.4 and Corollaries 2.3.1–2.3.3). We hope there is some physical meaning related to our cohomological anomaly cancellation formulas.

In dimension $4k+3$, we derive some results for the reduced η–invariants of family odd signature operators and family tangent twisted Dirac operators, which are interesting in spectral geometry (Theorems 2.4.1 and 2.4.2 and Corollaries 2.4.1–2.4.3). Moreover, functions of the form

$$\exp\{2\pi\sqrt{-1}(\text{linear combination of reduced eta invariants})\}$$

have appeared in physics (see Diaconescu, Moore and Witten [13]) as a phase of effective action of M–theory in 11 dimensions. We hope our results for the reduced η–invariants can also find applications in physics.

We obtain our anomaly cancellation formulas by combining the family index theory and modular invariance of characteristic forms. The gravitational and gauge anomaly cancellations are important in superstring theories. It is quite interesting to notice that these cancellation formulas are consequences of the modular properties of characteristic forms which are rooted in elliptic genera.

2 Results

In this section, we will first prepare some geometric settings in Section 2.1 and then present our results in Sections 2.2–2.4. The proofs of the theorems in Sections 2.2–2.4 will be given in Section 3.

2.1 Geometric settings

Following Bismut [9], we define some geometric data on a fiber bundle as follows. Let $\pi\colon M\to Y$ be a smooth fiber bundle with compact fibers Z and connected base Y. Let TZ be the vertical tangent bundle of the fiber bundle and g^Z be a metric on TZ. Let T^HM be a smooth subbundle of TM transversal to TZ such that $TM=T^HM\oplus TZ$. Assume that TY is endowed with a metric g^Y. We lift the metric of TY to T^HM and by assuming that T^HM and TZ are orthogonal, TM is endowed with a metric which we denote $g^Y \oplus g^Z$. Let ∇^L be the Levi-Civita connection of TM for the metric $g^Y \oplus g^Z$ and P_Z denote the orthogonal projection from TM to TZ. Let ∇^Z denote the connection on TZ defined by the relation $U \in TM, V \in TZ$, $\nabla^Z_U V = P_Z \nabla^L_U V$. We have that ∇^Z preserves the metric g^Z. Let $R^Z = (\nabla^Z)^2$ be the curvature of ∇^Z.

Let E, F be two Hermitian vector bundles over M carrying Hermitian connections ∇^E, ∇^F respectively. Let $R^E = (\nabla^E)^2$ (resp. $R^F = (\nabla^F)^2$) be the curvature of ∇^E (resp. ∇^F). If we set the formal difference $G = E - F$, then G carries an induced Hermitian connection ∇^G in an obvious sense. We define the associated Chern character form as (cf Zhang [25])

$$\mathrm{ch}(G, \nabla^G) = \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi} R^E\right)\right] - \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi} R^F\right)\right]. \tag{2-1}$$

For any complex variable t, let

$$\Lambda_t(E) = \mathbb{C} + tE + t^2\Lambda^2(E) + \cdots, \quad S_t(E) = \mathbb{C} + tE + t^2 S^2(E) + \cdots,$$

denote respectively the total exterior and symmetric powers of E, in $K(M)[\![t]\!]$. The following relations between these two operations hold (see Atiyah [3, Chapter 3]):

$$S_t(E) = \frac{1}{\Lambda_{-t}(E)}, \quad \Lambda_t(E - F) = \frac{\Lambda_t(E)}{\Lambda_t(F)}. \tag{2-2}$$

The connections ∇^E, ∇^F naturally induce connections on $S_t E, \Lambda_t E$, etc. Moreover, if $\{\omega_i\}$, $\{\omega_j{}'\}$ are formal Chern roots for Hermitian vector bundles E, F respectively, then by [16, Chapter 1],

$$\mathrm{ch}(\Lambda_t(E), \nabla^{\Lambda_t(E)}) = \prod_i (1 + e^{\omega_i} t). \tag{2-3}$$

We have the following formulas for Chern character forms:

$$\mathrm{ch}(S_t(E), \nabla^{S_t(E)}) = \frac{1}{\mathrm{ch}(\Lambda_{-t}(E), \nabla^{\Lambda_{-t}(E)})} = \frac{1}{\prod_i (1 - e^{\omega_i} t)}, \tag{2-4}$$

$$\mathrm{ch}(\Lambda_t(E - F), \nabla^{\Lambda_t(E-F)}) = \frac{\mathrm{ch}(\Lambda_t(E), \nabla^{\Lambda_t(E)})}{\mathrm{ch}(\Lambda_t(F), \nabla^{\Lambda_t(F)})} = \frac{\prod_i (1 + e^{\omega_i} t)}{\prod_j (1 + e^{\omega_j{}'} t)} \tag{2-5}$$

If W is a real Euclidean vector bundle over M carrying a Euclidean connection ∇^W, then its complexification $W_{\mathbb{C}} = W \otimes \mathbb{C}$ is a complex vector bundle over M carrying a canonically induced Hermitian metric, as well as a Hermitian connection $\nabla^{W_{\mathbb{C}}}$ induced from ∇^W.

If E is a vector bundle (complex or real) over M, set $\widetilde{E} = E - F^{\operatorname{rk} E}$ in $K(M)$ or $KO(M)$, where F is $\mathbb{C}$ or $\mathbb{R}$.

Let $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbb{H}$, the upper half complex plane. Let $T_{\mathbb{C}}Z$ be the complexification of TZ. Set

$$\Theta_1(T_{\mathbb{C}}Z) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}Z}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{T_{\mathbb{C}}Z}), \tag{2-6}$$

$$\Theta_2(T_{\mathbb{C}}Z) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}Z}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-1/2}}(\widetilde{T_{\mathbb{C}}Z}), \tag{2-7}$$

which are elements in $K(M)[\![q^{1/2}]\!]$.

$\Theta_1(T_{\mathbb{C}}Z)$ and $\Theta_2(T_{\mathbb{C}}Z)$ admit formal Fourier expansion in $q^{1/2}$ as

$$\Theta_1(T_{\mathbb{C}}Z) = A_0(T_{\mathbb{C}}Z) + A_1(T_{\mathbb{C}}Z)q^{1/2} + \cdots, \tag{2-8}$$

$$\Theta_2(T_{\mathbb{C}}Z) = B_0(T_{\mathbb{C}}Z) + B_1(T_{\mathbb{C}}Z)q^{1/2} + \cdots, \tag{2-9}$$

where the A_j and B_j are virtual bundles formally generated by complex vector bundles over M. Moreover, they carry canonically induced connections denoted by ∇^{A_j} and ∇^{B_j} respectively. Let $\nabla^{\Theta_1(T_{\mathbb{C}}Z)}$, $\nabla^{\Theta_2(T_{\mathbb{C}}Z)}$ be the induced connections with $q^{1/2}$–coefficients on Θ_1, Θ_2 from the ∇^{A_j}, ∇^{B_j}.

