Algebraic & Geometric Topology 14 (2014) 283–297

The bumping set and the characteristic submanifold

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We show here that the Nielsen core of the bumping set of the domain of discontinuity of a Kleinian group Γ is the boundary of the characteristic submanifold of the associated 3–manifold with boundary. Some examples of interesting characteristic submanifolds are given. We also give a construction of the characteristic submanifold directly from the Nielsen core of the bumping set. The proofs are from "first principles", using properties of uniform domains and the fact that quasi-conformal discs are uniform domains.

30F40, 57M60

1 Notation and background

In this paper we show that a particular aspect of the 3–dimensional topology of a hyperbolic 3–manifold with incompressible boundary can be deduced from the domain of discontinuity on the sphere at infinity.

Let G be a Kleinian group without torsion where $\Lambda(G)$ is the limit set and $\Omega(G)$ is the domain of discontinuity. We denote the quotient 3–manifold with boundary, $(\mathbb{H}^3 \cup \Omega(G))/G$, by $M(G)$. We require throughout that $M(G)$ is geometrically finite with incompressible, quasi-Fuchsian boundary components. The group G is finitely generated and the components of the domain of discontinuity are all discs. The conformal boundary of $M(G)$, $\Omega(G)/G$, is a finite union of finite area surfaces by Ahlfors' finiteness theorem. Then since each surface subgroup is quasi-Fuchsian, each component of $\Omega(G)$ is a quasi-disc, the image of the standard unit disc in the complex plane under a quasi-conformal homeomorphism. The closure of any one component is a closed disc. In this case we say, by abuse of notation, that G is a *geometrically finite Kleinian group with incompressible boundary*.

We will be interested in where components of the domain of discontinuity meet. Accordingly, define $\mathrm{Bump}(C_1, C_2, \ldots, C_n)$ to be $\overline{C}_1 \cap \overline{C}_2 \cap \cdots \cap \overline{C}_n$ where $C_1, \ldots, C_n$ are components of the domain of discontinuity.

Published: 9 January 2014 DOI: 10.2140/agt.2014.14.283

Let C be a component of the domain of discontinuity $\Omega(G)$. The *Nielsen core* of a subset X of ∂C is the convex hull of X in the Poincaré metric on C. This is a subset of $\Omega(G)$. The Nielsen core is well defined as the Riemann map from the interior of the unit disc to C extends to the boundary by Osgood and Taylor [11]. Let $\mathrm{Niel}_{C_1}(\mathcal{C})$ denote the Nielsen core of the bumping set of components $\mathcal{C} = \{C_1, \dots, C_n\}$ in C_1. The *Nielsen core of the bumping set of* $\mathcal{C} = \{C_1, \dots, C_n\}$ is $\bigcup_i \mathrm{Niel}_{C_i}(\mathcal{C})$, which we denote simply by $\mathrm{Niel}(\mathcal{C})$.

If the Nielsen core of any bumping set is non-trivial, this is an obstruction to $M(G)$ admitting a hyperbolic metric with totally geodesic boundary, and by work of Thurston (see [8, Theorem 6.2.1]) this is the only obstruction. Maskit showed:

Lemma 1.1 (Maskit [9, Theorem 3]) $\Lambda(G_C \cap G_B) = \Lambda(G_C) \cap \Lambda(G_B) = \bar{C} \cap \bar{B}$

This was later generalized by Anderson [1].

In Section 2 we show that the image of these Nielsen cores in the quotient manifold with boundary is in general a subsurface of the boundary.

Theorem 2.1 *Let G be a geometrically finite Kleinian group with incompressible boundary and let ρ: $\mathbb{H}^3 \cup \Omega(G) \to M(G)$ be the covering map induced by the action of G. Suppose that $C_1, \dots, C_n$ are components of $\Omega(G)$ with non-trivial Nielsen core of the bumping set $\mathrm{Bump}(C_1, \dots, C_n)$. Then the image of $\mathrm{Niel}_{C_1}(C_1, \dots, C_n)$ in $\partial M(G)$ is either a simple geodesic or a subsurface of $\rho(C_1)$ bounded by geodesics.*

In Section 3, we show that these subsurfaces are exactly the boundary of the characteristic submanifold of $M(G)$. More precisely:

Theorem 3.1 *Let G be a geometrically finite Kleinian group with incompressible boundary and let ρ: $\mathbb{H}^3 \cup \Omega(G) \to M(G)$ be the covering map induced by the action of G. Then consider*

$$S' = \bigsqcup_C \rho(\mathrm{Niel}_C(\mathcal{C})),$$

where C ranges over the components of $\Omega(G)$ and $\mathcal{C}$ is a collection of components of $\Omega(G)$ containing C, which has non-trivial bumping set. We require that each collection be maximal in the sense that adding any other components would strictly decrease the bumping set. Let S be S' with any simple closed curves replaced by regular annular neighborhoods of these curves. Then S, considered as a disjoint union of components as above, is the boundary of the characteristic submanifold of $M(G)$.

Since the boundary of the characteristic submanifold is a union of convex hulls, this gives a geometric structure to this region, which is of course dependent on the representation. Some background and examples of characteristic submanifolds are given in Section 3.

