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Twisted equivariant K–theory and K–homology of $\mathrm{Sl}_3\mathbb{Z}$

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We use a spectral sequence to compute twisted equivariant K–theory groups for the classifying space of proper actions of discrete groups. We study a form of Poincaré duality for twisted equivariant K–theory studied by Echterhoff, Emerson and Kim in the context of the Baum–Connes conjecture with coefficients and verify it for the group $\mathrm{Sl}_3(\mathbb{Z})$.

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In this work, we examine computational aspects relevant to the computation of twisted equivariant K–theory and K–homology groups for proper actions of discrete groups.

Twisted K–theory was introduced by Donovan and Karoubi [9] assigning to a torsion element $\alpha \in H^3(X,\mathbb{Z})$ an abelian group ${}^{\alpha}K^*(X)$ defined on a space by using finite-dimensional matrix bundles. After the growing interest by physicists in the 1990s and 2000s, Atiyah and Segal [2] introduced a notion of twisted equivariant K–theory for actions of compact Lie groups. In another direction, orbifold versions of twisted K–theory were introduced by Adem and Ruan [1], and progress was made to develop computational tools for twisted equivariant K–theory with the construction of a spectral sequence by the authors, Espinoza and Uribe in [5].

In [4], the first author, Espinoza, Joachim and Uribe introduce twisted equivariant K–theory for proper actions, allowing a more general class of twists, classified by the third integral Borel cohomology group $H^3(X\times_G EG,\mathbb{Z})$.

We concentrate on the case of twistings given by discrete torsion, which is given by cocycles

$$\alpha \in Z^2(G,S^1)$$

representing classes in the image of the projection map

$$H^2(G,S^1)\xrightarrow{\cong} H^3(BG,\mathbb{Z})\to H^3(X\times_G EG,\mathbb{Z}).$$

Under this assumption on the twist, a version of Bredon cohomology with coefficients in twisted representations can be used to approximate twisted equivariant K–theory, by means of a spectral sequence studied in [5] and Dwyer [10].

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The Bredon (co)homology groups relevant to the computation of twisted equivariant K–theory, and its homological version, twisted equivariant K–homology, satisfy a universal coefficient theorem; Theorem 1.13. We state it more generally for a pair of coefficient systems satisfying Condition 1.12.

Theorem (Universal coefficient theorem) *Let X be a proper, finite G–CW complex. Let $M^?$ and $M_?$ be a pair of functors satisfying Condition 1.12. Then, there exists a short exact sequence of abelian groups*

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(H_{n-1}^G(X, M_?), \mathbb{Z}) \to H_G^n(X, M^?) \to \mathrm{Hom}_{\mathbb{Z}}(H_n^G(X, M_?), \mathbb{Z}) \to 0.$$

The main application for the cohomological methods described in this note are computations verifying a form of Poincaré duality in twisted equivariant K–theory for discrete groups, as studied by Echterhoff, Emerson and Kim [12] in the context of the Baum–Connes conjecture with coefficients.

Given a G–C*–algebra A, the Baum–Connes conjecture with coefficients in A for G predicts that an analytical assembly map

$$K_*^G(\underline{E}G, A) = KK_G(C_0(\underline{E}G), A) \longrightarrow K_*(A \rtimes G)$$

is an isomorphism, where $A \rtimes G$ is the crossed product C^*–algebra.

A discrete torsion twist defines an action of G on the C^*–algebra of compact operators on $l^2(G)$, which only depends on the cohomology class of ω and not on particular representing cocycles. We will denote this G–C^*–algebra by K_ω. Given a discrete torsion twist ω, there exist analytic versions of twisted equivariant K–homology and K–theory groups, $K_G^*(X, \omega)$, $K_*^G(X, \omega)$, which are defined in terms of the equivariant Kasparov KK–theory groups $KK_G^*(C_0(X), K_\omega)$, respectively $KK_G^*(K_\omega, C_0(X))$; see [12] for more details.

Our computational methods describe a duality relation, Corollary 7.3, between these groups as follows.

Corollary (Duality for twisted equivariant K–theory) *Let G be a discrete group with a finite model for $\underline{E}G$. Let $\omega \in Z^2(G, S^1)$. Assume that the Bredon cohomology groups $H_G^*(X, R^{-\alpha})$ relevant to the computation of the twisted equivariant K–theory are all free abelian and are concentrated in degree 0 and 1. Then, there exists a duality isomorphism*

$$K_G^*(\underline{E}G, K_{-\omega}) \longrightarrow K_G^*(K_\omega, \underline{E}G).$$

The family of groups G satisfying the previous assumptions on both the twist and the classifying space $\underline{E}G$ includes several examples. We pay attention to $\mathrm{Sl}_3\mathbb{Z}$.

We use the computation of the cohomology of $\mathrm{Sl}_3\mathbb{Z}$ due to Soulé [18], previous work by Sánchez-García [16], as well as the theory of projective representations of finite groups to compute the twisted equivariant K–theory on the classifying space $\underline{E}\,\mathrm{Sl}_3\mathbb{Z}$, by computing the Bredon cohomology group associated to a specific torsion twist. Using the spectral sequence we verify that the computation of twisted equivariant K–theory reduces to Bredon cohomology in Theorem 6.3.

Theorem (Calculation of twisted equivariant K–homology of $\mathrm{Sl}_3\mathbb{Z}$) *The equivariant K–homology groups of $\mathrm{Sl}_3\mathbb{Z}$ with coefficients in the $\mathrm{Sl}_3\mathbb{Z}$–C^* algebra $\mathcal{K}_{u_1}$ are given as follows:*

$$K_p^{\mathrm{Sl}_3\mathbb{Z}}(\underline{E}\,\mathrm{Sl}_3\mathbb{Z}, \mathcal{K}_{u_1}) = 0, \quad p \text{ odd}, \qquad K_p^{\mathrm{Sl}_3\mathbb{Z}}(\underline{E}\,\mathrm{Sl}_3\mathbb{Z}, \mathcal{K}_{u_1}) \cong \mathbb{Z}^{\oplus 13}, \quad p \text{ even}.$$

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1 Bredon cohomology

We recall briefly some definitions relevant to Bredon homology and cohomology; see Mislin and Valette [15] for more details. Let G be a discrete group. A G–CW complex is a CW complex with a G–action permuting the cells and such that if a cell is sent to itself, this is done by the identity map. We call the G–action proper if all cell stabilizers are finite subgroups of G.

Definition 1.1 A model for $\underline{E}G$ is a proper G–CW complex X such that for any proper G–CW complex Y there is a unique G–map $Y \to X$, up to G–homotopy equivalence.

One can prove that a proper G–CW complex X is a model of $\underline{E}G$ if and only if the subcomplex of fixed points X^H is contractible for each finite subgroup $H \subseteq G$. It can be shown that classifying spaces for proper actions always exist. They are clearly unique up to G–homotopy equivalence.

Let $\mathcal{O}_G$ be the orbit category of G; a category with one object G/H for each subgroup $H \subseteq G$ and where morphisms are given by G–equivariant maps. There exists a morphism $\phi\colon G/H \to G/K$ if and only if H is conjugate in G to a subgroup of K.

Definition 1.2 (Cellular chain complex associated to a G–CW complex) Let X be a G–CW complex. The contravariant functor $\underline{C}_*(X)\colon \mathcal{O}_G \to \mathbb{Z}\text{–CHCOM}$ assigns to every object G/H the cellular $\mathbb{Z}$–chain complex of the H–fixed point complex $\underline{C}_*(X^H) \cong C_*(\mathrm{Map}_G(G/H, X))$ with respect to the cellular boundary maps $\underline{\partial}_*$.

We will use homological algebra to define Bredon homology and cohomology functors.

A contravariant coefficient system is a contravariant functor

$$M\colon \mathcal{O}_G \to \mathbb{Z}\text{–MODULES}.$$

Given a contravariant coefficient system M, the Bredon cochain complex $C^*_G(X; M)$ is defined as the abelian group of natural transformations of functors defined on the orbit category $\underline{C}_*(X) \to M$. In symbols,

$$C^n_G(X; M) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}_G}}(\underline{C}_n(X), M),$$

where $\mathcal{F}_G$ is a family containing the isotropy groups of X.

Given a set $\{e_\lambda\}$ of orbit representatives of the n–cells of the G–CW complex X, and isotropy subgroups S_λ of the cells e_λ, the abelian groups $C^n_G(X, M)$ satisfy

$$C^n_G(X, M) = \bigoplus_\lambda \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[e_\lambda], M(G/S_\lambda))$$

with one summand for each orbit representative e_λ. The abelian groups afford a differential $\delta^n\colon C^n_G(X, M) \to C^{n-1}_G(X, M)$ determined by $\underline{\partial}_*$ and also afford maps $M(\phi)\colon M(G/S_\mu) \to M(G/S_\lambda)$ for morphisms $\phi\colon G/S_\lambda \to G/S_\mu$.

Definition 1.3 (Bredon cohomology) The Bredon cohomology groups with coefficients in M, denoted by $H^*_G(X, M)$ are the cohomology groups of the cochain complex $(C^*_G(X, M), \delta^*)$.