Consider the q–series

$$\delta_1(\tau) = \frac{1}{4} + 6\sum_{n=1}^{\infty} \sum_{d|n,\, d \text{ odd}} dq^n = \frac{1}{4} + 6q + 6q^2 + \cdots, \tag{2-10}$$

$$\varepsilon_1(\tau) = \frac{1}{16} + \sum_{n=1}^{\infty} \sum_{d|n} (-1)^d d^3 q^n = \frac{1}{16} - q + 7q^2 + \cdots, \tag{2-11}$$

$$\delta_2(\tau) = -\frac{1}{8} - 3\sum_{n=1}^{\infty} \sum_{d|n,\, d \text{ odd}} dq^{n/2} = -\frac{1}{8} - 3q^{1/2} - 3q + \cdots, \tag{2-12}$$

$$\varepsilon_2(\tau) = \sum_{n=1}^{\infty} \sum_{d|n,\, n/d \text{ odd}} d^3 q^{n/2} = q^{1/2} + 8q + \cdots. \tag{2-13}$$

When the dimension of the fiber is $8m+1, 8m+2$ or $8m+3$, define virtual complex vector bundles $b_r(T_{\mathbb{C}}Z)$ on M, $0 \le r \le m$, via the equality

$$\Theta_2(T_{\mathbb{C}}Z) \equiv \sum_{r=0}^{m} b_r(T_{\mathbb{C}}Z)(8\delta_2)^{2m+1-2r}\varepsilon_2^r \mod q^{(m+1)/2} \cdot K(M)[\![q^{1/2}]\!]. \tag{2-14}$$

When the dimension of the fiber is $8m-1, 8m-2$ or $8m-3$, define virtual complex vector bundles $z_r(T_{\mathbb{C}}Z)$ on M, $0 \le r \le m$, via the equality

$$\Theta_2(T_{\mathbb{C}}Z) \equiv \sum_{r=0}^{m} z_r(T_{\mathbb{C}}Z)(8\delta_2)^{2m-2r}\varepsilon_2^r \mod q^{(m+1)/2}\cdot K(M)[\![q^{1/2}]\!]. \tag{2-15}$$

It is not hard to show that each $b_r(T_{\mathbb{C}}Z), 0 \le r \le m$, is a canonical linear combination of $B_j(T_{\mathbb{C}}Z), 0 \le j \le r$. This is also true for the $z_r(T_{\mathbb{C}}Z)$. These $b_r(T_{\mathbb{C}}Z)$ and $z_r(T_{\mathbb{C}}Z)$ carry canonically induced metrics and connections.

We will use δ_1 and ε_1 later in the proofs of our results. We list them here for completeness.

From (2-14) and (2-15), we can calculate that

$$b_0(T_{\mathbb{C}}Z) = -\mathbb{C}, \quad b_1(T_{\mathbb{C}}Z) = T_{\mathbb{C}}Z + \mathbb{C}^{24(2m+1)-\dim Z}, \tag{2-16}$$

$$z_0(T_{\mathbb{C}}Z) = \mathbb{C}, \qquad z_1(T_{\mathbb{C}}Z) = -T_{\mathbb{C}}Z - \mathbb{C}^{48m-\dim Z}. \tag{2-17}$$

2.2 Determinant line bundles and anomaly cancellation formulas

Suppose the dimension of the fiber is $2n$ and the dimension of the base Y is p. Assume that TZ is oriented. Let T^*Z be the dual bundle of TZ.

Let $E = \bigoplus_{i=0}^{2n} E^i$ be the smooth infinite-dimensional $\mathbb{Z}$–graded vector bundle over Y whose fiber over $y \in Y$ is $C^\infty(Z_y, \Lambda_{\mathbb{C}}(T^*Z)|_{Z_y})$, ie

$$C^\infty(Y, E^i) = C^\infty(M, \Lambda^i_{\mathbb{C}}(T^*Z)),$$

where $\Lambda^i_{\mathbb{C}}(T^*Z)$ is the complexified exterior algebra bundle of TZ.

For $X \in TZ$, let $c(X)$ and $\hat{c}(X)$ be the Clifford actions on $\Lambda_{\mathbb{C}}(T^*Z)$ defined by $c(X) = X^* - i_X$, $\hat{c}(X) = X^* + i_X$, where $X^* \in T^*Z$ corresponds to X via g^Z.

Let $\{e_1, e_2, \dots, e_{2n}\}$ be an oriented orthogonal basis of TZ. Set

$$\Omega = (\sqrt{-1})^n c(e_1)\cdots c(e_{2n}).$$

Then Ω is a self-adjoint element acting on $\Lambda_{\mathbb{C}}(T^*Z)$ such that $\Omega^2 = \mathrm{Id}\,|_{\Lambda_{\mathbb{C}}(T^*Z)}$.

Let dv_Z be the Riemannian volume form on fibers Z associated to the metric g^Z (dv_Z is actually a section of $\Lambda^{\dim Z}_{\mathbb{C}}(T^*Z)$). Let $\langle\cdot,\cdot\rangle_{\Lambda_{\mathbb{C}}(T^*Z)}$ be metric on $\Lambda_{\mathbb{C}}(T^*Z)$ induced by g^Z. Then E has a Hermitian metric h^E such that for $\alpha, \alpha' \in C^\infty(Y, E)$ and $y \in Y$,

$$\langle \alpha, \alpha'\rangle_{h^E}(y) = \int_{Z_y} \langle \alpha, \alpha'\rangle_{\Lambda_{\mathbb{C}}(T^*Z)}\, dv_{Z_y}.$$

Let d^Z be the exterior differentiation along fibers. d^Z can be considered as an element of $C^\infty(Y, \mathrm{Hom}(E^\bullet, E^{\bullet+1}))$. Let d^{Z*} be the formal adjoint of d^Z with respect to the inner product $\langle\cdot,\cdot\rangle_{h^E}$. Define the family signature operator (cf Ma and Zhang [22]) D^Z_{sig} to be

$$D^Z_{\mathrm{sig}} = d^Z + d^{Z*} \colon C^\infty(M, \Lambda_{\mathbb{C}}(T^*Z)) \longrightarrow C^\infty(M, \Lambda_{\mathbb{C}}(T^*Z)). \tag{2-18}$$

The $\mathbb{Z}_2$–grading of D^Z_{sig} is given by the $+1$ and -1 eigenbundles of Ω. Clearly, for each $y \in Y$,

$$(D^Z_{\mathrm{sig}})_y : C^\infty(Z_y, \Lambda_{\mathbb{C}}(T^*Z)|_y) \longrightarrow C^\infty(Z_y, \Lambda_{\mathbb{C}}(T^*Z)|_y)$$

is the signature operator for the fiber Z_y.

Further assume that TZ is spin. Following [9], the family of Dirac operators are defined as follows.

Let O be the $SO(2n)$ bundle of oriented orthogonal frames in TZ. Since TZ is spin, the $SO(2n)$ bundle $O \xrightarrow[\varrho]{} M$ lifts to a $\mathrm{Spin}(2n)$ bundle

$$O' \xrightarrow[\sigma]{} O \xrightarrow[\varrho]{} M$$

such that σ induces the covering projection $\mathrm{Spin}(2n) \to SO(2n)$ on each fiber. Assume F, $F_\pm$ denote the Hermitian bundles of spinors

$$F = O' \times_{\mathrm{Spin}(2n)} S_{2n}, \quad F_\pm = O' \times_{\mathrm{Spin}(2n)} S_{\pm,2n},$$

where $S_{2n} = S_{+,2n} \oplus S_{-,2n}$ is the space of complex spinors. The connection ∇^Z on O lifts to a connection on O'. F, $F_\pm$ are then naturally endowed with a unitary connection, which we simply denote by ∇.

Let V be a l–dimensional Hermitian vector bundle on M. Assume that V is endowed with a unitary connection ∇^V whose curvature is R^V. The Hermitian bundle $F \otimes V$ is naturally endowed with a unitary connection which we still denote by ∇.

Let H^∞ and $H^\infty_\pm$ be the sets of C^∞ sections of $F \otimes V$ and $F_\pm \otimes V$ over M. We view H^∞ and $H^\infty_\pm$ as the sets of C^∞ sections over Y of infinite-dimensional bundles which are still denoted by $H^\infty, H^\infty_\pm$. For $y \in Y$, $H^\infty_y, H^\infty_{y,\pm}$ are the sets of C^∞ sections over Z_y of $F \otimes V, F_\pm \otimes V$.

