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The connective real K–theory of Brown–Gitler spectra

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We calculate the connective real K–theory homology of the mod 2 Brown–Gitler spectra. We use this calculation and the theory of Dieudonné rings and Hopf rings to determine the mod 2 homology of the spaces in the connective Ω–spectrum for topological real K–theory.

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1 Introduction

Suppose E is a ring spectrum, and let $\underline{E}_k = \Omega^\infty \Sigma^k E$ denote the k^{th} space in its Ω–spectrum. The mod p homology $H_*(\underline{E}_*)$ is a Hopf ring; a ring object in the category of coalgebras over $\mathbb{F}_p$. In [9], Goerss defined a category of Dieudonné rings over the p–adic integers $\hat{\mathbb{Z}}_p$. He showed that the Dieudonné functor $D_*(-)$ from Hopf rings to Dieudonné rings was symmetric monoidal, thereby establishing an equivalence of categories and, consequently, an isomorphism between any Hopf ring $H_*(\underline{E}_*)$ and its associated Dieudonné ring $D_*(H_*(\underline{E}_*))$. Building on his earlier work with Lannes and Morel [11], Goerss also showed there is a surjective map from the E–homology of Brown–Gitler spectra $E_*(B(*))$ to the Dieudonné ring $D_*(H_*(\underline{E}_*))$ that is periodically an isomorphism. When the mod p (co)homology of the spectrum E is known, the E–homology of the Brown–Gitler spectra $E_*(B(*))$ can be calculated via an Adams spectral sequence. Thus, it is often possible to calculate the Hopf ring $H_*(\underline{E}_*)$ via the Adams spectral sequence for $E_*(B(*))$.

Calculating the Hopf ring $H_*(\underline{E}_*)$ using an Adams spectral sequence for $E_*(B(*))$ is remarkable for several reasons. First, this method for calculating the homology of the *spaces* $\underline{E}_*$ is done using only the (co)homology of the *spectrum* E and the Brown–Gitler *spectra* $B(*)$ as input to the Adams spectral sequence. Second, this approach can be used even when the spaces $\underline{E}_*$ have not been identified in terms of already-known spaces. Third, unlike the bar spectral sequence, which computes $H_*(\underline{E}_k)$ inductively on k (ie, one space $\underline{E}_k$ at a time), this method computes $H_n(\underline{E}_*)$ inductively on n (ie, across all spaces $\underline{E}_*$ at once), and as a result it does well

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at identifying natural generators for the Hopf ring coming from the homotopy and homology of the spectrum E.

Every Hopf ring $H_*(\underline{E}_*)$ has a unique suspension class $e \in H_1(\underline{E}_1)$ such that

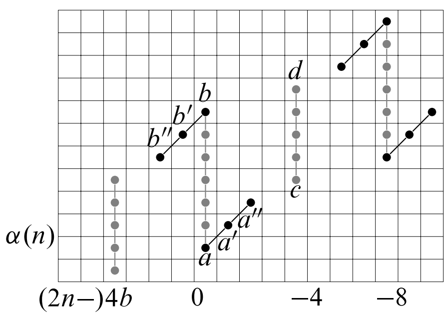
$$e \circ (-)\colon H_d(\underline{E}_k) \to H_{d+1}(\underline{E}_{k+1}), \tag{1-1}$$

$$e^{\infty}(-)\colon H_{d+k}(\underline{E}_k) \to H_d(E), \tag{1-2}$$

are the homology suspension and stabilization (ie, infinitely iterated homology suspension) homomorphisms, respectively. We call an element in a Hopf ring *unstable* if it is in the kernel of the stabilization homomorphism, and *stable* if it is not. There is also a destabilization function

$$e^{-\infty}(-)\colon H_d(E) \to H_{d+k}(\underline{E}_k) \tag{1-3}$$

that takes a stable homology class back to its space of origin, ie, for each $x \in H_d(E)$ it finds the smallest k such that $e^{-\infty}(x)$ is nonzero in $H_{d+k}(\underline{E}_k)$.



Bidegree $(t-s, s)$:
$a = (2n, \alpha(n))$
$b = (2n, \alpha(n) + \nu(2n))$
$c = (2n+4, \alpha(n)+3)$ if $\nu(n) \geq 1$
$d = (2n+4, \alpha(n) + \nu(4n))$ if $\nu(n) \geq 1$
c and d do not exist if $\nu(n) = 0$

Figure 1: The chart $C^{s,t}(2n)$. Height of gray vertical towers varies with n, and the tower connecting c to d does not exist when $\nu(n) = 0$.

In this paper we calculate the ko homology of the mod 2 Brown–Gitler spectra via an Adams spectral sequence. Recall from Ravenel [20] that

$$H_*(\mathrm{ko}) = P(\zeta_1^4, \zeta_2^2, \zeta_i \mid i \geq 3), \tag{1-4}$$

where $\zeta_n = \chi(\xi_n)$ is the conjugate of the Milnor generator in the dual of the Steenrod algebra, and $P(x, y, \dots)$ denotes a polynomial algebra over $\mathbb{F}_2$. Define the ξ weight of a monomial to be

$$wt_\xi(\xi_1^{i_1}\xi_2^{i_2}\cdots\xi_\ell^{i_\ell}) = \sum_{j=1}^{\ell} i_j 2^{j-1},$$

and define the weight on sums by taking the maximum weight among all terms, ie, $wt_\xi(\sum \xi^I) = \max_I\{wt_\xi(\xi^I)\}$. If $\sum \zeta^J = \sum \xi^I$ under change of basis, set $wt_\xi(\sum \zeta^J)$ equal to $wt_\xi(\sum \xi^I)$.

Let A_1 denote the sub-Hopf algebra of the mod 2 Steenrod algebra generated by Sq^1 and Sq^2. Let $C^{*,*}(0) = \mathrm{Ext}^{*,*}_{A_1}(\mathbb{F}_2, \mathbb{F}_2)$, and define $C^{*,*}(2n)$ for $n \geq 1$ by the chart in Figure 1 extended to the left and to the right by $(8,4) = (t-s, s)$ real Bott periodicity, deleting everything below Adams filtration $s = 0$. Let $\alpha(n)$ be the number of ones in the 2–adic expansion of n, and let $\nu(n)$ be the number of times that 2 divides n.

Our main result is the calculation of $\mathrm{ko}_*(B(2n))$, the connective real K–theory of a mod 2 Brown–Gitler spectrum $B(2n)$.

Theorem 1.1 *Consider the Adams spectral sequence*

$$\mathrm{Ext}^{s,t}_{A_1}(H^*(B(2n)), \mathbb{F}_2) \Longrightarrow \mathrm{ko}_{t-s}(B(2n)).$$

(1) *For $s = 0$, there is a bijection*

$$\mathrm{Ext}^{0,*}_{A_1}(H^*(B(2n)), \mathbb{F}_2) \leftrightarrow \left\{\sum \zeta^I \in H_*(\mathrm{ko}) \mid wt_\xi\left(\sum \zeta^I\right) \leq n\right\}.$$

(2) *There is a map*

$$C^{s,t}(2n) \to \mathrm{Ext}^{s,t}_{A_1}(H^*(B(2n)), \mathbb{F}_2)$$

that is injective if $s = 0$ and an isomorphism if $s \geq 1$.

(3) *Finally, the Adams spectral sequence for* $\mathrm{ko}_*(B(2n))$ *collapses, ie, $E_2 \cong E_\infty$.*

Part (1) of this theorem is presented in Theorem 7.10, while parts (2) and (3) are presented in Theorem 8.2. The mod 2 Hopf ring for ko was calculated by Morton using the bar spectral sequence [18; 19]. However, the results are so lengthy and detailed that discerning the global structure of the Hopf ring for ko is difficult. In contrast, part (1) of Theorem 1.1 determines all of the stable classes in this Dieudonné ring by name and part (2) reveals a "lightning bolt and tower" pattern that describes the unstable classes and the relations among them in the Dieudonné ring $D_*(H_*(\underline{\mathrm{ko}}_*))$. One limitation of our approach is that the map $\mathrm{ko}_*(B(n)) \to H_n(\underline{\mathrm{ko}}_*)$, given in Equation (5-3), is an isomorphism when n is even, but only a surjection when n is odd. However, since $B(2n) \simeq B(2n+1)$ and $\mathrm{ko}_*(B(2n+1)) \to H_{2n+1}(\underline{\mathrm{ko}}_*)$ is surjective, every generator in $H_{2n+1}(\underline{\mathrm{ko}}_*)$ is a homology suspension of a generator in $H_{2n}(\underline{\mathrm{ko}}_*)$. Consequently, the results of Theorem 1.1 calculate $H_{2n}(\underline{\mathrm{ko}}_*)$ completely, and $H_{2n+1}(\underline{\mathrm{ko}}_*)$ up to determination of elements in the kernel of the homology suspension $e \circ (-)\colon H_{2n}(\underline{\mathrm{ko}}_k) \to H_{2n+1}(\underline{\mathrm{ko}}_{k+1})$.

The organization of this paper is as follows. In Section 2 we define categories of Hopf rings and Dieudonné rings and recall their equivalence. In Section 3, we define a trigrading on the dual of the mod 2 Steenrod algebra and recall the action of the

Steenrod algebra on its dual. In Section 4 we recall the Lambda algebra and the Adams spectral sequence. In Section 5 we give the connection between Brown–Gitler spectra and Dieudonné rings. In Section 6 we recall the Dieudonné ring and Hopf ring for the mod 2 Eilenberg–Mac Lane spectrum. In Section 7 we calculate $s = 0$ line of the Adams spectral sequence for ko homology, thereby determining the stable classes in the Dieudonné ring for ko. In Section 8 we calculate the E_2 term of the Adams spectral sequence for the ko homology of $B(2n)$ up to stable isomorphism, thereby determining the unstable classes in the Dieudonné ring for ko.

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2 Hopf rings and Dieudonné rings

Fix a prime $p > 0$. Let $\mathbb{F}_p$ be the finite field of p elements, and let $\mathcal{C}$ be the category of graded connected cocommutative, coassociative coalgebras with counit over $\mathbb{F}_p$. Graded abelian group and ring objects in $\mathcal{C}$ comprise the categories of Hopf algebras $\mathcal{HA}$ and Hopf rings $\mathcal{HR}$ over $\mathbb{F}_p$, respectively. A Hopf algebra (or coalgebraic group) is an algebra with addition $+$, multiplication $*$, conjugation χ, and coproduct ψ. Its multiplication $*$ is a categorical addition with inverse χ and zero element $[0] = 1$. A Hopf ring (or coalgebraic ring) has an additional product $\circ$, which is a categorical multiplication with unit element $[1]$. For detailed information about Hopf rings and coalgebraic algebra, please see Ravenel and Wilson [21], Strickland [24], Hunton and Turner [12] and Wilson [25].

Example 2.1 Let E be a ring spectrum and let $\underline{E}_k = \Omega^\infty \Sigma^k E$ be the k^{th} space in its Ω–spectrum. Write $H_{n,k}$ for $H_n(\underline{E}_k)$. Then $H_{*,k} = H_*(\underline{E}_k)$ is a Hopf algebra over $\mathbb{F}_p$ for each k, and $H_{*,*} = H_*(\underline{E}_*)$ is a Hopf ring over $\mathbb{F}_p$. Three sub-Hopf rings of $H_{*,*}$ are $H_{0,*}$, $H_{*,0}$ and $\{H_{n,k}\}_{(n,k)\in\mathbb{N}\times\mathbb{N}}$.

Every Hopf algebra $H = H_{*,k}$ over $\mathbb{F}_p$ has Frobenius and Verschiebung maps. Let $H^\vee = \mathrm{Hom}_{\mathbb{F}_p\text{–mod}}(H, \mathbb{F}_p)$ denote the $\mathbb{F}_p$–linear dual of H.

Definition 2.2 Fix $k \in \mathbb{Z}$. The *Frobenius* $F\colon H_{n,k} \to H_{pn,k}$ is defined by $F(x) = x^{*p}$. The *Verschiebung* $V\colon H_{n,k} \to H_{n/p,k}$ is the $\mathbb{F}_p$–linear dual of the p^{th} power map

$f\colon H^\vee \to H^\vee$, $f\colon x \mapsto x^p$, and V is zero when $p \nmid n$. If H is of finite type, it is equivalent to define the Verschiebung by $V(x) = ax'$ if the iterated coproduct is

$$\psi^{(p-1)}(x) = a \underbrace{x' \otimes x' \otimes \cdots \otimes x'}_{p \text{ factors}} + (\text{other terms}),$$

where each of the other terms have at least one tensor factor different from the others.

