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# A weak Zassenhaus Lemma for discrete subgroups of Diff($I$)

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We prove a weaker version of the Zassenhaus Lemma for subgroups of Diff($I$). We also show that a group with commutator subgroup containing a non-Abelian free subsemigroup does not admit a $C_0$–discrete faithful representation in Diff($I$).

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In this paper, we continue our study of discrete subgroups of $\mathrm{Diff}_+(I)$; the group of orientation-preserving diffeomorphisms of the closed interval $I = [0, 1]$. Following recent trends, we try to view the group $\mathrm{Diff}_+(I)$ as an analogue of a Lie group, and we study still basic questions about discrete subgroups of it. This paper can be viewed as a continuation of Akhmedov [1] although the proofs of the results of this paper are independent of [1].

Throughout the paper, the letter $G$ will denote the group $\mathrm{Diff}_+(I)$. Assume $G$ has the metric induced by the standard norm of the Banach space $C^1[0, 1]$. We will denote this metric by $d_1$. Sometimes, we also will consider the metric on $G$ that comes from the standard sup norm $\|f\|_0 = \sup_{x\in[0,1]} |f(x)|$ of $C[0, 1]$, which we will denote by $d_0$. However, unless specified, the metric in all the groups $\mathrm{Diff}^r_+(I), r \in \mathbb{R}, r \geq 1$ will be assumed to be $d_1$.

The central theme of the paper is the Zassenhaus Lemma. This lemma states that in a connected Lie group $H$ there exists an open non-empty neighborhood $U$ of the identity such that any discrete subgroup generated by elements from $U$ is nilpotent (see Raghunathan [6]). For example, if $H$ is a simple Lie group (such as $\mathrm{SL}_2(\mathbb{R})$), and $\Gamma \leq H$ is a lattice, then $\Gamma$ cannot be generated by elements too close to the identity.

In this paper we prove weak versions of the Zassenhaus Lemma for the group $G = \mathrm{Diff}_+(I)$. Our study leads us to showing that finitely generated groups with exponential growth that satisfy a very mild condition do not admit faithful $C_0$–discrete representations in $G$:

**Theorem A** *Let $\Gamma$ be a subgroup of $G$, and $f, g \in [\Gamma, \Gamma]$ such that $f$ and $g$ generate a non-Abelian free subsemigroup. Then $\Gamma$ is not $C_0$–discrete.*

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We also study the Zassenhaus Lemma for the relatives of $G$ such as $\mathrm{Diff}_+^{1+c}(I)$, $c \in \mathbb{R}$, $c > 0$; the group of orientation-preserving diffeomorphisms of regularity $1+c$. In the case of $\mathrm{Diff}_+^{1+c}[0,1]$, combining Theorem A with the results of [3], we show that $C_0$–discrete subgroups are more rare.

**Theorem B** *Let $\Gamma$ be a $C_0$–discrete subgroup of* $\mathrm{Diff}_+^{1+c}[0,1]$. *Then $\Gamma$ is solvable with solvability degree at most $k(c)$.*

Theorem B can be strengthened if the regularity is increased further; combining Theorem A with the results of Navas [3], Plante and Thurston [5] and Szekeres [7] we obtain the following:

**Theorem C** *If $\Gamma$ is $C_0$–discrete subgroup of* $\mathrm{Diff}_+^{2}[0,1]$ *then $\Gamma$ is meta-Abelian.*

It follows from the results of [1], as remarked there, that the Zassenhaus Lemma does not hold either for $\mathrm{Diff}_+(I)$ or for $\mathrm{Homeo}_+(I)$ in metrics $d_1$ and $d_0$ respectively.

In the increased regularity the lemma still fails: given an arbitrary open neighborhood $U$ of the identity diffeomorphism in $G$, it is easy to find two $C^\infty$ "bump functions" in $U$ that generate a discrete group isomorphic to $\mathbb{Z} \wr \mathbb{Z}$; thus the lemma fails for $\mathrm{Diff}_+^\infty(I)$.

Because of the failure of the lemma, it is natural to consider strongly discrete subgroups, which we have defined in [1]. Indeed, for strongly discrete subgroups, we are able to obtain positive results that are natural substitutes for the Zassenhaus Lemma.