The elements of TZ acts by Clifford multiplication on $F \otimes V$. Suppose $\{e_1, e_2, \dots e_{2n}\}$ is a local orthogonal basis of TZ. Define the family of Dirac operator twisted by V

to be $D^Z \otimes V = \sum_{i=1}^{2n} e_i \nabla_{e_i}$. Let $(D^Z \otimes V)_\pm$ denote the restriction of $D^Z \otimes V$ to $H_\pm^\infty$. For each $y \in Y$,

$$(D^Z \otimes V)_y = \begin{bmatrix} 0 & (D^Z \otimes V)_{y,-} \\ (D^Z \otimes V)_{y,+} & 0 \end{bmatrix} \in \mathrm{End}^{\mathrm{odd}}(H_{y,+}^\infty \oplus H_{y,-}^\infty) \tag{2-19}$$

is the twisted Dirac operator on the fiber Z_y.

The family signature operator is a twisted family of Dirac operator. Actually, we have $D_{\mathrm{sig}}^Z = D^Z \otimes F$ (cf Berline, Getzler and Vergne [8]).

Let $\mathcal{L}_{D^Z \otimes V} = \det(\mathrm{Ker}(D^Z \otimes V)_+)^* \otimes \det(\mathrm{Ker}(D^Z \otimes V)_-)$ be the determinant line bundle of the family operator $D^Z \otimes V$ over Y; see Quillen [23] and [10]. The nontriviality of $\mathcal{L}_{D^Z \otimes V}$ corresponds to an anomaly in physics.

The determinant line bundle carries the Quillen metric $g^{\mathcal{L}_{D^Z \otimes V}}$ as well as the Bismut–Freed connection $\nabla^{\mathcal{L}_{D^Z \otimes V}}$ compatible to $g^{\mathcal{L}_{D^Z \otimes V}}$, the curvature $R^{\mathcal{L}_{D^Z \otimes V}}$ of which is equal to the two-form component of the integration of $\widehat{A}(TZ, \nabla^Z)\,\mathrm{ch}(V, \nabla^V)$ along the fiber in the Atiyah–Singer family index theorem [10; 11]. $\sqrt{-1}/(2\pi) R^{\mathcal{L}_{D^Z \otimes V}}$ is a representative of the local anomaly.

For the global anomaly, in [11], Bismut and Freed give a heat equation proof of the holonomy theorem in the form suggested by Witten in [24]. To detect information for the integral first Chern class of $\mathcal{L}_{D^Z \otimes V}$, Freed uses $\mathbb{Z}/k$ manifolds in [15]. $\mathbb{Z}/k$ manifold is introduced by Sullivan in his studies of geometric topology. A closed $\mathbb{Z}/k$ manifold (cf [15]) consists of (1) a compact manifold Q with boundary; (2) a closed manifold P; (3) a decomposition $\partial Q = \coprod_{i=1}^k (\partial Q)_i$ of the boundary of Q into k disjoint manifolds and diffeomorphisms $\alpha_i \colon P \to (\partial Q)_i$. The identification space $\bar{Q}$, formed by attaching Q to P by α_i is more properly called $\mathbb{Z}/k$ manifolds. $\bar{Q}$ is singular at identification points. If Q and P are compatibly oriented, then $\bar{Q}$ carries a fundamental class $[\bar{Q}] \in H_*(\bar{Q}, \mathbb{Z}/k)$. In [15], the first Chern class of the determinant line bundle is evaluated for all $\mathbb{Z}/k$ surfaces $\bar{\Sigma}$ and all maps $\bar{\Sigma} \to Y$ to detect the rest information not contained in the real cohomology.

For local anomalies, we have the following cancellation formulas.

Theorem 2.2.1 *If the fiber is $(8m+2)$–dimensional, then the following local anomaly cancellation formula holds:*

$$R^{\mathcal{L}_{D_{\mathrm{sig}}^Z}} - 8 \sum_{r=0}^{m} 2^{6m-6r} R^{\mathcal{L}_{D^Z \otimes b_r(T_{\mathbb{C}} Z)}} = 0 \tag{2-20}$$

Theorem 2.2.2 *If the fiber be* $(8m-2)$*–dimensional, then the following local anomaly cancellation formula holds:*

$$R^{\mathcal{L}_{D^Z_{\mathrm{sig}}}} - \sum_{r=0}^{m} 2^{6m-6r} R^{\mathcal{L}_{D^Z \otimes z_r(T_{\mathbb{C}}Z)}} = 0 \tag{2-21}$$

For global anomalies, we have the following cancellation formulas concerning the holonomies of the Bismut–Freed connections on the determinant line bundles.

Theorem 2.2.3 *If the fiber is* $(8m+2)$*–dimensional,* (Σ, S) *is a* $\mathbb{Z}/k$ *surface and* $f\colon \bar{\Sigma} \to Y$ *is a map, then*

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\operatorname{lnhol}_{\mathcal{L}_{D^Z_{\mathrm{sig}}}}(S) - 8\sum_{r=0}^{m} 2^{6m-6r}\frac{\sqrt{-1}}{2\pi}\operatorname{lnhol}_{\mathcal{L}_{D^Z\otimes b_r(T_{\mathbb{C}}Z)}}(S) \\ \equiv c_1(f^*(\mathcal{L}_{D^Z_{\mathrm{sig}}}))[\bar{\Sigma}] - 8\sum_{r=0}^{m} 2^{6m-6r} c_1(f^*(\mathcal{L}_{D^Z\otimes b_r(T_{\mathbb{C}}Z)}))[\bar{\Sigma}] \mod 1,\end{aligned} \tag{2-22}$$

where we view $\mathbb{Z}/k \cong \mathbb{Z}[1/k]/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$.

Theorem 2.2.4 *If the fiber is* $(8m-2)$*–dimensional,* (Σ, S) *is a* $\mathbb{Z}/k$ *surface and* $f\colon \bar{\Sigma} \to Y$ *is a map, then*

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\operatorname{lnhol}_{\mathcal{L}_{D^Z_{\mathrm{sig}}}}(S) - \sum_{r=0}^{m} 2^{6m-6r}\frac{\sqrt{-1}}{2\pi}\operatorname{lnhol}_{\mathcal{L}_{D^Z\otimes z_r(T_{\mathbb{C}}Z)}}(S) \\ \equiv c_1(f^*(\mathcal{L}_{D^Z_{\mathrm{sig}}}))[\bar{\Sigma}] - \sum_{r=0}^{m} 2^{6m-6r} c_1(f^*(\mathcal{L}_{D^Z\otimes z_r(T_{\mathbb{C}}Z)}))[\bar{\Sigma}] \mod 1,\end{aligned} \tag{2-23}$$

where we view $\mathbb{Z}/k \cong \mathbb{Z}[1/k]/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$.

Putting $m=0$ and $m=1$ in the above theorems and using (2-16) as well as (2-17), we have the following.

