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The minimal genus problem in $\mathbb{CP}^2 \# \mathbb{CP}^2$

MOHAMED AIT NOUH

In this paper, we give two infinite families of counterexamples and finite positive examples to a conjecture on the minimal genus problem in $\mathbb{CP}^2 \# \mathbb{CP}^2$ proposed by Lawson [10].

57Q25; 57Q45, 57N70

This paper is dedicated to the memory of my PhD thesis advisor Yves Mathieu

1 Introduction

Let X be a smooth, closed, oriented, simply connected 4–manifold, and let $b_2^+(X)$ (resp. $b_2^-(X)$) be the rank of the positive (resp. negative) part of the intersection form of X. The minimal genus problem is concerned with finding the genus function G_X defined on $H_2(X;\mathbb{Z})$ as follows: For $\alpha \in H_2(X;\mathbb{Z})$, consider

$$G_X(\alpha) = \min\{\text{genus}(\Sigma) \mid \Sigma \subset X \text{ represents } \alpha,\ \text{ie}, [\Sigma] = \alpha\},$$

where Σ ranges over closed, connected, oriented surfaces smoothly embedded in the 4–manifold X. Note that $G_X(-\alpha) = G_X(\alpha)$ and $G_X(\alpha) \geq 0$ for all $\alpha \in H_2(X;\mathbb{Z})$ (cf Gompf and Stipsicz [5]).

The minimal genus problem has been solved for the 4–manifolds $\mathbb{CP}^2$, $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$; see Kronheimer and Mrowka [8] and Ruberman [15]. For more results of this kind, we refer to Lawson's expository paper [10]. The minimal genus problem in the case of $\mathbb{CP}^2$ is well known. In this paper, we treat $\mathbb{CP}^2 \# \mathbb{CP}^2$, which has $b_2^+ = 2$ and admits no algebraic structure since a simple characteristic class argument shows that its tangent bundle admits no complex structure (cf Gompf and Stipsicz [5]).

Conjecture 1.1 (Lawson [10]) *The minimal genus of $(m,n) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2) = H_2(\mathbb{CP}^2) \oplus H_2(\mathbb{CP}^2)$ is given by $\binom{m-1}{2} + \binom{n-1}{2}$, and it is the genus realized by the connected sum of the complex projective curves in each factor.*

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Taking the connected sum of the complex projective curves in each factor representing respectively $m\gamma_1 \in H_2(\mathbb{CP}^2;\mathbb{Z})$ and $n\gamma_2 \in H_2(\mathbb{CP}^2;\mathbb{Z})$, where γ_1 and γ_2 are the standard generators of $H_2(\mathbb{CP}^2 \# \mathbb{CP}^2)$, yields a surface representing $(m,n) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2;\mathbb{Z})$. Then, for any $(m,n) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2;\mathbb{Z})$, the minimal genus problem function satisfies

$$G_{\mathbb{CP}^2\#\mathbb{CP}^2}((m,n)) \leq G_{\mathbb{CP}^2}(m) + G_{\mathbb{CP}^2}(n).$$

The minimal genus of $(m,n) \in H_2(\mathbb{CP}^2\#\mathbb{CP}^2;\mathbb{Z})$ is bounded above by $\binom{m-1}{2}+\binom{n-1}{2}$, by the positive answer to Thom's conjecture; see Kronheimer and Mrowka [7]. This bound is sharp if $|m| \leq 2$ and $|n| \leq 2$ since each class can be represented by a sphere in $\mathbb{CP}^2\#\mathbb{CP}^2$. The simplest remaining case is the class $(3,2) \in H_2(\mathbb{CP}^2\#\mathbb{CP}^2)$, which is still unresolved. This class can be represented by an embedded torus, but it is unknown whether it can be represented by an embedded sphere [10]. Surprisingly enough, even if Conjecture 1.1 seems to be far from being true, there are some nontrivial positive examples of it. Therefore, it would be rather interesting to find the complex projective curves in $\mathbb{CP}^2 \# \mathbb{CP}^2$ for which Lawson's conjecture holds.

In Section 2, we prove Theorem 1.2 which exhibits two infinite families of counterexamples.

Theorem 1.2 *Conjecture 1.1 fails for the following infinite families:*

(1) $(2p,d) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2;\mathbb{Z})$, *where* d *is a possible degree of* $T(p,4p-1)$ *in* $\mathbb{CP}^2$, *for any* $p \geq 2$, *and* $T(p,4p-1)$ *denotes the* $(p,4p-1)$*–torus knot.*

(2) $(m,0) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2;\mathbb{Z})$ *for any* $m \geq 3$.

In Section 3, we prove Proposition 1.1 that exhibits two nontrivial positive examples.

Proposition 1.1 *The minimal genera of the pairs* $(3,3)$ *and* $(6,6) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2)$ *are respectively* 2 *and* 20.

Throughout this paper, we work in the smooth category. All orientable manifolds will be assumed to be oriented unless otherwise stated. In particular, all knots are oriented. Recall that $\mathbb{CP}^2$ is the closed 4–manifold obtained by the free action of $\mathbb{C}^* = \mathbb{C} - \{0\}$ on $\mathbb{C}^3 - \{(0,0,0)\}$ defined by $\lambda(x,y,z) = (\lambda x, \lambda y, \lambda z)$, where $\lambda \in \mathbb{C}^*$, ie $\mathbb{CP}^2 = (\mathbb{C}^3 - \{(0,0,0)\})/\mathbb{C}^*$. An element of $\mathbb{CP}^2$ is denoted by its homogeneous coordinates $[x:y:z]$, which are defined up to the multiplication by $\lambda \in \mathbb{C}^*$. The fundamental class of the submanifold $H = \{[x:y:z] \in \mathbb{CP}^2 \mid x = 0\}$ $(H \cong \mathbb{CP}^1)$ generates the second homology group $H_2(\mathbb{CP}^2;\mathbb{Z})$ (cf [5]). Since $H \cong \mathbb{CP}^1$, then the standard generator of $H_2(\mathbb{CP}^2;\mathbb{Z})$ is denoted, from now on, by $\gamma = [\mathbb{CP}^1]$. Therefore, the standard generator of $H_2(\mathbb{CP}^2 - B^4; \mathbb{Z})$ is $\mathbb{CP}^1 - B^2 \subset \mathbb{CP}^2 - B^4$ with the complex orientations. A class $\xi \in H_2(\mathbb{CP}^2 - B^4, \partial(\mathbb{CP}^2 - B^4); \mathbb{Z})$ is identified with its image by the homomorphism