Dually to the cohomological situation, given a covariant functor

$$N\colon \mathcal{O}_G \to \mathbb{Z}\text{–MODULES},$$

the chain complex

$$C^G_* = \underline{C}^n(X) \otimes_{\mathcal{O}_G} N = \bigoplus_\lambda \mathbb{Z}[e_\lambda] \otimes N(G/S_\lambda)$$

admits differentials $\delta_* = \partial_* \otimes N(\phi)$ for morphisms $\phi\colon G/S_\lambda \to G/S_\mu$ in $\mathcal{O}_G$.

Definition 1.4 (Bredon homology) The Bredon homology groups with coefficients in N, denoted by $H_*^G(X,N)$, are defined as the homology groups of the chain complex $(C_*^G(X,N),\delta_*)$.

Remark 1.5 (Determination of Bredon (co)homology in practice) The coefficient systems considered in this note yield chain complexes, respectively cochain complexes of free abelian groups with preferred bases to compute both Bredon homology and cohomology.

Notice that if we have a complex of free abelian groups

$$\cdots \to \mathbb{Z}^{\oplus n} \xrightarrow{f} \mathbb{Z}^{\oplus m} \xrightarrow{g} \mathbb{Z}^{\oplus k} \to \cdots$$

with f and g represented by matrices A and B for some fixed basis, then the homology at $\mathbb{Z}^{\oplus m}$ is

$$\ker(g)/\operatorname{im}(f) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_s\mathbb{Z} \oplus \mathbb{Z}^{\oplus(m-s-r)},$$

where $r = \operatorname{rank}(B)$ and $d_1,\dots,d_s$ are the elementary divisors of A.

Given a discrete group G, the complex representation ring defines two functors defined on the subcategory generated by objects G/H with H finite:

$$\mathcal{R}^?, \quad \mathcal{R}_?.$$

For every object G/H, the groups $R^?(G/H)$, $R_?(G/H)$ agree with the Grothendieck group of isomorphism classes of complex representations $R_{\mathbb{C}}(H)$ of the finite subgroup H.

The contravariant functor $\mathcal{R}^?$ assigns to a G–map $\phi\colon G/H \to G/K$ the restriction map $R_{\mathbb{C}}(K) \to R_{\mathbb{C}}(gHg^-1) \cong R_{\mathbb{C}}(H)$ to the subgroup gKg^{-1} of H determined by the morphism $\phi\colon G/H \to G/K$, where $g \in G$ is such that $\phi(eH) = gK$.

The covariant functor $\mathcal{R}_?$ assigns the morphism ϕ the induction map $R_{\mathbb{C}}(H) \cong R_{\mathbb{C}}(gHg^{-1}) \to R_{\mathbb{C}}(K)$.

Given a finite group H the group $R_{\mathbb{C}}(H)$ is free abelian, isomorphic to the free abelian group generated by the set $\rho_1,\dots,\rho_s$ of irreducible characters. For any representation ρ, there exists a unique expression $\rho = n_1\rho_1,\dots,n_s\rho_s$, where $n_i = (\rho \mid \rho_i)$, and $(\cdot \mid \cdot)$ is the scalar product of characters.

Recall that due to Frobenius reciprocity, given a subgroup K of H, a representation τ of K and a representation ρ of H, the equation $(\tau\uparrow\mid\rho)_H = (\tau\mid\rho\downarrow)_K$ holds, where $\downarrow$ denotes restriction and $\uparrow$ denotes induction.

1.1 Bredon (co)homology with coefficients in twisted representations

Definition 1.6 Let H be a finite group and V be a complex vector space. Given a cocycle $\alpha\colon H \times H \to S^1$ representing a class in $H^2(H, S^1) \cong H^3(H, \mathbb{Z})$, an α–twisted representation is a function $P\colon H \to \mathrm{Gl}(V)$ satisfying

$$P(e) = 1,$$
$$P(x)P(y) = \alpha(x, y)P(xy).$$

The isomorphism type of an α–twisted representation only depends on the cohomology class in $H^2(H, S^1)$.

Definition 1.7 Let H be a finite group and $\alpha\colon H \times H \to S^1$ be a cocycle representing a class in $H^2(H, S^1) \cong H^3(H, \mathbb{Z})$. The α–twisted representation group of H, denoted by ${}^{\alpha}\mathcal{R}(H)$ is the Grothendieck group of isomorphism classes of complex, α–twisted representations with direct sum as binary operation.

Let H be a finite group. Given a cocycle $\alpha \in H^2(H, S^1)$ representing a torsion class of order n, the normalization procedure gives a cocycle β cohomologous to α such that $\beta\colon H \times H \to S^1$ takes values in the subgroup $\mathbb{Z}/n \subset S^1$ generated by a primitive n^{th} root of unity η. Associated to a normalized cocycle, there exists a central extension

$$1 \to \mathbb{Z}/n \to H^* \to H \to 1$$

with the property that any twisted representation of H is a linear representation of H^*, with the additional property that $\mathbb{Z}/n$ acts by multiplication with η. Such a group is called a Schur covering group for H.

Definition 1.8 Let $\rho\colon H \to \mathrm{Gl}(V)$ be an α–twisted representation. The character of ρ is the map $H \to \mathbb{C}$ given by $\chi(h) = \mathrm{trace}(\rho(h))$.

Given a cocycle α, an element $h \in H$ is said to be α–regular if $\alpha(h, x) = \alpha(x, h)$ for all $x \in H$. For a choice of representatives of conjugacy classes of α–regular elements, $\{x_1, \dots, x_k\}$ in H, an α–character table gives the values of the character of irreducible α–twisted representations by evaluating $\chi(x_i)$. An α–character is not constant in conjugacy classes of elements in H; it depends on the choice of representatives of the conjugacy classes of α–regular elements. The following result shows the relation to different choices of a set of representatives by comparing it to the linear characters of a Schur covering group H^*. It is proven in Karpilovsky [14, Theorem 1.1(i), Chapter 5, page 205].

Theorem 1.9 (Rectification procedure for characters of a Schur covering group) *Let*

$$1 \to A \to H^* \xrightarrow{f} H \to 1$$

be a finite central group extension and let $\mu\colon H \to H^*$ *be a fixed section of* f. *For any given* $\xi \in \mathrm{Hom}(A, \mathbb{C}^*)$, *let* $\alpha = \alpha_\xi \in Z^2(H, \mathbb{C}^*)$ *be defined by*

$$\alpha(x, y) = \xi(\mu(x)\mu(y)\mu(xy)^{-1}) \quad \textit{for all } x, y \in H.$$

Then if $\lambda_1^*, \dots, \lambda_n^*$ *are all distinct irreducible* $\mathbb{C}$*–characters of* H^* *whose restriction to* A *has* ξ *as an irreducible constituent and if* $\lambda_i\colon H \to \mathbb{C}$ *is defined by*

$$\lambda_i(g) := \lambda_i^*(\mu(g)) \quad \textit{for all } g \in H,$$

then $\lambda_1, \dots, \lambda_r$ *are all distinct irreducible* α*–characters of* H. *Moreover, if* μ *is a conjugation preserving section, then each* λ_i *is a class function.*

We can define contravariant and covariant coefficient systems for the family $\mathcal{F}_G = \mathcal{FIN}$ of finite subgroups agreeing on objects by using the α–twisted representation group functor $\mathcal{R}_\alpha(\cdot)$.

Definition 1.10 Let G be a discrete group and let $\alpha \in Z^2(G, S^1)$ be a cocycle. Define $\mathcal{R}_\alpha$ on objects by

$$\mathcal{R}_{\alpha ?}(G/H) = \mathcal{R}_\alpha^?(G/H) := {}^{i^*(\alpha)}\mathcal{R}(H),$$

where $i\colon H \to G$ is the inclusion, and induction of α–twisted representations for the covariant part $\mathcal{R}_{\alpha ?}$, respectively restriction of α–representations for the contravariant part $\mathcal{R}_\alpha^?$.

Orthogonality relations between irreducible α–characters and Frobenius reciprocity go over the setting of α–twisted representations; compare [14, Proposition 11.7, page 73] and [14, Theorem 11.8, page 73].

Definition 1.11 Let G be a discrete group, let X be a proper G–CW complex, and let $\alpha \in Z^2(G, S^1)$ be a cocycle. The α–twisted Bredon cohomology, respectively α–twisted homology groups of X are the Bredon cohomology, respectively homology groups with respect to the functors described in Definition 1.10.

We will consider coefficient systems which consist of abelian groups with preferred bases. This condition produces a convenient duality situation, and produces based (co)chain complexes as input for the computation of Bredon (co)homology groups.

