The proper geometric dimension
of the mapping class group

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We show that the mapping class group of a closed surface admits a cocompact classifying space for proper actions of dimension equal to its virtual cohomological dimension.

1 Introduction

Let $\Gamma_{g,n}$ be the mapping class group of a connected orientable surface of genus g with n marked points. In this note we are interested in the minimal dimension $\underline{\mathrm{gd}}(\Gamma_{g,n})$ of a classifying space $\underline{E}\Gamma_{g,n}$ for proper actions of $\Gamma_{g,n}$. Recall that, given a discrete group G, the space $\underline{E}G$ is a contractible space on which G acts properly, and such that the fixed point set of a subgroup $H < G$ is contractible if H is finite, and is empty otherwise.

Since $\Gamma_{g,n}$ is virtually torsion-free, its virtual cohomological dimension $\mathrm{vcd}(\Gamma_{g,n})$ is a lower bound for $\underline{\mathrm{gd}}(\Gamma_{g,n})$; we remark, however, that there are groups for which the inequality is strict (Leary and Nucinkis [12]). In [7], Harer computed $\mathrm{vcd}(\Gamma_{g,n})$ for all $g, n \geq 0$; see Theorem 2.1 below. A central ingredient of Harer's argument is the construction, for $n > 0$, of a cocompact $\Gamma_{g,n}$–equivariant deformation retract (a *spine*) of Teichmüller space $\mathcal{T}_{g,n}$, of dimension $\mathrm{vcd}(\Gamma_{g,n})$. Work of Penner [18] and Hensel, Osajda and Przytycki [8] shows that Harer's spine is in fact an $\underline{E}\widehat{\Gamma}_{g,n}$ for the *pure* mapping class group $\widehat{\Gamma}_{g,n}$.

On the other hand, the case of closed surfaces of genus $g \geq 2$ is far from well-understood. Ji and Wolpert [10] used the fact that the Teichmüller space $\mathcal{T}_{g,n}$ is an $\underline{E}\Gamma_{g,n}$ to prove that the *thick part* $\mathcal{T}_{g,n}^{\geq \epsilon}$ of Teichmüller space is a cocompact $\underline{E}\Gamma_{g,n}$ for all $g, n \geq 0$ (see also Broughton [4] and Mislin [17] for an alternative construction). Ji [9] has recently exhibited cocompact spines of $\mathcal{T}_{g,0}$ of dimension less than $\dim(\mathcal{T}_{g,0})$ – but also greater than $\mathrm{vcd}(\Gamma_{g,0})$ – that serve as cocompact models of $\underline{E}\Gamma_{g,0}$. However, it is not known whether there is a cocompact spine of $\mathcal{T}_{g,0}$ of optimal dimension; see [3, Question 1.1].

More generally, Bridson and Vogtmann have asked whether if it possible to construct a cocompact $\underline{E}\Gamma_{g,0}$ of dimension equal to vcd$(\Gamma_{g,0})$; see [3, Question 2.3]. The purpose of this note is to prove the existence of such $\underline{E}\Gamma_{g,0}$:

Theorem 1.1 *For any $g \geq 0$ there exists a cocompact $\underline{E}\Gamma_{g,0}$ of dimension equal to* vcd$(\Gamma_{g,0})$. *In other words,* $\underline{\mathrm{gd}}(\Gamma_{g,0}) = \mathrm{vcd}(\Gamma_{g,0})$.