$$H_2(\mathbb{CP}^2 - B^4, \partial(\mathbb{CP}^2 - B^4); \mathbb{Z}) \cong H_2(\mathbb{CP}^2 - \mathrm{int}(B^4); \mathbb{Z}) \longrightarrow H_2(\mathbb{CP}^2; \mathbb{Z}).$$

Let d be an integer, then the degree d smooth slice genus of a knot K in $\mathbb{CP}^2$ is defined as

$$\begin{aligned} &g_{\mathbb{CP}^2}(d, K) \\ &\quad = \min\{\mathrm{genus}(\Sigma) \mid \partial\Sigma = K \text{ and } [\Sigma, \partial\Sigma] = d\gamma \in H_2(\mathbb{CP}^2 - B^4, \partial(\mathbb{CP}^2 - B^4); \mathbb{Z})\}, \end{aligned}$$

where Σ ranges over connected, oriented, smooth surfaces properly embedded in $\mathbb{CP}^2 - B^4$.

If such a surface exists, then we call d a *possible degree* of K in $\mathbb{CP}^2$. By the above identification, we also have $[\Sigma] = d\gamma \in H_2(\mathbb{CP}^2 - B^4; \mathbb{Z})$. Then the $\mathbb{CP}^2$–genus of a knot K is defined as

$$g_{\mathbb{CP}^2}(K) = \min\{g_{\mathbb{CP}^2}(d, K) \mid\ d \text{ is a possible degree of } K\}.$$

A similar definition could be made for any 4–manifold and that this is a generalization of the 4–ball genus; see the author [13].

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2 Proof of Theorem 1.2

Our counterexamples to Conjecture 1.1 are based on twisting operations of knots defined as follows.

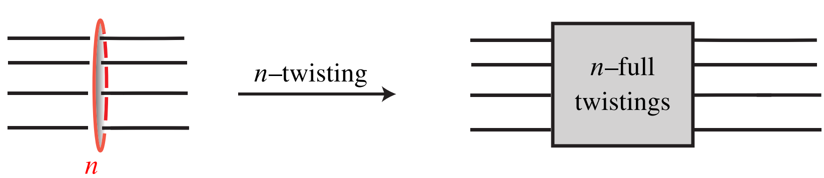


Figure 1

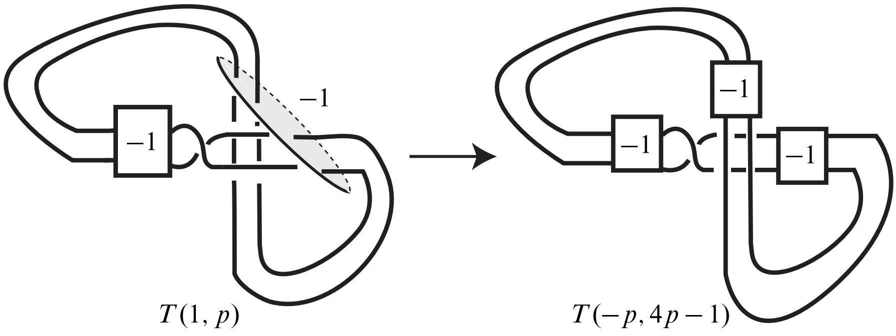


Figure 2: $T(1,p) \cong U \xrightarrow{(-1,2p)} T(-p,4p-1)$

Let K be a knot in the 3–sphere S^3, and D^2 a disk intersecting K in its interior. Let n be an integer. A $(-\frac{1}{n})$–Dehn surgery along ∂D^2 changes K into a new knot K_n in S^3. Let $\omega = \mathrm{lk}(\partial D^2, K)$. We say that K_n is obtained from K by (n,ω)*–twisting* (or simply *twisting*). Then we write

$$K \xrightarrow{(n,\omega)} K_n.$$

We say that K_n is n*-twisted* if K is the trivial knot (see Figure 1). An example of interest is illustrated in Figure 2, where $T(p,q)$ ($0 < p < q$ and p and q are coprime) denotes the (p,q)*–torus knot*; see Burde and Zieschang [3].

The 4–ball genus (resp. 3–genus) of a knot k in S^3, denoted by $g^*(k)$ (resp. $g(k)$), is the minimum genus of all smooth compact connected and orientable surfaces bounded by $k \subset \partial B^4 = S^3$ in B^4 (resp. S^3). A knot is called positive if it has a positive diagram, ie a diagram with all crossings positive. To deny Conjecture 1.1, we need the following four lemmas.