Condition 1.12 Let G be a discrete group, Let $M_?$ and $M^?$ be covariant, respectively contravariant functors defined on a subcategory $\mathcal{O}_G$ of the orbit category $\mathcal{O}$ agreeing on objects. Suppose that:

- There exists for every object G/H a choice of a finite basis $\{\beta_{iH}\}$ expressing $M_?(G/H) = M^?(G/H)$ as the finitely generated, free abelian group on $\{\beta_{iH}\}$ and isomorphisms
$$a_H \colon M^?(G/H) \xrightarrow{\cong} \mathbb{Z}[\{\beta_{iH}\}] \xleftarrow{\cong} M_?(G/H) : b_H.$$
- For the covariant functor $\widehat{M} := \mathrm{Hom}_{\mathbb{Z}}(M^?(\cdot), \mathbb{Z})$, the dual basis $\{\widehat{\beta}_{iH}\}$ of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\{\beta_{iH}\}], \mathbb{Z})$ and the isomorphisms a_H and b_H, the following diagram is commutative:
$$\begin{tikzcd} M_?(G/H) \arrow[r, "b_H"] \arrow[d, "M_?(\phi)"'] & \mathbb{Z}[\{\beta_{iH}\}] \arrow[r, "D_H"] & \mathbb{Z}[\{\widehat{\beta}_{iH}\}] \arrow[r, "\widehat{a_H}"] & \widehat{M}(G/H) \arrow[d, "\widehat{M}(\phi)"] \\ M_?(G/K) \arrow[r, "b_K"'] & \mathbb{Z}[\{\beta_{jK}\}] \arrow[r, "D_K"'] & \mathbb{Z}[\{\widehat{\beta}_{jK}\}] \arrow[r, "\widehat{a_K}"'] & \widehat{M}(G/K), \end{tikzcd}$$
where D_H, D_K are the duality isomorphisms associated to the bases and $\phi \colon G/H \to G/K$ is a morphism in the orbit category.

Condition 1.12 is satisfied in the following cases.

- Constant coefficients $\mathbb{Z}$ yield Condition 1.12.
- The complex representation ring functors defined on the family $\mathcal{FIN}$ of finite subgroups, $\mathcal{R}^?$, $\mathcal{R}_?$. A computation using characters as bases and Frobenius reciprocity yields Condition 1.12.
- Consider a discrete group G and a normalized torsion cocycle
$$\alpha \in Z^2(G, S^1),$$
take the $\alpha-$ and α^{-1}–twisted representation ring functors $\mathcal{R}_{-\alpha?}$ $\mathcal{R}_{\alpha}{}^?$ defined on the objects G/H, where H belongs to the family $\mathcal{FIN}$ of finite subgroups. Consider for every object G/H the cocycles $i_H^*(\alpha)$, where $i_H \colon H \to G$ is the inclusion, and assume without loss of generality that they are normalized and correspond to a family of Schur covering groups in central extensions $1 \to \mathbb{Z}/n_H \to H^* \to H \to 1$.
We select the set $\{\beta_H\}$ given as the set of characters of irreducible representations of H^* where $\mathbb{Z}/n_H$ acts by multiplication with a primitive n_H^{th} root of unity. Given a choice of sections for the quotient maps $H^* \to H$, one

can construct isomorphisms ${}^{i^*(\alpha)}\mathcal{R}(G/H) \xrightarrow{\cong} \mathbb{Z}[\{\beta_H\}]$. The orthogonality relations and Frobenius reciprocity for their twisted characters guarantee that Condition 1.12 holds.

Theorem 1.13 *Let X be a proper, finite G–CW complex. Let $M^?$ and $M_?$ be a pair of functors satisfying Condition 1.12. Then, there exists a short exact sequence of abelian groups*

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(H^G_{n-1}(X, M_?), \mathbb{Z}) \to H^n_G(X, M^?) \to \mathrm{Hom}_{\mathbb{Z}}(H^G_n(X, M_?), \mathbb{Z}) \to 0.$$

Proof The proof consists of two steps:

- Construction of chain homotopy equivalences
$$(C^G_n(X, M_?), \delta_n) \to (C^G_n(X, \widehat{M}), \hat{\delta}^n),$$
$$(C^n_G(X, M^?), \delta^n) \to ((\mathrm{Hom}_{\mathbb{Z}}(C^G_n(X, \widehat{M}), \mathbb{Z}); \mathrm{Hom}_{\mathbb{Z}}(\delta_n, \mathbb{Z})).$$
- Construction of an exact sequence
$$0 \to \ker h \to H^n_G(X, M^?) \xrightarrow{h} \mathrm{Hom}_{\mathbb{Z}}(H^G_n(X, \widehat{M}), \mathbb{Z}) \to 0$$
and identification of the kernel as $\mathrm{Ext}_{\mathbb{Z}}(H^G_{n-1}(X, M_?), \mathbb{Z})$.

For the first chain homotopy equivalence, notice that given a set $\{e_\lambda\}$ of orbit representatives of the n–cells of the G–CW complex X, and isotropy subgroups S_λ of the cells e_λ, the chain complex for computing the n^{th} Bredon homology with coefficients in $M_?$ is given by $\bigoplus_\lambda \mathbb{Z}[e_\lambda] \otimes M(G/S_\lambda)$. Condition 1.12 gives for every λ an isomorphism of free abelian groups $M_?(G/S_\lambda) \to \widehat{M}(G/S_\lambda)$ which gives a chain map

$$C^G_n(X, M_?) \cong \bigoplus_\lambda \mathbb{Z}[e_\lambda] \otimes M_?(G/S_\lambda) \longrightarrow \bigoplus_\lambda \mathbb{Z}[e_\lambda] \otimes \widehat{M}(G/S_\lambda) \cong C^G_n(X, \widehat{M})$$

giving a chain homotopy equivalence and subsequently an isomorphism in homology groups $H^G_n(X, M_?) \cong H^G_n(X, \widehat{M})$.

For the second isomorphism, consider the chain map

$$A\colon C^n_G(X, M^?) \to \mathrm{Hom}_{\mathbb{Z}}(C^G_n(X, \widehat{M}), \mathbb{Z}),$$
$$A\colon \mathrm{Hom}_{\mathbb{Z}}\Bigg(\bigoplus_\lambda \mathbb{Z}[e_\lambda], M^?(G/S_\lambda)\Bigg) \to \mathrm{Hom}_{\mathbb{Z}}\Bigg(\bigoplus_\lambda \mathbb{Z}[e_\lambda] \otimes \widehat{M}(G/S_\lambda), \mathbb{Z}\Bigg),$$

assigning to $\varphi \in C^n_G(X, M^?)$ the homomorphism ψ which is defined on basis elements $[e_\lambda] \otimes \widehat{\beta}_{i_{S_\lambda}}$ as $\widehat{\beta}_{i_{S_\lambda}}(\varphi([e_\lambda])) \in \mathbb{Z}$. A chain homotopy inverse for this map is given by the chain map

$$B\colon \operatorname{Hom}_{\mathbb{Z}}\left(\bigoplus_\lambda \mathbb{Z}[e_\lambda] \otimes \widehat{M}(G/S_\lambda), \mathbb{Z}\right) \to \operatorname{Hom}_{\mathbb{Z}}\left(\bigoplus_\lambda \mathbb{Z}[e_\lambda], M^?(G/S_\lambda)\right),$$

assigning to a homomorphism $\psi \in \operatorname{Hom}_{\mathbb{Z}}(\bigoplus_\lambda \mathbb{Z}[e_\lambda] \otimes \widehat{M}(G/S_\lambda), \mathbb{Z})$ the homomorphism defined on the basis $[e_\lambda]$ as $\psi([e_\lambda]) = \sum_{i_{S_\lambda}} \psi([e_\lambda] \otimes \widehat{\beta}_{i_{S_\lambda}})\beta_{i_{S_\lambda}}$.

For the second part, consider the cochain complex given in degree n by

$$C^n_G(X; M^?) = (\operatorname{Hom}_{\mathcal{F}_G}(\underline{C}_n(X), M), \delta^n)$$

and the chain complex $C^G_n(X, \widehat{M}) = (\underline{C}^n(X) \otimes_{\mathcal{O}_G} \widehat{M}, \delta_n)$. Denote by $Z^n = \ker \delta^n$, $B^n = \operatorname{im} \delta^{n-1}$ and $Z_n = \ker \delta_n$, $B_n = \operatorname{im} \delta_{n+1}$.

A class in $H^G_n(X, \widehat{M})$ is represented by a homomorphism $\varphi\colon C^G_n(X, \widehat{M}) \to \mathbb{Z}$ such that $\operatorname{Hom}_{\mathbb{Z}}(\delta_n, \mathbb{Z})(\varphi) = 0$, that means $\varphi|B_n = 0$. Then if $\varphi_0 = \varphi|Z_n$ we have a map defined in the quotient $\overline{\varphi}_0\colon Z_n/B_n \to \mathbb{Z}$, or in others words an element of $\operatorname{Hom}_{\mathbb{Z}}(H^G_n(X, \widehat{M}), \mathbb{Z})$. We have defined a map

$$h\colon H^n_G(X, \widehat{M}) \to \operatorname{Hom}_{\mathbb{Z}}(H^G_n(X, \widehat{M}), \mathbb{Z}).$$

This map is surjective. Now we proceed to identify $\ker(h)$.