Let us recall the definition of strongly discrete subgroup from [1]:

**Definition 1** Let $\Gamma$ be a subgroup of $\mathrm{Diff}_+(I)$. $\Gamma$ is called *strongly discrete* if there exists $C > 0$ and $x_0 \in (0,1)$ such that $|g'(x_0) - 1| > C$ for all $g \in \Gamma \backslash \{1\}$. Similarly, we say $\Gamma$ is *$C_0$–strongly discrete* if $|g(x_0) - x_0| > C$ for all $g \in \Gamma \backslash \{1\}$.

Let us note that a strongly discrete subgroup of $G$ is discrete, and a $C_0$–strongly discrete subgroup of $G$ is $C_0$–discrete.

For the convenience of the reader, let us recall several basic notions on the growth of groups: if $\Gamma$ is a finitely generated group, and $S$ a finite generating set, we will define $\omega(\Gamma, S) = \lim_{n\to\infty} \sqrt[n]{|B_n(1; S, \Gamma)|}$, where $B_n(1; S, \Gamma)$ denotes the ball of radius $n$ around the identity element. (Often we will denote this ball simply by $B_n(1)$.) We will also write $\omega(\Gamma) = \inf_{|S|<\infty, \langle S\rangle = \Gamma} \omega(\Gamma, S)$, where the infimum is taken over all finite generating sets $S$ of $\Gamma$. If $\omega(\Gamma) > 1$ then one says that $\Gamma$ has uniform exponential growth.

Now we are ready to state weak versions of the Zassenhaus Lemma for the group $G$. First, we state a theorem about $C_0$–strongly discrete subgroups.

**Theorem 2** *Let* $\omega > 1$. *Then there exists an open non-empty neighborhood* $U$ *of the identity* $1 \in \mathrm{Diff}^1_+[0,1]$ *such that if* $\Gamma$ *is a finitely generated* $C_0$*–strongly discrete subgroup of* $\mathrm{Diff}^1_+[0,1]$ *with* $\omega(\Gamma) \geq \omega$, *then* $\Gamma$ *cannot be generated by elements from* $U$.

By increasing the regularity, we can prove a similar version for strongly discrete subgroups

**Theorem 3** *Let* $\omega > 1$. *Then there exists an open non-empty neighborhood* $U$ *of the identity* $1 \in \mathrm{Diff}^1_+[0,1]$ *such that if* $\Gamma$ *is a finitely generated strongly discrete subgroup of* $\mathrm{Diff}^2_+[0,1]$ *with* $\omega(\Gamma) \geq \omega$, *then* $\Gamma$ *cannot be generated by elements from* $U$.

**Remark 4** In regard to the Zassenhaus Lemma, it is interesting to ask a reverse question, ie, given an arbitrary open neighborhood $U$ of the identity in $G$, is it true that any finitely generated torsion free nilpotent group $\Gamma$ admits a faithful discrete representation in $G$ generated by elements from $U$? In Farb and Franks [2], it is proved that any such $\Gamma$ does admit a faithful representation into $G$ generated by diffeomorphisms from $U$. Also, it is proved in Navas [4] that any finitely generated nilpotent subgroup of $G$ indeed can be conjugated to a subgroup generated by elements from $U$.

**Remark 5** Because of the assumptions about uniform exponential growth in Theorem 2 and Theorem 3, it is natural to ask whether or not every finitely generated subgroup of $G$ of exponential growth has uniformly exponential growth. This question has already been raised in [3].

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# Proofs of Theorem 2 and Theorem 3

**Proof of Theorem 2** We can choose $\lambda > 1$ such that $\lambda < \omega(\Gamma)$. Then the cardinality of the sphere of radius $n$ of $\Gamma$ with respect to any fixed finite generating set is bigger than the exponential function $\lambda^n$, for infinitely many $n$.

Then let $\epsilon > 0$ such that $(1-10\epsilon)\lambda > 1$. We let $U$ be the $\epsilon$–neighborhood of the identity in $G$ with respect to $d_1$ metric (we always assume $d_1$ metric in $G$ unless otherwise stated).

Let $\Gamma$ be generated by finitely many non-trivial diffeomorphisms $f_1, f_2, \dots, f_s \in U$. We fix this generating set and denote it by $S$, ie, $S = \{f_1, f_1^{-1}, \dots, f_s, f_s^{-1}\}$.

