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On compact hyperbolic manifolds of Euler characteristic two

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We prove that for $n > 4$ there is no compact arithmetic hyperbolic n–manifold whose Euler characteristic has absolute value equal to 2. In particular, this shows the nonexistence of arithmetically defined hyperbolic rational homology n–spheres with n even and different than 4.

22E40; 55C35, 51M25

Dedicated to the memory of Colin Maclachlan

1 Main result and discussion

1.1 Smallest hyperbolic manifolds

Let $\mathbb{H}^n$ be the hyperbolic n–space. By a *hyperbolic n–manifold* we mean an orientable manifold $M = \Gamma\backslash\mathbb{H}^n$, where Γ is a torsion free discrete subgroup $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$. The set of volumes of hyperbolic n–manifolds being well ordered, it is natural to try to determine for each dimension n the hyperbolic manifolds of smallest volume. For $n = 3$ this problem has recently been solved by Gabai, Meyerhoff and Milley in [15], the smallest volume being achieved by a unique compact manifold, the Weeks manifold. When n is even the volume is proportional to the Euler characteristic, and this allows us to formulate the problem in terms of finding the hyperbolic manifolds M with smallest $|\chi(M)|$. In particular this observation solves the problem in the case of surfaces. For $n > 3$, noncompact hyperbolic n–manifolds M with $|\chi(M)| = 1$ have been found for $n = 4, 6$; see Everitt, Ratcliffe and Tschantz [13].

In the present paper we consider the case of compact manifolds of even dimension. In particular, such manifolds have even Euler characteristic (see Kellerhals and Zehrt [17, Theorem 1.2]). We restrict ourselves to the case of *arithmetic* manifolds, where Prasad's formula [20] can be used to study volumes. We complete the proof of the following result.

Theorem 1 *Let $n > 5$. There is no compact arithmetic manifold $M = \Gamma\backslash\mathbb{H}^n$ with $|\chi(M)| = 2$.*

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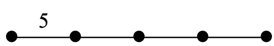
The result for $n > 10$ already follows from the work of Belolipetsky [2; 3], also based on Prasad's volume formula. More precisely, Belolipetsky determined the smallest Euler characteristic $|\chi(\Gamma)|$ for arithmetic orbifold quotients $\Gamma\backslash\mathbb{H}^n$ (n even). This smallest value grows fast with the dimension n, and for compact quotients we have $|\chi(\Gamma)| > 2$ for $n > 10$. That the result of nonexistence holds for n high enough is already a consequence of Borel and Prasad's general finiteness result [7], which was the first application of Prasad's formula. The proof of Theorem 1 for $n = 6, 8, 10$ requires a more precise analysis of the Euler characteristic of arithmetic subgroups $\Gamma \subset \mathrm{PO}(n,1)$, and in particular of the special values of Dedekind zeta functions that appear as factors of $\chi(\Gamma)$.

For $n = 4$, the corresponding problem is not solved, but there is the following result [3].

Theorem 2 (Belolipetsky) *If $M = \Gamma\backslash\mathbb{H}^4$ is a compact arithmetic manifold with $\chi(M) \le 16$, then Γ arises as a (torsion free) subgroup of the following hyperbolic Coxeter group:*

(1) $W_1 =$ 

An arithmetic (orientable) hyperbolic 4–manifold of Euler characteristic 16 was first constructed by Conder and Maclachlan in [11], using the presentation of W_1 to obtain a torsion free subgroup with the help of a computer. Further examples with $\chi(M) = 16$ have been obtained by Long in [18] by considering a homomorphism from W_1 onto the finite simple group $\mathrm{PSp}_4(4)$.

1.2 Hyperbolic homology spheres

Our original motivation for Theorem 1 was the problem of existence of hyperbolic homology spheres. A *homology n–sphere* (resp. *rational* homology n–sphere) is a n–manifold M that possesses the same integral (resp. rational) homology as the n–sphere S^n. This forces M to be compact and orientable.

Rational homology n–spheres M have $\chi(M) = 2$ if n is even. On the other hand, for $M = \Gamma\backslash\mathbb{H}^n$ with $n = 4k + 2$ we have $\chi(M) < 0$ (cf Serre [25, Proposition 23]), and this excludes the possibility of hyperbolic rational homology spheres for those dimensions. For n even, Wang's finiteness theorem [28] implies that there is only a finite number of hyperbolic rational homology n–spheres. Theorem 1 shows the nonexistence of arithmetic rational homology spheres for $n > 5$ even.

For odd dimensions, $\chi(M)=0$ and *a priori* the volume is not a limitation for the existence of hyperbolic (rational) homology spheres. In fact, an infinite tower of covers by hyperbolic integral homology 3–spheres has been constructed by Baker, Boileau and Wang in [1]. In [8] Calegari and Dunfield constructed an infinite tower of hyperbolic rational homology 3–spheres that are arithmetic and obtained by congruence subgroups. Note that a recent conjecture of Bergeron and Venkatesh predicts a lot of torsion in the homology groups of such a "congruence tower" of arithmetic n–manifolds with n odd [5].