The Verschiebung is a homomorphism of Hopf rings (2-1), but the Frobenius is not. Instead, these maps satisfy the Frobenius reciprocity relations (2-2) and (2-3):

$$V(x \circ y) = V(x) \circ V(y) \tag{2-1}$$

$$F(x \circ V(y)) = F(x) \circ y \tag{2-2}$$

$$F(V(x) \circ y) = x \circ F(y) \tag{2-3}$$

The following Hopf algebras are used to define the Dieudonné functor.

Definition 2.3 Let $CW(0) = \widehat{\mathbb{Z}}_p[\mathbb{Z}]$ be the Hopf algebra over $\widehat{\mathbb{Z}}_p$ concentrated in degree 0 with coproduct $\psi([r]) = [r] \otimes [r]$. For $n > 0$, write $n = p^a b$, where $\gcd(p, b) = 1$, and let $CW(n) = \widehat{\mathbb{Z}}_p[x_0, \dots, x_a]$. For $0 \leq i \leq a$, give $CW(n)$ the grading $|x_i| = p^i b$ and the unique coproduct such that the Witt polynomials

$$w_i = x_0^{*p^i} + p x_1^{*p^{i-1}} + p^2 x_2^{*p^{i-2}} + \cdots + p^i x_i$$

are primitive.

Let $H(n)$ be the Hopf algebra over $\mathbb{F}_p$ that is the mod p reduction of $CW(n)$. Let $v\colon H(n) \to H(pn)$ be the identity map if $n = 0$ and the inclusion $v\colon x_i \mapsto x_i$ if $n > 0$. Let $f\colon H(pn) \to H(n)$ be $f([r]) = [rp]$ if $n = 0$ and the map $f\colon x_i \mapsto (x_{i-1})^p$ if $n > 0$, where $x_{-1} = 0$.

We now define the categories $\mathcal{DM}$ and $\mathcal{DR}$ of graded Dieudonné modules and rings over $\widehat{\mathbb{Z}}_p$, respectively. Then, we show that the Dieudonné functor D establishes an equivalence between the categories of Hopf rings over $\mathbb{F}_p$ and Dieudonné rings over $\widehat{\mathbb{Z}}_p$. For more details on Dieudonné rings, please see Goerss [9], and Buchstaber and Lazarev [6].

Definition 2.4 Fix $k \in \mathbb{Z}$. A graded *Dieudonné module* $M = M_{*,k}$ over $\widehat{\mathbb{Z}}_p$ is a non-negatively graded abelian group with a Frobenius map $F\colon M_{n,k} \to M_{pn,k}$ and a Verschiebung map $V\colon M_{n,k} \to M_{n/p,k}$, which is zero when $p \nmid n$, such that $F(V(x)) = V(F(x)) = px$, V is the identity on $M_{0,k}$, and $p^{a+1} M_{p^a b,k} = 0$ if $\gcd(p, b) = 1$.

Definition 2.5 Fix $k \in \mathbb{Z}$. Let $H = H_{*,k}$ be a Hopf algebra over $\mathbb{F}_p$. The Dieudonné module $D_*(H)$ is the graded abelian group $\{D_n(H_{*,k})\}_{n\in\mathbb{N}}$ with

$$D_n(H_{*,k}) = \mathrm{Hom}_{\mathcal{HA}}(H(n), H_{*,k}).$$

The Frobenius and Verschiebung

$$F = f^*\colon D_n(H_{*,k}) \to D_{pn}(H_{*,k}),$$
$$V = v^*\colon D_{pn}(H_{*,k}) \to D_n(H_{*,k})$$

are induced by the maps f and v of Definition 2.3.

Remark 2.6 The fixed integer $k \in \mathbb{Z}$ in Definitions 2.4 and 2.5 plays no role for an individual Dieudonné module and could be omitted from these definitions. The index $k \in \mathbb{Z}$ was inserted into Definitions 2.4 and 2.5 because it will be used later to assemble an indexed collection of Dieudonné modules into a Dieudonné ring.

Theorem 2.7 (Schoeller's Theorem [22; 9, Theorem 4.7]) *The Dieudonné functor D has a right adjoint U, and the pair (D, U) is an equivalence between the category $\mathcal{HA}$ of Hopf algebras and $\mathcal{DM}$ of Dieudonné modules.*

We now define the category $\mathcal{DR}$ of Dieudonné rings.

Definition 2.8 A graded commutative *Dieudonné ring* over $\widehat{\mathbb{Z}}_p$ is a collection of Dieudonné modules $\{M_{*,k}\}_{k\in\mathbb{Z}}$ together with bilinear maps

$$\circ\colon M_{m,j} \otimes_{\widehat{\mathbb{Z}}_p} M_{n,k} \to M_{m+n,j+k}$$

such that equations (2-1)–(2-3) are satisfied. Graded commutativity is expressed by $x \circ y = (-1)^{mn+jk} y \circ x$ for $x \in M_{m,j}$ and $y \in M_{n,k}$.

In [9], Goerss constructed symmetric monoidal products $\boxtimes_{\mathcal{HA}}$ and $\boxtimes_{\mathcal{DM}}$ for the categories of Hopf algebras and Dieudonné modules. He showed that the Dieudonné functor was symmetric monoidal, and thus established an equivalence between the category of Hopf rings over $\mathbb{F}_p$ that are group rings in degree zero and Dieudonné rings over $\widehat{\mathbb{Z}}_p$.

Theorem 2.9 (Goerss's Theorem [9, Theorem 7.7]) *For any $H, K \in \mathcal{HA}$ such that $H_{0,k}$ and $K_{0,k}$ are group rings for every integer k, there is a natural isomorphism of Dieudonné modules*

$$D_*(H) \boxtimes_{\mathcal{DM}} D_*(K) \to D_*(H \boxtimes_{\mathcal{HA}} K).$$

Example 2.10 Let E be a ring spectrum. Then

$$H_0(\underline{E}_k) = \mathbb{F}_p[\pi_0(\underline{E}_k)] = \mathbb{F}_p[\pi^S_{-k}(E)]$$

is a group ring for each k. By Theorem 2.9, the Hopf ring $H_*(\underline{E}_*)$ over $\mathbb{F}_p$ is equivalent to the Dieudonné ring $D_*(H_*(\underline{E}_*))$ over $\widehat{\mathbb{Z}}_p$, and under this equivalence $H_n(\underline{E}_k)$ corresponds to $D_n(H_*(\underline{E}_k))$.

3 The dual of the Steenrod algebra

In this section, we recall the dual of the Steenrod algebra at the prime 2 as a trigraded object. In this section and the remainder of the paper, denote the mod 2 Eilenberg–Mac Lane spectrum by $\mathrm{H}\mathbb{F}$. Recall from Milnor [17] that the $\mathbb{F}_2$–linear dual of the mod 2 Steenrod algebra is $A^\vee = H_*(\mathrm{H}\mathbb{F}) = \mathbb{F}_2[\xi_i \mid i \geq 0]/(\xi_0 = 1)$. Let $\zeta_i = \chi(\xi_i)$, where χ is the canonical antiautomorphism. Then $A^\vee = \mathbb{F}_2[\zeta_i \mid i \geq 1]$, and χ is a change of basis. For $I = (i_1, \dots, i_n)$, let $\xi^I = \xi_1^{i_1} \cdots \xi_n^{i_n}$.

Definition 3.1 The *degree function* $\deg\colon A^\vee \to \mathbb{N}$ is given by $\deg(1) = 0$ and $\deg(\xi_n) = \deg(\zeta_n) = 2^n - 1$ and satisfies $\deg(xy) = \deg(x) + \deg(y)$. Sums of monomials in $A^\vee$ must have homogeneous degree.

Definition 3.2 The ξ *weight function* $\mathrm{wt}_\xi\colon A^\vee \to \mathbb{N}$ is given by $\mathrm{wt}_\xi(1) = 0$ and $\mathrm{wt}_\xi(\xi_n) = 2^{(n-1)}$ and satisfies $\mathrm{wt}_\xi(xy) = \mathrm{wt}_\xi(x) + \mathrm{wt}_\xi(y)$. If $\sum \xi^I = \sum \zeta^J$ under change of basis, then the value of $\mathrm{wt}_\xi(\sum \zeta^J)$ is set equal to $\mathrm{wt}_\xi(\sum \xi^I)$.

Definition 3.3 The ζ *weight function* $\mathrm{wt}_\zeta\colon A^\vee \to \mathbb{N}$ is given by $\mathrm{wt}_\zeta(1) = 0$ and $\mathrm{wt}_\zeta(\zeta_n) = 2^{(n-1)}$ and satisfies $\mathrm{wt}_\zeta(xy) = \mathrm{wt}_\zeta(x) + \mathrm{wt}_\zeta(y)$. If $\sum \xi^I = \sum \zeta^J$ under change of basis, then the value of $\mathrm{wt}_\zeta(\sum \xi^I)$ is set equal to $\mathrm{wt}_\zeta(\sum \zeta^J)$.

Definition 3.4 Define the ξ *factors function* $\mathrm{fact}_\xi\colon A^\vee \to \mathbb{N}$ is given by $\mathrm{fact}_\xi(1) = 0$ and $\mathrm{fact}_\xi(\xi_n) = 1$ and satisfies $\mathrm{fact}_\xi(xy) = \mathrm{fact}_\xi(x) + \mathrm{fact}_\xi(y)$. If $\sum \zeta^J = \sum \xi^I$ under change of basis, then the value of $\mathrm{fact}_\xi(\sum \zeta^J)$ is set equal to $\mathrm{fact}_\xi(\sum \xi^I)$.

The degree, ξ weight, and ξ factors satisfy a linear dependence relation.

Lemma 3.5 *Given $\sum \xi^I \in H_d(\mathrm{H}\mathbb{F})$, the degree $d = \deg(\sum \xi^I)$, the maximum number of factors $k = \mathrm{fact}_\xi(\sum \xi^I)$, and the maximum weight $n = \mathrm{wt}_\xi(\sum \xi^I)$ satisfy $d + k = 2n$.*

Proof Let $\sum \xi^I \in H_d(\mathrm{H}\mathbb{F})$. For any monomial ξ^I, we have

$$\begin{aligned} d &= \deg(\xi_1^{i_1}\xi_2^{i_2}\cdots\xi_\ell^{i_\ell}) \\ &= \sum_{j=1}^{\ell} i_j(2^j-1) = 2\Big(\sum_{j=1}^{\ell} i_j 2^{j-1}\Big) - \sum_{j=1}^{\ell} i_j = 2\,\mathrm{wt}_\xi(\xi^I) - \mathrm{fact}_\xi(\xi^I) \end{aligned}$$

and thus

$$\deg(\xi^I) + \mathrm{fact}_\xi(\xi^I) = 2\,\mathrm{wt}_\xi(\xi^I). \tag{3-1}$$

Since every monomial in the sum $\sum \xi^I$ has the same fixed degree $d = \deg(\xi^I)$, any monomial ξ^J in the sum that has the maximum number of ξ–factors must also have the maximum ξ–weight by Equation (3-1). Thus, $d + k = 2n$. □

We recall the action of the Steenrod algebra on its dual, which will be used in Adams spectral sequence computations.

Lemma 3.6 [16, Theorem 6.17; 5, Lemma 6.1] *Let* $\mathrm{Sq} = \sum_{i\geq 0} \mathrm{Sq}^i$ *be the total Steenrod square. The canonical right action of the Steenrod algebra on its dual* $A^\vee$ *is*

$$\xi_n \cdot \mathrm{Sq} = \xi_n + \xi_{n-1}, \quad \zeta_n \cdot \mathrm{Sq} = \sum_{i=0}^{n} \zeta_{n-i}^{2^i}.$$

4 The Lambda algebra and the Adams spectral sequence

We recall the Lambda algebra, the Araki–Kudo (or Dyer–Lashof) algebra, and the Adams spectral sequence at the prime $p = 2$.

Definition 4.1 The *Lambda algebra* Λ is the associative bigraded differential algebra over $\mathbb{F}_2$ with generators λ_a, $a \geq -1$, of bidegree $(1, a+1) = (s,t)$ modulo the two-sided ideal generated by the relations

$$\lambda_a\lambda_b = \sum_{\lceil (b-2a)/2\rceil \leq c < b-2a} \binom{c-1}{2c-b+2a} \lambda_{a+c}\lambda_{b-c}, \quad \text{if } 0 \leq 2a < b, \tag{4-1}$$

and the left ideal $\Lambda\{\lambda_{-1}\}$. Its differential $d_1(\lambda_b) = \lambda_{-1}\lambda_b$ is a derivation.