We want to prove that $\Gamma$ is not $C_0$–strongly discrete. Assuming the opposite, let $x_0 \in (0,1)$ such that for some $C > 0$, $|g(x_0) - x_0| > C$ for all $g \in \Gamma \backslash \{1\}$.

Let $B_n(1)$ be the ball of radius $n$ around the identity in the Cayley graph of $\Gamma$ with respect to $S$. Then $\mathrm{Card}(B_n(1) \backslash B_{n-1}(1)) > \lambda^n$ for infinitely many $n \in \mathbb{N}$. Let $A$ denote the set of all such $n$.

Let $\Delta$ be a closed subinterval of $(0,1)$ of length less than $C$ such that $x_0$ is the left end of $\Delta$.

We denote the right-invariant Cayley metric of $\Gamma$ with respect to $S$ by $|\cdot|$. For all $g \in \Gamma$, let $\Delta_g = g(\Delta)$. Thus we have a collection $\{\Delta_g\}_{g \in G}$ of closed subintervals of $(0,1)$.

Notice that if $g = sw, s \in S$ then by mean value theorem, $|\Delta_{sw}| > (1-10\epsilon)|s(\Delta_w)|$. Then, necessarily, for all $n \in A$, we have $\sum_{|g|=n} |\Delta_g| > (1-10\epsilon)^n \lambda^n |\Delta| \to \infty$ as $n \to \infty$.

Then there exist $g_1, g_2 \in \Gamma, g_1 \neq g_2$ such that $g_2(x_0) \in \Delta_{g_1}$. Then $g_1^{-1} g_2(x_0) \in \Delta$. Since $|\Delta| < C$, we obtain a contradiction. □

Now we prove a better result by assuming higher regularity for the representation.

**Proof of Theorem 3** Let $\lambda, \lambda_1, \lambda_2$ be constants such that $1 < \lambda < \lambda_1 < \lambda_2 < \omega(\Gamma)$. Then the cardinality of the sphere of radius $n$ of $\Gamma$ with respect to any fixed finite generating set is bigger than the exponential function $\lambda_2^n$, for infinitely many $n$.

We choose $\epsilon > 0, \eta > 0$ to be such that $1 < \eta < \frac{\lambda}{1+\epsilon}$ and $\frac{1+\epsilon}{1-\epsilon} < \frac{\lambda_1}{\lambda}$. Let $U$ be the ball of radius $\epsilon$ around the identity diffeomorphism.

We again assume that $\Gamma$ is generated by finitely many non-trivial diffeomorphisms $f_1, f_2, \dots, f_s \in U$, and we fix the generating set $S = \{f_1, f_1^{-1}, \dots, f_s, f_s^{-1}\}$. Let $B_n(1)$ be the ball of radius $n$ around the identity in the Cayley graph of $\Gamma$ with respect to $S$. Then we have $\mathrm{Card}(B_n(1) \backslash B_{n-1}(1)) > \lambda_2^n$ for infinitely many $n \in \mathbb{N}$. Let $A$ denote the set of all such $n$.

We need to show that $\Gamma$ is not strongly discrete. Assuming the opposite, let $x_0 \in (0,1)$ such that for some $C > 0$, $|g'(x_0) - 1| > C$ for all $g \in \Gamma \backslash \{1\}$.

Let $C_1$ be a positive number such that

$$1 - C < (1 - C_1)^2 \quad \text{and} \quad 1 + C > (1 + C_1)^2.$$

Let also $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$1 - C_1 < \left(1 - \frac{1}{\eta^n}\right)^n < \left(1 + \frac{1}{\eta^n}\right)^n < 1 + C_1.$$

Notice that for all $n \in A, g \in B_n(1) \backslash B_{n-1}(1)$, and $x \in [0,1]$, we have $(1-\epsilon)^n < g'(x) < (1+\epsilon)^n$. Since $\frac{1+\epsilon}{1-\epsilon} < \frac{\lambda_1}{\lambda}$, there exists $n \in A$ and $g_1, g_2 \in \Gamma$ such that

$$n > N_1, \quad g_1 \neq g_2, \quad |g_1| = |g_2| = n$$

but

$$|g_1(x_0) - g_2(x_0)| \leq \frac{1}{\lambda^n} (\star_1) \quad \text{and} \quad 1 - C_1 < \frac{g_1'(x_0)}{g_2'(x_0)} < 1 + C_1 (\star_2).$$