1.3 Locally symmetric homology spheres

Instead of considering hyperbolic homology spheres, one can more generally look for homology spheres that are locally isometric to a given symmetric space of nonpositive nonflat sectional curvature. Such a symmetric space X is said to be of noncompact type, and it is classical that X can be written as G/K, where G is a connected real semisimple Lie group with trivial center with $K \subset G$ a maximal compact subgroup. Moreover, G identifies as a finite index subgroup in the group of isometries of X (of index two if G is simple).

Let us explain why the case $X=\mathbb{H}^n$ is the main source of locally symmetric rational homology spheres (among X of noncompact type). Let M be a compact orientable manifold locally isometric to X. Then M can be written as $\Gamma\backslash X$, where $\Gamma\cong\pi_1(M)$ is a discrete subgroup of isometries of X. We will suppose that $\Gamma\subset G$, for G as above. Let X_u be the compact dual of X. We have the following general result (see Borel [6, Sections 3.2 and 10.2]).

Proposition 3 *There is an injective homomorphism $H^j(X_u,\mathbb{C})\to H^j(\Gamma\backslash X,\mathbb{C})$, for each j.*

In particular, if $\Gamma\backslash X$ is a rational homology sphere, then so is X_u. Note that the compact dual of $X=\mathbb{H}^n$ is the genuine sphere S^n. By looking at the classification of compact symmetric spaces, Johnson showed the following in [16, Theorem 7].

Corollary 4 *If $M=\Gamma\backslash X$ is a rational homology n–sphere with $\Gamma\subset G$, then X is either the hyperbolic n–space $\mathbb{H}^n$ (with $n\neq 4k+2$), or $X=\mathrm{PSL}_3(\mathbb{R})/\mathrm{PSO}(3)$ (which has dimension 5).*

Proposition 3 shows that the correct problem to look at — rather than homology spheres — is the existence of locally symmetric spaces $\Gamma\backslash X$ with the same (rational) homology as the compact dual X_u. When X is the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$, the compact dual is the projective plane $\mathbb{P}^2_{\mathbb{C}}$, and the quotients $\Gamma\backslash X$ are compact complex surfaces called *fake projective planes*. Their classification was recently obtained by the work of Prasad and Yeung [21], together with Cartwright and Steger [9] who performed the necessary computer search. Later, Prasad and Yeung also considered the problem of the existence of more general arithmetic fake Hermitian spaces [22; 23].

The present paper uses the same methodology as in Prasad and Yeung's work, the main ingredient being the volume formula.

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2 Proof of Theorem 1

Let $G = \mathrm{PO}(n,1)^\circ \cong \mathrm{Isom}^+(\mathbb{H}^n)$ and consider the universal covering $\phi\colon \mathrm{Spin}(n,1) \to G$. For our purpose it will be easier to work with lattices in $\mathrm{Spin}(n,1)$. A lattice $\overline{\Gamma} \subset G$ is arithmetic exactly when $\Gamma = \phi^{-1}(\overline{\Gamma})$ is an arithmetic subgroup of $\mathrm{Spin}(n,1)$. Since the covering ϕ is twofold, we have $\chi(\Gamma) = \frac{1}{2}\chi(\overline{\Gamma})$, where χ is the Euler characteristic in the sense of Wall. In particular, if $M = \overline{\Gamma}\backslash\mathbb{H}^n$ is a manifold with $|\chi(M)| = 2$, then $|\chi(\Gamma)| = 1$. Thus, Theorem 1 is an obvious consequence of the following proposition. The proof relies on the description of arithmetic subgroups with the help of Bruhat–Tits theory, as done for instance in [7; 20]. An introduction can be found in the author's work [12]. We also refer to [27] for the needed facts from Bruhat–Tits theory.

Proposition 5 *Let $n > 4$. There is no cocompact arithmetic lattice $\Gamma \subset \mathrm{Spin}(n,1)$ such that $\chi(\Gamma)$ is a reciprocal integer, ie, such that $\chi(\Gamma) = 1/q$ for some $q \in \mathbb{Z}$.*