2 The image of the Nielsen core of the bumping set

Here we show that the image of the Nielsen core of the bumping set of two or more components is a union of simple closed essential curves and essential subsurfaces of the boundary components. This image, with any simple closed curves thickened, will form the boundary of the characteristic submanifold of the quotient $M(G)$. Recall that the stabilizer $G_C \subset G$ of a component C of $\Omega(G)$ has a representation ϕ into $\mathrm{PSL}(2,\mathbb{R})$ induced by the uniformization map from the unit disc. An *accidental parabolic* is an element $g \in G_C$ that is parabolic in G but where $\phi(g)$ is hyperbolic. Note that our definition of Kleinian group with incompressible boundary, requiring that the closure of any component is a disc, rules out accidental parabolics.

Theorem 2.1 *Let G be a geometrically finite Kleinian group with incompressible boundary and let ρ: $\mathbb{H}^3 \cup \Omega(G) \to M(G)$ be the covering map induced by the action of G. Suppose that $C_1, \dots, C_n$ are components of $\Omega(G)$ with non-trivial Nielsen core of the bumping set* $\mathrm{Bump}(C_1, \dots, C_n)$*. Then the image of* $\mathrm{Niel}_{C_1}(C_1, \dots, C_n)$ *in $\partial M(G)$ is either a simple geodesic or a subsurface of $\rho(C_1)$ bounded by geodesics.*

We note that a similar theorem is proven in Maskit [9, Theorem 1] although this uses the deep work of his decomposition theorem, which we do not use. There is also a similar statement in Lecuire [6]. Bill Thurston understood the characteristic submanifold from this point of view in his discussion of the window in [12]. However, we wish to emphasize that this result follows directly from properties of a group of quasi-conformal maps acting on the sphere at infinity and is self-contained.

Proof We will consider the image of a boundary curve β of $\mathrm{Niel}_{C_1}(C_1, \dots, C_n)$. Recall that $\mathrm{Niel}_{C_1}(C_1, \dots, C_n)$ is the Nielsen core in C_1 of the bumping set $\overline{C}_1 \cap \overline{C}_2 \cap \cdots \cap \overline{C}_n$. The curve β is necessarily a geodesic since it is the boundary curve of a convex hull. If the convex hull consists of a geodesic going between the two points of the bumping set, we consider this to be a boundary curve. The strategy of the proof is the following. We show that the image of β (1) is simple, (2) does not accumulate, and (3) does not exit a cusp. Therefore the image of a boundary curve β is an essential simple closed curve on the surface $\rho(C_1)$. Since the map is a covering map, the image will be bounded by essential simple closed curves, and hence will be either a simple closed curve or a subsurface of $\rho(C_1)$.

(1) The image of a boundary curve β is simple. The pre-image of the curve $\rho(\beta)$ in C_1 is the orbit of β under the action of G_{C_1}. If the image were not simple, then its pre-images in C_1 would intersect. Thus assume that there is a γ in the stabilizer of C_1 such that $\gamma(\beta)$ intersects β transversely. Since β is the boundary curve of a convex set,

this implies $\gamma \notin \mathrm{Stab}(C_i)$ for some $i \in \{2, 3, \dots, n\}$. Two circles on the two-sphere that intersect transversely must intersect more than once. Consider the convex hulls of the endpoints of β in $\overline{C}_1$ and $\overline{C}_i$ and the endpoints of $\gamma(\beta)$ in $\overline{C}_1$ and $\gamma(\overline{C}_i)$. These are two circles in $\overline{C}_1 \cup \overline{C}_i \cup \gamma(\overline{C}_i)$ that intersect exactly once transversely in C_1 and that do not intersect on the boundaries of the components. Thus the circles intersect again, which implies C_i and $\gamma(C_i)$ intersect in their interior. This contradicts the fact that they are distinct components.

(2) The image of a boundary curve does not accumulate. Here we use that a quasi-disc is a uniform domain.

Definition 2.2 A domain A is *uniform* if there are constants a and b such that every pair of points $z_1, z_2 \in A$ can be joined by an arc α in A with the following properties:

(1) The Euclidean length of α satisfies $l(\alpha) \leq a|z_1 - z_2|$.

(2) For every $z \in \alpha$, $\min(l(\alpha_1), l(\alpha_2)) \leq bd(z, \partial A)$, where α_1 and α_2 are the components of $\alpha \setminus z$.

By Lehto [7, Part I, Theorem 6.2], a K–quasi-disc is a uniform domain with constants a and b, which depend only on K. Now suppose that the image of a boundary curve β of $\mathrm{Niel}_{C_1}(C_1, \dots, C_n)$ accumulates in $\rho(C_1)$. Then the images of β in C_1 under the action of $\mathrm{Stab}(C_1)$ accumulate. Since β is geodesic, there is a sequence $\{\gamma_i\}$ in $\mathrm{Stab}(C_1)$ such that the endpoints of $\gamma_i(\beta)$ accumulate in $\partial(C_1)$ in the Poincaré metric to two points p and q. Since β is a boundary curve, there is some component $B \in \{C_2, \dots, C_n\}$ such that for an infinite number of γ_i, $\gamma_i(B)$ are all distinct from each other and from B. We continue to call this subsequence $\{\gamma_i\}$. Since the γ_i all act conformally on S^2_∞, each $\partial\gamma_i(B)$ is a K_B–quasi-circle where K_B is the quasi-conformal constant associated to B. The point is that the $\gamma_i(B)$ will eventually be too skinny to satisfy a fixed b in condition (2) above.