Indeed, by the pigeonhole principle, for all $n \in A$, there exists $j \in \{0, 1, \ldots, [\lambda^n]\}$ such that

$$\mathrm{Card}\left\{ g \in B_n(1) \backslash B_{n-1}(1) \;\middle|\; g(x_0) \in \left[\frac{j}{\lambda^n}, \frac{j+1}{\lambda^n}\right) \right\} \geq \frac{\lambda_2^n}{\lambda^n + 1}.$$

For all $n \in A$, $j \in \{0, 1, \ldots, [\lambda^n]\}$, let

$$D(n, j) = \left\{ g \in B_n(1) \backslash B_{n-1}(1) \;\middle|\; g(x_0) \in \left[\frac{j}{\lambda^n}, \frac{j+1}{\lambda^n}\right) \right\}.$$

Then, for sufficiently big $n \in A$, there exists $j \in \{0, 1, \ldots, [\lambda^n]\}$ such that

$$\mathrm{Card}(D(n, j)) \geq \frac{\lambda_1^n}{\lambda^n} (\star_3).$$

For all $n \in A$, let

$$J(n) = \left\{ j \in \{0, 1, \ldots, [\lambda^n]\} \;\middle|\; \mathrm{Card}(D(n, j)) \geq \frac{\lambda_1^n}{\lambda^n} \right\}.$$

Recall also that for all $g \in D(n, j)$, we have

$$(1-\epsilon)^n < g'(x_0) < (1+\epsilon)^n$$

Then, since $\frac{1+\epsilon}{1-\epsilon} < \frac{\lambda_1}{\lambda}$, for sufficiently big $n \in A$ and $j \in J(n)$, applying the pigeonhole principle to the set $D(n, j)$, we obtain that (besides the inequality $(\star_3)$) there exist distinct $g_1, g_2 \in D(n, j)$ such that the inequality

$$1 - C_1 < \frac{g_1'(x_0)}{g_2'(x_0)} < 1 + C_1$$

holds. On the other hand, by definition of $D(n,j)$, we have $|g_1(x_0)-g_2(x_0)| \le \frac{1}{\lambda^n}$; thus we established the desired inequalities $(\star_1)$ and $(\star_2)$.

Let now $y_0=g_1(x_0)$, $z_0=g_2(x_0)$, $W=g_1^{-1}$, $V=g_1^{-1}g_2$, and let $W=h_nh_{n-1}\cdots h_1$ where $W$ is a reduced word in the alphabet $S$ of length $n$ and $h_i\in S, 1\le i\le n$.

Let also $W_k$ be the suffix of $W$ of length $k$, $y_k=W_k(y_0)$, $z_k=W_k(z_0)$, $1\le k\le n$.

Furthermore, let

$$\max_{1\le i\le s}\ \sup_{0\le y\ne z\le 1}\frac{|f_i'(y)-f_i'(z)|}{|y-z|}=M \quad \text{and} \quad L=1+\epsilon.$$

Then we have $|y_k-z_k|\le L^k/\lambda^n$, $|h'_{k+1}(y_k)-h'_{k+1}(z_k)|\le ML^k/\lambda^n$, $0\le k\le n-1$. Then

$$1-\frac{ML^{k+1}}{\lambda^n}\le\frac{h'_{k+1}(y_k)}{h'_{k+1}(z_k)}\le 1+\frac{ML^{k+1}}{\lambda^n}$$

for all $0\le k\le n-1$. From here we obtain that

$$\prod_{k=0}^{n-1}\Big(1-\frac{ML^{k+1}}{\lambda^n}\Big)\le\prod_{k=0}^{n}\frac{h'_{k+1}(y_k)}{h'_{k+1}(z_k)}\le\prod_{k=0}^{n-1}\Big(1+\frac{ML^{k+1}}{\lambda^n}\Big).$$

Then, for sufficiently big $n$ in $A$

$$\Big(1-\frac{1}{\eta^n}\Big)^n=\prod_{k=0}^{n-1}\Big(1-\frac{1}{\eta^n}\Big)\le\prod_{k=0}^{n-1}\frac{h'_{k+1}(y_k)}{h'_{k+1}(z_k)}\le\prod_{k=0}^{n-1}\Big(1+\frac{1}{\eta^n}\Big)=\Big(1+\frac{1}{\eta^n}\Big)^n.$$