The main tool of our proof is the algebraic invariant $\underline{\mathrm{cd}}(\Gamma_{g,n})$ (see Section 3), which serves as the algebraic counterpart of $\underline{\mathrm{gd}}(\Gamma_{g,n})$. These two invariants are related in the same way as the ordinary cohomological dimension of a group G is related to the minimal dimension of an EG. For example, generalizing what happens in the torsion-free case, Lück [13] proved the following Eilenberg–Ganea-type theorem, which will play a central role in our proof:

Theorem 1.2 [13] *Let G be a group with $\underline{\mathrm{cd}}(G) = d \geq 3$. Then there is a d–dimensional $\underline{E}G$. Moreover, if G has a cocompact $\underline{E}G$ then it also admits a cocompact $\underline{E}G$ of dimension d.*

In the light of Lück's theorem, we will prove that $\underline{\mathrm{cd}}(\Gamma_{g,0}) = \mathrm{vcd}(\Gamma_{g,0})$ whenever $g \geq 3$, using a result of the second author stated as Theorem 3.3 below. The case $g \leq 2$ will require separate treatment.

Recall that the mapping class group $\Gamma_{g,n}^b$ of a surface with $b > 0$ boundary components is torsion-free, and therefore $\underline{\mathrm{gd}}(\Gamma_{g,n}^b) = \underline{\mathrm{cd}}(\Gamma_{g,n}^b) = \mathrm{vcd}(\Gamma_{g,n}^b)$, by Lück's Theorem 1.2. This fact, together with Theorem 1.1, plus a minor extension of our arguments (see Remark 4.5 below) yields the following:

Corollary 1.3 *For all $g, n, b \geq 0$, there exists a cocompact $\underline{E}\Gamma_{g,n}^b$ of dimension equal to* vcd$(\Gamma_{g,n}^b)$.

Acknowledgements

2 Preliminaries

Let $S_{g,n}$ be a connected, orientable surface of genus $g \geq 0$, with empty boundary and $n \geq 0$ marked points. The mapping class group $\Gamma_{g,n}$ is the group of isotopy classes of orientation-preserving homeomorphisms of $S_{g,n}$, where homeomorphisms and isotopies are required to map the set of marked points to itself. For simplicity, we will write $S_g := S_{g,0}$ and $\Gamma_g := \Gamma_{g,0}$.

2.1 Virtual cohomological dimension

Recall that $\Gamma_{g,n}$ has a torsion-free subgroup of finite index. As mentioned earlier, Harer [7] computed the virtual cohomological dimension $\mathrm{vcd}(\Gamma_{g,n})$ of $\Gamma_{g,n}$:

Theorem 2.1 (Harer) *If $2g + n > 2$, then*

$$\mathrm{vcd}(\Gamma_{g,n}) = \begin{cases} 4g + n - 4 & \text{if } g, n > 0, \\ 4g - 5 & \text{if } n = 0, \\ n - 3 & \text{if } g = 0. \end{cases}$$

Remark 2.2 If $n \leq 1$ then $\Gamma_{0,n}$ is trivial. Also, $\Gamma_{0,2} \cong \mathbb{Z}$ and $\Gamma_{1,0} \cong \mathrm{SL}_2(\mathbb{Z})$ (Farb and Margalit [6]). Therefore, $\mathrm{vcd}(\Gamma_{0,n}) = 0$ for $n \leq 1$, and $\mathrm{vcd}(\Gamma_{0,2}) = \mathrm{vcd}(\Gamma_{1,0}) = 1$.

2.2 Riemann–Hurwitz formula

Let $g \geq 2$. By the Nielsen realization theorem [11], every finite subgroup of Γ_g may be realized as a group of isometries with respect to some hyperbolic metric on S_g. Therefore, given a finite subgroup $L \leq \Gamma_g$, and slightly abusing notation, we may consider the (hyperbolic) orbifold S_g/L. We denote by g_L the genus of S_g/L; similarly, let k_L be the number of orbifold points of S_g/L, of orders $p_1^L, \ldots, p_{k_L}^L$, respectively. The tuple $(g_L; p_1^L, \ldots, p_{k_L}^L)$ is called the *signature* of L. Since $S_g \to S_g/L$ is an orbifold cover of degree $|L|$, the multiplicativity of the orbifold Euler characteristic implies that g and the signature of S_g/L are related by the so-called *Riemann–Hurwitz formula*; see eg [6]:

$$(1) \qquad \frac{2g - 2}{|L|} = 2g_L - 2 + l_L,$$

where

$$(2) \qquad l_L = \sum_{i=1}^{k_L} \left(1 - \frac{1}{p_i^L}\right).$$

Observe that (2) implies

$$(3) \qquad \frac{k_L}{2} \le l_L \le k_L.$$

We will need the following surely well-known observation:

Lemma 2.3 *Let $L < T$ be two distinct finite subgroups of Γ_g, with $g \ge 2$, and denote by $(g_L; q_1, \ldots, q_{k_L})$ and $(g_T; p_1, \ldots, p_{k_T})$ the signatures of S/L and S/T, respectively. Then:*

(i) *If $g_T > 1$ then $g_T < g_L$.*

(ii) *If $g_T \le 1$ then $g_T \le g_L$; moreover, if $g_T = g_L$ then $k_T < k_L$.*

Proof The map $S_g/L \to S_g/T$ is an orbifold cover of degree $d = [T : L] > 1$. By the Riemann–Hurwitz formula, we have

$$2 - 2g_L + \sum_{i=1}^{k_L} 1/q_i - k_L = d \left(2 - 2g_T + \sum_{i=1}^{k_T} 1/p_i - k_T \right).$$

Now

$$\sum_{i=1}^{k_L} 1/q_i = d \sum_{i=1}^{k_T} 1/p_i.$$

(see, for instance, [6, Section 7.2.2]). Hence

$$(4) \qquad 2g_L + k_L - 2 = d(2g_T - 2) + dk_T.$$

We prove the first claim of the lemma. Assume that $g_T > 1$ and suppose, for contradiction, that $g_L \le g_T$. From (4),

$$d(2g_T - 2) + dk_T = 2g_L - 2 + k_L \le 2g_T - 2 + dk_T,$$

which implies that $d(2g_T - 2) \le 2g_T - 2$, a contradiction since $d > 1$. We have thus proved part (i).

Moving on to the second claim of the lemma, assume $g_T \le 1$. We first prove that $g_T \le g_L$. Arguing again by contradiction, the only case to rule out is $g_T = 1$ and $g_L = 0$. From (4), we get: $k_L - 2 = dk_T \ge k_L$, which is impossible. Hence $g_L \le g_T$, as claimed.

Finally, we prove that $k_T < k_L$ whenever $g_T = g_L$; recall that the latter implies $g_T \in \{0, 1\}$. First, if $g_T = 1$, (4) gives $k_L = dk_T$, and thus $k_T < k_L$, as desired. If $g_T = 0$, again (4) yields $k_L - 2 = d(k_T - 2)$, which gives $k_T < k_L$ as well (observe that $k_L, k_T > 2$ since $g_T = g_L = 0$). This finishes the proof of the lemma.   □

3 Preliminaries on classifying spaces for proper actions

As mentioned in the introduction, we will determine $\underline{\mathrm{gd}}(\Gamma_{g,n})$ using the algebraic invariant $\underline{\mathrm{cd}}(\Gamma_{g,n})$, which is defined along the same lines as the ordinary cohomological dimension but in the setting of proper actions. Informally, it is the length of the shortest projective resolution of the trivial object in a certain category, whose objects are called Bredon modules. Here, we will only need to make use of two facts about $\underline{\mathrm{cd}}(G)$, referring the reader to Lück [13; 14] for a discussion on $\underline{\mathrm{cd}}(G)$.

The first fact about $\underline{\mathrm{cd}}(G)$ that we will need is Lück's Theorem 1.2, which is a consequence of [13, Theorem 13.19]. A proof of the existence of the model in Theorem 1.2 was given by Brady, Leary and Nucinkis [2]; we now explain how to adapt their argument to produce a cocompact one.