Lemma 2.1 *Let K_0 be a knot in S^3 with 4–ball genus g^*.*

(a) *If K is a knot obtained by a $(-1,\omega)$–twisting from the knot K_0, then K bounds a properly embedded genus g^* surface in $\mathbb{CP}^2$ with possible degree ω.*

(b) *If $K_0 \xrightarrow{(-1,m)} K_m \xrightarrow{(-1,n)} K$, then K bounds a properly embedded genus g^* in $\mathbb{CP}^2 \# \mathbb{CP}^2 - B^4$ representing $[\Sigma_{g^*}] = m\gamma_1 + n\gamma_2 \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2, S^3; \mathbb{Z})$.*

Proof (a) As shown in Figure 3, let D be a disk on which the $(-1,\omega)$–twisting is performed. Note that the $(+1)$–Dehn surgery on ∂D changes K_0 to K. Regard K_0 and D as contained in the boundary of a 4–dimensional handle h^0. Then attach a

2–handle h^2, to h^0 along ∂D with framing $+1$. The resulting 4–manifold $h^0 \cup h^2$ is $\mathbb{CP}^2 - B^4$ (see Figure 3). Let $(\Sigma_{g^*}, \partial\Sigma_{g^*}) \subset (B^4, \partial B^4 \cong S^3)$ be the orientable and compact surface with $\partial\Sigma_{g^*} = K_0$. Since $\mathrm{lk}(K_0, \partial D) = \omega$, then we can check that $[\Sigma_{g^*}] = \omega\gamma \in H_2(\mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$.

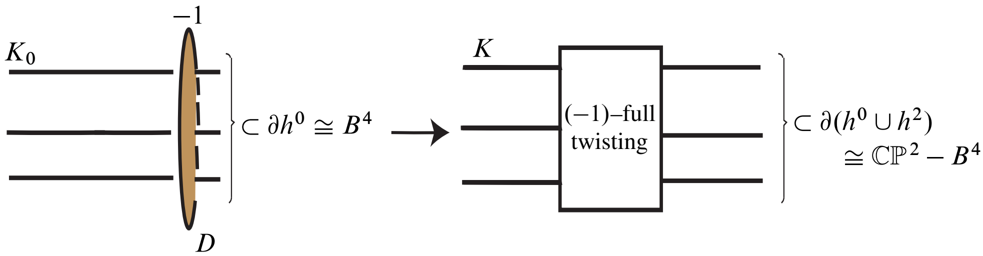


Figure 3

(b) As shown in Figure 4, let D_1 and D_2 be the disks on which the $(-1, m)$–twisting and $(-1, n)$–twisting are respectively performed. Note that the $(+1)$–Dehn surgery on respectively ∂D_1 and ∂D_2 changes K_0 to K. Regard K_0, D_1 and D_2 as contained in the boundary of a 4–dimensional handle h^0. Then attach the 2–handles h_1^2 and h_2^2 along $\partial D_1 \cup \partial D_2$ with the same respective framing $+1$. The 4–manifold $h^0 \cup h_1^2 \cup h_2^2$ is $\mathbb{CP}^2 \# \mathbb{CP}^2 - B^4$. Let $(\Sigma_{g^*}, \partial\Sigma_{g^*}) \subset (B^4, \partial B^4 \cong S^3)$ be the orientable and compact surface with $\partial\Sigma_{g^*} = K_0$. Since $\mathrm{lk}(\partial D_1, K_0) = m$ and $\mathrm{lk}(\partial D_2, K_0) = n$, then $[\Sigma_{g^*}] = m\gamma_1 + n\gamma_2 \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$.

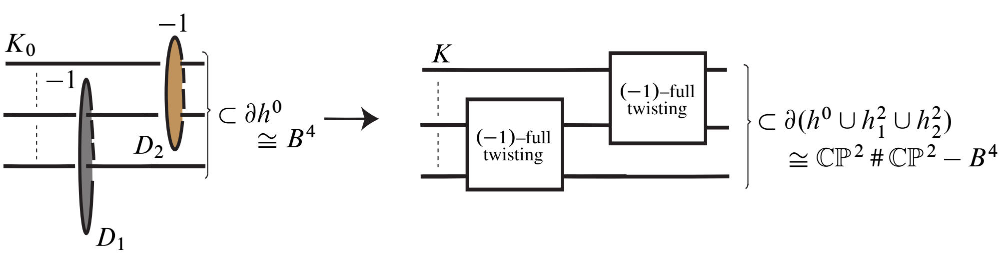
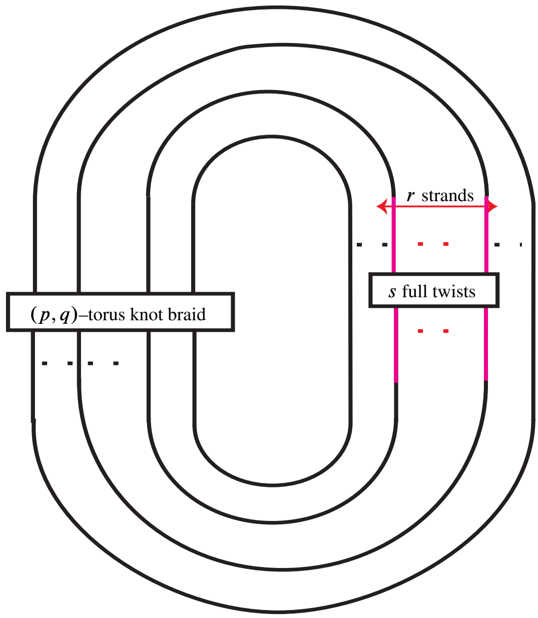


Figure 4

This completes the proof. □

Lemma 2.2 *$T(-p, 4p \pm 1)$ for $p \geq 2$ is smoothly slice in $\mathbb{CP}^2$ with a possible degree $d = 2p$.*

Proof Figure 2 proves that $T(-p, 4p-1)$ is obtained from the trivial knot $T(-1, p)$ by a single $(-1, 2p)$–twisting. Then, the proof of Lemma 2.2 is a straightforward consequence of Lemma 2.1. □



$T(p,q,r,s)$

Figure 5: Twisted torus knot $T(p,q,r,s)$

Lemma 2.3 *We have* $g_{\mathbb{CP}^2}(T(p,q)) \leq \frac{1}{2}(p-1)(q-1)-1$.