Since, by the chain rule,

$$\prod_{k=0}^{n-1}\frac{h'_{k+1}(y_k)}{h'_{k+1}(z_k)}=\frac{(g_1^{-1})'(y_0)}{(g_1^{-1})'(z_0)},$$

we obtain that $1-C_1<(g_1^{-1})'(y_0)/(g_1^{-1})'(z_0)<1+C_1$. Then

$$\begin{aligned}V'(x_0)&=(g_1^{-1})'(g_2(x_0))g_2'(x_0)\\&=\frac{(g_1^{-1})'(g_2(x_0))g_2'(x_0)}{(g_1^{-1})'(g_1(x_0))g_1'(x_0)}(g_1^{-1})'(g_1(x_0))g_1'(x_0)\\&=\frac{(g_1^{-1})'(g_2(x_0))g_2'(x_0)}{(g_1^{-1})'(g_1(x_0))g_1'(x_0)}\\&=\frac{(g_1^{-1})'(g_2(x_0))}{(g_1^{-1})'(g_1(x_0))}\frac{g_2'(x_0)}{g_1'(x_0)}\in((1-C_1)^2,(1+C_1)^2)\subset(1-C,1+C).\end{aligned}$$

Thus we proved that $1-C<V'(x_0)<1+C$, which contradicts our assumption. □

**Remark 6** The same proof, with slight changes, works for representations of $C^{1+c}$–regularity for any real $c > 0$.

# Proofs of Theorems A, B, C

In the proofs of Theorem 2 and of Theorem 3, we consider the orbit of the point $x_0$ under the action of $\Gamma$. By using exponential growth, we find two distinct elements $g_1, g_2$ such that $g_1(x_0)$ and $g_2(x_0)$ are very close. Then we "pull back" $g_2(x_0)$ by $g_1^{-1}$, ie, we consider the point $g_1^{-1}g_2(x_0)$ and show that this point is sufficiently close to $x_0$. It is at this stage that we heavily use the condition that $\Gamma$ is generated by elements from the small neighborhood of $1 \in G$, ie, derivatives of the generators are uniformly close to 1. However, if $\Gamma$ is an arbitrary subgroup of the commutator group $[G, G]$, not necessarily generated by elements close to the identity element, then for any $x_0 \in (0, 1)$, $f \in \Gamma$ and for any $\epsilon > 0$, there exists $W \in \Gamma$ such that $|f'(W(x_0)) - 1| < \epsilon$; we simply need to find $W$ such that $W(x_0)$ is sufficiently close to 1 (or to 0). This fact provides a new idea of taking $x_0$ close to 1, then considering the part of the orbit that lies in a small neighborhood of 1, then using exponential growth to find points close to each other in that neighborhood, and then perform the "pull back".

The following proposition is a special case of Theorem A, and answers [1, Question 2]. For simplicity, we give a separate proof of it.

**Proposition 7** *$\mathbb{F}_2$ does not admit a faithful $C_0$–discrete representation in $G$.*

**Proof** Since the commutator subgroup of $\mathbb{F}_2$ contains an isomorphic copy of $\mathbb{F}_2$, it is sufficient to prove that $\mathbb{F}_2$ does not admit a faithful $C_0$–discrete representation in $G^{(1)} = [G, G]$.

Let $\Gamma$ be a subgroup of $G^{(1)}$ isomorphic to $\mathbb{F}_2$ generated by diffeomorphisms $f$ and $g$. Without loss of generality we may assume that $\Gamma$ has no fixed point on $(0, 1)$. Let also $\epsilon > 0$ and $M = \max_{0 \le x \le 1}(|f'(x)| + |g'(x)|)$.

We choose $N \in \mathbb{N}, \delta > 0$ and $\theta_N$ such that $1/N < \epsilon$, $1 < \theta_N < \sqrt[2N]{2}$, and for all $x \in [1-\delta, 1]$, the inequality $1/\theta_N < \phi'(x) < \theta_N$ holds where $\phi \in \{f, g, f^{-1}, g^{-1}\}$.

Let $W = W(f, g)$ be an element of $\Gamma$ such that $W(1/N) \in [1-\delta, 1]$, $m$ be the length of the reduced word $W$. Let also $x_i = i/N, 0 \le i \le N$.