As we have a projective resolution of $H^G_n(X, \widehat{M})$,

$$0 \to B_n \to Z_n \to H^G_n(X, \widehat{M}),$$

the group $\operatorname{Ext}(H^G_n(X, \widehat{M}, \mathbb{Z})$ can be calculated using the exact sequence

$$\operatorname{Hom}_{\mathbb{Z}}(Z_n, \mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(B_n, \mathbb{Z}) \to \operatorname{Ext}(H^G_n(X, \widehat{M}, \mathbb{Z}) \to 0. \tag{1.14}$$

On the other hand, we have the following long exact sequences:

$$0 \to Z_n \to C^G_n(X, \widehat{M}) \xrightarrow{\delta} B_{n-1} \to 0 \tag{1.15}$$

$$0 \to H^G_n(X, \widehat{M}) \to C^G_n(X, \widehat{M})/B_n \xrightarrow{\delta} B_{n-1} \to 0 \tag{1.16}$$

The sequences (1.15) and (1.16) are split because B_{n-1} is free abelian.

We can decompose δ as the map

$$
\begin{aligned}
(1.17)\quad 0 \longrightarrow C_n^G(X,\widehat{M}) & \\
\longrightarrow C_n^G(X,\widehat{M})/B_n \longrightarrow C_n^G(X,\widehat{M})/Z_n & \\
\xrightarrow{\cong} B_{n-1} \subset Z_{n-1} \subset C_{n-1}^G(X,\widehat{M}). &
\end{aligned}
$$

We have the following diagram of exact sequences (here $H_n = H_n^G(X,\widehat{M})$ and $C_n = C_n^G(X,\widehat{M})$):

$$
\begin{tikzcd}
 & & 0 & \\
 & & \mathrm{Hom}_{\mathbb{Z}}(H_n,\mathbb{Z}) \arrow[u] & \\
\mathrm{Hom}_{\mathbb{Z}}(B_n,\mathbb{Z}) & \mathrm{Hom}_{\mathbb{Z}}(C_n,\mathbb{Z}) \arrow[l] & \mathrm{Hom}_{\mathbb{Z}}(C_n/B_n,\mathbb{Z}) \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & \mathrm{Ext}(H_n,\mathbb{Z}) \arrow[l] & \mathrm{Hom}_{\mathbb{Z}}(B_{n-1},\mathbb{Z}) \arrow[l] \arrow[u] & \mathrm{Hom}_{\mathbb{Z}}(Z_{n-1},\mathbb{Z}) \arrow[l] \arrow[u] \\
 & & 0 \arrow[u] & \mathrm{Hom}_{\mathbb{Z}}(C_{n-1},\mathbb{Z}) \arrow[u] \arrow[ul]
\end{tikzcd}
\tag{1.18}
$$

The right vertical sequence is exact because (1.15) splits, the left vertical sequence is split exact because (1.16) is. The upper horizontal sequence is obtained from applying Hom to the projective resolution of $H_n^G(X,\widehat{M})$ and the lower horizontal sequence is (1.14).

Analyzing the diagram (1.18) and decomposition (1.17) we obtain

$$Z^n \cong \mathrm{Hom}_{\mathbb{Z}}(C_n(X,\widehat{M})/B_n,\mathbb{Z}),$$

and on the other hand

$$B^n \cong \mathrm{im}(\mathrm{Hom}_{\mathbb{Z}}(Z_{n-1},\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(C_n(X,\widehat{M})/B_n,\mathbb{Z}).$$

Then

$$H_G^n(X,M^?) \cong \mathrm{coker}(\mathrm{Hom}_{\mathbb{Z}}(Z_{n-1},\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(C_n(X,\widehat{M})/B_n,\mathbb{Z})).$$

The maps $\mathrm{Hom}_{\mathbb{Z}}(Z_{n-1},\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(B_{n-1},\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(C_n(X,\widehat{M})/B_n,\mathbb{Z})$ together with the section $\mathrm{Hom}_{\mathbb{Z}}(C_n(X,\widehat{M})/B_n,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(B_{n-1},\mathbb{Z})$ induce an exact sequence of cokernels identifying $\ker h$ with $\mathrm{Ext}_{\mathbb{Z}}(H_{n-1}^G(X,M_?),\mathbb{Z})$. $\square$

2 Spectral sequences for twisted equivariant K–theory

Twisted equivariant K–theory for proper and discrete actions has been defined in a variety of ways. For a torsion cocycle $\alpha \in Z^2(G, S^1)$, it is possible to define twisted equivariant K–theory in terms of finite-dimensional, so called α–twisted vector bundles, an α–twisted equivariant K–theory for proper actions of discrete groups on finite, proper G–CW complexes.

Definition 2.1 Let $\alpha \in Z^2(G, S^1)$ be a normalized torsion cocycle of order n for the discrete group G, with associated central extension $0 \to \mathbb{Z}/n \to G_\alpha \to G$. An α–twisted vector bundle is a finite-dimensional G_α–equivariant complex vector bundle such that $\mathbb{Z}/n$ acts by multiplication with a primitive root of unity. The groups ${}^{\alpha}\mathbb{K}^0_{G_\alpha}(X)$ are defined as the Grothendieck groups of the isomorphism classes of α–twisted vector bundles over X.

Given a proper G–CW complex X, define the α–twisted equivariant K–theory groups ${}^{\alpha}K^{-n}_G(X)$ as the kernel of the induced map

$$ {}^{\alpha}\mathbb{K}^0_{G_\alpha}(X \times S^n) \xrightarrow{\text{incl}^*} {}^{\alpha}\mathbb{K}^0_{G_\alpha}(X). $$

The α–twisted equivariant K–theory catches relevant information to the class of twistings coming from the torsion part of the group cohomology of the group, in the sense that the K–groups trivialize for cocycles representing nontorsion classes.

As noted in [5], there is a spectral sequence connecting the α–twisted Bredon cohomology and the α–twisted equivariant K–theory of finite proper G–CW complexes. When the twisting is discrete this spectral sequence is a special case of the Atiyah–Hirzebruch spectral sequence for *untwisted* G–cohomology theories constructed by Davis and Lück [8]. In particular, it collapses rationally.

Theorem 2.2 *Let X be a finite proper G–CW complex for a discrete group G, and let $\alpha \in Z^2(G, S^1)$ be a normalized torsion cocycle. Then there is a spectral sequence with*

$$ E_2^{p,q} = \begin{cases} H^p_G(X, \mathcal{R}^?_\alpha) & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd,} \end{cases} $$

so that $E_\infty^{p,q} \Rightarrow {}^{\alpha}K_G^{p+q}(X)$.

In [4], a definition for twisted equivariant K–theory is proposed, where the class of twistings is extended to include arbitrary elements in the third Borel cohomology group with integer coefficients $H^3(X \times_G EG, \mathbb{Z})$. Extending the work by [10], this theory gives nontrivial twisted equivariant K–theory groups for cocycles which are nontorsion. In the work [5], a spectral sequence is developed to compute the twisted equivariant K–theory under these conditions, in terms of generalizations of Bredon cohomology which capture more general twisting data. Specializing to the trivial group $\{e\}$, the spectral sequence of [5] yields explicit descriptions of both the E_2 term and the differentials of the nonequivariant Atiyah–Hirzebruch spectral sequence described in Atiyah and Segal [3]. In contrast to the spectral sequence constructed in [10], the spectral sequence in [5] does not collapse rationally in general.

3 Cohomology of $\mathrm{Sl}_3\mathbb{Z}$ and twists

We concentrate now in the example of the group $\mathrm{Sl}_3\mathbb{Z}$. This group is particularly accessible due to the existence of a convenient model for the classifying space of proper actions and the fact that the group cohomology relevant to the twists is completely determined by finite subgroups.

3.1 Twist data

We will describe the twist data, which reduce completely to torsion classes. After the work of Soulé in [18, Theorem 4, page 14] the integral cohomology of $\mathrm{Sl}_3\mathbb{Z}$ only consists of 2 and 3–torsion. The 3–primary part is isomorphic to the graded algebra

$$\mathbb{Z}[x_1, x_2] \mid 3x_1 = 3x_2 = 0$$

with both generators in degree 4.