There are points p_i and q_i of the $\gamma_i(B)$ that are accumulating to p and q. Consider circles centered at p that separate p and q. Then there is some such circle $C(p, r)$ that separates infinitely many p_i from the associated q_i, and that separates p from q. Now consider arcs α_i in $\gamma_i(B)$ that connect p_i and q_i. Let z_i be a point on $\alpha_i \cap C(p, r)$. Then the distances $d(z_i, \partial\gamma_i(B))$ are going to zero since the $\partial\gamma_i(B) \cap C(p, r)$ are accumulating in $C(p, r)$. However, the distances from z_i to p_i and from z_i to q_i are bounded strictly above zero. This is because we may assume that the p_i are contained in a closed disc, which has positive distance from $C(p, r)$. We may assume the same thing for the q_i. Since the lengths of the arcs of $\alpha_i \setminus z_i$ are bounded below by the distances $d(z_i, p_i)$ and $d(z_i, q_i)$, this contradicts property (2) above of a uniform domain in Definition 2.2. Since quasi-discs are uniform domains [7, Part I, Theorem 6.2], the image of a boundary curve cannot accumulate.

(3) Next we claim that the image of a boundary curve c of $\mathrm{Niel}_{C_1}(C_1,\dots,C_n)$ does not exit a cusp.

Suppose that the image of a boundary curve β of $\mathrm{Niel}_{C_1}(C_1,\dots,C_n)$ does exit a cusp. Then in C_1, one of the endpoints of β in ∂C_1 is the fixed point of a parabolic element $\gamma_p \in \mathrm{Stab}(C_1)$. Call this point p. Let B be another component of $\Omega(\Gamma)$ whose boundary contains p.

We first claim that γ_p also stabilizes B. Indeed, suppose not and conjugate so that γ_p fixes the point at infinity and translates by the action $z \to z+1$. If γ_p does not stabilize B, the interiors of $\gamma_p^n(B)$ are distinct for $n \in \mathbb{Z}$. The quasi-circles $\partial\gamma_i(B)$ all go through infinity. Therefore, since the transformations γ_p and γ_p^{-1} are translations that take B off of itself, any point of B is at most distance 1 from ∂B. Since ∂B goes through ∞, there are points z_1 and z_2 of B such that $|z_1 - z_2| > 2b$, for any constant b. Then any arc α connecting z_1 and z_2 contains a point z (the midpoint) such that $\min(l(\alpha_1), l(\alpha_2)) \geq \frac{1}{2}|z_1 - z_2| > b \geq b d(z, \partial B)$, where α_1 and α_2 are the components of $\alpha \setminus z$. This is a contradiction, since B is a quasi-disc and hence a uniform domain.

Thus γ_p stabilizes B, where B is any component of $\Omega(G)$ such that $p \in \partial B$. But this contradicts the assumption that β is a boundary curve. Indeed, let q be the other endpoint of β. Then $\gamma_p^n(q)$ and $\gamma_p^{-n}(q)$ will approach p from both sides. Since γ_p stabilizes $\mathrm{Bump}(C_1,\dots,C_n)$, β cannot be a boundary curve of the convex hull of this set. Therefore, the image of a boundary curve β of $\mathrm{Niel}_{C_1}(C_1,\dots,C_n)$ does not exit a cusp.

Since the image of a boundary curve is simple, does not accumulate, and does not exit a cusp, it is a simple closed curve. It is a geodesic in the Poincaré metric on C_1 as it is the boundary of a convex hull of points on the boundary ∂C_1. Since $\rho\colon \mathbb{H}^3 \cup \Lambda(G) \to M(G)$ is a covering map, the image of boundary curves are boundary curves of the image. This proves Theorem 2.1. $\square$

3 The characteristic submanifold

The characteristic submanifold of $(M(G), \partial M(G))$ is a 3–submanifold (X_M, S_M) of $(M(G), \partial M(G))$ such that all of the essential tori and annuli in $M(G)$ can be properly isotoped into (X_M, S_M). It was defined and studied extensively by Jaco and Shalen [3] and Johannson [4]. See also Kapovich [5, 1.8]. It is defined by the following properties.

(1) Each component (X, S) of (X_M, S_M) is an I–bundle over a surface or a solid torus equipped with a Seifert fibered structure.
(2) The components of $\partial X \setminus \partial M(G)$ are essential annuli.
(3) Any essential annulus (or Möbius band) is properly homotopic into (X_M, S_M).
(4) (X, S) is unique up to isotopy.

A component of the characteristic submanifold that is a Seifert fibered solid torus can be described from the point of view of the domain of discontinuity as in the following example. Suppose that there are three components A, B and C of the domain of discontinuity $\Omega(G)$ such that $\overline{A} \cap \overline{B} \cap \overline{C} = \{p, q\}$ and ϕ is a pseudo-Anosov element of G that fixes p and q such that $\phi(A) = B$, $\phi(B) = C$ and $\phi(C) = A$. Thus $\phi^3 \in \mathrm{Stab}(A)$. Now consider the solid torus T_ϕ that is the quotient of a regular ϕ–invariant neighborhood of the geodesic g_ϕ in $\mathbb{H}^3$ with endpoints p and q. There is also an annulus in $\rho(A)$ that is the quotient of a regular neighborhood of the image of the geodesic in the Poincaré metric on A, g_A, with endpoints p and q. This g_A is stabilized by ϕ^3. Suppose further that A, B and C are the only components of $\Omega(G)$ whose closures meet p and q. This annulus in $\rho(A)$ and an annulus on ∂T_ϕ co-bound an annulus $\times I$. There is a preimage of this annulus $\times I$ invariant under ϕ in the universal cover that has three pieces, all of which meet the boundary of a regular ϕ–invariant neighborhood of g_ϕ. One piece meets A, the other B, and the third C. The annulus on ∂T_ϕ wraps three times around in the direction invariant by ϕ. Then there is a component X of the characteristic submanifold of $M(G)$ that is T_ϕ union the annulus $\times I$. This is a solid torus fibered by that which wind three times around the core. The boundary of X is an annulus on $\partial M(G)$ union an annulus in the interior of $M(G)$. Thus $\partial X \setminus \partial M(G)$ is an annulus. This is one of the cases described in the proof of Theorem 3.1 below.