If $I = (i_1, \ldots, i_s)$ is an s–tuple of nonnegative integers, set $\lambda_I = \lambda_{i_1}\cdots\lambda_{i_s}$ and $\lambda_{(\,)} = 1$. We say that λ_I is admissible if $2i_j \geq i_{j+1}$ for $1 \leq j < s$. The admissible monomials form a basis for Λ.

The Lambda algebra provides an E_1–term for the Adams spectral sequence (Bousfield, Curtis, Kan, Quillen, Rector and Schlesinger [3]). Let A denote the mod 2 Steenrod algebra.

Theorem 4.2 (Adams spectral sequence [2; 3]) *Let X be a complex or spectrum of finite type, and let E be a spectrum. The E_1–term of the Adams spectral sequence for E homology is the differential right (Λ, d_1)–module*

$$E_1^{*,*}(A, E \wedge X) = H_*(E \wedge X) \otimes_{\mathbb{F}_2} \Lambda \tag{4-2}$$

with differential $d_1(z \otimes \lambda_I) = \sum_{i\geq 0} z \cdot \mathrm{Sq}^i \otimes \lambda_{i-1}(\lambda_I)$. Its E_2–term is

$$E_2^{s,t}(A, E \wedge X) = \mathrm{Ext}_A^{s,t}(H^*(E \wedge X), \mathbb{F}_2) \Longrightarrow \pi_{t-s}^S(E \wedge X) = E_{t-s}(X). \tag{4-3}$$

If $H^*(E)$ is a Hopf algebra quotient of the Steenrod algebra, the Adams spectral sequence for calculating $E_*(X)$ can be simplified by a change of rings theorem.

Theorem 4.3 (Change of rings [2]) *If E is a ring spectrum such that $H^*(E) = A/\!/C = A \otimes_C \mathbb{F}_2$ for some sub-Hopf algebra $C \subset A$, then there is an isomorphism, natural in X,*

$$\mathrm{Ext}_C^{s,t}(H^*(X), \mathbb{F}_2) \cong \mathrm{Ext}_A^{s,t}(H^*(E \wedge X), \mathbb{F}_2).$$

Define a sub-Hopf algebra $A_h \subset A$ by $A_h = \langle \mathrm{Sq}^{2^i} \mid 0 \leq i \leq h\rangle$ and set $A_{-1} = \mathbb{F}_2$. Then when $E = \mathrm{eo}_h$, we may use the change of rings theorem since $H^*(\mathrm{eo}_h) = A/\!/A_h = A \otimes_{A_h} \mathbb{F}_2 = A/A\{\mathrm{Sq}^{2^i} \mid 0 \leq i \leq h\}$.

5 Brown–Gitler spectra and Dieudonné rings

Brown and Gitler constructed a family of spectra at the prime 2 in [4]. Analogues of these spectra at odd primes were later constructed by R Cohen [7]. In this section we specialize to the prime 2, although analogous results also exist for odd primes [9].

The n^{th} mod 2 Brown–Gitler spectrum, which was originally denoted $B(n)$ and indexed by $n \in \frac{1}{2}\mathbb{N}$ in [4], will be denoted $B(2n)$ and indexed by $\mathbb{N}$. There is a homotopy equivalence $B(2n) \simeq B(2n+1)$ for all $n \in \mathbb{N}$, and $B(0)$ and $B(2)$ are the 2 complete sphere spectrum and mod 2 Moore spectrum, respectively. The Brown–Gitler spectra realize certain cyclic modules over the Steenrod algebra. They are characterized up to homotopy 2–equivalence by the following theorem.

Theorem 5.1 [4] *For each $n \in \mathbb{N}$ there is a 2–complete spectrum $B(2n)$ satisfying:*

(1) $H^*(B(2n)) = A/A\{\chi(\mathrm{Sq}^i) \mid i > n\}$ *as left A modules.*

(2) *If $\iota\colon B(2n) \to \mathrm{H}\mathbb{F}$ classifies the element $1 \in H^0(B(2n))$, then the induced map of reduced homology theories $\iota_*\colon B(2n)_i(X) \to H_i(X)$ is an epimorphism for all complexes X and $0 \le i \le 2n+1$.*

The homology of Brown–Gitler spectra can be described as a right A submodule of $A^\vee$ using the ζ weight function. This weight function is induced by the May filtration of $\Omega^2 S^3$ by identifying the Thom spectrum of the canonical bundle on $\Omega^2 S^3$ with $\mathrm{H}\mathbb{F}$ (Mahowald [14]).

Lemma 5.2 [14] *There is an isomorphism of right A–modules*

$$H_*(B(2n)) \cong \mathbb{F}_2\{\zeta^I \in A^\vee \mid \mathrm{wt}_\zeta(\zeta^I) \le n\}.$$

Remark 5.3 The right A–module structure of $A^\vee = H_*(\mathrm{H}\mathbb{F})$ is given in Lemma 3.6.

The following Mahowald cofiber sequence is very useful for computations.

Lemma 5.4 [9] *For each integer $n \ge 1$, there is a cofiber sequence of spectra*

$$B(2n-2) \longrightarrow B(2n) \xrightarrow{v} \Sigma^n B(n), \tag{5-1}$$

which induces a short exact sequence of left A–modules

$$0 \longleftarrow H^*(B(2n-2)) \longleftarrow H^*(B(2n)) \xleftarrow{v^*} \Sigma^n H^*(B(n)) \longleftarrow 0 \tag{5-2}$$

in which $v^*(\Sigma^n 1) = \chi(\mathrm{Sq}^n)$.

We now show that $E_*(B(*))$ is a Dieudonné ring.

Example 5.5 Let $E_*(-)$ be a generalized homology theory. There are pairings $B(m) \wedge B(n) \to B(m+n)$ that make $B(*) = \{B(n)\}_{n\in\mathbb{N}}$ a graded commutative ring spectrum, and $B(*)_*(E)$ a graded commutative ring. Additionally, there are maps $f\colon \Sigma^n B(n) \to B(2n)$ and $v\colon B(2n) \to \Sigma^n B(n)$ so that fv and vf are multiplication by 2. The map v is the map in the Mahowald cofiber sequence of Equation (5-1). The maps f and v induce the Frobenius and Verschiebung maps in the Dieudonné ring $E_*(B(*))$. For more details, please see [9].

The next theorem states that Brown–Gitler spectra are, in some sense, the representing objects for the Dieudonné functor.

Theorem 5.6 [9, Proposition 11.3] *For any ring spectrum E and all $(n,k) \in \mathbb{N} \times \mathbb{Z}$, the map*

$$T\colon E_{n-k}(B(n)) \to D_n(H_*(\underline{E}_k))$$

is a surjective homomorphism of Dieudonné rings that respects the Frobenius and Verschiebung, and is an isomorphism when n is even.

To calculate Dieudonné ring and Hopf ring for a ring spectrum E, we use the composite

$$E_2^{*,*}(A, E \wedge B(*)) \Longrightarrow E_*(B(*)) \xrightarrow{T} D_*(H_*(\underline{E}_*)) \xrightarrow{U} H_*(\underline{E}_*) \tag{5-3}$$

of the Adams spectral sequence, the canonical antiautomorphism χ induced by the transposition map, the surjective map T which is an isomorphism half of the time, and the right adjoint U of the Dieudonné functor D.

We also need integral versions of Brown–Gitler spectra for our later calculations. The n^{th} integral Brown–Gitler spectrum, which was originally denoted $B_1(n)$ and indexed by $n \in \frac{1}{2}\mathbb{N}$ in Shimamoto [23], and Goerss, Jones and Mahowald [10], will be denoted $B_0(4n)$. For all $n \in \mathbb{N}$ and $1 \le i \le 3$, set $B_0(4n) = B_0(4n+i)$ and then index $B_0(n)$ by $n \in \mathbb{N}$. The integral Brown–Gitler spectra realize certain cyclic modules over the Steenrod algebra, and are characterized up to homotopy 2–equivalence by the following theorem.

Theorem 5.7 [10; 23] *For $n \in \mathbb{N}$ there is a 2–complete spectrum $B_0(4n)$ satisfying:*

1. $H^*(B_0(4n)) = H^*(B(4n)) \otimes_{A_0} \mathbb{F}_2 = A/A\{\chi(\mathrm{Sq}^i), \mathrm{Sq}^1 \mid i > 2n\}$ *as left A–modules.*
2. *If $\iota\colon B_0(4n) \to \mathrm{H}\widehat{\mathbb{Z}}_2$ classifies the element $1 \in H^0(B_0(4n); \widehat{\mathbb{Z}}_2)$, then the induced map of reduced homology theories $\iota_*\colon B_0(4n)_i(X) \to H_i(X; \widehat{\mathbb{Z}}_2)$ is an epimorphism for all complexes X and $0 \le i \le 4n+1$.*

The following Shimamoto cofiber sequences relate the mod 2 and integral Brown–Gitler spectra and are very useful for computations.

Lemma 5.8 [23, Theorem 2.15] *There are cofiber sequences of spectra*

$$B_0(4n-4) \longrightarrow B_0(4n) \longrightarrow B(4n), \quad \text{for } n \ge 1, \tag{5-4}$$

$$B_0(4n) \xrightarrow{\cdot 2} B_0(4n) \longrightarrow B(4n+2), \quad \text{for } n \ge 0, \tag{5-5}$$

that induce short exact sequences of left A–modules

$$0 \longleftarrow H^*(B_0(4n)) \longleftarrow H^*(B(4n)) \longleftarrow \Sigma^1 H^*(B_0(4n-4)) \longleftarrow 0, \tag{5-6}$$

$$0 \longleftarrow H^*(B_0(4n)) \longleftarrow H^*(B(4n+2)) \longleftarrow \Sigma^1 H^*(B_0(4n)) \longleftarrow 0. \tag{5-7}$$

6 The Hopf ring $H_*(\underline{\mathrm{H}\mathbb{F}}_*)$

In this section we recall the structure of the Hopf ring $H_*(\underline{\mathrm{H}\mathbb{F}}_*)$, define its conjugate generators z_i and define its destabilization function.

Recall that $H_*(\underline{\mathrm{H}\mathbb{F}}_1) = H_*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{F}_2\{b_i \mid i \geq 1\}$, where $|b_i| = i$. The product is $b_i * b_j = \binom{i+j}{i} b_{i+j}$, the Frobenius is $F(b_i) = b_i^{*2} = 0$, and the indecomposables are the module $*\!-\mathrm{Ind}(H_*(\underline{\mathrm{H}\mathbb{F}}_1)) = \mathbb{F}_2\{x_i \mid i \geq 0\}$, where $x_i = b_{2^i}$. The coproduct is $\psi(b_n) = \sum_{0 \leq i \leq n} b_{n-i} \otimes b_i$, the Verschiebung is $V(b_{2i}) = b_i$ and $V(b_{2i+1}) = 0$, and the primitives are given by Newton polynomials, which are defined recursively by $N_i = N_i(b_1, \dots, b_i) = i b_i + \sum_{j=1}^{i-1} b_j * N_{i-j}(b_1, \dots, b_{i-j}) \bmod 2$ for $i \geq 1$ (see [2, pages 93–94; 13, Section 3] for more details). The suspension class is $e = x_0 = b_1$. The stabilization homomorphism is $e^\infty(x_i) = \xi_i$, and satisfies $e^\infty(x_i * x_j) = 0$ and $e^\infty(x_i \circ x_j) = \xi_i \xi_j$. Note: by definition

$$H_n(\mathrm{H}\mathbb{F}) = \lim_{k \to \infty} H_{n+k}(\underline{\mathrm{H}\mathbb{F}}_k),$$

where the limit is taken by iterating the homology suspension $e \circ (-)\colon H_{n+k}(\underline{\mathrm{H}\mathbb{F}}_k) \to H_{n+k+1}(\underline{\mathrm{H}\mathbb{F}}_{k+1})$.