Corollary 2.2.1 *If the fiber is* 2*–dimensional, then the following local anomaly cancellation formula holds:*

$$R^{\mathcal{L}_{D^Z_{\mathrm{sig}}}} + 8R^{\mathcal{L}_{D^Z}} = 0 \tag{2-24}$$

If (Σ, S) *is a* $\mathbb{Z}/k$ *surface and* $f\colon \bar{\Sigma} \to Y$ *is a map, then*

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\operatorname{lnhol}_{\mathcal{L}_{D^Z_{\mathrm{sig}}}}(S) + 8\frac{\sqrt{-1}}{2\pi}\operatorname{lnhol}_{\mathcal{L}_{D^Z}}(S) \\ \equiv c_1(f^*(\mathcal{L}_{D^Z_{\mathrm{sig}}}))[\bar{\Sigma}] + 8c_1(f^*(\mathcal{L}_{D^Z}))[\bar{\Sigma}] \mod 1.\end{aligned} \tag{2-25}$$

Corollary 2.2.2 *If the fiber is* 6*–dimensional, then the following local anomaly cancellation formula holds:*

$$R^{\mathcal{L}_{D^Z_{\mathrm{sig}}}}+R^{\mathcal{L}_{D^Z\otimes T_{\mathbb{C}}Z}}-22R^{\mathcal{L}_{D^Z}}=0 \tag{2-26}$$

If (Σ,S) *is a* $\mathbb{Z}/k$ *surface and* $f\colon\overline{\Sigma}\to Y$ *is a map, then*

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\operatorname{lnhol}_{\mathcal{L}_{D^Z_{\mathrm{sig}}}}(S)+\frac{\sqrt{-1}}{2\pi}\operatorname{lnhol}_{\mathcal{L}_{D^Z\otimes T_{\mathbb{C}}Z}}(S)-22\frac{\sqrt{-1}}{2\pi}\operatorname{lnhol}_{\mathcal{L}_{D^Z}}(S)\\ \equiv c_1(f^*(\mathcal{L}_{D^Z_{\mathrm{sig}}}))[\overline{\Sigma}]+c_1(f^*(\mathcal{L}_{D^Z\otimes T_{\mathbb{C}}Z}))[\overline{\Sigma}]\\ -22c_1(f^*(\mathcal{L}_{D^Z}))[\overline{\Sigma}]\mod 1.\end{aligned} \tag{2-27}$$

Corollary 2.2.3 *If the fiber is* 10*–dimensional, then the following local anomaly cancellation formula holds:*

$$R^{\mathcal{L}_{D^Z_{\mathrm{sig}}}}-8R^{\mathcal{L}_{D^Z\otimes T_{\mathbb{C}}Z}}+16R^{\mathcal{L}_{D^Z}}=0 \tag{2-28}$$

If (Σ,S) *is a* $\mathbb{Z}/k$ *surface and* $f\colon\overline{\Sigma}\to Y$ *is a map, then*

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\operatorname{lnhol}_{\mathcal{L}_{D^Z_{\mathrm{sig}}}}(S)-8\frac{\sqrt{-1}}{2\pi}\operatorname{lnhol}_{\mathcal{L}_{D^Z\otimes T_{\mathbb{C}}Z}}(S)+16\frac{\sqrt{-1}}{2\pi}\operatorname{lnhol}_{\mathcal{L}_{D^Z}}(S)\\ \equiv c_1(f^*(\mathcal{L}_{D^Z_{\mathrm{sig}}}))[\overline{\Sigma}]-8c_1(f^*(\mathcal{L}_{D^Z\otimes T_{\mathbb{C}}Z}))[\overline{\Sigma}]\\ +16c_1(f^*(\mathcal{L}_{D^Z}))[\overline{\Sigma}]\mod 1.\end{aligned} \tag{2-29}$$

We find that we have the following.

Theorem 2.2.5 *In dimensions* $2,6,10$, *our anomaly cancellation formulas* (2-24), (2-26) *and* (2-28) *give the gravitational anomaly cancellation formulas* (1-1)–(1-3) *of Alvarez-Gaumé and Witten.*

The proof this theorem will also be given in Section 3.

2.3 Index gerbes and anomaly cancellation formulas

Now we still assume that TZ is oriented but the dimension of the fiber is $2n+1$, ie we consider odd-dimensional fibers. We still adopt the geometric settings in Section 2.1.

Let $\{e_1,e_2,\dots,e_{2n+1}\}$ be an oriented orthogonal basis of TZ. Set

$$\Gamma=(\sqrt{-1})^{n+1}c(e_1)\cdots c(e_{2n+1}).$$

Then Γ is a self-adjoint element acting on $\Lambda_{\mathbb{C}}(T^*Z)$ such that $\Gamma^2=\mathrm{Id}|_{\Lambda_{\mathbb{C}}(T^*Z)}$.

Define the family odd signature operator B^Z_{sig} to be

$$B^Z_{\mathrm{sig}} = \Gamma d^Z + d^Z\Gamma\colon C^\infty(M, \Lambda^{\mathrm{even}}_{\mathbb{C}}(T^*Z)) \longrightarrow C^\infty(M, \Lambda^{\mathrm{even}}_{\mathbb{C}}(T^*Z)). \tag{2-30}$$

For each $y \in Y$,

$$(B^Z_{\mathrm{sig}})_y\colon C^\infty(Z_y, \Lambda^{\mathrm{even}}_{\mathbb{C}}(T^*Z)|_y) \longrightarrow C^\infty(Z_y, \Lambda^{\mathrm{even}}_{\mathbb{C}}(T^*Z)|_y) \tag{2-31}$$

is the odd signature operator B_{even} for the fiber Z_y in Atiyah, Patodi and Singer [5].

Now assume that TZ is spin and still let V be an l–dimensional Hermitian vector bundle with unitary connection ∇^V. One can still define the family of Dirac operator $D^Z \otimes V$ similar as the even-dimensional fiber case. The only difference is that now the spinor bundle F' associated to TZ is not $\mathbb{Z}_2$–graded. Let H^∞ be the set of C^∞ sections of $F' \otimes V$ over M. H^∞ is viewed as the set of C^∞ sections over Y of infinite-dimensional bundles which are still denoted by H^∞. For $y \in Y$, H^∞_y is the set of C^∞ sections over Z_y of $F' \otimes V$. For each $y \in Y$,

$$(D^Z \otimes V)_y \in \mathrm{End}(H^\infty_y)$$

is the twisted Dirac operator on the fiber Z_y.

The family of odd signature operator is a twisted family of Dirac operator. Actually, we have $B^Z_{\mathrm{sig}} = D^Z \otimes F'$.

As a higher analogue of the determinant line bundle, Lott [21] constructs the index gerbe $\mathcal{G}^{D^Z \otimes V}$ with connection on Y for the family twisted odd Dirac operator $D^Z \otimes V$, the curvature $R^{\mathcal{G}_{D\otimes V}}$ (a closed 3–form on Y) of which is equal to the three-form component of the integration of $\widehat{A}(TZ, \nabla^Z)\,\mathrm{ch}(V, \nabla^V)$ along the fiber in the Atiyah–Singer families index theorem. As remarked in [21], the curvature of the index gerbe is certain nonabelian gauge anomaly in physics (see Faddeev [14], cf [21]).

We have the following anomaly cancellation formulas for index gerbes.

Theorem 2.3.1 *If the fiber is $(8m+1)$–dimensional, then the following anomaly cancellation formula holds:*

$$R^{\mathcal{G}_{B^Z_{\mathrm{sig}}}} - 8\sum_{r=0}^{m} 2^{6m-6r}\, R^{\mathcal{G}_{D\otimes b_r(T_{\mathbb{C}}Z)}} = 0 \tag{2-32}$$

Theorem 2.3.2 *If the fiber is $(8m-3)$–dimensional, then the following anomaly cancellation formula holds:*

$$R^{\mathcal{G}_{B^Z_{\mathrm{sig}}}} - \sum_{r=0}^{m} 2^{6m-6r}\, R^{\mathcal{G}_{D\otimes z_r(T_{\mathbb{C}}Z)}} = 0 \tag{2-33}$$

If ω is a closed differential form on Y, denote the cohomology class that ω represents in the de Rham cohomology of Y by $[\omega]$.

We have the following cancellation formulas for cohomology anomalies.

Theorem 2.3.3 *If the fiber is $(8m+1)$–dimensional, then the following anomaly cancellation formula in cohomology holds:*

$$\sum_{r=0}^{m} 2^{6m-6r}[R^{\mathcal{G}_{D\otimes b_r(T_{\mathbb{C}}Z)}}]=0 \tag{2-34}$$

Theorem 2.3.4 *If the fiber is $(8m-3)$–dimensional, then the following anomaly cancellation formula in cohomology holds:*

$$\sum_{r=0}^{m} 2^{6m-6r}[R^{\mathcal{G}_{D\otimes z_r(T_{\mathbb{C}}Z)}}]=0 \tag{2-35}$$

Putting $m=0$ and $m=1$ in the above theorems and using (2-16) as well as (2-17), we have the following.