For now we give another example, where the union of the convex hulls of the bumping sets is all of $\Omega(G)$. Figure 1 shows the limit set of a quasi-Fuchsian free group of rank 2 acting on $\mathbb{C} \cup \infty = S^2_\infty$. This picture was made with Curt McMullen's lim program [10]. The stabilizer of either component is the whole group. Figure 2 shows what happens when we adjoin the square root of one of the generators. If we denote the square root by γ, then γ switches two components of $\Omega(G)$ that meet at the endpoints of the geodesic invariant by γ. These are the center and outer components in Figure 2. γ^2 is in the stabilizer of both. As above, there is a component of the characteristic submanifold which is a Seifert fibered torus and whose boundary is a union of two annuli. In this case, we can also think of this component as a twisted I–bundle.

We will show here that the Nielson core of the bumping set, taken over all components of $\Omega(G)$, is the boundary of the characteristic submanifold of $M(G)$.

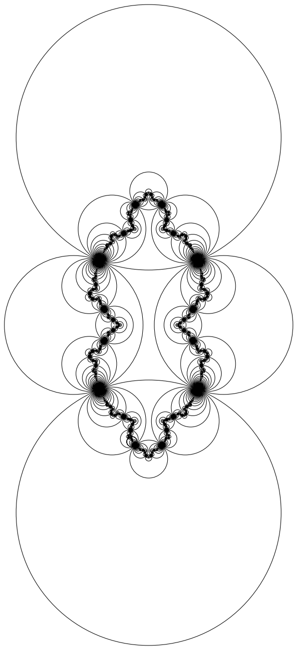

Figure 1: A quasi-Fuchsian punctured torus

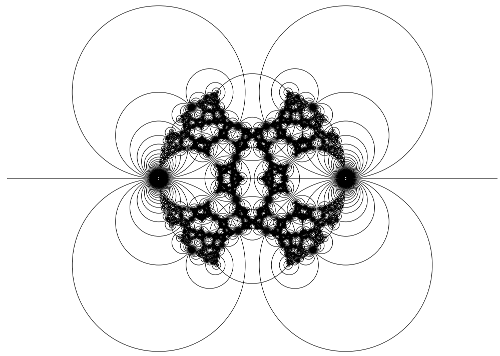

Figure 2: After adjoining a square root

Theorem 3.1 *Let G be a geometrically finite Kleinian group with incompressible boundary and let $\rho\colon \mathbb{H}^3 \cup \Omega(G) \to M(G)$ be the covering map induced by the action of G. Then consider*

$$S' = \bigsqcup_{C} \rho(\mathrm{Niel}_C(\mathcal{C})),$$

where C ranges over the components of $\Omega(G)$ and $\mathcal{C}$ is a collection of components of $\Omega(G)$ containing C which has non-trivial bumping set. We require that each collection be maximal in the sense that adding any other components would strictly decrease the bumping set. Let S be S' with any simple closed curves replaced by regular annular neighborhoods of these curves. Then S, considered as a disjoint union of components as above, is the boundary of the characteristic submanifold of $M(G)$.

To this end, we will need the following lemma:

Lemma 3.2 *Let G be a geometrically finite Kleinian group with incompressible boundary. Let C be a component of $\Omega(G)$. Assume $\phi \in \mathrm{Stab}(C)$ and that the fixed points of ϕ in ∂C are in $\mathrm{Bump}(C, B)$. Then $\phi \in \mathrm{Stab}(B)$.*

Proof The transformation ϕ is either parabolic or hyperbolic, by which we mean either strictly hyperbolic or loxodromic. The parabolic case is contained in (3) of the proof of Theorem 2.1.

Suppose that ϕ is hyperbolic. We may assume that ϕ fixes 0 and ∞. Then the boundary of B, and the boundary of $\phi^n(B)$ for all n, meet 0 and ∞.

If the $\phi^n(B)$ are all distinct, then they accumulate along the circle $|z| = 1$. Thus there are pairs of points in the $\phi^n(B)$ with one element of the pair close to 0 and the other on the circle $|z| = 2$ that cannot be connected by any arc satisfying (2) of Definition 2.2. This is because any arc connecting such a pair would have to pass through the circle $|z| = 1$, and these points are arbitrarily close to the boundary. Thus $\phi^n(B) = B$ for some n.