Proof Note that $T(p,q)$ is obtained from $T(2,3)$ by adding $(p-1)(q-1)-2$ half-twisted bands. Since $T(2,3)$ is (-1)–twisted (cf [13]), then $T(2,3)$ is smoothly slice in $\mathbb{CP}^2$. This implies that there is a genus $(p-1)(q-1)/2-1$ concordance between $T(2,3)$ and $T(p,q)$, which proves Lemma 2.3. □

This let us make progress on the following problem (cf [13]).

Problem 2.4 *Show that* $g_{\mathbb{CP}^2}(T(p,q)) = \frac{1}{2}(p-1)(q-1)-1$.

We gave positive examples of this problem for a finite family of $(\pm 2, q)$–torus knots in [13].

To prove Lemma 2.5, recall that a knot in the 3–sphere obtained from the torus knot $T(p,q)$ by performing s–times full twists on adjacent r–strands of the parallel p–strings of $T(p,q)$ is called a twisted torus knot, denoted by $T(p,q,r,s)$ as depicted in Figure 5 (we refer the reader to Callahan, Dean and Weeks [4] for more details).

We have:

(1) $u(K_i) = u - i$, $0 \le i \le u$ (in particular, K_u is the trivial knot).

(2) Two succeeding knots of the sequence are related by one crossing change.

(3) $u = u(K)$ is the unknotting number of K.

Furthermore, the set of respective crossings positions $\{C_1, C_2, \dots, C_{u-1}, C_u\}$ at which these crossing changes are performed in the following order:

$$K_0 \xrightarrow{C_1} K_1 \xrightarrow{C_2} K_2 \cdots \xrightarrow{C_u} K_u,$$

where $u = u(K)$, is called a minimal U–crossing data for the knot K. An example can be found in Vikas and Madeti [18] for the case of torus knots (see Figure 6 in the case of a $(5,4)$–torus knot).

Lemma 2.5 *Let K be a knot such that $u(K) = g^*(K)$, then $g^*(K_1) \le g^*(K) - 1$.*

Proof By the unknotting inequality we have $g^*(K_1) \le u(K_1)$. Since $g^*(K) = u(K)$, and by the above construction $u(K_1) = u(K) - 1$, then $g^*(K_1) \le g^*(K) - 1$. □

Remark 2.6 It is well-known that if K is a positive knot, then $u(K) = g^*(K)$ (see Nakamura [12], Shibuya [16] and Przytycki [14] for proofs). Also, Baader classified quasipositive knots for which this equality holds (cf [1]).

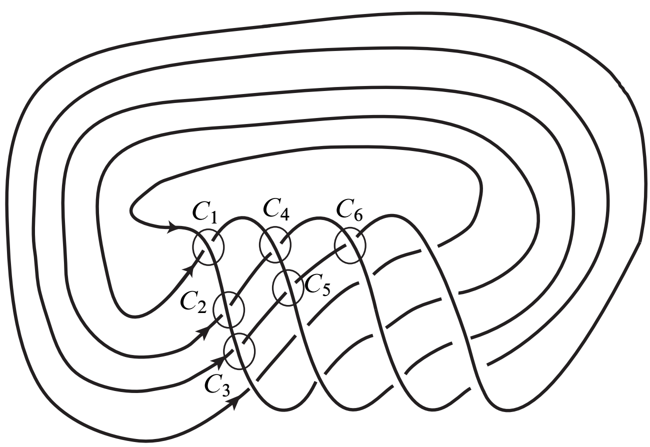


Figure 6: Minimal U–crossing data for $T(5,4)$

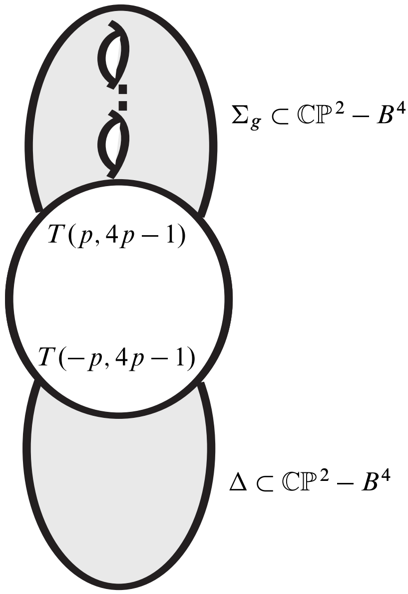


Figure 7: Gluing of surfaces technique

Proof of Theorem 1.2 By Lemma 2.2, $T(-p, 4p-1)$ for $p \geq 2$ is smoothly slice in $\mathbb{CP}^2$ with degree $d = 2p$. Then, there is a smooth disk $(\Delta, \partial\Delta) \subset (\mathbb{CP}^2 - B^4, S^3)$ such that $\partial\Delta = T(-p, 4p-1)$ and $[\Delta] = 2p\gamma$ in $H_2(\mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$. On the other hand, there is a surface $(\Sigma_g, \partial\Sigma_g) \subset (\mathbb{CP}^2 - B^4, S^3)$ such that $\partial\Sigma_g = T(-p, 4p-1)$ and $[\Sigma_g] = d\gamma \in H_2(\mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$, where $g = g_{\mathbb{CP}^2}(T(p, 4p-1))$. Let γ_1 and γ_2 be the standard generators of $H_2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{Z})$. Then, the genus g closed surface $\Sigma = \Delta \cup \Sigma_g$ in $\mathbb{CP}^2 \# \mathbb{CP}^2$ satisfies $[\Sigma] = 2p\gamma_1 + d\gamma_2$ in $H_2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{Z})$ (see Figure 7). If Conjecture 1.1 were true, then the genus of Σ which is equal to $g_{\mathbb{CP}^2}(T(p, 4p-1))$ would satisfy

$$\frac{(2p-1)(2p-2)}{2} + \frac{(|d|-1)(|d|-2)}{2} \leq g_{\mathbb{CP}^2}(T(p, 4p-1)).$$

By Lemma 2.3, we have

$$\frac{(2p-1)(2p-2)}{2} + \frac{(|d|-1)(|d|-2)}{2} \leq \frac{(p-1)(4p-2)}{2} - 1.$$

Or equivalently,

$$(2p-1)(p-1) + \frac{(|d|-1)(|d|-2)}{2} \leq (p-1)(2p-1) - 1,$$

and this contradicts the positivity of $(|d|-1)(|d|-2)/2 \geq 0$ for $d \in \mathbb{Z}$. $\square$

To prove that Conjecture 1.1 fails for $(m, 0) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{Z})$ for any $m \geq 3$, we have to treat two cases.