The two-primary component is isomorphic to the graded algebra

$$\mathbb{Z}[u_1, \dots, u_7]$$

with respective degrees 3,3,4, 4,5, 6,6, subject to the relations

$$\begin{gathered}
2u_1 = 2u_3 = 4u_3 = 4u_4 = 2u_5 = 2u_6 = 2u_7 = 0,\\
u_7u_1 = u_7u_4 = u_7u_5 = u_7u_6 = u_2u_5 = u_2u_6 = 0,\\
u_7^2 + u_7u_2^2 = u_3u_4 + u_1u_5 = u_3u_6 + u_3u_1^2 = u_3u_6 + u_5^2 = 0,\\
u_1u_6 + u_4u_5 = u_3^0u_4^2 + u_6^2 = u_5u_6 + u_5u_1^2 = 0.
\end{gathered}$$

The twists in equivariant K–theory are given by the classes in $H^3(\mathrm{Sl}_3\mathbb{Z}, \mathbb{Z})$. For this reason we shall restrict to the two-primary component in the integral cohomology. In order to have a local description of these classes, we describe the cohomology of some finite subgroups inside $\mathrm{Sl}_3\mathbb{Z}$. Again [18, Theorem 4, page 14] gives the following result. There exists an exact sequence of abelian groups ($n \in \mathbb{N}$)

$$0 \to H^n(\mathrm{Sl}_3\mathbb{Z})_{(2)} \xrightarrow{\phi} H^n(S_4)_{(2)} \oplus H^n(S_4)_{(2)} \oplus H^n(S_4)_{(2)} \xrightarrow{\delta} H^n(D_4) \oplus H^n(\mathbb{Z}_2) \to 0,$$

where ϕ and δ (see [18, Corollary 2.1.b, page 9]) are determined by the following system of inclusions:

$$\begin{tikzcd}
 & & & \mathrm{Sl}_3\mathbb{Z} & & & \\
 & S_4 \arrow[urr] & & S_4 \arrow[u] & & S_4 \arrow[ull] & \\
 & & D_4 \arrow[ul, "i_2"] \arrow[ur, "i_1"'] & & C_2 \arrow[ul, "j_1"] \arrow[ur, "j_2"'] & &
\end{tikzcd}$$

We use the notation $R = i_1^*(H^*(S_4)) \cap i_2^*(H^*(S_4))$. Then, the image of the morphism ϕ: $H^*(\mathrm{Sl}_3\mathbb{Z})_{(2)} \to H^*(S_4)_{(2)} \oplus (i_1^*)^{-1}(R)$ is the set of elements (y, z) such that $j_2^*(y) = j_1^*(z)$. From Soulé's work, we know that $H^*(S_4)_{(2)} = \mathbb{Z}[y_1, y_2, y_3]$, with $2y_1 = 2y_z = 4y_3 = y_1^4 + y_2^2 y_1 + y_3 y_1^2 = 0$, and, if R is as above, then we have $(i_1^*)^{-1}(R) = \mathbb{Z}[z_1, z_2, z_3]$, with $2z_1 = 4z_2 = 2z_3 = z_3^2 + z_3 z_1^2 = 0$. Furthermore $j_2^*(y_1) = t$, $j_2^*(y_2) = 0$, $j_2^*(y_3) = t^2$, $j_1^*(z_1) = 0$, $j_1 * (z_2) = t^2$, and $j_1^*(z_3) = 0$. Then the elements $u_1 = y_2$, $u_2 = z_1$, $u_3 = y_1^2 + z_2$, $u_4 = y_1^2 + y_3$, $u_5 = y_1 y_2$, $u_6 = y_1 y_3 + y_1^3$ and $u_7 = z_3$ generate $H^*(\mathrm{Sl}_3\mathbb{Z})_{(2)}$.

In $H^3(\cdot)$ the above discussion can be summarized in the following diagram:

$$\begin{tikzcd}
 & \langle u_1, u_2 \rangle = H^3(\mathrm{Sl}_3\mathbb{Z}) \arrow[dl, "i^*"'] \arrow[d, "i^*"] \arrow[dr, "i^*"] \arrow[drr, "i^*"] & & \\
\langle z_1 \rangle \subseteq H^3(S_4) \arrow[d, "i_1^*"] & \langle z_1 \rangle \subseteq H^3(S_4) \arrow[dl, "i_2^*"] \arrow[d, "j_1^*"] & \langle y_2 \rangle \subseteq H^3(S_4) \arrow[dl, "j_2^*"] \arrow[d] & \langle y_2 \rangle \subseteq H^3(D_6) \arrow[dl] \\
\langle x_3 \rangle \subseteq H^3(D_4) \arrow[d] & 0 & \langle y_2 \rangle \subseteq H^3(D_2) & \\
\langle x_3 \rangle \subseteq H^3(D_2) & & &
\end{tikzcd} \tag{3.1}$$

There are four twists up to cohomology, namely $0, u_1, u_2, u_1 + u_2$, we will work with u_1.

3.2 A model for the classifying space of proper actions

We recall the model for the classifying space for proper actions of $\mathrm{Sl}_3\mathbb{Z}$, as described in [18], but also in [16]. Let Q be the space of real, positive definite 3×3–square matrices. Multiplication by positive scalars gives an action whose quotient space $Q/\mathbb{R}^+$ is homotopy equivalent to $\mathrm{Sl}_3\mathbb{Z}/\underline{E}\mathrm{Sl}_3\mathbb{Z}$.

We now describe its orbit space. Let C be the truncated cube of $\mathbb{R}^3$ with centre $(0,0,0)$ and side length 2, truncated at the vertices $(1,1,-1), (1,-1,1), (-1,1,1)$ and $(-1,-1,-1)$, trough the midpoints of the corresponding sides. As stated in [18], every matrix A admits a representative of the form

$$\begin{pmatrix} 2 & z & y \\ z & 2 & x \\ y & x & 2 \end{pmatrix}$$

which may be identified with the corresponding point (x,y,z) inside the truncated cube. We introduce the following notation for the vertices of the cube:

$$\begin{aligned} O &= (0,0,0), & Q &= (1,0,0) \\ M &= (1,1,1), & N &= (1,1,\tfrac{1}{2}) \\ M' &= (1,1,0), & N' &= (1,\tfrac{1}{2},-\tfrac{1}{2}) \\ P &= (\tfrac{2}{3},\tfrac{2}{3},-\tfrac{2}{3}) \end{aligned}$$

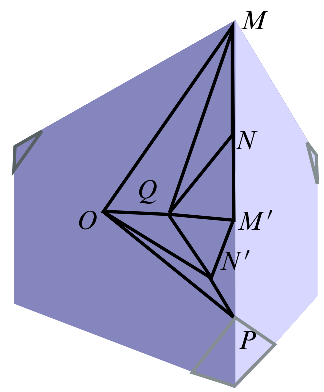

Figure 1: Triangulation for the fundamental region

Note that the elements of $\mathrm{Sl}_3\mathbb{Z}$

$$q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad q_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix},$$

send the triangle (M, N, Q) to $(M', N'Q)$ and the quadrilateral (N, N', M', Q) to (N', N, M', Q). Thus, the following identification must be performed in the quotient $M \cong M'$, $N \cong N'$, $QM \cong QM'$, $QN \cong QN'$, $MN \cong M'N' \cong M'N$ and $QMN \cong QM'N \cong QM'N'$.

Following [16] we now describe the orbits of cells and corresponding stabilizers. This can be found also in Soulé's article [18, Theorem 2] (although we use a cellular structure instead of a simplicial one). We have changed the chosen generators so that they agree with the presentations on section 4. We summarize the information in Table 1. We use the following notation: $\{1\}$ denotes the trivial group, C_n the cyclic group of n elements, D_n the dihedral group with $2n$ elements and S_n the symmetric group of permutations on n objects.

vertices				2–cells			
v_1	O	g_2, g_3	S_4	t_1	OQM	g_2	C_2
v_2	Q	g_4, g_5	D_6	t_2	$QM'N$	g_1	$\{1\}$
v_3	M	g_6, g_7	S_4	t_3	$MN'P$	g_{12}, g_{14}	$C_2 \times C_2$
v_4	N	g_6, g_8	D_4	t_4	$OQN'P$	g_5	C_2
v_5	P	g_5, g_9	S_4	t_5	$OMM'P$	g_6	C_2
edges				3–cells			
e_1	OQ	g_2, g_5	$C_2 \times C_2$	T_1	g_1	$\{1\}$	
e_2	OM	g_6, g_{10}	D_3				
e_3	OP	g_6, g_5	D_3				
e_4	QM	g_2	C_2				
e_5	QN'	g_5	C_2				
e_6	MN	g_6, g_{11}	$C_2 \times C_2$				
e_7	$M'P$	g_6, g_{12}	D_4				
e_8	$N'P$	g_5, g_{13}	D_4				

Table 1

The first column is an enumeration of equivalence classes of cells; the second lists a representative of each class; the third column gives generating elements for the stabilizer of the given representative; and the last one is the isomorphism type of the stabilizer. The generating elements referred to above are

$$g_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad g_2=\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix},\quad g_3=\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}$$

$$g_4=\begin{pmatrix}-1&0&0\\0&1&1\\0&0&-1\end{pmatrix},\quad g_5=\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad g_6=\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{pmatrix},$$

$$g_7=\begin{pmatrix}0&0&-1\\-1&0&0\\1&1&1\end{pmatrix},\quad g_8=\begin{pmatrix}-1&0&0\\0&1&0\\0&-1&-1\end{pmatrix},\quad g_9=\begin{pmatrix}0&0&-1\\-1&0&-1\\0&1&1\end{pmatrix},$$

$$g_{10}=\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix},\quad g_{11}=\begin{pmatrix}-1&0&0\\0&-1&0\\1&1&1\end{pmatrix},\quad g_{12}=\begin{pmatrix}0&-1&-1\\0&-1&0\\-1&1&0\end{pmatrix},$$

$$g_{13}=\begin{pmatrix}0&1&1\\1&0&1\\0&0&-1\end{pmatrix},\quad g_{14}=\begin{pmatrix}-1&0&0\\-1&0&-1\\1&-1&0\end{pmatrix}.$$

Finally we describe the cells coboundary, we fix an orientation; namely, the ordering of the vertices $O<Q<M<M'<N<N'<P$ induces an orientation in E and also in $\underline{B}\mathrm{Sl}_3\mathbb{Z}=E/\equiv$. We resume this in Figure 1.