Now suppose that $\phi(C) = C$. This does not preclude ϕ from being loxodromic. However, ϕ leaves $S^2_\infty \setminus \overline{C}$ invariant, and this domain is conformally equivalent to the hyperbolic plane. Then ϕ is conjugate in the isometry group of $S^2_\infty \setminus \overline{C}$ in its Poincaré metric to $z \to \lambda z$, where λ is real. By abuse of notation, we continue to denote the transformation $z \to \lambda z$ by ϕ and we denote the image of B by $\phi(B)$. The transformation ϕ^n leaves rays from the origin invariant. As $\overline{B}$ is invariant under ϕ^n, there is a leftmost ray r that meets $\overline{B}$ in a point p other than 0 or ∞. As $\overline{B}$ is connected and invariant under ϕ^n, there is an arc X in B connecting p and $\phi^n(p)$ which must lie to the right of r. The component $\phi(B)$ meets the ray r in $\phi(p)$ and $\phi^{n+1}(p)$. These points are connected in $\phi(B)$ by an arc that must lie to the right of r. The points p and $\phi^n(p)$ are linked with $\phi(p)$ and $\phi^{n+1}(p)$ along the ray r. Therefore, if $B \neq \phi(B)$, B and $\phi(B)$ must intersect, contradicting the assumption that they are different components. □

We now state the annulus theorem in this setting. See Cannon and Feustel [2] for the general case. The proper immersed image $\mathcal{A}$ of an annulus or Möbius band in a hyperbolic manifold M with boundary is *essential* if it induces an injection on the level of fundamental groups, and if it is not properly homotopic into a cusp neighborhood.

Theorem 3.3 [2] *Let G be a Kleinian group with incompressible boundary and let ρ: $\mathbb{H}^3 \cup \Omega(G) \to M(G)$ be the covering map. Let A be a proper immersed essential annulus or Möbius band in $M(G)$ with embedded boundary. Then there is a proper embedded essential annulus or Möbius band $\mathcal{A}$ with the same boundary and a pre-image $\widetilde{\mathcal{A}}$ in $\mathbb{H}^3 \cup \Omega(G)$ with boundary in two different components of $\Omega(G)$.*

That any pre-image has boundary in two different components of $\Omega(G)$ follows immediately from the fact that $\mathcal{A}$ is essential. We add this to the statement only because it is important for our point of view.

We now give the proof of Theorem 3.1. Given a Kleinian group G with incompressible boundary, we will form a submanifold N of $M(G)$ that is a characteristic submanifold. We form this submanifold in pieces, considering maximal collections of components of $\Omega(G)$ that meet in a given bumping set. Note that if there are more than two components in such a collection, the bumping set must be exactly two points, as every circle on S^2_∞ is separating. (We ignore collections that bump in exactly one point, since the convex hull of the bumping set will be trivial in this case.) In constructing these pieces, we will show that they satisfy properties (1) and (2) of the definition of characteristic submanifold above. Then we will show (3) that any essential annulus or Möbius band is properly homotopic into one of these components. That the result is unique up to isotopy follows from the fact that it is a characteristic submanifold.

Note that we are considering the disjoint union of the components in the statement of Theorem 3.1. To have the union in $M(G)$ consist of disjoint components, the components may need to be pushed slightly off of each other.

Components of the characteristic submanifold obtained by the bumping of two components We first consider two components C and D that bump, and that bump in exactly two points p and q. We assume maximality in that there are no other components of the domain of discontinuity that meet both p and q. In this case $\mathrm{Niel}_C(C,D)$ consists of one arc in C, $\widetilde{l}$, which is invariant under some element $g \in G$ by Theorem 2.1. The element g fixes both p and q on S^2_∞. We choose g so that it is primitive in the stabilizer of C. By Lemma 3.2 D is also stabilized by g, g is primitive in $\mathrm{Stab}(D)$, and the arc $\widetilde{l}' = \mathrm{Niel}_D(C,D)$ is also invariant under g. Then $l = \rho(\widetilde{l})$ and $l' = \rho(\widetilde{l}')$ are freely homotopic through the manifold $M(G)$ to the closed geodesic l_{int} in $M(G)$ that lifts to a geodesic $\widetilde{l}_{\mathrm{int}}$ in $\mathbb{H}^3$ with endpoints p and q that is invariant under g. If $l = l'$, then this homotopy will define an immersed Möbius strip. (In this case there is an $f \in G$ such that $f^2 = g$.) If $l \neq l'$, this homotopy defines an immersed annulus. In either case, by the annulus theorem, there is an embedded essential Möbius strip or annulus A with the same boundary. When A is an annulus, taking a regular neighborhood of this annulus gives us $(A \times I, S^1 \times S^0 \times I)$ as a component (X,S) of the characteristic submanifold (X_M, S_M). When A is a Möbius strip, we get a component (X,S) that is a twisted I–bundle over an annulus. We can also realize this case as (T,S), where T is a solid torus with a natural Seifert fibered structure and S is an annulus. There is a such a solid torus component of the characteristic submanifold of the 3–manifold illustrated in Figure 2.

We now consider two components C and D which bump, and whose bumping set contains more than two, and hence infinitely many, points. When $\mathrm{Bump}(C,D)$ contains more than 2 points, $\mathrm{Niel}_C(C,D)$ contains more than just a single geodesic.

If $\mathrm{Bump}(C, D) = \partial C = \partial D$, then the characteristic submanifold is the entire manifold $M(G)$, which is a I–bundle over a surface.

Otherwise the image $\rho(\mathrm{Niel}_C(C, D))$ of the convex hull of the bumping set is a subsurface of $\rho(C)$ bounded by geodesics, by Theorem 2.1. We claim that for each boundary curve l of $\rho(\mathrm{Niel}_C(C, D))$, there is a boundary curve l' of $\rho(\mathrm{Niel}_D(C, D))$ and an essential annulus A with $\partial A = l \cup l'$. Indeed, there is a lift $\widetilde{l}$ of l that is a boundary curve of $\mathrm{Niel}_C(C, D)$.