Proof We can assume that n is even. Let $\Gamma \subset \mathrm{Spin}(n,1)$ be a cocompact lattice. Clearly, it suffices to prove the proposition for Γ maximal. In this case, Γ can be written as the normalizer $\Gamma = N_{\mathrm{Spin}(n,1)}(\Lambda)$ of some *principal* arithmetic subgroup Λ (see [7, Proposition 1.4]). By definition, there exists a number field $k \subset \mathbb{R}$ and a k–group $\boldsymbol{G}$ with $\boldsymbol{G}(\mathbb{R}) \cong \mathrm{Spin}(n,1)$ such that $\Lambda = \boldsymbol{G}(k) \cap \prod_{v \in V_{\mathrm{f}}} P_v$, for some coherent collection $(P_v)_{v \in V_{\mathrm{f}}}$ of parahoric subgroups $P_v \subset \boldsymbol{G}(k_v)$ (indexed by the set V_{f} of finite places of k). It follows from the classification of algebraic groups (cf Tits [26]) that $\boldsymbol{G}$ is of type B_r with $r = n/2$ (> 2), the field k is totally real, and (using Godement's criterion) $k \neq \mathbb{Q}$. Let us denote by d the degree $[k : \mathbb{Q}]$.

Let $T \subset V_{\mathrm{f}}$ be the set of places where P_v is not hyperspecial. By Prasad's volume formula (see [20] and [7, Section 4.2]), we have

$$|\chi(\Lambda)| = 2|D_k|^{r^2+r/2} C(r)^d \prod_{j=1}^{r} \zeta_k(2j) \prod_{v \in T} \lambda_v, \tag{2}$$

with D_k (resp. ζ_k) the discriminant (resp. Dedekind zeta function) of k; the constant $C(r)$ is given by

$$C(r) = \prod_{j=1}^{r} \frac{(2j-1)!}{(2\pi)^{2j}}, \tag{3}$$

and each λ_v is given by the formula

$$\lambda_v = \frac{1}{(q_v)^{(\dim \mathcal{M}_v - \dim \boldsymbol{M}_v)/2}} \frac{|\mathcal{M}(\mathfrak{f}_v)|}{|\boldsymbol{M}_v(\mathfrak{f}_v)|}, \tag{4}$$

where $\mathfrak{f}_v$ is the residue field of k_v, of size q_v, and the reductive $\mathfrak{f}_v$–groups $\boldsymbol{M}_v$ and $\mathcal{M}_v$ associated with P_v are those described in [20]. By definition $\mathcal{M}_v$ is semisimple of type B_r.

$\boldsymbol{G}/k_v$	isogeny type of $\boldsymbol{M}_v$	λ_v
split	$\mathrm{B}_{r-1} \times (\text{split } \mathrm{GL}_1)$	$\frac{q^{2r}-1}{q-1}$
	$\mathrm{D}_i \times \mathrm{B}_{r-i}$ $(i = 2, \dots, r-1)$	$\frac{(q^i+1)\prod_{k=i+1}^{r}(q^{2k}-1)}{\prod_{k=1}^{r-i}(q^{2k}-1)}$
	${}^1\mathrm{D}_r$	q^r+1
nonsplit	$\mathrm{B}_{r-1} \times (\text{nonsplit } \mathrm{GL}_1)$	$\frac{q^{2r}-1}{q+1}$
	${}^2\mathrm{D}_{i+1} \times \mathrm{B}_{r-i-1}$ $(i = 1, \dots, r-2)$	$\frac{(q^{i+1}-1)\prod_{k=i+2}^{r}(q^{2k}-1)}{\prod_{k=1}^{r-i-1}(q^{2k}-1)}$
	${}^2\mathrm{D}_r$	q^r-1

Table 1: λ_v for P_v of maximal type

A necessary condition for $\Gamma = N_{\boldsymbol{G}(\mathbb{R})}(\Lambda)$ to be maximal is that each P_v defining Λ has maximal type in the sense of Ryzhkov and Chernousov [24]. We list in Table 1 the factors λ_v corresponding to parahoric subgroups P_v of maximal types (to improve the readability we set $q_v = q$ in the formulas). This list of maximal type and the formulas for λ_v are essentially the same as in [2, Table 1]: the only difference is a factor of 2 in the denominator of some λ_v, which can be explained from the fact that Belolipetsky did not work with $\boldsymbol{G}$ simply connected.

From [7, Section 5] (cf also [12, Chapter 12]) we can deduce that the index $[\Gamma : \Lambda]$ of Λ in its normalizer has the following property:

$$[\Gamma : \Lambda] \quad \text{divides} \quad h_k 2^d 4^{\#T} \tag{5}$$

Moreover, a case by case analysis of the possible factor λ_v shows that $\lambda_v > 4$, so that $4^{-\#T} \prod_{v \in T} \lambda_v \geq 1$ (with equality exactly when T is empty). We thus have the following lower bound for the Euler characteristic of any maximal arithmetic subgroup $\Gamma \subset \mathrm{Spin}(n,1)$:

$$|\chi(\Gamma)| \geq \frac{2}{h_k} \left(\frac{C(r)}{2} \right)^d |D_k|^{r^2 + r/2} \zeta_k(2) \cdots \zeta_k(2r) \tag{6}$$