4 Representation theoretical input

4.1 Cyclic group C_2

The cyclic group has no third-dimensional integer cohomology, hence the twisted representation ring coincides with the usual one. The character table is as follows:

C_2	1	g_i
ρ_1	1	-1
ρ_2	1	-1

4.2 Dihedral group $D_n=\langle g_i,g_j\rangle=\langle g_i,g_j\mid g_i^2=g_j^2=(g_ig_j)^n=1\rangle$

The dihedral group of order six has only the trivial class in 3–dimensional integer cohomology. Thus the projective representations do agree with the linear ones. In the even case, the subgroups inside $\mathrm{Sl}_3\mathbb{Z}$ are $C_2\times C_2=D_2$ and D_4.

The following is the linear character table for D_n:

D_n	$\langle (g_i, g_j)^k \rangle$	$\langle g_j (g_i g_j)^k \rangle$
ξ_1	1	1
ξ_2	1	-1
$\widehat{\xi}_3$	-1^k	-1^k
$\widehat{\xi}_4$	-1^k	-1^{k+1}
$\widehat{\phi}_p$	$2\cos(2\pi pk/n)$	0

where $0 \le k \le n-1$, p varies from 1 to $(n/2)-1$ (n even) or $(n-1)/2$ (n odd) and the hat denotes a representation which only appears in the case n is even. Most of the information is taken from [14, Chapter 5, Section 7], especially Theorem 7.1 and Corollary 7.2 on pages 258–261. The group $H^2(D_n, S^1)$ is isomorphic to $\mathbb{Z}/2$ if n is even and 0 if n is odd. From now on we concentrate in the case n even. For simplicity consider the presentation

$$D_n = \langle a, b \mid a^n = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle.$$

Given a primitive n^{th} root of unity $\epsilon \in S^1$, one can normalize a nontrivial cocycle $\alpha\colon D_n \times D_n \to S^1$ to one satisfying $\alpha(a^i, a^j b^k) = 1$ and $\alpha(a^i b, a^j b^k) = \epsilon^j$. For $r \in \{1, 2 \dots, n/2\}$ let

$$A_r = \begin{pmatrix} \epsilon^r & 0 \\ 0 & \epsilon^{1-r} \end{pmatrix}, \quad B_r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, the irreducible α–twisted representations of D_n are given by $\rho_r(a^i b_j) = A_r^i B_r^j$ for $i \in \{0, \dots, n-1\}$ $j = 0, 1$, and $r \in \{1, \dots, n/2\}$. The projective representations ρ_j for $j \in \{1, \dots, \frac{n}{2}\}$ are nonequivalent irreducible projective representations.

Consider the group

$$D_2^* = \langle h_1, h_3, z \rangle$$

which is isomorphic to the quaternions. A linear character table is given by:

D_2^*	1	z	$\{h_1, h_1^{-1}\}$	$\{h_3, h_3^{-1}\}$	$\{h_1 h_3, (h_1 h_3)^{-1}\}$
η_1	1	1	1	1	1
η_2	1	1	1	-1	-1
η_3	1	1	-1	1	-1
η_4	1	1	-1	-1	1
η_5	2	-2	0	0	0

The linear character table for a Schur covering group of D_6 is

D_6^*	1	z	a	za	a^2	za^2	a^3	b	ab
γ_1	1	1	1	1	1	1	1	1	1
γ_2	1	1	1	1	1	1	1	-1	1
γ_3	1	1	-1	-1	1	1	-1	1	-1
γ_4	1	1	-1	-1	1	1	-1	-1	1
γ_5	2	2	-1	-1	-1	-1	-2	0	0
γ_6	2	2	-1	-1	-1	-1	2	0	0
γ_7	2	-2	$\frac{3+i\sqrt{3}}{2}$	$-\frac{3+i\sqrt{3}}{2}$	$\frac{1+i\sqrt{3}}{2}$	$-\frac{1+i\sqrt{3}}{2}$	0	0	0
γ_8	2	-2	0	0	$-1-i\sqrt{3}$	$1+i\sqrt{3}$	0	0	0
γ_9	2	-2	$-\frac{3+i\sqrt{3}}{2}$	$\frac{3+i\sqrt{3}}{2}$	$\frac{1+i\sqrt{3}}{2}$	$-\frac{1+i\sqrt{3}}{2}$	0	0	0

4.3 Symmetric group S_4

The projective representation theory of the Symmetric groups goes back to the foundational work of Schur [17]. The information concerning the representation theory of the Symmetric group in four letters is taken from Hoffman and Humphreys [13, page 46] and [14, pages 215–243]. Recall that the conjugacy classes inside the group S_4 are determined by their cycle type. The cycle type of a permutation π is a sequence $(1^{a_1}, 2^{a_2}, \dots, k^{a_k})$, where the cycle factorization of π contains a_i–cycles of length i.

The symmetric group admits the presentation

$$S_4 = \langle g_1, g_2, g_3 \mid g_i^2 = (g_j g_{j+1})^3 = (g_k g_l)^2 = 1, \quad 1 \le i \le 2,\ j = 1,\ k \le l-2, \rangle,$$

where g_i is the transposition given by $(i-1, i)$.

The linear character table of S_4 is as follows:

S_4	(1^4)	$(2,1^2)$	$(3,1)$	(4)	(2^2)
θ_1	1	1	1	1	1
θ_2	1	-1	1	-1	1
θ_3	2	0	-1	0	2
θ_4	3	1	0	-1	-1
θ_5	3	-1	0	1	-1

The representations θ_1 and θ_2 are induced from the 1–dimensional trivial, respectively the sign representation. θ_3 is obtained from the 2–dimensional representation of the quotient group S_3 the character θ_4 is given as the $\xi - \theta_1$, where ξ is induced from the S_3–trivial representation, and the character θ_5 is the character assigned to the representation $g \mapsto \theta_2(g)V_4(g)$, where V_4 is the irreducible representation associated with the character θ_4. The linear character table of a Schur covering group S_4^* is obtained in [14, page 254] by considering the group with the presentation

$$S_4^* = \langle h_1, h_2, h_3, z \mid h_i^2 = (h_j h_{j+1})^3 = (h_k h_l)^2 = z, z^2 = [z, h_i] = 1, \\ 1 \le i \le 3,\ j = 1,\ k \le l-2 \rangle$$

and the central extension

$$1 \to \langle z \rangle \to S_4^* \xrightarrow{f} S_4 \to 1,$$

given by $f(h_i) = g_i$, as well as the choice of representatives of regular conjugacy classes as below,

S_4^*	(1^4)	$(1^4)'$	$(2,1^2)$	(2^2)	$(3,1)$	$(3,1)'$	(4)	$(4)'$
ϵ_1	1	1	1	1	1	1	1	1
ϵ_2	1	1	-1	1	1	1	-1	-1
ϵ_3	2	2	0	2	-1	-1	0	0
ϵ_4	3	3	1	-1	0	0	-1	-1
ϵ_5	3	3	-1	-1	0	0	1	1
ϵ_6	2	-2	0	0	1	-1	$\sqrt{2}$	$-\sqrt{2}$
ϵ_7	2	-2	0	0	1	-1	$-\sqrt{2}$	$\sqrt{2}$
ϵ_8	4	-4	0	0	-1	1	0	0

where the first five lines are characters associated to S_4, ϵ_6 is the *Spin* representation.

5 Untwisted equivariant K–theory

In this section we use the Atiyah–Hirzebruch spectral sequence to calculate the equivariant K–theory groups $K^*_{\mathrm{Sl}_3\mathbb{Z}}(\underline{E}\,\mathrm{Sl}_3\mathbb{Z})$. The cochain complex associated to the $\mathrm{Sl}_3\mathbb{Z}$–CW complex structure of $\underline{E}\,\mathrm{Sl}_3\mathbb{Z}$ described in Section 3 is

$$0 \to \bigoplus_{k=1}^{5} R(\mathrm{stab}(v_k)) \xrightarrow{\Phi_1} \bigoplus_{j=1}^{8} R(\mathrm{stab}(e_j)) \xrightarrow{\Phi_2} \bigoplus_{i=1}^{5} R(\mathrm{stab}(t_i)) \xrightarrow{\Phi_3} R(\mathrm{stab}(T)) \to 0,$$

where the Φ_i is the coboundary given by restriction over representations rings.

Frobenius reciprocity implies that if H' is a subgroup of H and

$$\uparrow_H^{\prime H}\colon R(H') \to R(H)$$

is represented by a matrix A (in the basis given by irreducible representations), then

$$\downarrow'^{H}_{H}\colon R(H) \to R(H')$$

is represented by the transpose matrix A^T (in the same basis). Given A_i are the matrix representing the morphism for K–homology calculated in [16], one has that A_i^T are the matrices representing the morphism Φ_i.