By Theorem 2.1 and Lemma 3.2, $\widetilde{l}$ is stabilized by some element g of G that stabilizes C. Let p and q be the fixed points of g on S^2_∞. By Lemma 3.2, the boundary geodesic $\widetilde{l}'$ of $\mathrm{Niel}_D(C, D)$ in D with endpoints p and q is also stabilized by g. Let $l' = \rho(\widetilde{l}')$. Then l and l' are freely homotopic, since they are both homotopic to the geodesic representing g.

If $l \neq l'$, then by the annulus theorem, there is an embedded annulus A with boundary l and l'.

If $l = l'$, then there is an $f \in G$ such that $f(\widetilde{l}) = \widetilde{l}'$, where f has the same fixed points as g. Since l is a boundary curve of $\mathrm{Niel}_C(C, D)$, $f(D) \neq C$, so p and q are contained in a bumping set involving at least C, D and $f(D)$. In this case, we replace the boundary arcs $\widetilde{l}$ and $\widetilde{l}'$ with g–equivariant arcs also called $\widetilde{l}$ and $\widetilde{l}'$ that lie just in the interior of $\mathrm{Niel}_C(C, D)$ and $\mathrm{Niel}_D(C, D)$. We replace $\mathrm{Niel}_C(C, D)$ and $\mathrm{Niel}_D(C, D)$ with the new, shrunken regions. Then $l \neq l'$ and we can form our embedded annulus with these new curves.

We do this for each boundary curve of $\rho(\mathrm{Niel}_C(C, D))$. Note that some boundary curves will correspond to the same annulus if $\rho(\mathrm{Niel}_C(C, D)) = \rho(\mathrm{Niel}_D(C, D))$.

Consider the resulting union of annuli. We claim that we may assume the union is embedded. Firstly, the boundaries of the family of annuli do not intersect by construction. Secondly, we can remove any inessential circles of intersection by an innermost disc argument as $M(G)$ is irreducible. Thirdly, there can be no essential intersections. Indeed, any such curve of intersection must lift to an arc in $\mathbb{H}^3$ that meets the limit set of G in the same two points as the boundary components of two different annuli. But the lifts of the boundaries of the allegedly essentially intersecting annuli meet the limit set in different points of $\mathrm{Bump}(C, D)$. This is because they correspond to different pairs of boundary curves of $\rho(\mathrm{Niel}_C(C, D))$ and $\rho(\mathrm{Niel}_D(C, D))$. Therefore, we may assume that the collection of annuli connecting the boundary components of $\rho(\mathrm{Niel}_C(C, D))$ and $\rho(\mathrm{Niel}_D(C, D))$ is embedded. This family of annuli lifts to an embedded family of strips $\mathcal{R}^2 \times I$ in $\mathbb{H}^3 \cup \Omega(G)$. There will be some region R bounded by these strips that meets $\mathrm{Niel}_C(C, D)$. The image $\rho(R)$ is a component (X, S) of (X_M, S_M). It is an I–bundle over a surface. When $\rho(\mathrm{Niel}_C(C,D)) = \rho(\mathrm{Niel}_D(C,D))$, this will be a twisted I–bundle. The components of $\partial X \setminus \partial M$ are the annuli constructed above.

Components of the characteristic submanifold obtained by the bumping of more than two components We now consider the case when there are more than two components of the domain of discontinuity that bump non-trivially. In this case the closures of the components must meet in exactly two points. Again we assume that adding any components of $\Omega(G)$ to the collection results in a smaller bumping set. Let $C_1, \dots, C_n$ be this maximal collection whose closures meet in two points p and q. By Theorem 2.1 and Lemma 3.2, there is a $g \in G$ such that for each C_i, there is a geodesic $\widetilde{l}_i$ that is stabilized by g. Denote the geodesic in $\mathbb{H}^3$ stabilized by g with endpoints p and q by $\widetilde{l}_{\mathrm{int}}$, with image l_{int}. Either (1) all the $\rho(\widetilde{l}_i) = l_i$ are the same, (2) all the $\rho(\widetilde{l}_i) = l_i$ are different, or (3) the images fall into m classes, where $m \mid n$. We deal with each situation in turn.

(1) If all the $l_i = l_1$ are the same, then consider a regular neighborhood $N(l_1)$ of l_1. This lifts to regular neighborhoods of each of the $\widetilde{l}_i$. The boundary curves of these regular neighborhoods end in p and q.

Orient $N(l_1)$ so that there is a left side $\partial N(l_1)_-$ and a right side $\partial N(l_1)_+$. Since each of $\partial N(l_1)_+$ and $\partial N(l_1)_-$ is freely homotopic to l_{int}, the lifts of $\partial N(l_1)_+$ and $\partial N(l_1)_-$ in each C_i bound strips $\mathbb{R} \times [0,1]$ with $\widetilde{l}_{\mathrm{int}}$. Then consider two strips in $\mathbb{H}^3 \cup \Omega(G)$, one which is bounded by a lift of $\partial N(l_1)_+$ and $\widetilde{l}_{\mathrm{int}}$, and the other which is bounded by a lift of $\partial N(l_1)_-$ and $\widetilde{l}_{\mathrm{int}}$, where the lifts of $\partial N(l_1)_+$ and $\partial N(l_1)_-$ are in different components of $\Omega(G)$. Then the union of these two strips will map down to an essential annulus in $M(G)$, and by the annulus theorem there is an embedded essential annulus $\mathcal{A}$ with the same boundary, which is $\partial N(l_1)_+ \cup \partial N(l_1)_-$.