Theorem 6.1 [25; 16] *As Hopf algebras over $\mathbb{F}_2$ with addition $+$ and multiplication $*$,*

$$H_*(\underline{\mathrm{H}\mathbb{F}}_k) = \begin{cases} \mathbb{F}_2[\mathbb{F}_2] & \text{if } k = 0, \\ E(x_{i_1} \circ \cdots \circ x_{i_k} \mid 0 \leq i_1 \leq \cdots \leq i_k) & \text{if } k \geq 1. \end{cases}$$

Further, $*$-$\mathrm{Ind}(H_*(\underline{\mathrm{H}\mathbb{F}}_*)) = \mathrm{Sym}(x_i \mid i \geq 0)$, the bigraded symmetric algebra over $\mathbb{F}_2$ with addition $+$, multiplication $\circ$, and generators $x_i \in H_{2^i}(\underline{\mathrm{H}\mathbb{F}}_1)$.

We now define elements z_n that are the destabilization of the conjugate $\zeta_n = \chi(\xi_n)$ in the dual of the Steenrod algebra, following the definition by Milnor [17].

Definition 6.2 An ordered partition of n of length ℓ is a sequence $(\alpha_1, \alpha_2, \dots, \alpha_\ell)$ of positive integers whose sum is n. Let $\mathrm{Part}(n)$ denote the set of all 2^{n-1} ordered partitions of n. Given an ordered partition $(\alpha_1, \dots, \alpha_\ell) \in \mathrm{Part}(n)$, let

$$\sigma(i) = \begin{cases} \alpha_1 + \alpha_2 + \cdots + \alpha_{i-1} & \text{if } 1 \leq i \leq \ell, \\ 2^n - 1 - (2^{\sigma(1)} + \cdots + 2^{\sigma(\ell)}) & \text{if } i = 0. \end{cases}$$

Let $e = x_0 = z_0 \in H_1(\underline{\mathrm{H}\mathbb{F}}_1)$, and for $n \geq 1$ define $z_n \in H_{2^{n+1}-2}(\underline{\mathrm{H}\mathbb{F}}_{2^n-1})$ by

$$z_n = \sum_{\substack{(\alpha_1, \dots, \alpha_\ell) \\ \in \mathrm{Part}(n)}} x_0^{\circ\sigma(0)} \circ x_{\alpha_1}^{\circ 2^{\sigma(1)}} \circ x_{\alpha_2}^{\circ 2^{\sigma(2)}} \circ \cdots \circ x_{\alpha_\ell}^{\circ 2^{\sigma(\ell)}}. \tag{6-1}$$

Remark 6.3 Every term of the sum (6-1) has $2^n-1=\deg(\zeta_n)$ factors because for each $(\alpha_1,\dots,\alpha_\ell)\in\mathrm{Part}(n)$ the sum of the exponents in the term $x_0^{\circ\sigma(0)}\circ x_{\alpha_1}^{\circ 2^{\sigma(1)}}\circ\cdots\circ x_{\alpha_\ell}^{2^{\sigma(\ell)}}$ is 2^n-1 by construction.

We now define the destabilization function $e^{-\infty}\colon H_*(\mathrm{H}\mathbb{F})\to H_*(\underline{\mathrm{H}\mathbb{F}}_*)$. It will be constructed so that every element in the image of $e^{-\infty}$ cannot be desuspended any further and $e^{-\infty}$ is a right inverse for the stabilization homomorphism e^{∞}, ie, the composite map

$$H_*(\mathrm{H}\mathbb{F})\xrightarrow{e^{-\infty}}H_*(\underline{\mathrm{H}\mathbb{F}}_*)\xrightarrow{e^{\infty}}H_*(\mathrm{H}\mathbb{F}) \tag{6-2}$$

is the identity on $H_*(\mathrm{H}\mathbb{F})$. It is clear that the destabilization function ought to be defined as follows. Since $e^{\infty}([1])=1$ and $e^{\infty}(x^{\circ I})=\xi^I$, the destabilization function should satisfy $e^{-\infty}(1)=[1]$ and $e^{-\infty}(\xi^I)=x^{\circ I}$. Further, if a sum of monomials $\sum x^{\circ I}\in H_{d+k}(\underline{\mathrm{H}\mathbb{F}}_k)$ has one or more terms with no $\circ$–product factors of $e=x_0$, then it cannot be desuspended further and the destabilization should be $e^{-\infty}(\sum\xi^I)=\sum x^{\circ I}$.

Finding an explicit formula for the destabilization function in terms of the x_i and z_i remains to be done. In the next definition and lemma, we construct an explicit formula for the destabilization $e^{-\infty}(\sum\xi^I)$, show that all the terms in $e^{-\infty}(\sum\xi^I)$ have the same bidegree, and show that the destabilization function is well-defined. From the construction of the formula for the destabilization, it will be evident that elements in the image of the destabilization cannot be desuspended further and that the destabilization is a right inverse for the stabilization.

Definition 6.4 The *destabilization function* $e^{-\infty}\colon H_*(\mathrm{H}\mathbb{F})\to H_*(\underline{\mathrm{H}\mathbb{F}}_*)$ is given as follows. Set $e^{-\infty}(1)=[1]\in H_0(\underline{\mathrm{H}\mathbb{F}}_0)=\mathbb{F}_2[\mathbb{F}_2]$. Suppose $\sum\xi^I=\sum\zeta^J$ has $d=\deg(\sum\xi^I)>0$ and $k=\mathrm{fact}_\xi(\sum\xi^I)$. Then, in terms of the basis of the x_i,

$$e^{-\infty}\left(\sum\xi^I\right)=e^{-\infty}\left(\sum\zeta^J\right):=\sum x_0^{\circ(k-\mathrm{fact}_\xi(\xi^I))}\circ x^{\circ I}. \tag{6-3}$$

In terms of the basis of the z_i,

$$e^{-\infty}\left(\sum\xi^I\right)=e^{-\infty}\left(\sum\zeta^J\right):=z_0^{\circ-(d-k)}\circ\left(\sum z^{\circ J}\right). \tag{6-4}$$

Remark 6.5 By construction, every term in the sum (6-3) has exactly k $\circ$–product factors of x_i. In contrast, the sum (6-4) does not necessarily have the same number of $\circ$–product factors of z_i in each term.

Lemma 6.6 *The classes defined by equations* (6-3) *and* (6-4) *are in* $H_{d+k}(\underline{\mathrm{H}\mathbb{F}}_k)$ *and the destabilization function is well-defined.*

Proof Suppose $\sum \xi^I = \sum \zeta^J \in H_d(\mathrm{H}\mathbb{F})$ has $d > 0$ and $\mathrm{fact}_\xi(\sum \xi^I) = k$. First, we verify that $e^{-\infty}(\sum \xi^I) \in H_{d+k}(\underline{\mathrm{H}\mathbb{F}}_k)$. By construction, every term in the sum

$$e^{-\infty}\Big(\sum \xi^I\Big) = \sum x_0^{\circ(k-\mathrm{fact}_\xi(\xi^I))} \circ x^{\circ I}$$

has exactly k $\circ$–product factors of x_i and therefore $e^{-\infty}(\sum \xi^I) \in H_*(\underline{\mathrm{H}\mathbb{F}}_k)$. Also, from Equation (6-3) it is clear that $e^{\infty}(e^{-\infty}(\sum \xi^I)) = \sum \xi^I \in H_d(\mathrm{H}\mathbb{F})$, and therefore $e^{-\infty}(\sum \xi^I) \in H_{d+k}(\underline{\mathrm{H}\mathbb{F}}_k)$.

Next, we show that

$$\sum x_0^{k-\mathrm{fact}_\xi(\xi^I)} \circ x^{\circ I} = z_0^{\circ-(d-k)} \circ \Big(\sum z^{\circ J}\Big).$$

By Remark 6.3, $d \geq k$ and under the change of basis $\sum z^{\circ J} = \sum x^{\circ L}$, every term $x^{\circ L}$ has $d = \deg(\sum \zeta^J)$ $\circ$–product factors. Since $d = d - \mathrm{fact}_\xi(\xi^I) + \mathrm{fact}_\xi(\xi^I)$ and $d - k \geq 0$ and $k - \mathrm{fact}_\xi(\xi^I) \geq 0$ for all terms in $\sum \xi^I$, it follows that

$$\begin{aligned}
\sum z^{\circ J} &= \sum x^{\circ L} \\
&= \sum x_0^{\circ(d-\mathrm{fact}_\xi(\xi^I))} \circ x^{\circ I} = \sum x_0^{\circ(d-k+k-\mathrm{fact}_\xi(\xi^I))} \circ x^{\circ I} \\
&= x_0^{\circ(d-k)} \circ \Big(\sum x_0^{\circ(k-\mathrm{fact}_\xi(\xi^I))} \circ x^{\circ I}\Big) = x_0^{\circ(d-k)} \circ e^{-\infty}(\textstyle\sum \xi^I).
\end{aligned}$$

Thus, after desuspending $(d-k)$ times we obtain

$$e^{-\infty}(\textstyle\sum \xi^I) = \sum x_0^{\circ(k-\mathrm{fact}_\xi(\xi^I))} \circ x^{\circ I} = z_0^{\circ-(d-k)}(\sum z^{\circ J}).$$

Therefore, $e^{-\infty}(\sum \xi^I) = e^{-\infty}(\sum \zeta^J)$ and the desuspension function is well-defined. $\square$

Remark 6.7 The desuspension $z_0^{\circ-(d-k)}$ in Equation (6-4) occurs for the following reason. When $\sum z^{\circ J}$ is written in terms of the basis of the x_i there may be cancellation of terms mod 2, and in the sum that remains after cancellation $(\sum x^{\circ L})$, the greatest common factor of the x_0 is $x_0^{\circ(d-k)}$.

Example 6.8 The destabilization of the element $\xi_3^2 + \xi_1^2\xi_2^4 = \zeta_3^2 + \zeta_1^8\zeta_2^2 \in H_{14}(\mathrm{H}\mathbb{F})$ with degree $d = 14$ and $k = \mathrm{fact}_\xi\big(\xi_3^2 + \xi_1^2\xi_2^4\big) = 6$ is

$$e^{-\infty}\big(\xi_3^2 + \xi_1^2\xi_2^4\big) = x_0^{\circ 4} \circ x_3^{\circ 2} + x_1^{\circ 2} \circ x_2^{\circ 4} \in H_{20}(\underline{\mathrm{H}\mathbb{F}}_6),$$

which also equals

$$e^{-\infty}\big(\zeta_3^2 + \zeta_1^8\zeta_2^2\big) = z_0^{\circ(-8)} \circ \big(z_3^{\circ 2} + z_1^{\circ 8} \circ z_2^{\circ 2}\big) \in H_{20}(\underline{\mathrm{H}\mathbb{F}}_6).$$

7 Stable classes in $\mathrm{ko}_*(B(*))$

In this section, we calculate the $s=0$ line of the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{A_1}(H^*(B(2n)),\mathbb{F}_2) \Longrightarrow \mathrm{ko}_*(B(2n)) \tag{7-1}$$

for all $n \geq 0$, thereby determining the stable classes in $\mathrm{ko}_*(B(*))$.

We begin by defining the destabilization function $\epsilon^{-\infty}$ for Dieudonné rings that is equivalent to the destabilization function $e^{-\infty}$ for Hopf rings. In this section and the next, the destabilization function $\epsilon^{-\infty}$ will be used to show that a permanent cycle on the $s=0$ line of the Adams spectral sequence

$$H^{0,d}(H_*(\mathrm{ko}) \otimes H_*(B(d+k)) \otimes \Lambda, d_1)$$

determines a nonzero element in $D_{d+k}(H_*(\underline{\mathrm{ko}}_k))$ that corresponds to a stable class in $H_{d+k}(\underline{\mathrm{ko}}_k)$. We now define the function that induces the Dieudonné ring *destabilization function* $H_*(E) \to D_*(H_*(\underline{E}_*))$.

Definition 7.1 Let $E = \mathrm{H}\mathbb{F}$. Define a function

$$\epsilon^{-\infty}\colon H_*(E) \to H_*(E) \otimes H_*(B(\infty)) \otimes \Lambda \tag{7-2}$$

by $\epsilon^{-\infty}(y) = \tau(\chi \otimes 1(\psi(y))) \otimes 1$, where ψ is the coproduct, χ is the antiautomorphism, and $\tau(x \otimes y) = y \otimes x$ is the graded twist map, which has no sign mod 2.