We calculate the elementary divisors of the matrices representing the morphism Φ_i. Using that we obtain

$$H^p(\underline{E}\,\mathrm{Sl}_3\mathbb{Z}, \mathcal{R}) = 0 \quad \text{if } p > 0,$$
$$H^0(\underline{E}\,\mathrm{Sl}_3\mathbb{Z}, \mathcal{R}) \cong \mathbb{Z}^{\oplus 8}.$$

As the Bredon cohomology concentrates at low degree, using the same argument that in [16] we conclude that

$$K^0_{\mathrm{Sl}_3\mathbb{Z}}(\underline{E}\,\mathrm{Sl}_3\mathbb{Z}) \cong \mathbb{Z}^{\oplus 8},$$
$$K^1_{\mathrm{Sl}_3\mathbb{Z}}(\underline{E}\,\mathrm{Sl}_3\mathbb{Z}) = 0.$$

This agrees with the result predicted by Theorem 1.13.

6 Twisted equivariant K–theory

In this section we use the spectral described in Theorem 2.2 to calculate the equivariant twisted K–theory groups ${}^{u_1}K_{\mathrm{Sl}_3\mathbb{Z}}(\underline{E}\,\mathrm{Sl}_3\mathbb{Z})$.

The cochain complex associated to the $\mathrm{Sl}_3\mathbb{Z}$–CW complex structure of $\underline{E}\,\mathrm{Sl}_3\mathbb{Z}$ described in Section 3 is

$$0 \to \bigoplus_{k=1}^{5} R_{u_1}(\mathrm{stab}(v_k)) \xrightarrow{\Phi_1} \bigoplus_{j=1}^{8} R_{u_1}(\mathrm{stab}(e_j))$$
$$\xrightarrow{\Phi_2} \bigoplus_{i=1}^{5} R_{u_1}(\mathrm{stab}(t_i)) \xrightarrow{\Phi_3} R_{u_1}(\mathrm{stab}(T)) \to 0.$$

As the class u_1 restricts to 0 at the level of 2–cells and 3–cells, we have the following result.

Proposition 6.1 *For $* = 3$, we have a natural isomorphism*

$$H^*(\underline{E}\,\mathrm{Sl}_3\mathbb{Z}; \mathcal{R}_{u_1}) \cong H^*(\underline{E}\,\mathrm{Sl}_3\mathbb{Z}; \mathcal{R}).$$

We only have to determine Φ_1 and Φ_2, because Φ_3 was determined in Section 5.

6.1 Determination of Φ_1

We know that the class u_1 comes from the subgroup $\mathrm{stab}(v_1) = \langle g_2, g_3 \rangle \cong S_4$, from Section 3. We have an inclusion

$$\begin{aligned} \mathrm{stab}(e_1) &\xrightarrow{i} \mathrm{stab}(v_1), \\ \langle g_2, g_5 \rangle &\to \langle g_2, g_3 \rangle, \\ g_2 &\mapsto g_2, \\ g_5 &\mapsto g_3 g_2 g_3^{-1} g_2 g_3. \end{aligned}$$

Consider the central extension associated to a representing cocycle the unique nontrivial 3–cohomology class in S_4 (say $u_{1|}$) and its restriction to $C_2 \times C_2$:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & S_4^* \arrow[r] & S_4 \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[u] & G^* \arrow[r] \arrow[u, "j"] & C_2 \times C_2 \arrow[r] \arrow[u, "i"] & 0 \end{tikzcd}$$

To evaluate the morphism $i^*\colon R_{u_{1|}}(S_4) \to R_{u_{1|}}(C_2 \times C_2)$ is equivalent to determine the morphism $j^*\colon R(S_4^*) \to R(G^*)$.

We have a description of S_4^* in terms of generators and relations given in Section 4. In terms of these generators the group G^* can be described as

$$D_2^* = \langle h_1, h_3, z \rangle$$

and the homomorphism $j\colon D_2^* \to S_4^*$ is the inclusion.

Using the character tables from Section 4, the following fact is straightforward:

$$\begin{aligned} j^*(\epsilon_1) &= \eta_1, \\ j^*(\epsilon_2) &= \eta_4, \\ j^*(\epsilon_3) &= \eta_1 + \eta_4, \\ j^*(\epsilon_4) &= \eta_1 + \eta_2 + \eta_3, \\ j^*(\epsilon_5) &= \eta_2 + \eta_3 + \eta_4, \\ j^*(\epsilon_6) &= \eta_5, \\ j^*(\epsilon_7) &= \eta_5, \\ j^*(\epsilon_8) &= 2\eta_5. \end{aligned}$$

The matrix corresponding to the morphism $i^*\colon R_{u_1|}(\mathrm{stab}(v_1)) \to R_{u_1|}(\mathrm{stab}(e_1))$ is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

As the class u_1 restricts nontrivially to $\mathrm{stab}(e_1)$, we have to determine the morphism $i^*\colon R_{u_1|}(\mathrm{stab}(v_2)) \to R_{u_1|}(\mathrm{stab}(e_1))$.

Using the above character table one has

$$\begin{aligned} i^*(\gamma_1) &= \eta_1, \\ i^*(\gamma_2) &= \eta_4, \\ i^*(\gamma_3) &= \eta_3, \\ i^*(\gamma_4) &= \eta_2, \\ i^*(\gamma_5) &= \eta_2 + \eta_3, \\ i^*(\gamma_6) &= \eta_1 + \eta_4, \\ i^*(\gamma_7) &= \eta_5, \\ i^*(\gamma_8) &= \eta_5, \\ i^*(\gamma_9) &= \eta_5. \end{aligned}$$

The matrix corresponding to the morphism $i^*\colon R_{u_1|}(\mathrm{stab}(v_2)) \to R_{u_1|}(\mathrm{stab}(e_1))$ is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Now, we have to determine the morphisms

$$R_{u_1|}(\mathrm{stab}(v_1)) \to R_{u_1|}(\mathrm{stab}(e_i))$$

with $i=2,3$; note that $R_{u_1|}(\mathrm{stab}(e_i)) \cong R(\mathrm{stab}(e_i))$ because $H^3(D_3;\mathbb{Z})$ is trivial.

The inclusion $\mathrm{stab}(e_2) \to \mathrm{stab}(v_1)$ is given by

$$\begin{aligned}\langle g_6, g_{10}\rangle &\to \langle g_2, g_3\rangle,\\ g_6 &\mapsto g_3g_2g_3^{-1},\\ g_{10} &\mapsto g_2g_3g_2g_3^{-1}.\end{aligned}$$

This map induces a map $i\colon \mathrm{stab}(e_2)^* \to \mathrm{stab}(v_1)^*$, where G^* denotes the inverse image of $G \subseteq S_4$ by the covering map $S_4^* \to S_4$. Is easy to see that $i(g_6) \sim g_2$, $i(g_{10}) \sim zg_2$ and $i(g_6g_{10}) \sim zg_2g_3$. Note that S_4 has three nonisomorphic irreducible projective characters, namely ϵ_6, ϵ_7 and ϵ_8. We have that

$$i^*(\epsilon_6)(1) = 2, \quad i^*(\epsilon_6)(g_6) = 0, \quad i^*(\epsilon_6)(g_6g_{10}) = -1,$$

and then $i^*(\epsilon_6) = \lambda_3$. In the same way we obtain $i^*(\epsilon_7) = \lambda_3$. On the other hand

$$i^*(\epsilon_8)(1) = 4, \quad i^*(\epsilon_8)(g_6) = 0, \quad i^*(\epsilon_8)(g_6g_{10}) = 1,$$

and then $i^*(\epsilon_8) = \lambda_1 + \lambda_2 + \lambda_3$. Combining the above results with the obtained in [16] for linear characters we obtain that the matrix corresponding to the morphism

$$R_{u_1|}(\mathrm{stab}(v_1)) \to R(\mathrm{stab}(e_2))$$

is given by:

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & 1 \end{pmatrix}$$

Now we determine the morphism

$$i^* R_{u_1|}(\mathrm{stab}(v_1)) \to R(\mathrm{stab}(e_3)).$$

The inclusion $\mathrm{stab}(e_3) \to \mathrm{stab}(v_1)$ is given by

$$\begin{aligned} \langle g_6, g_5\rangle &\to \langle g_2, g_3\rangle, \\ g_6 &\mapsto g_3 g_2 g_3^{-1}, \\ g_5 &\mapsto g_3 g_2 g_3^{-1} g_2 g_3. \end{aligned}$$

This map induces a map $i\colon \mathrm{stab}(e_3)^* \to \mathrm{stab}(v_1)^*$. Is easy to see that $i(g_6) \sim g_2$, and $i(g_6 g_5) \sim z g_2 g_3$. We have that

$$i^*(\epsilon_6)(1) = 2, \quad i^*(\epsilon_6)(g_6) = 0, \quad i^*(\epsilon_6)(g_6 g_5) = -1,$$

and then $i^*(\epsilon_6) = \lambda_3$. In the same way we obtain $i^*(\epsilon_7) = \lambda_3$. On the other hand

$$i^*(\epsilon_8)(1) = 4, \quad i^*(\epsilon_8)(g_6) = 0, \quad i^*(\epsilon_8)(g_6 g_{10}) = 1,$$

and then $i^*(\epsilon_8) = \lambda_1 + \lambda_2 + \lambda_3$. Combining the above results with the obtained in [16] for linear characters we obtain that the matrix corresponding to the morphism

$$R_{u_1|}(\mathrm{stab}(v_1)) \to R(\mathrm{stab}(e_3))$$

is given by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We have to determine now the morphism

$$R_{u_1|}(\mathrm{stab}(v_2)) \to R(\mathrm{stab}(e_i)),$$

for $i = 4, 5$; note that $R_{u_1|}(\mathrm{stab}(e_i)) \cong R(\mathrm{stab}(e_i))$ because $H^3(C_2; \mathbb{Z})$ is trivial.