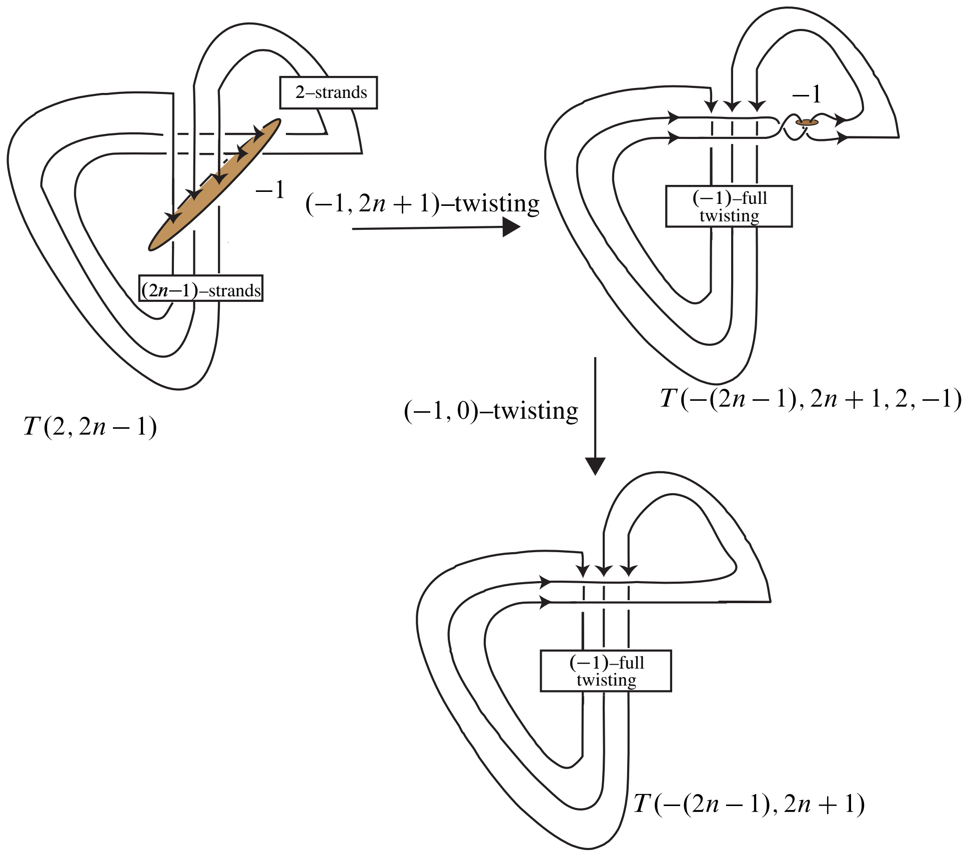


Figure 8: $T(2, 2n-1) \xrightarrow{(-1,2n+1)} T(-(2n-1), 2n+1, 2, -1) \xrightarrow{(-1,0)} T(-(2n-1), 2n+1)$

Case 1: $\boldsymbol{m = 2n + 1}$ **for** $\boldsymbol{n \geq 1}$ The proof of this case is based on Figure 8 showing that

$$T(2, 2n-1) \xrightarrow{(-1,2n+1)} T(-(2n-1), 2n+1, 2, -1) \xrightarrow{(-1,0)} T(-(2n-1), 2n+1).$$

By the positive answer to Milnor's Conjecture (cf Kronheimer and Mrowka [7]), the 4–ball genera of $T(2, 2n-1)$ and $T(2n-1, 2n+1)$ are respectively $n-1$ and $2n(n-1)$. As depicted in Figure 9, Lemma 2.1 yields the existence of a compact surface $(\Sigma_{n-1}, \partial\Sigma_{n-1}) \subset (B^4, \partial B^4 \cong S^3)$ with $\partial\Sigma_{n-1} = T(-(2n-1), 2n+1)$. As depicted in Figure 9, and by Lemma 2.1, we have

$$[\Sigma_{n-1}] = (2n+1)\gamma_1 \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2 - B^4, S^3; \mathbb{Z}).$$

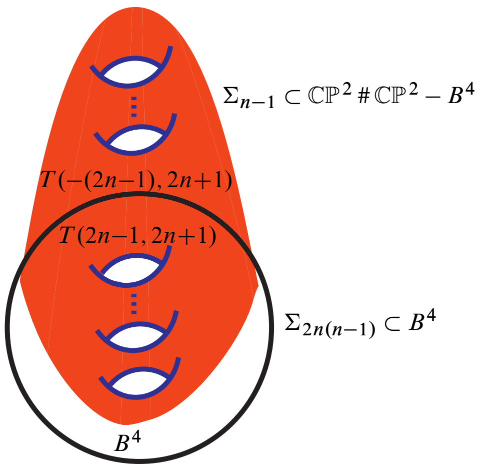


Figure 9: Gluing of surfaces technique

Let now $(\Sigma_{2n(n-1)}, \partial\Sigma_{2n(n-1)}) \subset (B^4, \partial B^4 \cong S^3)$ be a compact surface with

$$\partial\Sigma_{2n(n-1)} = T(2n-1, 2n+1).$$

Gluing Σ_{n-1} and $\Sigma_{2n(n-1)}$ along their boundaries yield a closed surface

$$\Sigma = \Sigma_{n-1} \cup \Sigma_{2n(n-1)} \subset \mathbb{CP}^2 \# \mathbb{CP}^2$$

representing $(2n+1)\gamma_1 \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2)$. If Conjecture 1.1 were true, then the genus of Σ, which is equal to $n-1+2n(n-1)$, would satisfy

$$\frac{(2n+1-1)(2n+1-2)}{2} \leq n-1+2n(n-1),$$

or equivalently, $2n^2-n \leq 2n^2-n-1$, an obvious contradiction.