We make use of the following upper bound for the class number (see for instance Belolipetsky and the author [4, Section 7.2]):

$$h_k \leq 16 (\tfrac{\pi}{12})^d |D_k| \tag{7}$$

which together with the basic inequality $\zeta_k(2j) > 1$ transforms (6) into

$$|\chi(\Gamma)| > \frac{1}{8} \left(\frac{6 \cdot C(r)}{\pi} \right)^d |D_k|^{r^2 + r/2 - 1}. \tag{8}$$

Moreover, according to Odlyzko [19, Table 4], we have that for a degree $d \geq 5$ the discriminant of k is larger than $(6.5)^d$. With this estimates we can check that for $r \geq 3$ and $d \geq 5$ we have $|\chi(\Gamma)| > 1$. For the lower degrees, if we suppose that $|\chi(\Gamma)| \leq 1$, we obtain upper bounds for $|D_k|$ from Equation (8). This upper bounds exclude the existence of such a Γ for $r \geq 6$ (which is already clear from the work of Belolipetsky [2]). For $r = 3$ (where the bounds are the worst) we obtain:

$$\begin{aligned} d = 2 &: |D_k| \leq 28 \\ d = 3 &: |D_k| \leq 134 \\ d = 4 &: |D_k| \leq 640 \end{aligned}$$

From the existing tables of number fields (eg, [10; 14]) we can list the possibilities this leaves us for k. We find that no field with $d = 4$ can appear, and for $d = 2, 3$ all possibilities have class number $h_k = 1$. Using Equation (7) with $h_k = 1$ we then improve the upper bounds for $|D_k|$ and thus shorten the list of possible fields. For $r = 5$ only $|D_k| = 5$ arises, and for $r = 4$ we have $|D_k| \leq 11$ (the possibility $d = 3$ is excluded here). For $r = 3$, we are left with $|D_k| \leq 20$ when $d = 2$, and $|D_k| = 49$ or 81 when $d = 3$.

With $h_k = 1$, using the functional equation for ζ_k and the property (5) for the index $[\Gamma : \Lambda]$, we can express the Euler characteristic of Γ as

$$|\chi(\Gamma)| = \frac{1}{2^a} \prod_{v \in T} \lambda_v \prod_{j=1}^{r} |\zeta_k(1-2j)| \tag{9}$$

for some integer a. The special values $\zeta_k(1-2j)$, which are rational by the Klingen–Siegel theorem, can be computed with the software Pari/GP (cf Remark 6). We list in Table 2 the values we need. We check that for every field k under consideration a prime factor greater than 2 appear in the numerator of the product $\prod_{j=1}^{r} |\zeta_k(1-2j)|$. A direct computation for $r = 3, 4, 5$ shows that the formula in Table 1 for each factor λ_v is actually given by a polynomial in q (this seems to hold for any r). In particular, we always have $\lambda_v \in \mathbb{Z}$, and we conclude from (9) that $|\chi(\Gamma)|$ cannot be a reciprocal integer.

degree	$\lvert D_k \rvert$	$\zeta_k(-1)$	$\zeta_k(-3)$	$\zeta_k(-5)$	$\zeta_k(-7)$	$\zeta_k(-9)$
$d=2$	5	1/30	1/60	67/630	361/120	412751/1650
	8	1/12	11/120	361/252	24611/240	
	12	1/6	23/60	1681/126		
	13	1/6	29/60	33463/1638		
	17	1/3	41/30	5791/63		
$d=3$	49	$-1/21$	79/210	$-7393/63$		
	81	$-1/9$	199/90	$-50353/27$		

Table 2: Special values of ζ_k

This completes the proof. □

Remark 6 The function `zetak` in Pari/GP allows us to obtain approximate values for $\zeta_k(1-2j)$. On the other hand the size of the denominator of the product $\prod_{j=1}^{m} \zeta_k(1-2j)$ can be bounded by the method described in [25, Section 3.7]. By recursion on m, this allows to ascertain that the values $\zeta_k(1-2j)$ correspond exactly to the fractions given in Table 2.

Remark 7 The fact that for $|D_k| = 5$ the value $\zeta_k(-1)\zeta_k(-3)$ has trivial numerator explains why the proof fails for $n = 4$ (ie, $r = 2$). And indeed there is a principal arithmetic subgroup $\Gamma \subset \mathrm{Spin}(4,1)$ with $|\chi(\Gamma)| = \frac{1}{14400}$ and whose image in $\mathrm{Isom}^+(\mathbb{H}^4)$ is contained as an index-2 subgroup of the Coxeter group W_1. On the other hand, for

$|D_k| > 5$ the appearance of a nontrivial numerator in $\zeta_k(-3)$ shows — at least for the fields considered in Table 2 — the impossibility of a Γ defined over k with $\chi(\Gamma)$ a reciprocal integer. This is the first step in Belolipetsky's proof of Theorem 2.

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