The pre-image of $\mathcal{A}$ is an embedded collection of strips, each of which meets $\Omega(G)$ in two different components. Order the components C_i cyclically around $\widetilde{l}_{\mathrm{int}}$. Since $\mathcal{A}$ is embedded, the strips must connect C_i to $C_{i+1} \bmod n$. Then the pre-image of $\mathcal{A}$ will partition $\mathbb{H}^3 \cup \Omega(G)$ into regions, one of which, R, will meet $\widetilde{l}_i$.

The region R is a regular neighborhood of $\widetilde{l}_{\mathrm{int}}$ union thickened strips which meet the lifts of $N(l_1)$. It is a naturally fibered by g–invariant lines. (Note that $\widetilde{l}_{\mathrm{int}}$ is f–invariant, where $f^n = g$.)

The image of R is a component of the characteristic submanifold,

$$(X, S) = (\rho(R), N(l_1)).$$

It is Seifert-fibered by the images of the g–invariant lines and $\partial X \setminus \partial M(G)$ is the annulus $\mathcal{A}$.

(2) We consider the case when the l_i are all distinct. Denote regular neighborhoods of these curves by $N(l_i)$ and lifts that meet p and q by $N(\widetilde{l_i})$. As above, order the C_i around $\widetilde{l}_{\mathrm{int}}$ and label the two boundary components of $N(\widetilde{l_i})$ by $\partial N(\widetilde{l_i})_+$ and $\partial N(\widetilde{l_i})_-$ so that $\partial N(\widetilde{l_i})_+$ is next to $\partial N(\widetilde{l}_{i+1})_-$ in this cyclic ordering. Note that the C_i may be swirling around p and q as they approach them (if g is loxodromic) but we can choose some circle on S^2_∞ separating p and q and cyclically order the components with respect to this circle, and this is well-defined up to the orientation of S^2_∞.

Then as above there are g–invariant strips in $\mathbb{H}^3 \cup \Omega(G)$ connecting each component of $\partial N(\widetilde{l_i})$ to $\widetilde{l}_{\mathrm{int}}$. The union of two such g–invariant strips, one from $\partial N(\widetilde{l_i})_+$ to $\widetilde{l}_{\mathrm{int}}$ and one from $\widetilde{l}_{\mathrm{int}}$ to $\partial N(\widetilde{l}_{i+1})_-$ map down to an immersed essential annulus in $M(G)$. By the annulus theorem, there is an embedded essential annulus with the same boundary, $\mathcal{A}_i$. This has a lift $\widetilde{\mathcal{A}}_i$ that meets p and q. Since all the l_i are distinct, this $\widetilde{\mathcal{A}}_i$ has boundary $\partial N(\widetilde{l_i})_+$ and $\partial N(\widetilde{l}_{i+1})_-$. We form such an embedded annulus $\mathcal{A}_i$ for each $i \bmod n$, with a lift $\widetilde{\mathcal{A}}_i$ with boundary $\partial N(\widetilde{l_i})_+$ and $\partial N(\widetilde{l}_{i+1})_-$ that approaches p and q.

We claim that we can choose such annuli so that the union is embedded. Firstly, the union of the boundaries is already embedded. Secondly, remove any circles of intersection which are trivial in some (hence any) annulus by incompressibility and irreducibility. Now consider any remaining circles of intersection between the $\mathcal{A}_i$ and $\mathcal{A}_1$ in $\mathcal{A}_1$. These are parallel, essential curves on $\mathcal{A}_1$. Hence in the lift $\widetilde{\mathcal{A}}_1$, the lifts of these intersections all approach p and q. This means that the lift $\widetilde{\mathcal{A}}_1$ only intersects the $\widetilde{\mathcal{A}}_i$ that approach p and q. As these are not linked in the cyclic ordering around p and q, there is some pair of intersection curves that bounds annuli on both $\mathcal{A}_1$ and some $\mathcal{A}_i$. Switching these two inner annuli and pushing off will reduce the number of intersection curves. Hence by choosing the collection $\mathcal{A}_i$ to minimize the number of intersection curves, the collection will be embedded.

All the pre-images of the embedded collection $\mathcal{A}_i$ will partition $\mathbb{H}^3 \cup \Omega(G)$ into regions that do not overlap in their interiors. One of these regions, R, will meet the $N(\widetilde{l_i})$. This region is naturally foliated by g–invariant lines. The image $(\rho(R), \bigcup N(l_i))$ is a component of the characteristic submanifold that is Seifert-fibered by the images of the g–invariant lines. $\rho(R)$ is a solid torus and $\partial(\rho(R)) \setminus \bigcup N(l_i)$ is the union of the annuli $\mathcal{A}_i$.

(3) Lastly we consider the case when the images of the $\widetilde{l_i}$ are m distinct curves, where $m \mid n$ and $m \neq 1, n$. Our first task is to show that in this case l_{int}, as defined above, is embedded.