Case 2: $m = 2p$ for $p \geq 2$ Figure 2 shows that $T(-p, 4p-1)$ is obtained from the trivial knot $T(-1, p)$ by a single $(-1, 2p)$–twisting. Let $\{C_1, C_2, \ldots, C_{u-1}, C_u\}$ be a U–crossing data for $T(-p, 4p-1)$. Changing the crossing C_1 from negative to positive is equivalent to performing a $(-1, 0)$–twisting along the crossing C_1 (see Figure 10) and this yields that

$$T(-1, p) \xrightarrow{(-1,p)} T(-p, 4p-1) \xrightarrow{(-1,0)} T(-p, 4p-1, 2, +1),$$

where $T(-p, 4p-1, 2, +1)$ is a twisted torus knot, as shown in Figure 10. By Lemma 2.5, we have that the 4–ball genus of $T(-p, 4p-1, 2, +1)$ satisfies the inequality $g^* \leq (p-1)(4p-2)/2-1$. Therefore, by a similar argument as in Case 1 above, if Conjecture 1.1 were true for $(2p, 0) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{Z})$ for any $p \geq 2$, then we would have $(2p-1)(2p-2)/2 \leq g^*$, which yields that

$$\frac{(2p-1)(2p-2)}{2} \leq \frac{(p-1)(4p-2)}{2} - 1,$$

or equivalently, $(2p-1)(p-1) \leq (p-1)(2p-1) - 1$, an obvious contradiction.

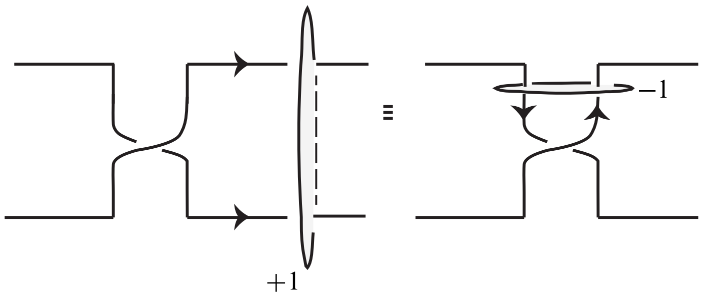


Figure 10

Corollary 2.7 *The class* $(3, 0) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2)$ *can be represented by a sphere, and therefore, it is the smallest counterexample to Conjecture 1.1.*

Proof This follows immediately from Case 1 if $n = 1$. □

3 Proof of Proposition 1.1

To prove Proposition 1.1, we need Lemma 3.1, Theorem 3.2 and Lemma 3.3 as well as Lemma 3.5. For this purpose, we recall some basic definitions. In what follows, let X be a smooth, closed, oriented, simply connected 4–manifold, then the second homology group $H_2(X; \mathbb{Z})$ is finitely generated (we refer to Spanier's book [17] for the details). The ordinary form $q_X\colon H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z}) \to \mathbb{Z}$ given by the intersection pairing for 2–cycles such that $q_X(\alpha, \beta) = \alpha \cdot \beta$ is a symmetric, unimodular bilinear form. The signature of this form, denoted $\sigma(X)$, is the difference between the number of positive and negative eigenvalues of a matrix representing q_X. Let $b_2^+(X)$ (resp. $b_2^-(X)$) be the rank of the positive (resp. negative) part of the intersection form of X. The second Betti number $b_2 = b_2^+ + b_2^-$ and the signature is $\sigma(X) = b_2^+ - b_2^-$.

A second homology class $\xi \in H_2(X; \mathbb{Z})$ is said to be characteristic provided that ξ is dual to the second Stiefel–Whitney class $w_2(X)$, or equivalently

$$\xi \cdot x \equiv x \cdot x \pmod 2 \tag{1}$$

for any $x \in H_2(X; \mathbb{Z})$ (we refer to Milnor and Stasheff's book [11] for the details).

Lemma 3.1 *$(a,b) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{Z})$ is characteristic if and only if a and b are both odd.*

Proof If $(a,b) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{Z})$ is characteristic, then $(a,b)\cdot(1,0) \equiv 1 \pmod 2$ and $(a,b)\cdot(0,1) \equiv 1 \pmod 2$. This yields that both a and b are odd. Conversely, let $\xi = (a,b) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{Z})$ and assume that a and b are both odd. Then for any $x = (x_1, x_2) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{Z})$, the identity (1) is equivalent to $ax_1 + bx_2 \equiv x_1^2 + x_2^2 \pmod 2$. Since $x_i \equiv x_i^2 \pmod 2$ for $i = 1,2$ and $a \equiv 1$ and $b \equiv 1 \pmod 2$, then (1) holds. This proves Lemma 3.1. □

Theorem 3.2 (Bryan [2]) *Let X be a smooth closed oriented and simply connected 4–manifold. We suppose Σ is an embedded surface in X of genus g and $[\Sigma]$ is divisible by 2. We assume that $\frac{1}{2}\Sigma$ is characteristic, $b_2^+ > 1$, and $\frac{1}{4}\Sigma\cdot\Sigma - \sigma(X) \geq 0$. Then*

$$g \geq \frac{5}{4}\left(\frac{\Sigma\cdot\Sigma}{4} - \sigma(X)\right) + 2 - b_2(X).$$

A proof of the following lemma can be found in [10, page 401].