Corollary 2.3.1 *If the fiber is 1–dimensional, ie for the circle bundle case, the following anomaly cancellation formula holds:*

$$R^{\mathcal{G}_{B^Z_{\mathrm{sig}}}}+8R^{\mathcal{G}_{D^Z}}=0 \tag{2-36}$$

The cohomology anomaly,

$$[R^{\mathcal{G}_{D^Z}}]=0. \tag{2-37}$$

Corollary 2.3.2 *The fiber is 5–dimensional, then the following anomaly cancellation formula holds:*

$$R^{\mathcal{G}_{B^Z_{\mathrm{sig}}}}+R^{\mathcal{G}_{D^Z\otimes T_{\mathbb{C}}Z}}-21R^{\mathcal{G}_{D^Z}}=0 \tag{2-38}$$

The cohomology anomaly,

$$[R^{\mathcal{G}_{D^Z\otimes T_{\mathbb{C}}Z}}]-21[R^{\mathcal{G}_{D^Z}}]=0. \tag{2-39}$$

Corollary 2.3.3 *If the fiber is 9–dimensional, then the following anomaly cancellation formula holds:*

$$R^{\mathcal{G}_{B^Z_{\mathrm{sig}}}}-8R^{\mathcal{G}_{D^Z\otimes T_{\mathbb{C}}Z}}+8R^{\mathcal{G}_{D^Z}}=0 \tag{2-40}$$

The cohomology anomaly,

$$[R^{\mathcal{G}_{D^Z\otimes T_{\mathbb{C}}Z}}]-[R^{\mathcal{G}_{D^Z}}]=0. \tag{2-41}$$

2.4 Results for η–invariants

For $y \in Y$, let $\eta_y(D^Z \otimes V)(s)$ be the eta function associated with $(D^Z \otimes V)_y$. Define as Atiyah, Patodi and Singer in [4]

$$\overline{\eta}_y(D^Z \otimes V)(s) = \frac{\eta_y(D^Z \otimes V)(s) + \ker(D^Z \otimes V)_y}{2}. \tag{2-42}$$

Denote $\overline{\eta}_y(D^Z \otimes V)(0)$ (a function on Y) by $\overline{\eta}(D^Z \otimes V)$.

We still adopt the setting of family odd signature operators and family twisted Dirac operators on a family of odd-dimensional manifolds in Section 2.3. We have the following theorems on the reduced η–invariants.

Theorem 2.4.1 *If the fiber is $(8m+3)$–dimensional, then*

$$\exp\Big\{2\pi\sqrt{-1}\Big(\overline{\eta}(B^Z_{\mathrm{sig}}) - 8\sum_{r=0}^{m} 2^{6m-6r}\overline{\eta}\big(D^Z \otimes b_r(T_{\mathbb{C}}Z)\big)\Big)\Big\} \tag{2-43}$$

is a constant function on Y.

Theorem 2.4.2 *If the fiber is $(8m-1)$–dimensional, then*

$$\exp\Big\{2\pi\sqrt{-1}\Big(\overline{\eta}(B^Z_{\mathrm{sig}}) - \sum_{r=0}^{m} 2^{6m-6r}\overline{\eta}\big(D^Z \otimes z_r(T_{\mathbb{C}}Z)\big)\Big)\Big\} \tag{2-44}$$

is a constant function on Y.

Putting $m=0$ and $m=1$ in the above theorems and using (2-16) as well as (2-17), we have the following.

Corollary 2.4.1 *If the fiber is 3–dimensional, then*

$$\exp\{2\pi\sqrt{-1}(\overline{\eta}(B^Z_{\mathrm{sig}}) + 8\overline{\eta}(D^Z))\} \tag{2-45}$$

is a constant function on Y.

Corollary 2.4.2 *If the fiber is 7–dimensional, then*

$$\exp\{2\pi\sqrt{-1}(\overline{\eta}(B^Z_{\mathrm{sig}}) + \overline{\eta}(D^Z \otimes T_{\mathbb{C}}Z) - 23\overline{\eta}(D^Z))\} \tag{2-46}$$

is a constant function on Y.

Corollary 2.4.3 *If the fiber is 11–dimensional, then*

$$\exp\{2\pi\sqrt{-1}(\overline{\eta}(B^Z_{\mathrm{sig}}) - 8\overline{\eta}(D^Z \otimes T_{\mathbb{C}}Z) + 24\overline{\eta}(D^Z))\} \tag{2-47}$$

is a constant function on Y.

3 Proofs

In this section, we prove the theorems stated in Sections 2.2–2.4.

3.1 Preliminaries

Let

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z}, \; ad - bc = 1 \right\}$$

as usual be the modular group. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

be the two generators of $\mathrm{SL}_2(\mathbb{Z})$. Their actions on $\mathbb{H}$ are given by

$$S\colon \tau \to -\frac{1}{\tau}, \quad T\colon \tau \to \tau + 1.$$

Let

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \, \middle| \, c \equiv 0 \pmod 2 \right\},$$

$$\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \, \middle| \, b \equiv 0 \pmod 2 \right\},$$

be two subgroups of $\mathrm{SL}_2(\mathbb{Z})$. It is known that the generators of $\Gamma_0(2)$ are T, ST^2ST and the generators of $\Gamma^0(2)$ are STS, T^2STS (cf Chandrasekharan [12]).

The four Jacobi theta functions are defined as (cf [12])

$$\theta(v,\tau) = 2q^{1/8}\sin(\pi v)\prod_{j=1}^{\infty}[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^j)(1-e^{-2\pi\sqrt{-1}v}q^j)],$$

$$\theta_1(v,\tau) = 2q^{1/8}\cos(\pi v)\prod_{j=1}^{\infty}[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^j)(1+e^{-2\pi\sqrt{-1}v}q^j)],$$

$$\theta_2(v,\tau) = \prod_{j=1}^{\infty}[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^{j-1/2})(1-e^{-2\pi\sqrt{-1}v}q^{j-1/2})],$$

$$\theta_3(v,\tau) = \prod_{j=1}^{\infty}[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^{j-1/2})(1+e^{-2\pi\sqrt{-1}v}q^{j-1/2})].$$

They are all holomorphic functions for $(v,\tau)\in\mathbb{C}\times\mathbb{H}$, where $\mathbb{C}$ is the complex plane and $\mathbb{H}$ is the upper half plane.

If we act on theta-functions by S and T, the theta functions obey the transformation laws (cf [12])

$$\theta(v,\tau+1)=e^{(\pi\sqrt{-1})/4}\theta(v,\tau),\ \theta(v,-1/\tau)=\frac{1}{\sqrt{-1}}\Big(\frac{\tau}{\sqrt{-1}}\Big)^{1/2}e^{\pi\sqrt{-1}\tau v^2}\theta(\tau v,\tau), \tag{3-1}$$

$$\theta_1(v,\tau+1)=e^{(\pi\sqrt{-1})/4}\theta_1(v,\tau),\ \theta_1(v,-1/\tau)=\Big(\frac{\tau}{\sqrt{-1}}\Big)^{1/2}e^{\pi\sqrt{-1}\tau v^2}\theta_2(\tau v,\tau), \tag{3-2}$$

$$\theta_2(v,\tau+1)=\theta_3(v,\tau),\ \theta_2(v,-1/\tau)=\Big(\frac{\tau}{\sqrt{-1}}\Big)^{1/2}e^{\pi\sqrt{-1}\tau v^2}\theta_1(\tau v,\tau), \tag{3-3}$$

$$\theta_3(v,\tau+1)=\theta_2(v,\tau),\ \theta_3(v,-1/\tau)=\Big(\frac{\tau}{\sqrt{-1}}\Big)^{1/2}e^{\pi\sqrt{-1}\tau v^2}\theta_3(\tau v,\tau). \tag{3-4}$$

Definition 3.1 Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A modular form over Γ is a holomorphic function $f(\tau)$ on $\mathbb{H}\cup\{\infty\}$ such that for any

$$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma,$$

the following property holds:

$$f(g\tau):=f\Big(\frac{a\tau+b}{c\tau+d}\Big)=\chi(g)(c\tau+d)^l f(\tau),$$

where $\chi\colon\Gamma\to\mathbb{C}^*$ is a character of Γ and l is called the weight of f.