For every $n \in \mathbb{N}$, let

$$S_n = \{H \in B_n(1) \mid u(W(x_1)) \ge W(x_1) \text{ for all suffixes } u \text{ of } H\}.$$

(Here we view $H$ as a reduced word in the alphabet $\{f, g, f^{-1}, g^{-1}\}$.) Then $|S_n| \ge 2^n$.

Then (assuming $N \geq 3$) we can choose and fix a sufficiently big $n$ such that the following two conditions are satisfied:

(i) There exist $g_1, g_2 \in S_n$ such that $g_1 \neq g_2$, and

$$|g_1 W(x_i) - g_2 W(x_i)| < \frac{1}{\sqrt[2N]{2}^n}, \quad 1 \leq i \leq N-1.$$

(ii) $M^m (\theta_N)^n \dfrac{1}{\sqrt[2N]{2}^n} < \epsilon.$

Indeed, let $(c_0, c_1, \dots, c_{N-1}, c_N)$ be a sequence of real numbers such that $\sqrt[2N]{2} = c_N < c_{N-1} < \cdots < c_1 < c_0 = 2$ and $c_i > \sqrt[2N]{2} c_{i+1}$, for all $i \in \{0, 1, \dots, N-1\}$. Then, by the pigeonhole principle, for sufficiently big $n$, there exists a subset $S_n(1) \subseteq S_n$ such that $|S_n(1)| \geq c_1^n$ and $|g_1 W(x_1) - g_2 W(x_1)| < 1/\sqrt[2N]{2}^n$, for all $g_1, g_2 \in S_n(1)$.

Suppose now $1 \leq k \leq N-2$, and $S_n \supseteq S_n(1) \supseteq \cdots \supseteq S_n(k)$ such that for all $j \in \{1, \dots, k\}$, $|S_n(j)| \geq c_j^n$ and for all $g_1, g_2 \in S_n(j)$ we have

$$|g_1 W(x_i) - g_2 W(x_i)| < \frac{1}{\sqrt[2N]{2}^n}, \quad 1 \leq i \leq j.$$

Then by applying the pigeonhole principle to the set $S_n(k)$ for sufficiently big $n$, we obtain $S_n(k+1) \subseteq S_n(k)$ such that $|S_n(k+1)| \geq c_{k+1}^n$, and for all $g_1, g_2 \in S_n(k+1)$ we have

$$|g_1 W(x_i) - g_2 W(x_i)| < \frac{1}{\sqrt[2N]{2}^n}, \quad 1 \leq i \leq k+1.$$

Then, for $k = N-2$, we obtain the desired inequality (condition (i)).

Now, let

$$h_1 = g_1 W, \quad h_2 = g_2 W, \quad y_i = W(x_i), \quad z_i' = g_1(y_i), \quad z_i'' = g_2(y_i), \quad 1 \leq i \leq N.$$

Without loss of generality, we may also assume that $g_2(y_1) \geq g_1(y_1)$.

Then for all $i \in \{1, \dots, N-1\}$, we have

$$\begin{aligned} |h_1^{-1} h_2(x_i) - x_i| &= |(g_1 W)^{-1} (g_2 W)(x_i) - x_i| \\ &= |(g_1 W)^{-1} (g_2 W)(x_i) - (g_1 W)^{-1} (g_1 W)(x_i)| \\ &= |W^{-1} g_1^{-1} g_2(y_i) - W^{-1} g_1^{-1} g_1(y_i)| \\ &= |W^{-1} g_1^{-1}(z_i'') - W^{-1} g_1^{-1}(z_i')|. \end{aligned}$$

Let $u$ be a prefix of the reduced word $g_1$, and $g_1 = uv$ (so a reduced word $v$ is a suffix of $g_1$). Then, since $g_1, g_2 \in S_n$, we have

$$u^{-1}(z_i') = v(y_i) \geq v(y_1) \geq y_1,$$
$$u^{-1}(z_i'') = u^{-1}(g_2(y_i)) \geq u^{-1}(g_2(y_1)) \geq u^{-1}(g_1(y_1)) \geq v(y_1) \geq y_1.$$