Lemma 3.3 (Kronheimer and Mrowka [9]) *Let X be a smooth closed, connected and oriented 4–manifold. Let $a(\Sigma) = 2g(\Sigma) - 2 - \Sigma\cdot\Sigma$. If $\xi \in H_2(X;\mathbb{Z})$ is a homology class with $\xi\cdot\xi \geq 0$ and Σ_ξ is a surface representing ξ and $g \geq 1$ when $\Sigma_\xi\cdot\Sigma_\xi = 0$, then for any $r > 0$, the class $r\xi$ can be represented by an embedded surface $\Sigma_{r\xi}$ with*

$$a(\Sigma_{r\xi}) = ra(\Sigma_\xi).$$

Remark 3.4 Note that in particular, if $X = \mathbb{CP}^2 \# \mathbb{CP}^2$, then $a(\Sigma_{2\xi}) = 2a(\Sigma_\xi)$ is equivalent to

$$g(\Sigma_{2\xi}) = 2g(\Sigma_\xi) + \Sigma_\xi\cdot\Sigma_\xi - 1.$$

Proof The computation

$$\begin{aligned} a(\Sigma_{2\xi}) = 2a(\Sigma_\xi) &\Longleftrightarrow 2g(\Sigma_{2\xi}) - 2 - \Sigma_{2\xi}\cdot\Sigma_{2\xi} = 2(2g - 2 - \Sigma_\xi\cdot\Sigma_\xi) \\ &\Longleftrightarrow 2g(\Sigma_{2\xi}) - 2 - 4\Sigma_\xi\cdot\Sigma_\xi = 2(2g - 2 - \Sigma_\xi\cdot\Sigma_\xi) \\ &\Longleftrightarrow g(\Sigma_{2\xi}) = 2g(\Sigma_\xi) + \Sigma_\xi\cdot\Sigma_\xi - 1 \end{aligned}$$

achieves the proof. □

Recall that the knot obtained from k by inverting the orientation is called the *inverted knot* and denoted $-k$. The *mirror image* of k or *mirrored knot* is denoted by k^*; it is obtained by a reflection of k in a plane [3, page 15]. In what follows, we let $\overline{k} = -k^*$ denote the inverse of the mirror image of k.

Lemma 3.5 (1) *The* 4*–ball genus of positive knots in* S^3 *is additive under connected sums.*

(2) *For any knot* k *in* S^3, $g^*(k) = g^*(\bar{k})$.

Proof It is well-known that $g^*(k) = g(k)$ for any positive knot [12]. Since the 3–ball genus of knots is additive under connected sums [3], and $g(k) = g(\bar{k})$ then the statements of the lemma are easily proven. □

Proof of Proposition 1.1 To prove Proposition 1.1 for $(3,3) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{Z})$, let Σ be a genus g surface such that $[\Sigma] = 3\gamma_1 + 3\gamma_2 \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2)$. Theorem 3.2 yields that $g \geq 2$. Indeed, Lemma 3.1 implies that $\xi = [\Sigma] \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2)$ is a characteristic class with $\Sigma \cdot \Sigma = 18$. In virtue of Lemma 3.3, the class $2\xi = 6\gamma_1 + 6\gamma_2 \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2)$ can be represented by an embedded surface $\Sigma_{2\xi}$ satisfying the identity $a(\Sigma_{2\xi}) = 2a(\Sigma)$. Since $\Sigma_{2\xi} \cdot \Sigma_{2\xi} = 4\Sigma \cdot \Sigma$, then the estimate in Theorem 3.2,

$$g(\Sigma_{2\xi}) \geq \frac{5}{4}\left(\frac{\Sigma_{2\xi} \cdot \Sigma_{2\xi}}{4} - \sigma(\mathbb{CP}^2 \# \mathbb{CP}^2)\right) + 2 - b_2(\mathbb{CP}^2 \# \mathbb{CP}^2),$$

is equivalent by Remark 2.6 to

$$2g + 17 \geq \tfrac{5}{4}(\Sigma \cdot \Sigma - \sigma(\mathbb{CP}^2 \# \mathbb{CP}^2)) + 2 - b_2(\mathbb{CP}^2 \# \mathbb{CP}^2),$$

This implies that $g \geq 2$.

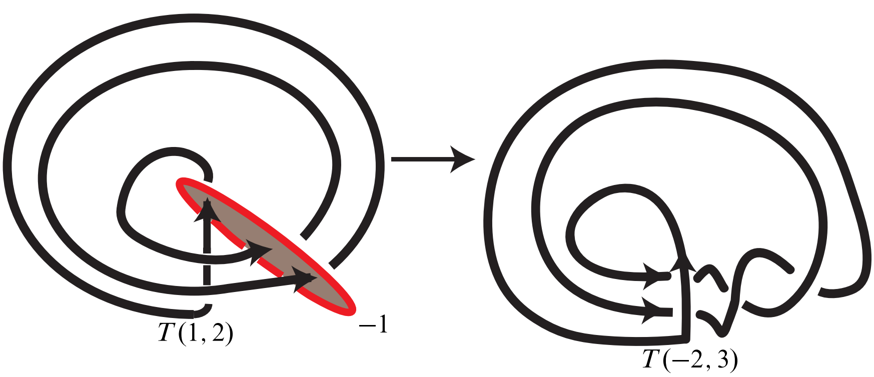


Figure 11: $T(1,2) \xrightarrow{(-1,3)} T(-2,3)$

To prove that $g \leq 2$, it is enough to exhibit a smooth closed genus two surface $\Sigma_2 \subset \mathbb{CP}^2 \# \mathbb{CP}^2$ representing $3\gamma_1 + 3\gamma_2 \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2)$. Indeed, Figure 11 shows that

$$T(1,2) \xrightarrow{(-1,3)} T(-2,3),$$

and therefore,

$$T(1,2)\,\#\,T(1,2) \xrightarrow{(-1,3)} T(1,2)\,\#\,T(-2,3) \xrightarrow{(-1,3)} T(-2,3)\,\#\,T(-2,3).$$