Let $M_{\mathbb{R}}(\Gamma)$ denote the ring of modular forms over Γ with real Fourier coefficients.

Writing simply $\theta_j=\theta_j(0,\tau)$, $1\le j\le 3$, we have (cf Hirzebruch, Berger and Jung [17] and the second author [20]),

$$\delta_1(\tau)=\tfrac{1}{8}(\theta_2^4+\theta_3^4),\qquad \varepsilon_1(\tau)=\tfrac{1}{16}\theta_2^4\theta_3^4,$$
$$\delta_2(\tau)=-\tfrac{1}{8}(\theta_1^4+\theta_3^4),\quad \varepsilon_2(\tau)=\tfrac{1}{16}\theta_1^4\theta_3^4.$$

We have transformation laws (cf Landweber [18] and [19]) given by

$$\delta_2\Big(-\frac{1}{\tau}\Big)=\tau^2\delta_1(\tau),\quad \varepsilon_2\Big(-\frac{1}{\tau}\Big)=\tau^4\varepsilon_1(\tau). \tag{3-5}$$

Let $\widehat{A}(TZ,\nabla^Z)$ and $L(TZ,\nabla^Z)$ be the Hirzebruch characteristic forms defined respectively by [25] for (TZ,∇^Z):

$$\begin{aligned}\widehat{A}(TZ,\nabla^Z)&=\det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^Z}{\sinh(\frac{\sqrt{-1}}{4\pi}R^Z)}\right),\\ \widehat{L}(TZ,\nabla^Z)&=\det{}^{1/2}\left(\frac{\frac{\sqrt{-1}}{2\pi}R^Z}{\tanh(\frac{\sqrt{-1}}{4\pi}R^Z)}\right)\end{aligned}\tag{3-6}$$

If ω is a differential form, denote the j–component of ω by $\omega^{(j)}$.

3.2 Proofs of Theorems 2.2.1, 2.2.3, 2.2.5, 2.3.1, 2.3.3 and 2.4.1

If the dimension of TZ is $8m+1, 8m+2$ or $8m+3$, for the vertical tangent bundle TZ, set

$$P_1(\nabla^Z,\tau):=\{\widehat{L}(TZ,\nabla^Z)\operatorname{ch}(\Theta_1(T_{\mathbb{C}}Z),\nabla^{\Theta_1(T_{\mathbb{C}}Z)})\}^{(8m+4)},\tag{3-7}$$

$$P_2(\nabla^Z,\tau):=\{\widehat{A}(TZ,\nabla^Z)\operatorname{ch}(\Theta_2(T_{\mathbb{C}}Z),\nabla^{\Theta_2(T_{\mathbb{C}}Z)})\}^{(8m+4)}.\tag{3-8}$$

Proposition 3.1 *We have $P_1(\nabla^Z,\tau)$ is a modular form of weight $4m+2$ over $\Gamma_0(2)$; $P_2(\nabla^Z,\tau)$ is a modular form of weight $4m+2$ over $\Gamma^0(2)$.*

Proof In terms of the theta functions, by the Chern–Weil theory, the following identities hold:

$$P_1(\nabla^Z,\tau)=\left\{\det{}^{1/2}\left(\frac{R^Z}{2\pi^2}\frac{\theta'(0,\tau)}{\theta(\frac{R^Z}{2\pi^2},\tau)}\frac{\theta_1(\frac{R^Z}{2\pi^2},\tau)}{\theta_1(0,\tau)}\right)\right\}^{(8m+4)},\tag{3-9}$$

$$P_2(\nabla^Z,\tau)=\left\{\det{}^{1/2}\left(\frac{R^Z}{4\pi^2}\frac{\theta'(0,\tau)}{\theta(\frac{R^Z}{4\pi^2},\tau)}\frac{\theta_2(\frac{R^Z}{4\pi^2},\tau)}{\theta_2(0,\tau)}\right)\right\}^{(8m+4)}\tag{3-10}$$

Applying the transformation laws of the theta functions, we have

$$P_1\left(\nabla^Z,-\frac{1}{\tau}\right)=2^{4m+2}\tau^{4m+2}P_2(\nabla^Z,\tau),\qquad P_1(\nabla^Z,\tau+1)=P_1(\nabla^Z,\tau).\tag{3-11}$$

Because the generators of $\Gamma_0(2)$ are T, ST^2ST and the generators of $\Gamma^0(2)$ are STS, T^2STS, the proposition follows easily. □

Lemma 3.1 [19] *One has that $\delta_1(\tau)$ (resp. $\varepsilon_1(\tau)$) is a modular form of weight 2 (resp. 4) over $\Gamma_0(2)$, $\delta_2(\tau)$ (resp. $\varepsilon_2(\tau)$) is a modular form of weight 2 resp. 4) over $\Gamma^0(2)$ and moreover $M_{\mathbb{R}}(\Gamma^0(2))=\mathbb{R}[\delta_2(\tau),\varepsilon_2(\tau)]$.*

We then apply Lemma 3.1 to $P_2(\nabla^Z,\tau)$ to get that

$$P_2(\nabla^Z,\tau)=h_0(T_{\mathbb{C}}Z)(8\delta_2)^{2m+1}+h_1(T_{\mathbb{C}}Z)(8\delta_2)^{2m-1}\varepsilon_2+\cdots+h_m(T_{\mathbb{C}}Z)(8\delta_2)\varepsilon_2^m. \tag{3-12}$$

Comparing with (2-14), we can see that

$$h_r(T_{\mathbb{C}}Z)=\{\widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(b_r(T_{\mathbb{C}}Z))\}^{(8m+4)},\quad 0\le r\le m.$$

By (3-5), (3-11) and (3-12), we have

$$P_1(\nabla^Z,\tau)=2^{4m+2}[h_0(T_{\mathbb{C}}Z)(8\delta_1)^{2m+1}+h_1(T_{\mathbb{C}}Z)(8\delta_1)^{2m-1}\varepsilon_1+\cdots+h_m(T_{\mathbb{C}}Z)(8\delta_1)\varepsilon_1^m]. \tag{3-13}$$

Comparing the constant term of the above equality, we have

$$\{\widehat{L}(TZ,\nabla^Z)\}^{(8m+4)}=8\sum_{r=0}^{m}2^{6m-6r}\{\widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(b_r(T_{\mathbb{C}}Z))\}^{(8m+4)}. \tag{3-14}$$

In the following, we will deal with the even case and odd case respectively.

3.2.1 The case of even-dimensional fibers We have the following theorem for the curvature of the Bismut–Freed connection on the determinant line bundle.

Theorem 3.2.1 (Bismut and Freed [11])

$$R^{\mathcal{L}_{D^Z\otimes V}}=2\pi\sqrt{-1}\left\{\int_Z\widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(V,\nabla^V)\right\}^{(2)} \tag{3-15}$$

To detect mod k information of the first Chern class of the determinant line bundle, Freed has the following result.