Let X be a cocompact $\underline{E}G$ of dimension d. The $(d-1)$–skeleton Z of X gives a chain complex of free Bredon modules, which is exact except possibly in degree $d-1$. Let M be the $(d-1)^{\mathrm{th}}$ homology group of Z. As in the classical case – see Brown [5, Section VIII, Lemma 2.1] – M is a projective Bredon module, which is finitely generated since X is cocompact. Note that M might not be free. However, the versions for Bredon modules, see [5, Section VIII, Lemma 4.4 and Proposition 6.5], together imply that there is some free Bredon module P such that $F := P \oplus M$ is free and that P can be taken to be finitely generated, so that F is also finitely generated. A finitely generated free Bredon module is determined by a finite family of representatives of conjugacy classes of finite subgroups. Denote by Λ_P, Λ_F, respectively, the families for P and F. Now, attach to Z orbits of $(d-1)$–cells of types $S^{d-1} \times G/H$ for $H \in \Lambda_P$, and use Hurewicz's Theorem to attach orbits of d–cells of types $S^d \times G/H$ for $H \in \Lambda_F$. This way we get a new cocompact CW–complex Y such that the fixed point set of $L < G$ is contractible whenever L is finite and empty otherwise. In other words, Y is the desired model for $\underline{E}G$.

Remark 3.1 Theorem 1.2 also holds if $d = 1$; see [2].

Before we describe the second property of $\underline{\mathrm{cd}}(G)$ that will be used, we need some definitions. Consider, for every finite subgroup $H < G$, the *Weyl group*

$$WH := N_G(H)/H,$$

where $N_G(H)$ denotes the normalizer of H in G. Observe that the centralizer $Z_G(H)$ of H has finite index in $N_G(H)$, and thus WH and $Z_G(H)$ are weakly commensurable. Let $\mathcal{F}_H = \{T \leq G \text{ finite} \mid H < T\}$, noting that the group WH acts on the poset $\mathcal{F}_H$ by conjugation.

Let $\mathcal{F}_{H\bullet}$ be the chain complex of G–modules associated to the geometric realization of $\mathcal{F}_H$, and let $\Sigma\widetilde{\mathcal{F}}_{H\bullet}$ be the result of augmenting and suspending $\mathcal{F}_{H\bullet}$. Finally, write $\mathrm{pd}_{WH}\,\Sigma\widetilde{\mathcal{F}}_{H\bullet}$ for the projective dimension of the chain complex $\Sigma\widetilde{\mathcal{F}}_{H\bullet}$, namely the shortest length of a chain complex $P_\bullet$ of projective G–modules such that there is a morphism $P_\bullet \to \Sigma\widetilde{\mathcal{F}}_{H\bullet}$ inducing an isomorphism in the homology groups.

A result of Connolly and Kozniewski, stated as [16, Theorem A], implies

$$\underline{\mathrm{cd}}(G) = \max_{H \leq G \text{ finite}} \mathrm{pd}_{WH}\,\Sigma\widetilde{\mathcal{F}}_{H\bullet}. \tag{5}$$

We will need:

Definition 3.2 (Length) The *length* $\lambda(L)$ of a finite group L is the largest number $i \in \mathbb{N} \cup \{0\}$ for which there is a sequence $1 = L_0 < L_1 < \cdots < L_i = L$.

We are finally ready to introduce the promised second fact about $\underline{\mathrm{cd}}$, which follows as an easy consequence of (5):

Theorem 3.3 *Let G be a virtually torsion-free group such that for any $H \leq G$ finite, $\mathrm{vcd}(WH) + \lambda(H) \leq \mathrm{vcd}(G)$. Then $\underline{\mathrm{cd}}(G) = \mathrm{vcd}(G)$.*