Lemma 7.2 *The destabilization function $\epsilon^{-\infty}$ is a ring homomorphism.*

Proof We verify $\epsilon^{-\infty}(ab) = \epsilon^{-\infty}(a)\epsilon^{-\infty}(b)$ and leave it to the reader to verify the remaining properties of a ring homomorphism are satisfied. First, note that τ and ψ are ring homomorphisms and that the antiautomorphism χ is also a ring homomorphism because $H_*(B(\infty)) = H_*(\mathrm{H}\mathbb{F})$ is commutative. Let $a, b \in H_*(\mathrm{H}\mathbb{F})$ and write $\psi(a) = \sum_i a_i' \otimes a_i''$ and $\psi(b) = \sum_j b_j' \otimes b_j''$. Then

$$\begin{aligned}
\epsilon^{-\infty}(ab) &= \tau(\chi \otimes 1(\psi(ab))) \otimes 1 = \tau(\chi \otimes 1(\psi(a)\psi(b))) \otimes 1 \\
&= \tau(\chi \otimes 1((\textstyle\sum_i a_i' \otimes a_i'')(\sum_j b_j' \otimes b_j''))) \otimes 1 \\
&= \tau(\chi \otimes 1(\textstyle\sum_{i,j} a_i'b_j' \otimes a_i''b_j'')) \otimes 1 = \sum_{i,j} a_i''b_j'' \otimes \chi(a_i')\chi(b_j') \otimes 1 \\
&= (\textstyle\sum_i a_i'' \otimes \chi(a_i') \otimes 1)(\sum_j b_j'' \otimes \chi(b_j') \otimes 1) \\
&= (\tau(\chi \otimes 1(\textstyle\sum_i a_i' \otimes a_i'')) \otimes 1)(\tau(\chi \otimes 1(\sum_j b_j' \otimes b_j'')) \otimes 1) \\
&= (\tau(\chi \otimes 1(\psi(a))) \otimes 1)(\tau(\chi \otimes 1(\psi(b))) \otimes 1) = \epsilon^{-\infty}(a)\epsilon^{-\infty}(b). \qquad \square
\end{aligned}$$

Example 7.3 On generators of $H_*(\mathrm{H}\mathbb{F})$, the destabilization $\epsilon^{-\infty}$ is

$$\epsilon^{-\infty}(\xi_n) = \sum_{i=0}^{n} \xi_i \otimes \zeta_{n-i}^{2^i} \otimes 1 \in H_*(\mathrm{H}\mathbb{F}) \otimes H_*(B(2^n)) \otimes \Lambda, \tag{7-3}$$

$$\epsilon^{-\infty}(\zeta_n) = \sum_{i=0}^{n} \zeta_{n-i}^{2^i} \otimes \xi_i \otimes 1 \in H_*(\mathrm{H}\mathbb{F}) \otimes H_*(B(2^{n+1}-2)) \otimes \Lambda, \tag{7-4}$$

where $\epsilon^{-\infty}(\xi_n)$ and $\epsilon^{-\infty}(\zeta_n)$ both have bidegree $(s,t) = (0, 2^n - 1)$.

Remark 7.4 Later, in Lemma 7.8, the element $\epsilon^{-\infty}(\xi_n)$ of Equation (7-3) will be shown to represent a Dieudonné ring generator in $D_{2^n}(H_*(\underline{\mathrm{H}\mathbb{F}}_1))$ that corresponds to the Hopf ring generator x_n in $H_{2^n}(\underline{\mathrm{H}\mathbb{F}}_1)$.

Lemma 7.5 *For the spectrum $E = \mathrm{H}\mathbb{F}$, every element in the image of the destabilization $\epsilon^{-\infty}$ is a permanent cycle.*

Proof We begin by showing that $\epsilon^{-\infty}(\xi_n)$ is a cycle for all $n \geq 0$. By the Cartan formula $(x \otimes y) \cdot \mathrm{Sq} = (x \cdot \mathrm{Sq}) \otimes (y \cdot \mathrm{Sq})$, we have

$$\Big(\sum_{0 \leq i \leq n} \xi_i \otimes \zeta_{n-i}^{2^i}\Big) \cdot \mathrm{Sq} = \sum_{0 \leq i \leq n} \Big(\sum_{i \leq j \leq n} \Big(\xi_j \otimes \zeta_{n-j}^{2^j} + \xi_{j-1} \otimes \zeta_{n-j}^{2^j}\Big)\Big) = \sum_{0 \leq i \leq n} \xi_i \otimes \zeta_{n-i}^{2^i},$$

because all terms cancel except when $i = j$, and thus Sq^0 is nonzero but Sq^k is zero for $k \geq 1$. Since $\lambda_{-1} \cdot 1 = 0$ and $\big(\sum_{i=0}^{n} \xi_i \otimes \zeta_{n-i}^{2^i}\big) \cdot \mathrm{Sq}^k = 0$ for $k \geq 1$, it follows that

$$d_1(\epsilon^{-\infty}(\xi_n)) = \sum_{k \geq 0} \Big(\Big(\sum_{0 \leq i \leq n} \xi_i \otimes \zeta_{n-i}^{2^i}\Big) \cdot \mathrm{Sq}^k\Big) \otimes \lambda_{k-1} \cdot 1 = 0.$$

Since the coproduct ψ and total Steenrod square Sq are ring homomorphisms, and d_1 is an $\mathbb{F}_2$–module homomorphism, it follows that $d_1(\epsilon^{-\infty}(\sum \xi^I)) = 0$ for all $\sum \xi^I \in H_*(\mathrm{H}\mathbb{F})$.

Finally, we show that $\epsilon^{-\infty}(\sum \xi^I)$ must be a permanent cycle. By change of rings (Theorem 4.3),

$$\mathrm{Ext}_A^{*,*}(\mathbb{F}_2, H_*(\mathrm{H}\mathbb{F}) \otimes H_*(B(\infty))) \cong \mathrm{Ext}_{\mathbb{F}_2}^{*,*}(\mathbb{F}_2, H_*(B(\infty))).$$

Thus, the spectral sequence is concentrated on the $s = 0$ line and collapses. □

In the next definition and lemma, we show that $\epsilon^{-\infty}$ preserves degree d, the maximum number of ξ factors k, and the maximum weight n. Note that the antiautomorphism χ and the twist map τ in the definition of $\epsilon^{-\infty}$ have the effect of mapping the ξ weight in $H_*(E)$ to the ζ weight in the second tensor factor of $H_*(E) \otimes H_*(B(\infty)) \otimes \Lambda$.

Definition 7.6 For each $\sum x_i \otimes y_i \otimes 1 \in H_*(E) \otimes H_*(B(\infty)) \otimes \Lambda$, let

$$\begin{aligned}\mathrm{fact}^1_\xi(\textstyle\sum x_i \otimes y_i \otimes 1) &= \max_i\{\mathrm{fact}_\xi(x_i)\},\\ \mathrm{wt}^2_\xi(\textstyle\sum x_i \otimes y_i \otimes 1) &= \max_i\{\mathrm{wt}_\xi(y_i)\},\\ \mathrm{wt}^2_\zeta(\textstyle\sum x_i \otimes y_i \otimes 1) &= \max_i\{\mathrm{wt}_\zeta(y_i)\}.\end{aligned}$$

Lemma 7.7 *If $\sum \xi^I = \sum \zeta^J \in H_*(\mathrm{H}\mathbb{F})$ has $d = \deg(\sum \xi^I)$, $k = \mathrm{fact}_\xi(\sum \xi^I)$ and $n = \mathrm{wt}_\xi(\sum \xi^I)$, then $\mathrm{fact}^1_\xi(\epsilon^{-\infty}(\sum \xi^I)) = k$ and*

$$\epsilon^{-\infty}(\textstyle\sum \xi^I) \in H^{0,d}(H_*(\mathrm{H}\mathbb{F}) \otimes H_*(B(2n)) \otimes \Lambda, d_1).$$

Proof It is clear that $\epsilon^{-\infty}$ preserves degree and that the lemma is true for $\epsilon^{-\infty}(1) = 1 \otimes 1 \otimes 1$. Suppose $\sum \xi^I \neq 1$. First, we show that $\epsilon^{-\infty}$ preserves k. From Equation (7-3), it is clear that $\mathrm{fact}^1_\xi(\epsilon^{-\infty}(\xi_r)) = 1$. Since $\epsilon^{-\infty}$ is a ring homomorphism and $\epsilon^{-\infty}(\sum \xi^I) = (\sum \xi^I) \otimes 1 \otimes 1 + (\mathit{other})$, it follows that $\mathrm{fact}^1_\xi(\epsilon^{-\infty}(\sum \xi^I)) = k$. Second, we show that $\epsilon^{-\infty}$ preserves n. From Equation (7-4), it is apparent that

$$\mathrm{wt}^2_\zeta(\epsilon^{-\infty}(\xi_n)) = \mathrm{wt}^2_\zeta(1 \otimes \zeta_n \otimes 1).$$

Since $\epsilon^{-\infty}$ is a ring homomorphism, it follows that

$$\mathrm{wt}^2_\zeta(\epsilon^{-\infty}(\textstyle\sum \xi^I)) = \mathrm{wt}^2_\zeta(1 \otimes (\sum \zeta^I) \otimes 1) = \mathrm{wt}^2_\xi(1 \otimes (\sum \xi^I) \otimes 1) = n.$$

Therefore, $\epsilon^{-\infty}(\sum \xi^I) \in H^{0,d}(H_*(\mathrm{H}\mathbb{F}) \otimes H_*(B(2n)) \otimes \Lambda, d_1)$. □

The following lemma shows that the Dieudonné ring destabilization $\epsilon^{-\infty}$ is, in fact, equivalent to the Hopf ring destabilization $e^{-\infty}$.

Lemma 7.8 *Suppose*

$$\textstyle\sum \xi^I = \sum \zeta^J \in H_d(\mathrm{H}\mathbb{F})$$

with $d = \deg(\sum \xi^I)$, $k = \mathrm{fact}_\xi(\sum \xi^I)$, and $n = \mathrm{wt}_\xi(\sum \xi^I)$. Then under the Dieudonné equivalence of Equation (5-3), *the destabilized element*

$$e^{-\infty}(\textstyle\sum \xi^I) = e^{-\infty}(\sum \zeta^J) \in H_{d+k}(\underline{\mathrm{H}\mathbb{F}}_k)$$

corresponds to

$$\epsilon^{-\infty}(\textstyle\sum \xi^I) = \epsilon^{-\infty}(\sum \zeta^J) \in E_2^{0,d}(A, \mathrm{H}\mathbb{F} \wedge B(2n)) \cong D_{2n}(H_*(\underline{\mathrm{H}\mathbb{F}}_k)).$$

In particular,

$$U(T(\epsilon^{-\infty}(\textstyle\sum \xi^I))) = \sum x^{\circ I},$$

where T and U are the maps in Equation (5-3).