Theorem 3.2.2 (Freed [15]) *If (Σ,S) is a $\mathbb{Z}/k$ surface and $f\colon\overline{\Sigma}\to Y$ is a map, then*

$$c_1(f^*(\mathcal{L}_{D^Z\otimes V}))[\overline{\Sigma}]=\frac{1}{k}\frac{\sqrt{-1}}{2\pi}\int_\Sigma f^*\left(\int_Z\widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(V,\nabla^V)\right)+\frac{\sqrt{-1}}{2\pi}\ln\mathrm{hol}_{\mathcal{L}_{D^Z\otimes V}}(S)\mod 1, \tag{3-16}$$

where we view $\mathbb{Z}/k\cong\mathbb{Z}[1/k]/\mathbb{Z}\subset\mathbb{Q}/\mathbb{Z}$.

If TZ is $(8m+2)$–dimensional, integrating both sides of (3.14) along the fiber, we have

$$\left\{\int_Z \widehat{L}(TZ,\nabla^Z)\right\}^{(2)} - 8\sum_{r=0}^{m} 2^{6m-6r}\left\{\int_Z \widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(b_r(T_{\mathbb{C}}Z))\right\}^{(2)} = 0. \tag{3-17}$$

By Theorem 3.2.1, we get

$$\begin{aligned} R^{\mathcal{L}_{D^Z_{\mathrm{sig}}}} - 8\sum_{r=0}^{m} 2^{6m-6r} R^{\mathcal{L}_{D^Z\otimes b_r(T_{\mathbb{C}}Z)}} &= 2\pi\sqrt{-1}\left\{\int_Z \widehat{L}(TZ,\nabla^Z)\right\}^{(2)} \\ &\qquad - 8\sum_{r=0}^{m} 2^{6m-6r} 2\pi\sqrt{-1}\left\{\int_Z \widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(b_r(T_{\mathbb{C}}Z))\right\}^{(2)} \\ &= 0. \end{aligned} \tag{3-18}$$

Therefore Theorem 2.2.1 follows.

Freed's Theorem 3.2.2 and (3-17) give us

$$\begin{aligned} c_1(f^*(\mathcal{L}_{D^Z_{\mathrm{sig}}}))[\overline{\Sigma}] - 8\sum_{r=0}^{m} 2^{6m-6r} c_1(f^*(\mathcal{L}_{D^Z\otimes b_r(T_{\mathbb{C}}Z)}))[\overline{\Sigma}] \\ - \Big(\frac{\sqrt{-1}}{2\pi}\,\mathrm{lnhol}_{\mathcal{L}_{D^Z_{\mathrm{sig}}}}(S) - 8\sum_{r=0}^{m} 2^{6m-6r}\frac{\sqrt{-1}}{2\pi}\,\mathrm{lnhol}_{\mathcal{L}_{D^Z\otimes b_r(T_{\mathbb{C}}Z)}}(S)\Big) \\ \equiv 0 \mod 1 \end{aligned} \tag{3-19}$$

and so Theorem 2.2.3 follows.

To prove Theorem 2.2.5, it is not hard to see from [1, (32), (38) and (56)] that, up to a same constant,

$$\widehat{I}_{1/2} = \left\{\int_Z \widehat{A}(TZ,\nabla^Z)\right\}^{(2)} = R^{\mathcal{L}_{D^Z}},$$

$$\widehat{I}_{3/2} = \left\{\int_Z \widehat{A}(TZ,\nabla^Z)(\mathrm{ch}(T_{\mathbb{C}}Z,\nabla^Z)-1)\right\}^{(2)} = R^{\mathcal{L}_{D^Z\otimes T_{\mathbb{C}}Z}} - R^{\mathcal{L}_{D^Z}},$$

$$\widehat{I}_A = -\frac{1}{8}\left\{\int_Z \widehat{L}(TZ,\nabla^Z)\right\}^{(2)} = -\frac{1}{8} R^{\mathcal{L}_{D^Z_{\mathrm{sig}}}},$$

where in the fiber bundle $Z \to M \to Y$, Z is a $(4k+2)$–dimensional spin manifold and Y is the quotient space of the space of metrics on Z by the action of certain subgroup of $\mathrm{Diff}(Z)$.

In dimension 2, by (2-24),

$$-\widehat{I}_{1/2}+\widehat{I}_A=-R^{\mathcal{L}_{D^Z}}-\tfrac{1}{8}R^{\mathcal{L}_{D^Z_{\mathrm{sig}}}}=-\tfrac{1}{8}(R^{\mathcal{L}_{D^Z_{\mathrm{sig}}}}+8R^{\mathcal{L}_{D^Z}})=0.$$

Therefore (1-1) follows.

In dimension 6, by (2-26),

$$\begin{aligned}21\widehat{I}_{1/2}-\widehat{I}_{3/2}+8\widehat{I}_A&=21R^{\mathcal{L}_{D^Z}}-(R^{\mathcal{L}_{D^Z\otimes T_{\mathbb{C}}Z}}-R^{\mathcal{L}_{D^Z}})-R^{\mathcal{L}_{D^Z_{\mathrm{sig}}}}\\&=22R^{\mathcal{L}_{D^Z}}-R^{\mathcal{L}_{D^Z\otimes T_{\mathbb{C}}Z}}-R^{\mathcal{L}_{D^Z_{\mathrm{sig}}}}=0.\end{aligned}$$

Therefore (1-2) follows.

In dimension 10, by (2-28),

$$\begin{aligned}-\widehat{I}_{1/2}+\widehat{I}_{3/2}+\widehat{I}_A&=-R^{\mathcal{L}_{D^Z}}+(R^{\mathcal{L}_{D^Z\otimes T_{\mathbb{C}}Z}}-R^{\mathcal{L}_{D^Z}})-\tfrac{1}{8}R^{\mathcal{L}_{D^Z_{\mathrm{sig}}}}\\&=-\tfrac{1}{8}(16R^{\mathcal{L}_{D^Z}}-8R^{\mathcal{L}_{D^Z\otimes T_{\mathbb{C}}Z}}+R^{\mathcal{L}_{D^Z_{\mathrm{sig}}}})=0.\end{aligned}$$

Therefore (1-3) follows.

3.2.2 The case of odd-dimensional fibers Lott has the following theorem for the curvature of index gerbes.

Theorem 3.2.3 (Lott [21])

$$R^{\mathcal{G}_{D^Z\otimes V}}=\left\{\int_Z\widehat{A}(TZ,\nabla^Z)\operatorname{ch}(V,\nabla^V)\right\}^{(3)} \tag{3-20}$$

If TZ is $(8m+1)$–dimensional, integrating both sides of (3-14) along the fiber, we get

$$\left\{\int_Z\widehat{L}(TZ,\nabla^Z)\right\}^{(3)}-8\sum_{r=0}^{m}2^{6m-6r}\left\{\int_Z\widehat{A}(TZ,\nabla^Z)\operatorname{ch}(b_r(T_{\mathbb{C}}Z))\right\}^{(3)}=0. \tag{3-21}$$

Note that we have $\widehat{A}(TZ,\nabla^Z)\operatorname{ch}(F',\nabla^{F'})=\widehat{L}(TZ,\nabla^Z)$. So by Theorem 3.2.3 and (3-21), we have

$$\begin{aligned}&R^{\mathcal{G}_{B^Z_{\mathrm{sig}}}}-8\sum_{r=0}^{m}2^{6m-6r}R^{\mathcal{G}_{D^Z\otimes b_r(T_{\mathbb{C}}Z)}}\\&\quad=\left\{\int_Z\widehat{L}(TZ,\nabla^Z)\right\}^{(3)}-8\sum_{r=0}^{m}2^{6m-6r}\left\{\int_Z\widehat{A}(TZ,\nabla^Z)\operatorname{ch}(b_r(T_{\mathbb{C}}Z))\right\}^{(3)}\\&\quad=0.\end{aligned} \tag{3-22}$$

Therefore Theorem 2.3.1 follows.