Proof For $i \geq 0$, the ith term of $\Sigma\widetilde{\mathcal{F}}_{H\bullet}$ is the permutation module associated to the action of G on the cells of the form $T := H_i > \cdots > H_1 > H_0 = H$, whose stabilizer is weakly commensurable with WT. Observe that $i + \lambda(H) \leq \lambda(T)$. Therefore

$$\mathrm{pd}_{WH}\,\Sigma\widetilde{\mathcal{F}}_{H\bullet} \leq \max\{\lambda(T) - \lambda(H) + \mathrm{vcd}\,WT \mid T \in \mathcal{F}_H \cup \{H\}\} \leq \mathrm{vcd}(G).$$

Using (5) we get $\underline{\mathrm{cd}}(G) \leq \mathrm{vcd}(G)$; the other inequality is well known. $\square$

4 Proof of Theorem 1.1

In the light of Theorem 3.3, we are going to need to understand the relation between $\mathrm{vcd}(\Gamma_g)$ and $\mathrm{vcd}(WL)$, for every finite subgroup $L < \Gamma_g$. The following is well-known; see, for instance, Maher [15, Proposition 2.3]:

Lemma 4.1 *Let $L \leq \Gamma_g$ be a finite subgroup of signature $(g_L; p_1^L, \ldots, p_{k_L}^L)$. Then WL has finite index in Γ_{g_L, k_L}. In particular, $\mathrm{vcd}(WL) = \mathrm{vcd}(\Gamma_{g_L, k_L})$.*

For notation purposes, it will be convenient to write

$$\nu(L) := 4g_L + k_L - 4.$$

Observe that, from Theorem 2.1 and Lemma 4.1, we have

$$
(6) \qquad \mathrm{vcd}(WL) = \begin{cases} \nu(L) & \text{if } g_L, k_L > 0, \\ \nu(L) - 1 & \text{if } k_L = 0, \\ \nu(L) + 1 & \text{if } g_L = 0. \end{cases}
$$

We will need:

Proposition 4.2 Let $L < T$ be finite subgroups of Γ_g, where $g \geq 2$. Assume that $g_T < g_L$. Then $\mathrm{vcd}(WT) < \mathrm{vcd}(WL)$, unless we are in one of the following two cases:

(i) $(g_L, k_L) = (2, 0)$ and $(g_T, k_T) = (0, 6)$.

(ii) $(g_L, k_L) = (1, r)$ and $(g_T, k_T) = (0, r + 3)$, for some $r \geq 1$.

Proof First, observe that $L < T$ implies $Z_{\Gamma_g}(T) \leq Z_{\Gamma_g}(L)$, and thus $\mathrm{vcd}(WT) \leq \mathrm{vcd}(WL)$. Using (3) and the Riemann–Hurwitz formula we deduce that

$$
(7) \qquad \nu(T) = 4g_T - 4 + k_T \leq 4g_T - 4 + 2l_T = \frac{4g_L - 4 + 2l_L}{|T : L|}
$$

$$
\leq \frac{4g_L - 4 + 2k_L}{|T : L|} = \frac{\nu(L) + k_L}{|T : L|} \leq \frac{\nu(L) + k_L}{2}.
$$

Armed with inequality (7), and noting that $g_L > 0$, we distinguish the following cases:

Case 1 $(g_T > 0)$ We have the following subcases:

(1a) $k_T = k_L = 0$: Since $g_T < g_L$ then $\mathrm{vcd}(WT) < \mathrm{vcd}(WL)$.

(1b) $k_L = 0, k_T \neq 0$: Since $g_L \geq 2$, we have that $\nu(L) \geq 4$. From (6),

$$
\mathrm{vcd}(WT) = \nu(T) \leq \frac{\nu(L)}{2} < \nu(L) - 1 = \mathrm{vcd}(WL).
$$

(1c) $k_L \neq 0, k_T = 0$: Note that $g_T \geq 2$, and so $g_L \geq 3$. In particular, $\mathrm{vcd}(WL) = 4g_L - 4 + k_L \geq k_L + 8$. Therefore, using (6) and (7),

$$
\mathrm{vcd}(WT) = \nu(T) - 1 \leq \frac{\nu(L) + k_L - 2}{2} < \mathrm{vcd}(WL).
$$

(1d) $k_L \neq 0, k_T \neq 0$: In this case, since $0 < g_T < g_L$, then $\mathrm{vcd}(WT) < \mathrm{vcd}(WL)$.