Then by the mean value theorem, we have

$$|h_1^{-1}h_2(x_i) - x_i| \leq M^m(\theta_N)^n |z_1' - z_1''| < M^m(\theta_N)^n \frac{1}{\sqrt[2N]{2}^n}.$$

Then, by condition (ii), we obtain $|h_1^{-1}h_2(x_i)-x_i| < \epsilon$. Then we have $|h_1^{-1}h_2(x)-x| < 2\epsilon$ for all $x \in [0,1]$. Indeed, let $x \in [x_i, x_{i+1}]$. Then

$$|h_1^{-1}h_2(x) - x| \leq \max\{|h_1^{-1}h_2(x_i) - x|, |h_1^{-1}h_2(x_{i+1}) - x|\}.$$

But $|h_1^{-1}h_2(x_i) - x| \leq |h_1^{-1}h_2(x_i) - x_i| + |x_i - x| < 2\epsilon$, and similarly,

$$|h_1^{-1}h_2(x_{i+1}) - x| \leq |h_1^{-1}h_2(x_{i+1}) - x_{i+1}| + |x_{i+1} - x| < 2\epsilon.$$

Since $\epsilon$ is arbitrary, we obtain that $\Gamma$ is not $C_0$–discrete. □

By examining the proof of Proposition 7, we will now prove Theorem A, thus obtaining a much stronger result. The inequality $|S_n| \geq 2^n$ is a crucial fact in the proof of Proposition 7; we need the cardinality of $S_n$ to grow exponentially. If $\Gamma$ is an arbitrary finitely generated group with exponential growth, this exponential growth of $S_n$ is not automatically guaranteed. But we can replace $S_n$ by another subset $\mathbb{S}_n$ that still does the job of $S_n$ and that grows exponentially, if we assume a mild condition on $\Gamma$.

First we need the following easy lemma.

**Lemma 8** *Let $\alpha, \beta \in G, z_0 \in (0,1)$ such that $z_0 \leq \alpha(z_0) \leq \beta\alpha(z_0)$. Then $U\beta\alpha(z_0) \geq z_0$, where $U = U(\alpha, \beta)$ is any positive word in letters $\alpha, \beta$.* □

Now we are ready to prove Theorem A.

**Proof of Theorem A** Without loss of generality, we may assume that $\Gamma$ has no fixed point on $(0,1)$. Let again $\epsilon > 0, N \in \mathbb{N}, \delta > 0, \theta_N > 0$,

$$M = 2 \sup_{0 \leq x \leq 1} (|f'(x)| + |g'(x)|)$$

such that $1/N < \epsilon, 1 < \theta_N < \sqrt[2N]{2}$, and for all $x \in [1-\delta, 1]$, the inequality $1/\theta_N < \phi'(x) < \theta_N$ holds where $\phi \in \{f, g, f^{-1}, g^{-1}\}$.

Let $W = W(f, g)$ be an element of $\Gamma$ such that

$$\{f^i W(1/N) \mid -2 \le i \le 2\} \cup \{g^i W(1/N) \mid -2 \le i \le 2\} \subset [1-\delta, 1]$$

and let $m$ be the length of the reduced word $W$. Let also $x_i = i/N, 0 \le i \le N$ and $z = W(1/N)$.

By replacing the pair $(f, g)$ with $(f^{-1}, g^{-1})$ if necessary, we may assume that $f(z) \ge z$. Then at least one of the following cases is valid:

**Case 1** $f(z) \le gf(z)$

**Case 2** $z \le gf(z)$

**Case 3** $gf(z) \le z$

If Case 1 holds then we let $\alpha = f, \beta = g, z_0 = z$. If Case 1 does not hold but Case 2 holds, then we let $\alpha = gf, \beta = f, z_0 = z$. Finally, if Case 1 and Case 2 do not hold but Case 3 holds, then we let $\alpha = f^{-1}g^{-1}, \beta = g^{-1}, z_0 = gf(z)$.

In all the three cases, we will have $z_0 \in [1-\delta, 1], z_0 \le z$, and $\alpha, \beta$ generate a free subsemigroup, and conditions of Lemma 8 are satisfied, ie, we have $z_0 \le \alpha(z_0) \le \beta\alpha(z_0)$. Moreover, we notice that $\sup_{0\le x\le 1}(|\alpha'(x)| + |\beta'(x)|) \le M^2$, and the length of $W$ in the alphabet $\{\alpha, \beta, \alpha^{-1}, \beta^{-1}\}$ is at most $2m$.