By Lemma 2.1, there is a disk $\Delta \subset \mathbb{CP}^2 \# \mathbb{CP}^2 - B^4$ so that $\partial\Delta = T(-2,3) \# T(-2,3)$ and $[\Delta] = 3\gamma_1 + 3\gamma_2 \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$. Since the 4–ball genus of $T(2,3)$ is one and $T(2,3)$ is a positive knot (see Kawauchi [6]), then Lemma 3.5 yields that the 4–ball genus of $\overline{k} = T(2,3) \# T(2,3)$ is two. Let $(\Sigma_2, \partial\Sigma_2) \subset (B^4, \partial B^4 \cong S^3)$ be an orientable and compact surface with $\partial\Sigma_2 = T(2,3) \# T(2,3)$. Gluing Δ and Σ_2 along their boundaries yield a closed genus 2 surface $\Sigma = \Delta \cup \Sigma_2 \subset \mathbb{CP}^2 \# \mathbb{CP}^2$ representing $3\gamma_1 + 3\gamma_2 \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2)$ (see Figure 12).

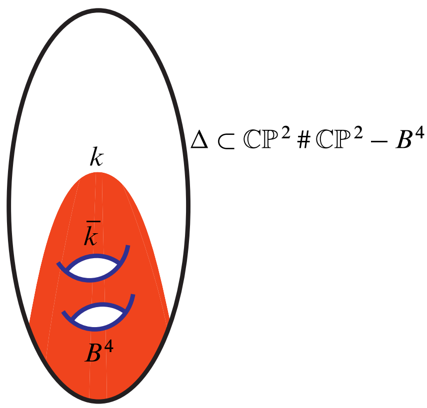


Figure 12: Gluing of surfaces technique

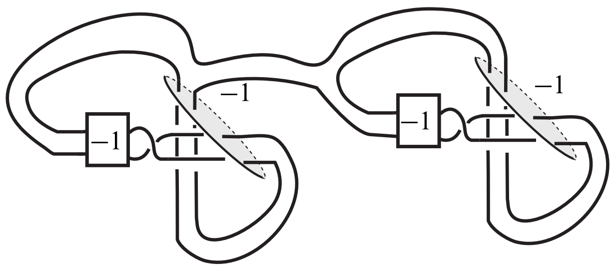


Figure 13: $T(1,p) \,\#\, T(1,q) \xrightarrow{(-1,2p)} T(-p,4p-1) \,\#\, T(1,q) \xrightarrow{(-1,2q)} T(-p,4p-1) \# T(-q,4q-1)$

To prove Proposition 1.1 for the pair $(6,6) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{Z})$, we first notice that if $T(p,q)$ denotes the (p,q)–torus knot for $0 < p < q$ with p and q are coprime, then the knot drawn in Figure 13 is ambient isotopic to the trivial knot $T(1,p) \# T(1,q)$, Thus

$$T(1,p) \# T(1,q) \xrightarrow{(-1,2p)} T(-p,4p-1) \# T(1,q) \xrightarrow{(-1,2q)} T(-p,4p-1) \# T(-q,4q-1).$$

By Lemma 2.1, there exists a properly embedded disk $\Delta \subset \mathbb{CP}^2 \# \mathbb{CP}^2 - B^4$ such that $\partial\Delta = T(-p,4p-1) \# T(-q,4q-1)$ and $[\Delta] = 2p\gamma_1 + 2q\gamma_2$, where γ_1 and γ_2 are the standard generators of $H_2(\mathbb{CP}^2 \# \mathbb{CP}^2 - B^4, S^3; \mathbb{Z})$. By the positive answer to Milnor's Conjecture by Kronheimer and Mrowka [7] and Lemma 3.5(1), the 4–ball genus of $T(p,4p-1) \# T(q,4q-1)$ is $(p-1)(2p-1) + (q-1)(2q-1)$. Let Σ_{g^*} be an oriented and compact surface properly embedded in B^4 and such that

$$\partial\Sigma_{g^*} = T(p,4p-1) \# T(q,4q-1),$$

and whose genus is $g^* = (p-1)(2p-1) + (q-1)(2q-1)$. Denote $\Sigma_{2p,2q} = \Delta \cup \Sigma_{g^*}$, then it is easily checked that $[\Sigma_{2p,2q}] = 2p\gamma_1 + 2q\gamma_2 \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{Z})$ and the genus of $\Sigma_{2p,2q}$ is $(p-1)(2p-1) + (q-1)(2q-1)$.

Assume now that $(2p,2q) = (6,6)$, or equivalently $(p,q) = (3,3)$. By Theorem 3.2, the genus of $(6,6) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{Z})$ can be shown to be greater or equal to twenty. Indeed, $\frac{1}{2}\Sigma = 3\gamma_1 + 3\gamma_2$ is characteristic (cf Lemma 3.1), $b_2^+(X) = b_2(X)(=2)$, and $(\Sigma \cdot \Sigma)/4 - \sigma(X) = 16$, where $[\Sigma] = 6\gamma_1 + 6\gamma_2 \in H_2(X; \mathbb{Z})$ and $X = \mathbb{CP}^2 \# \mathbb{CP}^2$. By virtue of Theorem 3.2, the inequality $g \geq \frac{5}{4}((\Sigma \cdot \Sigma)/4 - \sigma(X)) + 2 - b_2(X)$ holds. This is equivalent to $g \geq 20$. Therefore, it is sufficient to find a genus twenty surface representing $(6,6) \in H_2(\mathbb{CP}^2 \# \mathbb{CP}^2; \mathbb{Z})$, which is $\Sigma_{6,6}$ as constructed above. □

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Department of Mathematical Sciences, University of Texas at El Paso
500 West University Avenue, Bell Hall 144, El Paso, Texas 79968, USA

manouh@utep.edu

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