Case 2 $(g_T = 0)$ Note that $k_T > 0$. We have the following subcases:

(2a) $k_L = 0$: Again by (1), $g_L \geq 2$, and in particular $\mathrm{vcd}(WL) \geq 3$. From this, and using (6) and (7), we deduce

$$
\mathrm{vcd}(WT) - 1 \leq \frac{\mathrm{vcd}(WL) + 1}{2} < \mathrm{vcd}(WL) - 1
$$

unless $\text{vcd}(WL) = 3$. In the latter case, either $\text{vcd}(WT) < \text{vcd}(WL)$ or $(g_L, k_L) = (2, 0)$ and $(g_T, k_T) = (0, 6)$, as claimed.

(2b) $k_L \neq 0$: Suppose first that $g_L \geq 2$, in which case $\text{vcd}(WL) = 4g_L - 4 + k_L \geq k_L + 4$. From (6) and (7), we obtain

$$\text{vcd}(WT) - 1 \leq \frac{\text{vcd}(WL) + k_L}{2} \leq \frac{2\,\text{vcd}(WL) - 4}{2} < \text{vcd}(WL) - 1,$$

and thus the result follows. Suppose now that $g_L < 2$, and thus $g_L = 1$ as $0 = g_T < g_L$. As $\text{vcd}\,WT \leq \text{vcd}\,WL$, in the equality case we have

$$k_L = \text{vcd}(WL) = \text{vcd}(WT) = k_T - 3,$$

and we are in part (ii) of the theorem. $\qquad\square$

Remark 4.3 Cases (i) and (ii) in Proposition 4.2 do occur in practice. Indeed, there is a branched double-cover $S_{2,0} \to S_{0,6}^*$, where $S_{0,6}^*$ denotes a sphere with six cone points of angle π, induced by the hyperelliptic involution of $S_{2,0}$. By a result of Birman and Hilden [1], we may realize $\Gamma_{0,6}$ as a subgroup of index 2 in $\Gamma_{2,0}$. Along similar lines, $\Gamma_{0,5}$ is a subgroup of index 2 in $\Gamma_{1,2}$, arising from the hyperelliptic involution of $S_{1,2}$.

The next result is the key technical observation of this note:

Proposition 4.4 *If $g \geq 3$, then for any $T < \Gamma_g$ finite,*

$$\text{vcd}(WT) + \lambda(T) \leq \text{vcd}(\Gamma_g).$$

Proof Our first objective is to establish the following:

Claim Let $1 \neq T < \Gamma_g$ be finite, where $g \geq 3$. If $g_T > 0$, then

$$(8) \qquad\qquad \text{vcd}(WT) + \lambda(T) + 1 \leq \text{vcd}(\Gamma_g).$$

Proof of Claim Using (1) and (3), and since $g_T > 0$, we have

$$\frac{\text{vcd}(\Gamma_g) + 1}{|T|} = \frac{4g - 4}{|T|} = 4g_T - 4 + 2l_T \geq 4g_T - 4 + k_T$$

$$= \nu(T) \geq \text{vcd}(WT).$$

Rearranging, we obtain $\text{vcd}(\Gamma_g) \geq |T|\,\text{vcd}(WT) - 1$. In particular, observe that (8) is satisfied whenever

$$(9) \qquad\qquad \text{vcd}(WT) + \lambda(T) + 2 \leq |T|\,\text{vcd}(WT)$$

holds. We distinguish the following cases, depending on the value of $\mathrm{vcd}(WT)$:

- (i) If $\mathrm{vcd}(WT) \geq 3$, then 4 is true for all finite subgroups $T \leq \Gamma_g$, as $\lambda(G) \leq |G| - 1$ for every finite group G.
- (ii) If $\mathrm{vcd}(WT) = 2$, then 4 holds unless $|T| = 2$, again since $\lambda(T) \leq |T| - 1$. But if $|T| = 2$ then $\lambda(T) = 1$, and thus (8) follows because $\mathrm{vcd}(\Gamma_g) \geq 7$ as $g \geq 3$.
- (iii) If $\mathrm{vcd}(WT) = 1$ then 4 is satisfied unless $|T| \in \{2, 3, 4\}$. To see this, observe that if T has a maximal subgroup satisfying 4, then the same holds for T, and that groups of orders $8, 9, 6, p$ for p a prime $p > 3$ satisfy 4. In the remaining cases $\lambda(T) \leq 2$, and hence (8) follows as in the previous case since $g \geq 3$.

This finishes the proof of the claim. $\qquad\square$

Returning to the proof of the proposition, let $T \leq \Gamma_g$ be a finite subgroup. If $T = 1$ then the result is trivial, and if $g_T > 0$, then it follows from the claim above. Therefore, assume that $T \neq 1$ and $g_T = 0$. Let $L \leq T$ be such that $\lambda(T) = \lambda(L) + 1$. Suppose first that $L = 1$, noting that $\lambda(T) = 1$. Since $g_L = g \geq 3$, Proposition 4.2 implies that $\mathrm{vcd}(WT) < \mathrm{vcd}(WL) = \mathrm{vcd}(\Gamma_g)$, and so we are done. Thus assume that $L \neq 1$. If $g_L > 0$, the claim above yields

$$\mathrm{vcd}(WT) + \lambda(T) \leq \mathrm{vcd}(WL) + \lambda(L) + 1 \leq \mathrm{vcd}(\Gamma_g).$$

On the other hand, if $g_L = 0$ then $k_T < k_L$, by Lemma 2.3. Thus

$$\mathrm{vcd}(WT) + \lambda(T) = k_T - 3 + \lambda(T)$$
$$< \mathrm{vcd}(WL) + \lambda(L) + 1.$$

Hence $\mathrm{vcd}(WT) + \lambda(T) \leq \mathrm{vcd}(WL) + \lambda(L)$, and the result follows by induction on the length of T. $\qquad\square$

We are finally ready to prove Theorem 1.1:

Proof of Theorem 1.1 First, if $g \geq 3$, the result follows combining Lück's Theorem 1.2 with Theorem 3.3 and Proposition 4.4. If $g = 0$ then $\Gamma_{0,0} = 1$ so the result is trivial. Next, if $g = 1$ then $\Gamma_{1,0} = \mathrm{SL}_2(\mathbb{Z})$ and one can take the dual tree to the Farey graph as a model of $\underline{E}\Gamma_{1,0}$. Finally, $\Gamma_{2,0}$ is a central extension of $\Gamma_{0,6}$ by $\mathbb{Z}_2$. Let X be a cocompact model for $\underline{E}\Gamma_{0,6}$ of dimension $\mathrm{vcd}(\Gamma_{0,6})$ – Harer's spine, for instance. Then the action of $\Gamma_{0,6}$ on X can be lifted to an action of $\Gamma_{2,0}$, and thus X is a cocompact model for $\underline{E}\Gamma_{2,0}$ too. Since $\mathrm{vcd}(\Gamma_{2,0}) = \mathrm{vcd}(\Gamma_{0,6})$, we are done. $\qquad\square$

Remark 4.5 An inductive argument along the lines of [17, Section 4], using the Birman short exact sequence, [14, Theorem 5.16] and Harer's formula yields the analogue of Theorem 1.1 for surfaces with a non-empty set of marked points.

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