Now, for every $n \in \mathbb{N}$, let

$$\mathbb{S}_n = \{U(\alpha, \beta)\beta\alpha W \mid U(\alpha, \beta) \text{ is a positive word in } \alpha, \beta \text{ of length at most } n\}.$$

Applying Lemma 8 to the pair $\{\alpha, \beta\}$ we obtain that $VW^{-1}(z_0) \ge z_0$ for all $V \in \mathbb{S}_n$.

Then $|\mathbb{S}_n| \ge 2^n$. After achieving this inequality, we proceed as in the proof of Proposition 7 with just a slight change: there exists a sufficiently big $n$ such that the following two conditions are satisfied:

(i) There exist $g_1, g_2 \in \mathbb{S}_n$ such that $g_1 \ne g_2$, and

$$|g_1 W(x_i) - g_2 W(x_i)| < \frac{1}{\sqrt[2N]{2}^n}, \quad 1 \le i \le N-1.$$

(ii) $M^{2m+4}(\theta_N)^n \frac{1}{\sqrt[2N]{2}^n} < \epsilon$.

Let

$$h_1 = g_1 W, \quad h_2 = g_2 W, \quad y_i = W(x_i), \quad z_i' = g_1(y_i), \quad z_i'' = g_2(y_i), \quad 1 \le i \le N.$$

Without loss of generality, we may also assume that $g_2(y_1) \ge g_1(y_1)$.

Then for all $i \in \{1, \ldots, N-1\}$, we have

$$\begin{aligned} |h_1^{-1}h_2(x_i) - x_i| &= |(g_1W)^{-1}(g_2W)(x_i) - x_i| \\ &= |(g_1W)^{-1}(g_2W)(x_i) - (g_1W)^{-1}(g_1W)(x_i)| \\ &= |W^{-1}g_1^{-1}g_2(y_i) - W^{-1}g_1^{-1}g_1(y_i)| \\ &= |W^{-1}g_1^{-1}(z_i'') - W^{-1}g_1^{-1}(z_i')|. \end{aligned}$$

Since $g_1, g_2 \in \mathbb{S}_n$, by the mean value theorem, we have

$$|h_1^{-1}h_2(x_i) - x_i| \leq M^{2m+4}(\theta_N)^n |z_1' - z_1''| < M^{2m+4}(\theta_N)^n \frac{1}{\sqrt[2N]{2}^n}.$$

By condition (ii), we obtain that $|h_1^{-1}h_2(x_i) - x_i| < \epsilon$. Then we have $|h_1^{-1}h_2(x) - x| < 2\epsilon$ for all $x \in [0,1]$. Since $\epsilon$ is arbitrary, we obtain that $\Gamma$ is not $C_0$–discrete. □

**Proof of Theorem B** Let $H$ be an arbitrary finitely generated subgroup of $[\Gamma, \Gamma]$. If $H$ contains a non-Abelian free subsemigroup then we are done by Theorem A. If $H$ does not contain a non-Abelian free subsemigroup then by the result from [3], $H$ is virtually nilpotent. Then again by the result of [3], $H$ is solvable of solvability degree at most $l(c)$. Since the natural number $l(c)$ depends only on $c$, and not on $H$, and since $H$ is an arbitrary finitely generated subgroup of $[\Gamma, \Gamma]$ we obtain that $[\Gamma, \Gamma]$ is solvable of solvability degree at most $l(c)$. Hence $\Gamma$ is solvable with a solvability degree at most $l(c) + 1$. □

**Proof of Theorem C** Let again $H$ be an arbitrary finitely generated subgroup of $[\Gamma, \Gamma]$. Again, if $H$ contains a non-Abelian free subsemigroup then we are done by Theorem A. If $H$ does not contain a non-Abelian free subsemigroup then by the result from [3] $H$ is virtually nilpotent. Then, by the result of Plante and Thurston [5], $H$ is virtually Abelian. Then, by the result of Szekeres [7], $H$ is Abelian. Since $H$ is an arbitrary finitely generated subgroup of $[\Gamma, \Gamma]$, we conclude that $[\Gamma, \Gamma]$ is Abelian, hence $\Gamma$ is meta-Abelian. □

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