Proof From Lemma 3.5, $d+k=2n$ and thus $H_{d+k}(\underline{\mathrm{H}\mathbb{F}}_k)=H_{2n}(\underline{\mathrm{H}\mathbb{F}}_k)$ is equivalent to $D_{2n}(H_*(\underline{\mathrm{H}\mathbb{F}}_k))$. From Equation (5-3), it is clear that for any generator $\xi_n \in H_{2^n-1}(\mathrm{H}\mathbb{F})$,

$$U(T(\epsilon^{-\infty}(\xi_n))) = x_n$$

in the rank 1 module $H_{2^n}(\underline{\mathrm{H}\mathbb{F}}_1)$. The one-to-one correspondence follows for any $\sum \xi^I \in H_*(\mathrm{H}\mathbb{F})$ since $\epsilon^{-\infty}$ is a ring homomorphism and the right adjoint U to the Dieudonné functor preserves $+$ and $\circ$. $\square$

Example 7.9 The destabilization of the element $\zeta_3^2+\zeta_1^8\zeta_2^2=\xi_3^2+\xi_1^2\xi_2^4 \in H_{14}(\mathrm{H}\mathbb{F})$ with degree $d=14$, $k=f_\xi(\xi_3^2+\xi_1^2\xi_2^4)=6$, and $n=\mathrm{wt}_\xi(\xi_3^2+\xi_1^2\xi_2^4)=\mathrm{wt}_\zeta(\zeta_3^2+\zeta_1^2\zeta_2^4)=10$ is

$$\begin{aligned}
\epsilon^{-\infty}(\xi_3^2+\xi_1^2\xi_2^4) &= (\xi_3^2+\xi_1^2\xi_2^4)\otimes 1\otimes 1+\xi_2^4\otimes\zeta_1^2\otimes 1+(\xi_2^2+\xi_1^6)\otimes\zeta_1^8\otimes 1\\
&\qquad + \xi_1^4\otimes\zeta_1^{10}\otimes 1+1\otimes(\zeta_3^2+\zeta_1^2\zeta_2^4)\otimes 1\\
&= (\zeta_3^2+\zeta_1^8\zeta_2^2)\otimes 1\otimes 1+(\zeta_2^4+\zeta_1^{12})\otimes\zeta_1^2\otimes 1+\zeta_2^2\otimes\zeta_1^8\otimes 1\\
&\qquad + \zeta_1^4\otimes\zeta_1^{10}\otimes 1+1\otimes(\zeta_3^2+\zeta_1^2\zeta_2^4)\otimes 1\\
&\in H^{0,14}(H_*(\mathrm{H}\mathbb{F})\otimes H_*(B(20))\otimes\Lambda, d_1)
\end{aligned}$$

and corresponds to the class

$$x_0^{\circ 4}\circ x_3^{\circ 2}+x_1^{\circ 2}\circ x_2^{\circ 4} = z_0^{\circ(-8)}\circ\left(z_3^{\circ 2}+z_1^{\circ 8}\circ z_2^{\circ 2}\right)\in H_{20}(\underline{\mathrm{H}\mathbb{F}}_6).$$

Theorem 7.10 *There is a bijection*

$$\begin{aligned}
\text{(7-5)}\quad \epsilon^{-\infty}: \left\{\textstyle\sum \zeta^I \in H_*(\mathrm{ko}) \mid \mathrm{wt}_\xi(\sum\zeta^I)\le n\right\}&\\
\longrightarrow H^{0,*}(H_*(\mathrm{ko})\otimes H_*(B(2n))\otimes\Lambda, d_1)&.
\end{aligned}$$

Proof First, we show the map $\epsilon^{-\infty}$ in (7-5) is well-defined. Take $E=\mathrm{ko}$ in Equation (7-2).

From Equation (7-4), it is clear that the elements in the first tensor factor of $\epsilon^{-\infty}(\zeta_n)$ are in $H_*(\mathrm{ko})$ when $n\ge 3$, and the same is true for $\epsilon^{-\infty}(\zeta_1^4)$ and $\epsilon^{-\infty}(\zeta_2^2)$. Since $\epsilon^{-\infty}$ is a ring homomorphism, it follows that if $\sum\zeta^I\in H_*(\mathrm{ko})$, then the first tensor factor of $\epsilon^{-\infty}(\sum\zeta^I)$ is also in $H_*(\mathrm{ko})$.

Now suppose $\sum\zeta^I\in H_*(\mathrm{ko})$ satisfies $\mathrm{wt}_\xi(\sum\zeta^I)\le n$. From (7-4), $\epsilon^{-\infty}(\zeta_n)=1\otimes\xi_n\otimes 1+(\mathit{other})$, where all of the other terms have second tensor factor of smaller ζ weight than $\mathrm{wt}_\zeta(\xi_n)=2^n-1$. Since $\epsilon^{-\infty}$ is a ring homomorphism, it follows that $\epsilon^{-\infty}(\sum\zeta^I)=1\otimes(\sum\xi^I)\otimes 1+(\mathit{other})$, where all of the other terms have second tensor factor of smaller ζ–weight than $\mathrm{wt}_\zeta(\sum\xi^I)$, and none of the other terms cancel with $1 \otimes (\sum \xi^I) \otimes 1$. Since $\mathrm{wt}_\zeta(\sum \xi^I) = \mathrm{wt}_\xi(\sum \zeta^I)$, it follows that $\epsilon^{-\infty}(\sum \zeta^I)$ is in $H^{0,*}(H_*(\mathrm{ko}) \otimes H_*(B(2n)) \otimes \Lambda, d_1)$.

Second, we show that $\epsilon^{-\infty}$ in (7-5) is injective. Since $\epsilon^{-\infty}(\sum \zeta^I) = (\sum \zeta^I) \otimes 1 \otimes 1 +$ (*other*), where none of the other terms cancels with $(\sum \zeta^I) \otimes 1 \otimes 1$, $\epsilon^{-\infty}$ is injective.

Third and finally, we show $\epsilon^{-\infty}$ in (7-5) is surjective. Suppose $z \in H^{0,*}(H_*(\mathrm{ko}) \otimes H_*(B(2n)) \otimes \Lambda, d_1)$. Since $H_*(\mathrm{ko}) \subset H_*(\mathrm{H}\mathbb{F})$, it is clear that diagram (7-6) commutes.

$$
\begin{tikzcd}
H_*(\mathrm{ko}) \arrow[r, "\epsilon^{-\infty}"] \arrow[d, hook] & H^{0,*}(H_*(\mathrm{ko}) \otimes H_*(B(\infty)) \otimes \Lambda, d_1) \arrow[d, hook] \\
H_*(\mathrm{H}\mathbb{F}) \arrow[r, "\sim", "\epsilon^{-\infty}"'] & H^{0,*}(H_*(\mathrm{H}\mathbb{F}) \otimes H_*(B(\infty)) \otimes \Lambda, d_1)
\end{tikzcd}
\tag{7-6}
$$

From diagram (7-6), z can be included as an element $H^{0,*}(H_*(\mathrm{H}\mathbb{F}) \otimes H_*(B(2n)) \otimes \Lambda, d_1)$, which determines an element $\sum \zeta^I \in H_*(\mathrm{H}\mathbb{F})$. Thus, $z = \epsilon^{-\infty}(\sum \zeta^I) = (\sum \zeta^I) \otimes 1 \otimes 1 +$ (*other*) where none of the other terms cancel with $(\sum \zeta^I) \otimes 1 \otimes 1$, and thus $\sum \zeta^I \in H_*(\mathrm{ko})$ by the definition of $\epsilon^{-\infty}$. Since $z \in H^{0,*}(H_*(\mathrm{ko}) \otimes H_*(B(2n)) \otimes \Lambda, d_1)$ and $z = \epsilon^{-\infty}(\sum \zeta^I) = 1 \otimes (\sum \xi^I) \otimes 1 +$ (*other*) where none of the other terms cancel with $1 \otimes (\sum \xi^I) \otimes 1$, we must have $\mathrm{wt}_\zeta(\sum \xi^I) \le n$. But, $\mathrm{wt}_\zeta(\sum \xi^I) = \mathrm{wt}_\xi(\sum \zeta^I)$, and therefore $\epsilon^{-\infty}$ in Equation (7-5) is also surjective. □

Corollary 7.11 *The destabilization* $e^{-\infty}\colon H_*(\mathrm{ko}) \rightarrowtail H_*(\underline{\mathrm{ko}}_*)$ *is a restriction of the destabilization* $e^{-\infty}\colon H_*(\mathrm{H}\mathbb{F}) \rightarrowtail H_*(\underline{\mathrm{H}\mathbb{F}}_*)$*, ie, diagram* (7-7) *is commutative.*

$$
\begin{tikzcd}
H_*(\mathrm{ko}) \arrow[r, tail, "e^{-\infty}"] \arrow[d, hook] & H_*(\underline{\mathrm{ko}}_*) \arrow[d, hook] \\
H_*(\mathrm{H}\mathbb{F}) \arrow[r, tail, "e^{-\infty}"] & H_*(\underline{\mathrm{H}\mathbb{F}}_*)
\end{tikzcd}
\tag{7-7}
$$

8 Unstable classes in $\mathrm{ko}_*(B(*))$

We now complete the calculation of the Adams spectral sequences

$$\mathrm{Ext}^{s,t}_{A_1}(H^*(B(2n)), \mathbb{F}_2) \Longrightarrow \mathrm{ko}_{t-s}(B(2n))$$

for $n \ge 0$ begun in Section 7. We begin by calculating the ko homology of the integral Brown–Gitler spectra, and then use the cofiber sequences of Lemma 5.8 that relate integral and mod 2 Brown–Gitler spectra to calculate $\mathrm{ko}_*(B(*))$.

The Adams spectral sequences for the ko homology of integral Brown–Gitler spectra are stably isomorphic to truncations (Adams covers) of the spectral sequences for ko and bsp. The spectral sequences for ko and bsp in Figure 2 collapse at E_2 and have $(t-s,s) = (8,4)$ real Bott periodicity. All spectral sequence diagrams in this paper are indexed by stem $t-s$ along the horizontal and Adams filtration s along the vertical. Note that $\mathrm{ko} \simeq \mathrm{ko} \wedge B_0(0)$ and $\mathrm{bsp} \simeq \mathrm{ko} \wedge B_0(4)$. Let $\alpha(n)$ be the number of ones in the 2–adic expansion of n, and let $\nu(n)$ be the number of times that 2 divides n. These functions satisfy $n = \alpha(n) + \nu(n!)$ and $\nu(n!) = \sum_{1 \le i \le n} \nu(i)$.

Lemma 8.1 [15, Theorem 2.7] *For $n > 0$, the maps*

$$E_2^{s+\nu((4n)!),\, t+\nu((4n)!)}(A_1, B_0(0)) \to E_2^{s,t}(A_1, B_0(8n)), \tag{8-1}$$

$$E_2^{s+\nu((4n)!),\, t+\nu((4n)!)}(A_1, B_0(4)) \to E_2^{s,t}(A_1, B_0(8n+4)). \tag{8-2}$$

are injective for $s = 0$ and an isomorphism for $s > 0$. (Note: $\nu((4n)!) = 4n - \alpha(n)$.)

An Adams k–cover of an Ext chart is the chart obtained by deleting rows below Adams filtration $s = k$ and reindexing the remaining rows so that row $s = k$ in the original becomes row $s = 0$. Lemma 8.1 says that Adams $\nu((4n)!)$–covers of the Ext charts for ko and bsp are stably isomorphic to the Ext charts for $\mathrm{ko}_*(B_0(8n))$ and $\mathrm{ko}_*(B_0(8n+4))$.

Using the Mahowald and Shimamoto cofiber sequences, we determine the Ext chart for $\mathrm{ko}_*(B(2n))$ up to stable isomorphism.

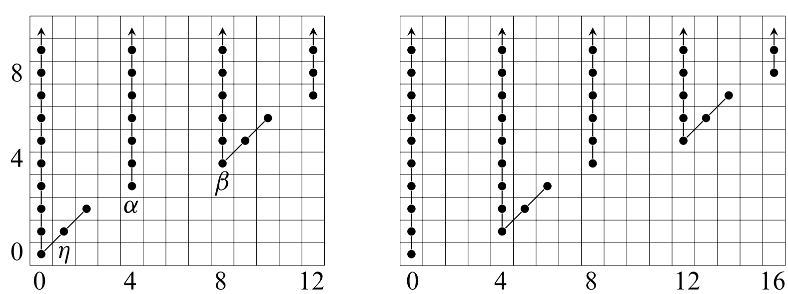


Figure 2: Left: $E_2^{s,t}(A_1, B_0(0)) \Rightarrow \pi_{t-s}^S(\mathrm{ko})$. Right: $E_2^{s,t}(A_1, B_0(4)) \Rightarrow \pi_{t-s}^S(\mathrm{bsp})$.

Theorem 8.2 *For $n \ge 0$, the map $C^{s,t}(2n) \to E_2^{s,t}(A_1, B(2n))$ is injective for $s = 0$ and the identity for $s > 0$. For $n \ge 0$ and $s \ge 0$, $E_2^{s,t}(A_1, B(2n)) = E_\infty^{s,t}(A_1, B(2n))$.*

Proof We use strong induction on $n \geq 1$ to calculate $E_2^{s,t}(A_1, B(2n))$ up to stable isomorphism. Since ko_* and bsp_* are 8–periodic, each step of the induction falls into one of four cases ($2n \equiv 0, 2, 4, 6 \bmod 8$) by Lemma 8.1. Although we present case 0 first, the induction begins with case $n = 1$ and cycles through all of the cases thereafter. The reason for this is to make the indexing consistent among all cases and to make case 0 simpler to state.