The inclusion $\mathrm{stab}(e_4) \to \mathrm{stab}(v_2)$ is given by

$$\begin{aligned} \langle g_6\rangle &\to \langle g_4, g_5\rangle, \\ g_6 &\mapsto g_5 (g_4 g_5)^3. \end{aligned}$$

This map induces a map $i\colon \mathrm{stab}(e_4)^* \to \mathrm{stab}(v_2)^*$. It is clear that $i(g_6) \sim ab$, in the notation used for character table of D_6^*,

$$i^*(\gamma_7)(1) = 2, \quad i^*(\gamma_7)(g_6) = 0,$$

and then $i^*(\gamma_7) = \rho_1 + \rho_2$, in the same way we obtain $i^*(\gamma_8) = \rho_1 + \rho_2$ and $i^*(\gamma_9) = \rho_1 + \rho_2$. Combining the above results with the obtained in [16] for linear characters we obtain that the matrix corresponding to the morphism

$$R_{u_1|}(\mathrm{stab}(v_2)) \to R(\mathrm{stab}(e_4))$$

is given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The inclusion $\mathrm{stab}(e_5) \to \mathrm{stab}(v_2)$ is given by

$$\begin{gathered} \langle g_5 \rangle \to \langle g_4, g_5 \rangle, \\ g_5 \mapsto g_5. \end{gathered}$$

This map induces a map $i\colon \mathrm{stab}(e_5)^* \to \mathrm{stab}(v_2)^*$. Again in the notation used for the character table of D_6^*, $i(g_5) \sim b$. We have that

$$i^*(\gamma_7)(1) = 2, \quad i^*(\gamma_7)(g_5) = 0,$$

and then $i^*(\gamma_7) = \rho_1 + \rho_2$, in the same way we obtain $i^*(\gamma_8) = \rho_1 + \rho_2$ and $i^*(\gamma_9) = \rho_1 + \rho_2$. Combining the above results with the obtained in [16] for linear characters we obtain that the matrix corresponding to the morphism

$$R_{u_1|}(\mathrm{stab}(v_2)) \to R(\mathrm{stab}(e_4))$$

is given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The elementary divisors of the matrix representing the morphism ϕ are 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1. The rank of this matrix is 19.

6.2 Determination of Φ_2

In order to determine Φ_2 notice that the unique morphisms that differ from the untwisted case are $i^*\colon\ R_{u_1|}(\mathrm{stab}(e_1)) \to R_{u_1|}(\mathrm{stab}(t_1))$ and $i^*\colon\ R_{u_1|}(\mathrm{stab}(e_1)) \to R_{u_1|}(\mathrm{stab}(t_4))$ in both cases we have that $i^*(\eta_5) = \rho_1 + \rho_2$. We obtain that the elementary divisors of the matrix representing the morphism ϕ are 1, 1, 1, 1, 1, 1, 1, 1. The rank of this matrix is 10.

As the cochain complexes involved are free, the computation of ranks and elementary divisors yield the following.

Theorem 6.2 *The Bredon cohomology with coefficients in twisted representations satisfies*

$$H^p(\underline{E}\,\mathrm{Sl}_3\mathbb{Z}, \mathcal{R}_{u_1}) = 0 \quad \text{if } p > 0, \qquad H^0(\underline{E}\,\mathrm{Sl}_3\mathbb{Z}, \mathcal{R}_{u_1}) \cong \mathbb{Z}^{\oplus 13}.$$

Since the Bredon cohomology concentrates at low degree, the spectral sequence described in Theorem 2.2 collapses at level 2 and we conclude the following.

Theorem 6.3 *We have*

$$\begin{aligned} {}^{u_1}K^0_{\mathrm{Sl}_3\mathbb{Z}}(\underline{E}\,\mathrm{Sl}_3\mathbb{Z}) &\cong \mathbb{Z}^{\oplus 13}, \\ {}^{u_1}K^1_{\mathrm{Sl}_3\mathbb{Z}}(\underline{E}\,\mathrm{Sl}_3\mathbb{Z}) &= 0. \end{aligned}$$

7 Twisted equivariant K–homology and the relation to the Baum–Connes conjecture with coefficients

The Baum–Connes conjecture [6; 15] predicts for a discrete group G the existence of an isomorphism

$$\mu_i\colon\ K_i^G(\underline{E}G) \to K_i(C_r^*(G))$$

given by the (analytical) assembly map, where $C_r^*(G)$ is the reduced C^*–algebra of the group G.

More generally, given any G–C^*–algebra, the Baum–Connes conjecture with coefficients predicts that a map

$$\mu_i\colon\ K_i^G(\underline{E}G, A) \to K_i(A \rtimes G)$$

is an isomorphism, where $K_i^G(\underline{E}G, A)$ is defined in terms of equivariant and bivariant *KK*–groups,

$$K_*^G(\underline{E}G, A) = \operatorname*{colim}_{G\text{–compact} X \subset \underline{E}G} KK_*(C_0(X), A),$$

and $A \rtimes G$ denotes the crossed product C^*–algebra. See Chabert and Echterhoff [7] and Echterhoff [11] for more details.

The class of twists considered in the example of $\mathrm{Sl}_3\mathbb{Z}$ let define a particular choice of coefficients for this assembly map.

Definition 7.1 Let G be a discrete group. Given a cocycle $\omega \in Z^2(G, S^1)$, an ω–representation on a Hilbert space $\mathcal{H}$ is a map $V\colon G \to \mathcal{U}(\mathcal{H})$ which satisfies $v(s)V(t) = \omega(s,t)V(st)$.

Composing with the quotient map $\mathcal{U}(\mathcal{H}) \to PU(\mathcal{H}) = \mathcal{U}(\mathcal{H})/S^1$, together with the identification of the group $PU(\mathcal{H})$ as the automorphisms of the C^*–algebra of compact operators on $\mathcal{H}$, denoted by $\mathcal{K}$ gives an action of G on $\mathcal{K}$. This algebra is denoted as $\mathcal{K}_\omega$. Theorem 1.13 together with the Atiyah–Hirzebruch spectral sequence yield the following.

Corollary 7.2 *Let G be a discrete group with a finite model for $\underline{E}G$ and also let $\omega \in Z^2(G, S^1)$. Assume that the Bredon homology groups $H_*^G(X, R^{-\alpha})$ are all free abelian and are concentrated in degree 0 and 1. Then there exists a duality isomorphism*

$$K_G^*(C_0(\underline{E}G), K_{-\omega}) \longrightarrow K_G^*(K_\omega, C_0(\underline{E}G)).$$

Related duality isomorphisms have been deduced for almost connected groups by Echterhoff [11] using methods from equivariant Kasparov *KK*–theory (particularly the Dirac–dual Dirac method) and positive results for the Baum–Connes conjecture with coefficients [12].

As a consequence of Theorem 1.13, and Theorem 6.2, the α–twisted Bredon homology of $\underline{E}\mathrm{Sl}_3\mathbb{Z}$ is given by

$$H_p(\underline{E}\mathrm{Sl}_3\mathbb{Z}, \mathcal{R}_{u_1}) = 0 \quad \text{if } p > 0, \quad H_0(\underline{E}\mathrm{Sl}_3\mathbb{Z}, \mathcal{R}_{u_1}) \cong \mathbb{Z}^{\oplus 13}.$$

A spectral sequence argument and the particular shape of the twists let us conclude that these groups agree with the equivariant K–homology groups with coefficients in $\mathcal{K}_{u_1}$, appearing in the left hand side of the conjecture with coefficients.

Corollary 7.3 *The equivariant K–homology groups of $\mathrm{Sl}_3\mathbb{Z}$ with coefficients in the $\mathrm{Sl}_3\mathbb{Z}$–C^* algebra $\mathcal{K}_{u_1}$ are given as follows:*

$$K_p^{\mathrm{Sl}_3\mathbb{Z}}(\underline{E}\mathrm{Sl}_3\mathbb{Z}, \mathcal{K}_{u_1}) = 0 \quad p \text{ odd}, \quad K_p^{\mathrm{Sl}_3\mathbb{Z}}(\underline{E}\mathrm{Sl}_3\mathbb{Z}, \mathcal{K}_{u_1}) \cong \mathbb{Z}^{\oplus 13} \quad p \text{ even}.$$

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