The following theorem for the odd Chern form of a family of self-adjoint Dirac operators is due to Bismut and Freed.

Theorem 3.2.4 (Bismut and Freed [11]) $\int_Z \widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(V,\nabla^V)$ *represents the odd Chern character of* $\mathrm{ind}(D^Z\otimes V)$.

However, as $\mathrm{ind}(B^Z_{\mathrm{sig}})$ vanishes in odd K theory, we have

$$\mathrm{ch}(\mathrm{ind}(B^Z_{\mathrm{sig}}))=0\in H^{\mathrm{odd}}(Y)$$

and therefore Theorem 3.2.4 tells us that $[\int_Z \widehat{L}(TZ,\nabla^Z)]=0\in H^{\mathrm{odd}}(Y)$. In particular, by Theorem 3.2.3, $[R^{\mathcal{G}_{B^Z_{\mathrm{sig}}}}]=0\in H^3(Y)$. Therefore, (3-22) implies that

$$\sum_{r=0}^{m}2^{6m-6r}[R^{\mathcal{G}_{D^Z\otimes b_r(T_{\mathbb{C}}Z)}}]=0. \tag{3-23}$$

So Theorem 2.3.3 follows.

If r is a real number, let $\{r\}$ denote the image of r in $\mathbb{R}/\mathbb{Z}$. As noted in Atiyah, Patodi and Singer [4; 6], $\overline{\eta}_y(D^Z\otimes V)(0)$ has integer jumps and therefore $\{\overline{\eta}((D^Z\otimes V))\}$ is a C^∞ function of on Y with values in $\mathbb{R}/\mathbb{Z}$ [4; 6]. For odd-dimensional fibers, we have the following Bismut–Freed theorem for the reduced η–invariants.

Theorem 3.2.5 (Bismut and Freed [11])

$$d\{\overline{\eta}(D^Z\otimes V)\}=\left\{\int_Z \widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(V,\nabla^V)\right\}^{(1)} \tag{3-24}$$

If TZ is $(8m+3)$–dimensional, integrating both sides of (3-14) along the fiber, we get

$$\left\{\int_Z \widehat{L}(TZ,\nabla^Z)\right\}^{(1)}-8\sum_{r=0}^{m}2^{6m-6r}\left\{\int_Z \widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(b_r(T_{\mathbb{C}}Z))\right\}^{(1)}=0. \tag{3-25}$$

Then the Bismut–Freed Theorem 3.2.5 gives us

$$\begin{aligned} &d\{\overline{\eta}(B^Z_{\mathrm{sig}})\}-8\sum_{r=0}^{m}2^{6m-6r}d\{\overline{\eta}(D^Z\otimes b_r(T_{\mathbb{C}}Z))\}\\ &=\left\{\int_Z \widehat{L}(TZ,\nabla^Z)\right\}^{(1)}-8\sum_{r=0}^{m}2^{6m-6r}\left\{\int_Z \widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(b_r(T_{\mathbb{C}}Z))\right\}^{(1)}\\ &=0. \end{aligned} \tag{3-26}$$

Therefore we have

$$d(\{\overline{\eta}(B^Z_{\mathrm{sig}})\} - 8\sum_{r=0}^{m} 2^{6m-6r}\{\overline{\eta}(D^Z \otimes b_r(T_{\mathbb{C}}Z))\}) = 0. \tag{3-27}$$

Since Y is connected,

$$\{\overline{\eta}(B^Z_{\mathrm{sig}})\} - 8\sum_{r=0}^{m} 2^{6m-6r}\Big\{\overline{\eta}(D^Z \otimes b_r(T_{\mathbb{C}}Z))\Big\}$$

must be a constant function on Y. Therefore it is not hard to see that Theorem 2.4.1 follows.

3.3 Proofs of Theorems 2.2.2, 2.2.4, 2.3.2, 2.3.4 and 2.4.2

The proofs are similar to the proofs of Theorems 2.2.1, 2.2.3, 2.3.1, 2.3.3 and 2.4.1. If the dimension of TZ is $8m-1, 8m-2$ or $8m-3$, for the vertical tangent bundle TZ, set

$$Q_1(\nabla^Z,\tau) := \{\widehat{L}(TZ,\nabla^Z)\,\mathrm{ch}(\Theta_1(T_{\mathbb{C}}Z), \nabla^{\Theta_1(T_{\mathbb{C}}Z)})\}^{(8m)}, \tag{3-28}$$

$$Q_2(\nabla^Z,\tau) := \{\widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(\Theta_2(T_{\mathbb{C}}Z), \nabla^{\Theta_2(T_{\mathbb{C}}Z)})\}^{(8m)}. \tag{3-29}$$

Similar to Proposition 3.1, we have

$$Q_1(\nabla^Z,\tau) = \left\{\det{}^{1/2}\left(\frac{R^Z}{2\pi^2}\frac{\theta'(0,\tau)}{\theta(\frac{R^Z}{2\pi^2},\tau)}\frac{\theta_1(\frac{R^Z}{2\pi^2},\tau)}{\theta_1(0,\tau)}\right)\right\}^{(8m)}, \tag{3-30}$$

$$Q_2(\nabla^Z,\tau) = \left\{\det{}^{1/2}\left(\frac{R^Z}{4\pi^2}\frac{\theta'(0,\tau)}{\theta(\frac{R^Z}{4\pi^2},\tau)}\frac{\theta_2(\frac{R^Z}{4\pi^2},\tau)}{\theta_2(0,\tau)}\right)\right\}^{(8m)}. \tag{3-31}$$

Also $Q_1(\nabla^Z,\tau)$ is a modular form of weight $4m$ over $\Gamma_0(2)$ and $Q_2(\nabla^Z,\tau)$ is a modular form of weight $4m$ over $\Gamma^0(2)$. Moreover,

$$Q_1\Big(\nabla^Z,-\frac{1}{\tau}\Big) = 2^{4m}\tau^{4m}Q_2(\nabla^Z,\tau), \quad Q_1(\nabla^Z,\tau+1) = Q_1(\nabla^Z,\tau). \tag{3-32}$$

Similar to (3-12) and (3-13), by using Lemma 3.1 and (3-32), we have

$$\begin{aligned}
Q_2(\nabla^Z,\tau) = \{\widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(z_0(T_{\mathbb{C}}Z))\}^{(8m)}(8\delta_2)^{2m} \\
+ \{\widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(z_1(T_{\mathbb{C}}Z))\}^{(8m)}(8\delta_2)^{2m-2}\varepsilon_2 \\
+\cdots+ \{\widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(z_m(T_{\mathbb{C}}Z))\}^{(8m)}\varepsilon_2^m,
\end{aligned}$$

$$\begin{aligned}
Q_1(\nabla^Z,\tau) = 2^{4m}[\{\widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(z_0(T_{\mathbb{C}}Z))\}^{(8m)}(8\delta_1)^{2m} + \cdots \\
+ \{\widehat{A}(TZ,\nabla^Z)\,\mathrm{ch}(z_m(T_{\mathbb{C}}Z))\}^{(8m)}\varepsilon_1^m].
\end{aligned}$$

Comparing the constant term of the above equality, we have

$$\{\widehat{L}(TZ,\nabla^Z)\}^{(8m)}=\sum_{r=0}^{m}2^{6m-6r}\{\widehat{A}(TZ,\nabla^Z)\operatorname{ch}(z_r(T_{\mathbb{C}}Z))\}^{(8m)}. \tag{3-33}$$

Then one can integrate both sides of (3-33) along the fiber and combine the theorems of Bismut and Freed and Freed and Lott to obtain Theorems 2.2.2, 2.2.4, 2.3.2, 2.3.4 and 2.4.2.

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