As before, each $\widetilde{l_i}$ is invariant under $g \in G$, where g is hyperbolic and fixes p and q on S^2_∞. Recall that we denote the geodesic in $\mathbb{H}^3$ invariant under g by $\widetilde{l}_{\mathrm{int}}$. Then, as there are $w = n/m$ curves in $\widetilde{l}_i$ that are identified in the quotient, $\widetilde{l}_{\mathrm{int}}$ is invariant under f, where $f^w = g$. Now let $\widetilde{l}_1, \widetilde{l}_2, \dots, \widetilde{l}_w$ be the f–orbit of $\widetilde{l}_1$, cyclically ordered around p and q as above. Let $l_1 = \rho(\widetilde{l}_1)$ and let $N(l_1)$ be a regular neighborhood of this image. Then label the two boundary components of $N(l_1)$ by $\partial N(l_1)_+$ and $\partial N(l_1)_-$ so that the induced labeling of the boundary components of the lifts has $\partial N(\widetilde{l}_i)_+$ next to $\partial N(\widetilde{l}_{i+1})_-$ mod w in the cyclic ordering around p and q.

Then there is a g–invariant strip connecting $\partial N(\widetilde{l}_1)_+$ and $\widetilde{l}_{\mathrm{int}}$ and another connecting $\widetilde{l}_{\mathrm{int}}$ with $\partial N(\widetilde{l}_2)_-$. The union of these two invariant strips maps down to an immersed essential annulus in $M(G)$ and by the annulus theorem, there is an embedded essential annulus $\mathcal{A}_{\mathrm{temp}}$ with the same boundary.

The lifts of $\mathcal{A}_{\mathrm{temp}}$ do not intersect and hence the lifts meeting p and q consist of w strips connecting each $\partial N(\widetilde{l}_i)_+$ to $\partial N(\widetilde{l}_{i+1})_-$. The action of f permutes these w strips cyclically and takes $\widetilde{l}_{\mathrm{int}}$ to itself. Therefore, $\widetilde{l}_{\mathrm{int}}$ is on the inside of these strips. That is, there is a region R bounded by preimages of $\mathcal{A}_{\mathrm{temp}}$ that meets the $N(\widetilde{l}_i)$ and R contains $\widetilde{l}_{\mathrm{int}}$. Thus $\widetilde{l}_{\mathrm{int}}$ intersects its images under G in either itself or the empty set, which implies $\rho(\widetilde{l}_{\mathrm{int}}) = l_{\mathrm{int}}$ is embedded.

Now consider the whole set of $\widetilde{l}_i$. Order them cyclically around p and q. Each $\widetilde{l}_i$ is connected to $\widetilde{l}_{\mathrm{int}}$ by a g–invariant strip. Since l_{int} is embedded, it has a regular neighborhood $N(l_{\mathrm{int}})$, which is a solid torus. The image of such a g–invariant strip in $M(G)$ will restrict to an essential proper immersed annulus in $M(G) \setminus N(l_{\mathrm{int}})$. By the annulus theorem, there is an embedded annulus $\mathcal{A}_{i,\mathrm{int}}$ in $M(G) \setminus N(l_{\mathrm{int}})$ with the same boundary. We form such an annulus for each l_i. Note that each boundary on $\partial N(l_{\mathrm{int}})$ is a curve of the same slope. Therefore, we can arrange so that the boundaries of the $\mathcal{A}_{i,\mathrm{int}}$ are disjoint, and cyclically ordered in the same order as the $\widetilde{l}_i$. Then since the boundary of the annuli are not linked, by choosing a collection that intersect minimally, the $\mathcal{A}_{i,\mathrm{int}}$ will be disjoint.

Now take a regular neighborhood of $N(l_{\mathrm{int}}) \cup \bigcup_i \mathcal{A}_{i,\mathrm{int}}$. This will be a solid torus W, which meets the boundary of $M(G)$ in n parallel annuli $N(l_i)$, where $N(l_i)$ is a regular neighborhood of l_i. The component of the characteristic submanifold will be $(W, \bigcup N(l_i))$. This has a pre-image R in $\mathbb{H}^3 \cup \Omega(G)$, which meets the $N(\widetilde{l}_i)$ and which is naturally foliated by g–invariant lines. The images of these lines in W are a Seifert-fibering of the solid torus W. The components of $\partial W \setminus \bigcup N(l_i)$ are annuli on ∂W. The pre-images in R, as before, connect neighboring boundary components of the $N(\widetilde{l}_i)$.

Now let $\mathcal{B}$ be an essential annulus or Möbius strip in $M(G)$. We will show that $\mathcal{B}$ is properly homotopic into the submanifold constructed above. Pick a basepoint on $\mathcal{B}$ and let g generate the fundamental group of $\mathcal{B}$ in G. Since $\mathcal{B}$ is essential, a lift

$\widetilde{\mathcal{B}}$ of $\mathcal{B}$ must meet two different components, C and D, of $\Omega(G)$, both of which are g–invariant and which meet the fixed points p and q of g on S^2_∞. Thus C and D bump at p and q. From our construction, there is a g–invariant strip $\widetilde{A}$ contained in some component (X, S) that meets C and D in the convex hull of p and q in each component. Consider the solid torus $T = (\mathbb{H}^3 \cup S^2_\infty \setminus \{p, q\})/\langle g \rangle$. This is a solid torus and the images of $\widetilde{B}$ and $\widetilde{A}$ are two embedded essential annuli with the same slope. They are therefore parallel by a proper isotopy. This isotopy maps down to a proper homotopy of $\mathcal{B}$ into a component (X, S) of the submanifold we have constructed. $\square$

Acknowledgements

I would like to thank Curt McMullen and Ian Biringer for encouragement. I am also grateful to the NSF for partial support via grants 0805908 and 1207644.

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Received: 10 August 2012 Revised: 31 May 2013