	Case 0	Case 1	Case 2	Case 3
a	$(8m, \alpha(m))$	$(8m+2, \alpha(m)+1)$	$(8m+4, \alpha(m)+1)$	$(8m+6, \alpha(m)+2)$
b	$(8m, \alpha(m-1)+4)$	$(8m+2, \alpha(m)+2)$	$(8m+4, \alpha(m)+3)$	$(8m+6, \alpha(m)+3)$
c	$(8m+4, \alpha(m)+3)$	DNE	$(8m+8, \alpha(m)+4)$	DNE
d	$(8m+4, \alpha(m-1)+5)$	DNE	$(8m+8, \alpha(m)+4)$	DNE

Table 1: Bidegrees $(t-s, s)$ of the elements a, b, c, and d used in the proof of Theorem 8.2

Case 0 $(2n = 8m)$ To calculate $E_2^{s,t}(A_1, B(8m))$ for $s > 0$ we use the Mahowald and Shimamoto cofiber sequences

$$B(8m-2) \longrightarrow B(8m) \longrightarrow \Sigma^{4m} B(4m), \tag{8-3}$$

$$B_0(8m) \longrightarrow B(8m) \longrightarrow \Sigma B_0(8m-4), \tag{8-4}$$

of Lemmas 5.4 and 5.8. These cofiber sequences induce the long exact sequences of Ext_{A_1} groups in (8-5) and the top row of (8-6):

$$\cdots \xrightarrow{d_1} E_2^{s,t}(B(8m-2)) \longrightarrow E_2^{s,t}(B(8m)) \longrightarrow E_2^{s,t-4m}(B(4m)) \xrightarrow{d_1} \cdots \tag{8-5}$$

$$\begin{array}{ccccccc} \cdots \xrightarrow{d_1} & E_2^{s,t}(B_0(8m)) & \longrightarrow & E_2^{s,t}(B(8m)) & \longrightarrow & E_2^{s,t-1}(B_0(8m-4)) & \xrightarrow{d_1} \cdots \\ & \uparrow & & \| & & \uparrow & \\ \cdots \xrightarrow{d_1} & E_2^{s+x,t+x}(B(0)) & \longrightarrow & E_2^{s,t}(B(8m)) & \longrightarrow & E_2^{s+y,t+y-1}(B_0(4)) & \xrightarrow{d_1} \cdots \end{array} \tag{8-6}$$

The top and bottom rows of diagram (8-6) are equal for $s > 0$ by Lemma 8.1, where $x = \nu((4m)!) = 4m - \alpha(m)$ and $y = \nu((4(m-1))!) = 4(m-1) - \alpha(m-1)$.

On the top of Figure 3 we display the Mahowald long exact sequence (8-5) by superimposing $E_2^{s,t}(A_1, B(8m-2))$ and $E_2^{s,t-4m}(A_1, B(4m))$. Similarly, on the bottom of Figure 3 we display the Shimamoto long exact sequence (8-6) by superimposing $E_2^{s+x,t+x}(A_1, B(0))$ and $E_2^{s+y,t+y-1}(A_1, B_0(4))$. The charts in Figure 3 have $(t-s, s) = (8, 4)$ Bott periodicity. Consequently, there are effectively four different possible ways that the bottom edges of these spectral sequences could be truncated, and we have shown only one of them. The diligent reader is encouraged to verify that nothing unexpected happens in the other three ways of truncating the bottom edges of

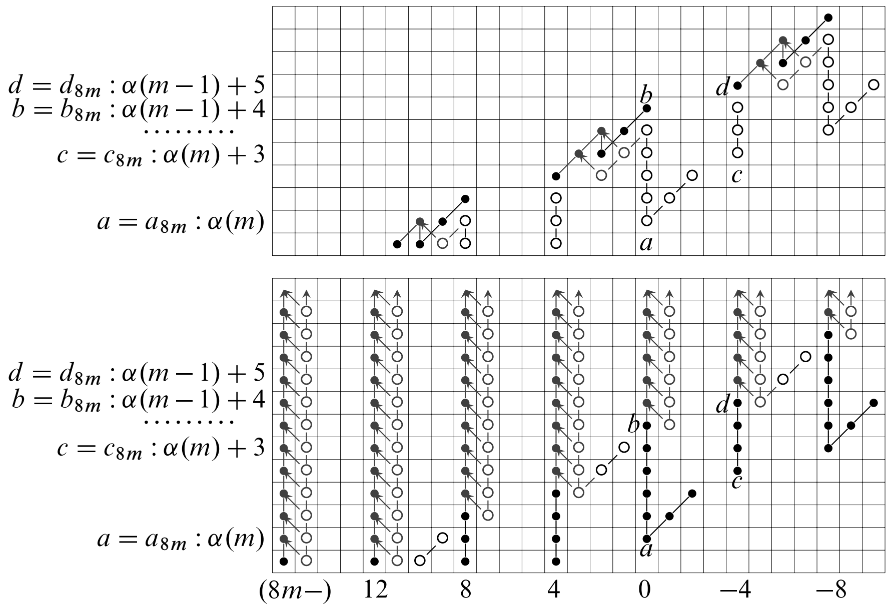


Figure 3: Case 0: Calculating $E_2^{s,t}(A_1, B(8m))$ for $s > 0$. Top figure: $E_2^{s,t}(A_1, B(8m-2))$ displayed using $\bullet$, $E_2^{s,t}(A_1, \Sigma^{4m}B(4m))$ displayed using $\circ$. Bottom figure: $E_2^{s,t}(A_1, B_0(8m))$ displayed using $\bullet$, $E_2^{s,t}(A_1, \Sigma B_0(8m-4))$ displayed using $\circ$. Gray means killed by differential $d_1 = \nwarrow$. Vertical axis label separator $\cdots$ indicates that height of towers connecting a to b and c to d varies with m.

these charts. The bidegrees of the elements a, b, c and d in Figure 3 can be determined using either of the long exact sequences and are given in Table 1.

Because the Mahowald long exact sequence has no infinite $\lambda_0 = h_0$ towers, the connecting homomorphism d_1 in the Shimamoto long exact sequence must be an isomorphism where indicated in Figure 3 on bottom. The d_1 differentials in the Mahowald long exact sequence are then forced by comparison with the Shimamoto long exact sequence. The unsolved $\lambda_0 = h_0$ and $\lambda_1 = h_1$ extensions in each long exact sequence are solved by comparison with the other long exact sequence.

It remains to show that the lightning flash containing a and b and the tower containing c and d in $E_2^{s,t}(A(1), B(8m))$ have the same size those as in $C^{s,t}(8m)$. From Table 1, the lightning bolts in $E_2^{s,t}(A(1), B(8m))$ have height (that is, difference in Adams filtration)

$$\begin{aligned} AF(b) - AF(a) &= 4 + \alpha(m-1) - \alpha(m) = 4 + [m - 1 - \nu((m-1)!) - (m - \nu(m!))] \\ &= 4 + [\nu(m) - 1] = \nu(8m), \end{aligned}$$

and the towers have height $AF(d) - AF(c) = (\alpha(m-1)+5) - (\alpha(m)+3) = \nu(2m)$. For $C^{s,t}(8m)$ the height of the lightning bolts is $AF(b) - AF(a) = \nu(8m)$, and the height of the towers is $AF(d) - AF(c) = \nu(16m) - 3 = \nu(2m)$. Thus, the two charts are stably isomorphic.

Case 1 $(2n = 8m+2)$ To calculate $E_2^{s,t}(A_1, B(8m+2))$ we use the following long exact sequences in Ext_{A_1} induced by the Mahowald and Shimamoto cofiber sequences (Lemmas 5.4 and 5.8) together with Lemma 8.1.

$$\cdots \xrightarrow{d_1} E_2^{s,t}(B(8m)) \to E_2^{s,t}(B(8m+2)) \to E_2^{s,t-(4m+1)}(B(4m)) \xrightarrow{d_1} \cdots \tag{8-7}$$

$$\begin{tikzcd}
\cdots \arrow[r, "d_1"] & E_2^{s,t}(B_0(8m)) \arrow[r] & E_2^{s,t}(B(8m+2)) \arrow[r] & E_2^{s,t-1}(B_0(8m)) \arrow[r, "d_1"] & \cdots \\
\cdots \arrow[r, "d_1"] & E_2^{s+x,t+x}(B(0)) \arrow[r] \arrow[u] & E_2^{s,t}(B(8m+2)) \arrow[r] \arrow[u, equal] & E_2^{s+y,t+y-1}(B_0(0)) \arrow[r, "d_1"] \arrow[u] & \cdots
\end{tikzcd} \tag{8-8}$$

Here, $x = y = \nu((4m)!) = 4m - \alpha(m)$. These long exact sequences are displayed in Figure 4, and the bidegrees of the elements a and b are given in Table 1. Comparing the two charts stem by stem, the differentials in one chart are forced by the other chart. The unsolved λ_0 and λ_1 extensions will be solved later in Corollary 8.4. From Table 1, the lightning bolts in $E_2^{*,*}(A_1, B(8m+2))$ have height $AF(b) - AF(a) = 1$, and those in $C^{*,*}(8m+2)$ also have height $AF(b) - AF(a) = \nu(8m+2) = 1$, so the charts are stably isomorphic.

Case 2 $(2n = 8m+4)$ To calculate $E_2^{s,t}(A_1, B(8m+4))$ we use the following long exact sequences in Ext_{A_1} induced by the Mahowald and Shimamoto cofiber sequences (Lemmas 5.4 and 5.8) together with Lemma 8.1.

$$\begin{aligned}\cdots \xrightarrow{d_1} E_2^{s,t}(B(8m+2)) &\longrightarrow E_2^{s,t}(B(8m+4)) \\ &\longrightarrow E_2^{s,t-(4m+2)}(B(4m+2)) \xrightarrow{d_1} \cdots\end{aligned} \tag{8-9}$$

$$\begin{tikzcd}
\cdots \arrow[r, "d_1"] & E_2^{s,t}(B_0(8m+4)) \arrow[r] & E_2^{s,t}(B(8m+4)) \arrow[r] & E_2^{s,t-1}(B_0(8m)) \arrow[r, "d_1"] & \cdots \\
\cdots \arrow[r, "d_1"] & E_2^{s+x,t+x}(B(4)) \arrow[r] \arrow[u] & E_2^{s,t}(B(8m+4)) \arrow[r] \arrow[u, equal] & E_2^{s+y,t+y-1}(B_0(0)) \arrow[r, "d_1"] \arrow[u] & \cdots
\end{tikzcd} \tag{8-10}$$

where $x = y = \nu((4m)!) = 4m - \alpha(m)$. These long exact sequences are displayed in Figure 5, and the bidegrees of the elements a, b, c and d are given in Table 1. Comparing the two charts stem by stem, the differentials in one chart are forced by

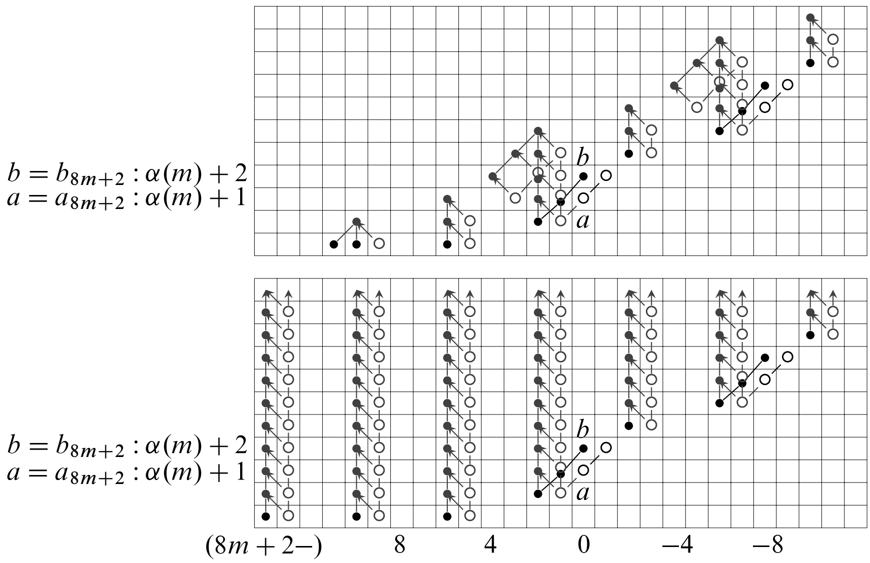


Figure 4: Case 1: Calculating $E_2^{s,t}(A_1, B(8m+2))$ for $s > 0$. Top: $E_2^{s,t}(A_1, B(8m))$ displayed using $\bullet$, $E_2^{s,t}(A_1, \Sigma^{4m+1}B(4m))$ displayed using $\circ$. Bottom: $E_2^{s,t}(A_1, B_0(8m))$ displayed using $\bullet$, $E_2^{s,t}(A_1, \Sigma B_0(8m))$ displayed using $\circ$. Gray means killed by differential $d_1 = \nwarrow$.

the other chart. As in case 0, the unsolved λ_0 and λ_1 extensions can be solved by comparing the charts. From Table 1, the lightning bolts in $E_2^{*,*}(A_1, B(8m+4))$ have height $AF(b) - AF(a) = 2$, and the towers have height $AF(d) - AF(c) = 0$. In $C^{*,*}(8m+4)$ the lightning bolt height is $AF(b) - AF(a) = \nu(8m+4) = 2$ and the tower height is $AF(d) - AF(c) = \nu(16m+8) - 3 = 0$. Thus, the charts are stably isomorphic.

Case 3 $(2n = 8m+6)$ To calculate $E_2^{s,t}(A_1, B(8m+6))$ we use the following long exact sequences in Ext_{A_1} induced by the Mahowald and Shimamoto cofiber sequences (Lemmas 5.4 and 5.8) together with Lemma 8.1.

$$\text{(8-11)} \quad \cdots \xrightarrow{d_1} E_2^{s,t}(B(8m+4)) \longrightarrow E_2^{s,t}(B(8m+6)) \longrightarrow E_2^{s,t-(4m+3)}(B(4m+2)) \xrightarrow{d_1} \cdots$$

$$\text{(8-12)} \quad \begin{array}{ccccc} \cdots \xrightarrow{d_1} E_2^{s,t}(B_0(8m+4)) & \longrightarrow & E_2^{s,t}(B(8m+6)) & \longrightarrow & E_2^{s,t-1}(B_0(8m+4)) \xrightarrow{d_1} \cdots \\ \uparrow & & \| & & \uparrow \\ \cdots \xrightarrow{d_1} E_2^{s+x,t+x}(B(4)) & \longrightarrow & E_2^{s,t}(B(8m+6)) & \longrightarrow & E_2^{s+y,t+y-1}(B_0(4)) \xrightarrow{d_1} \cdots \end{array}$$

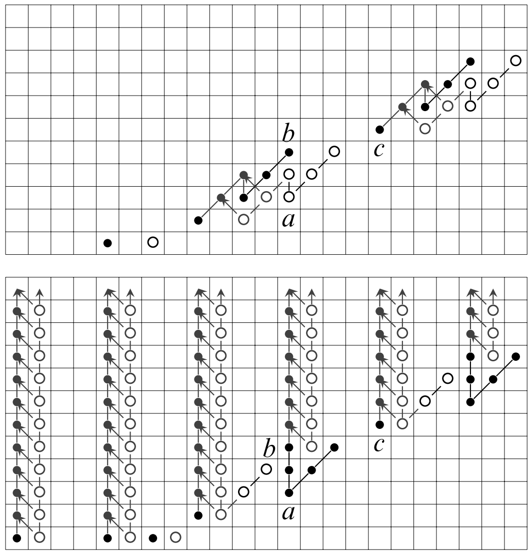


Figure 5: Case 2: Calculating $E_2^{s,t}(A_1, B(8m+4))$ for $s > 0$. Top: $E_2^{s,t}(A_1, B(8m+2))$ displayed using $\bullet$, $E_2^{s,t}(A_1, \Sigma^{4m+2}B(4m+2))$ displayed using $\circ$. Bottom: $E_2^{s,t}(A_1, B_0(8m+4))$ displayed using $\bullet$, $E_2^{s,t}(A_1, \Sigma B_0(8m))$ displayed using $\circ$. Gray means killed by $d_1 = \nwarrow$.

where $x = y = \nu((4m)!) = 4m - \alpha(m)$. These long exact sequences are displayed in Figure 6, and the bidegrees of the elements a, b, c and d are given in Table 1. Comparing the two charts stem by stem, the differentials in one chart are forced by the other chart. The unsolved λ_0 and λ_1 extensions will be solved later in Corollary 8.4. From Table 1, the lightning bolts in $E_2^{*,*}(A_1, B(8m+6))$ have height $AF(b) - AF(a) = 1$, and those in $C^{*,*}(8m+6)$ have height is $AF(b) - AF(a) = \nu(8m+6) = 1$. Thus, the charts are stably isomorphic and the proof by induction is finished.

The spectral sequence $E_2^{s,t}(A(1), B(2n))$ collapses (ie, $E_2 = E_\infty$) because any possibly nontrivial differentials are incompatible with the multiplicative structure. $\square$

In the next lemma and corollary, we determine $H^*(B(4n+2))$ as a stable A_1–module and calculate $E_2^{s,t}(A_1, B(4n+2))$ for $s > 0$. The purpose for this is to resolve the λ_0 and λ_1 extensions in cases 1 and 3 in the proof of Theorem 8.2.

In [8, page 50], four types of A_1–modules $Q_{i,j}$, $i \in \mathbb{Z}/(4)$ and $j \geq 0$, are constructed so that $Q_{i,j}$ contains no free A_1–submodules and $H^*(B_0(4n)) \cong Q_{i,j} \oplus F$ as left A_1–modules for some i, j and some free A_1–module F. Define $Q_{i,j}$ by induction on j by the nontrivial extension of left A_1–modules

$$0 \longrightarrow W \longrightarrow Q_{i,j} \longrightarrow Q_{i,j-1} \longrightarrow 0$$

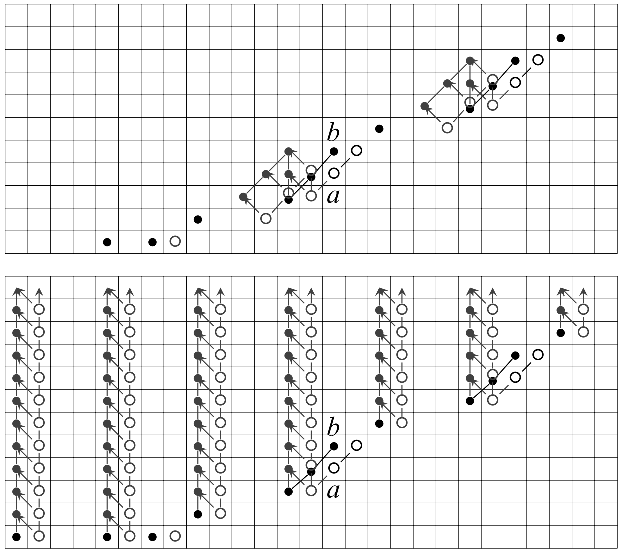


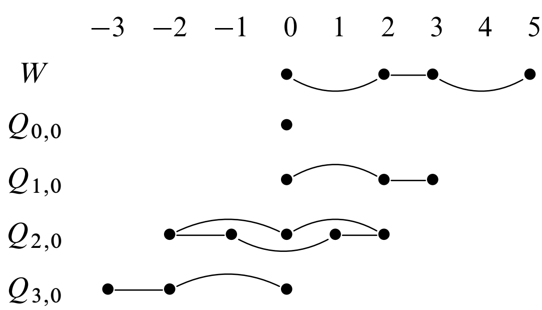
Figure 6: Case 3: Calculating $E_2^{s,t}(A_1, B(8m+6))$ for $s > 0$. Top: $E_2^{s,t}(A_1, B(8m+4))$ displayed using $\bullet$, $E_2^{s,t}(A_1, \Sigma^{4m+3} B(4m+2))$ displayed using $\circ$. Bottom: $E_2^{s,t}(A_1, B_0(8m+4))$ displayed using $\bullet$, $E_2^{s,t}(A_1, \Sigma B_0(8m+4))$ displayed using $\circ$. Gray means killed by $d_1 = \nwarrow$.

with $W = \mathbb{F}_2\{1, \mathrm{Sq}^2, \mathrm{Sq}^3, \mathrm{Sq}^2\,\mathrm{Sq}^3\}$, and

$$\begin{aligned} Q_{0,0} &= \mathbb{F}_2\{1\}, \quad Q_{1,0} = \mathbb{F}_2\{1, \mathrm{Sq}^2, \mathrm{Sq}^3\} = H^*(B_0(4)), \\ Q_{2,0} &= J = \Sigma^{-2}\mathbb{F}_2\{1, \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^2\,\mathrm{Sq}^1, \mathrm{Sq}^3\,\mathrm{Sq}^1\}, \\ Q_{3,0} &= \Sigma^{-3}\mathrm{F}_2\{1, \mathrm{Sq}^1, \mathrm{Sq}^2\,\mathrm{Sq}^1\} = H^*(DB_0(4)), \end{aligned}$$

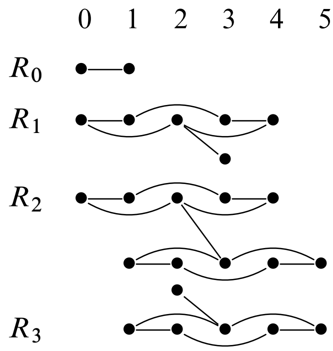
as shown in (8-13). Here, W is a bow-shaped module, J denotes the Adams joker, and DX denotes the Spanier–Whitehead dual of X.

(8-13)



Lemma 8.3 *$H^*(B(4n+2))$ is stably A_1 isomorphic to a suspension of R_i for some $i \in \mathbb{Z}/(4)$, where R_i is defined as in Equation* (8-14)*.*

(8-14)



Note that $R_0 \cong H^*(B(2))$, $R_1 \cong H^*(B(6))$, and R_3 is isomorphic to a suspension of $H^*(DB(6))$.

Proof From [23], $B(4n+2) \simeq B(2) \wedge B_0(4n)$ for all n. Thus, $H^*(B(4n+2)) \cong H^*(B(2)) \otimes H^*(B_0(4n))$ as left A–modules, and as a left A_1–module

$$\begin{aligned} H^*(B(2)) \otimes H^*(B_0(4n)) &\cong H^*(B(2)) \otimes (Q_{i,j} \oplus F) \\ &\cong (H^*(B(2)) \otimes Q_{i,j}) \oplus (H^*(B(2)) \otimes F) \end{aligned}$$

for some $i \in \mathbb{Z}/(4)$ and $j \geq 0$. Since $H^*(B(2)) \otimes F$ is free and $H^*(B(2)) \otimes W \cong A_1$, by direct calculation there is an isomorphism of left A_1–modules $H^*(B(2)) \otimes Q_{i,j} \cong R_i \oplus F'$ for some free module F'. □

Corollary 8.4 *For $n \geq 0$, the maps*

$$E_2^{s+\nu((4n)!),\, t+\nu((4n)!)}(A_1, B(2)) \to E_2^{s,t}(A_1, B(8n+2)), \tag{8-15}$$

$$E_2^{s+\nu((4n)!),\, t+\nu((4n)!)}(A_1, B(6)) \to E_2^{s,t}(A_1, B(8n+6)), \tag{8-16}$$

are injective for $s = 0$ and the identity for $s > 0$.

Proof By direct calculation, $E_2^{*,*}(A_1, B(2)) = C^{*,*}(2)$ and $E_2^{*,*}(A_1, B(6)) = \mathbb{F}_2 \oplus C^{*,*}(6)$, where $\mathbb{F}_2$ is in $(s,t) = (0,0)$.

From [1, Theorem 5.1], since $H^*(B(2n))$ is a free left A_0–module, left multiplication by the Bott element β: $E_2^{s,t}(A_1, B(2n)) \to E_2^{s+4,t+12}(A_1, B(2n))$ is an isomorphism for $s > 0$. In particular, if

$$\cdots \longrightarrow P_4 \xrightarrow{d_4} P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} H^*(B(2)) \longrightarrow 0$$

is an A_1 free minimal resolution of $H^*(B(2))$, then $\ker(d_{4i+j}) \cong \ker(d_j)$ for all $i, j > 0$. By direct calculation, up to suspension $\ker(d_i) \cong R_{3-i}$ for $1 \le i \le 4$. Thus, by Lemma 8.3, $E_2^{*,*}(A_1, B(8n+2))$ is an Adams k–cover of $E_2^{*,*}(A_1, B(2))$, and from Case 1 in the proof of Theorem 8.2, we find $k = \nu((4n)!)$.

Similarly, if $P_* \to H^*(B(6)) \to 0$ is an A_1 free minimal resolution, then up to suspension $\ker(d_i) \cong R_{4-i}$ for $1 \le i \le 4$. Thus, by Lemma 8.3, $E_2^{*,*}(A_1, B(8n+6))$ is an Adams k–cover of $E_2^{*,*}(A_1, B(6))$, and from Case 3 in the proof of Theorem 8.2, we find $k = \nu((4n)!)$. □

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