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CAT(0) spaces with boundary the join of two Cantor sets

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We will show that if a proper complete CAT(0) space X has a visual boundary homeomorphic to the join of two Cantor sets, and X admits a geometric group action by a group containing a subgroup isomorphic to $\mathbb{Z}^2$, then its Tits boundary is the spherical join of two uncountable discrete sets. If X is geodesically complete, then X is a product, and the group has a finite index subgroup isomorphic to a lattice in the product of two isometry groups of bounded valence bushy trees.

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1 Introduction

CAT(0) spaces with homeomorphic visual boundaries can have very different Tits boundaries. However, if X admits a proper and cocompact group action by isometries, or a geometric group action in short, then this places a restriction on the possible Tits boundaries for a given visual boundary. (We follow the definition of a proper group action in Bridson–Haefliger [3, Chapter I.8]; some use the term "properly discontinuous" for this.) Kim Ruane has showed in [13] that for a CAT(0) space X with boundary ∂X homeomorphic to the suspension of a Cantor set, if it admits a geometric group action, then the Tits boundary $\partial_{\mathrm{T}} X$ is isometric to the suspension of an uncountable discrete set. In this paper we will show the following.

Theorem 1.1 *If a* CAT(0) *space X has a boundary ∂X homeomorphic to the join of two Cantor sets C_1 and C_2 and if X admits a geometric group action by a group containing a subgroup isomorphic to $\mathbb{Z}^2$, then its Tits boundary $\partial_{\mathrm{T}} X$ is isometric to the spherical join of two uncountable discrete sets. So if X is geodesically complete, then $X = X_1 \times X_2$ with ∂X_i homeomorphic to C_i, $i = 1, 2$.*

As for the group acting on X, we will prove the following.

Theorem 1.2 *Let X be a geodesically complete* CAT(0) *space such that ∂X is homeomorphic to the join of two Cantor sets. Then for a group $G < \mathrm{Isom}(X)$ acting*

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geometrically on X and containing a subgroup isomorphic to $\mathbb{Z}^2$, either G or a subgroup of G of index 2 is a uniform lattice in $\mathrm{Isom}(X_1)\times\mathrm{Isom}(X_2)$. *Furthermore, a finite index subgroup of G is a lattice in* $\mathrm{Isom}(T_1)\times\mathrm{Isom}(T_2)$, *where T_i is a bounded valence bushy tree quasi-isometric to X_i, $i=1,2$.*

Remark The assumption that G contains a subgroup isomorphic to $\mathbb{Z}^2$ is only used to obtain a hyperbolic element in G with endpoints in $\partial X\setminus(C_1\cup C_2)$, which we use in Section 4 to prove Theorem 1.1. It is conjectured that a CAT(0) group is either Gromov hyperbolic or it contains a subgroup isomorphic to $\mathbb{Z}^2$. Without using the assumption on G, we can show that G cannot be hyperbolic, which follows from Lemma 2.3 and the flat plane theorem [3, Theorem III.H.1.5]. Thus if the conjecture is shown to be true for general CAT(0) groups, the assumption on G will not be necessary. The conjecture has been proved for some classes of CAT(0) groups; see Kapovich–Klein [8] and Caprace–Haglund [5] for examples.

If X_i are proper geodesically complete, one might hope that they are trees, so G will be a uniform lattice in the product of two isometry groups of trees. Surprisingly, this may not be the case. Ontaneda constructed a 2–complex Z which is non-positively curved and geodesically complete with free group F_n as its fundamental group (see Ontaneda [10, Proposition 1]). Its universal cover is quasi-isometric to F_n, so it is a Gromov hyperbolic space with Cantor set boundary, while being also a CAT(0) space. Under an additional condition that the isotropy subgroup of $\mathrm{Isom}(X_i)$ of every boundary point of X_i acts cocompactly on X_i, then X_i is a tree (see Caprace–Monod [6, Theorem 1.3]).

There are irreducible lattices in a product of two trees, so G may not have a finite index subgroup which splits as a product. See Burger–Mozes [4] for a detailed investigation.

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2 Preliminaries

First we fix the notations. For a CAT(0) space X, its (visual) boundary with the cone topology is ∂X. For a subset $H\subset X$, we denote by $\partial H := \bar{H}\cap\partial X$, where the closure $\bar{H}$ is taken in $\bar{X} := X\cup\partial X$. The angular and the Tits metrics on the boundary are denoted as $\angle(\cdot\,,\cdot)$ and $\mathrm{d_T}(\cdot\,,\cdot)$ respectively. We denote the boundary with the Tits metric by $\partial_{\mathrm{T}}X$. The identity map from the Tits boundary $\partial_{\mathrm{T}}X$ to the visual boundary ∂X is continuous but usually not a homeomorphism (see Bridson–Haefliger [3, Proposition II.9.7]). If g is a group element acting on X by isometry, we denote by $\overline{g}$ the action of g extended to ∂X by homeomorphism. If g acts on X by a hyperbolic isometry, the two endpoints of its axes on ∂X are denoted by $g^{\pm\infty}$. We refer to [3] for details on basic facts about $\mathrm{CAT}(0)$ spaces.

Let X be a complete $\mathrm{CAT}(0)$ space with ∂X homeomorphic to the join of two Cantor sets C_1 and C_2, and $G < \mathrm{Isom}(X)$ be a group acting on X geometrically. We will not assume that G contains a subgroup isomorphic to $\mathbb{Z}^2$ until Section 4. By the following lemma, we can assume that G stabilizes C_1 and C_2.

Lemma 2.1 *Either G or a subgroup of G of index 2 stabilizes each of C_1 and C_2.*

Proof Consider ∂X as a complete bipartite graph with C_1, C_2 as the two sets of vertices. For any $g \in G$, if $\overline{g} \cdot x_1 \in C_1$ for some $x_1 \in C_1$, then $\overline{g} \cdot C_i = C_i$, $i = 1, 2$; otherwise $\overline{g} \cdot C_1 = C_2$ and $\overline{g} \cdot C_2 = C_1$. So the homomorphism from G to symmetric group on two elements is well-defined and its kernel is the subgroup of G which stabilizes each of C_1 and C_2. □

By an arc we specifically mean a segment from a point in C_1 to a point in C_2 which does not pass through any other point of C_1 or C_2, and by open (closed) segment a segment on the boundary excluding (including) its two endpoints. We will investigate the positions of the endpoints of hyperbolic elements in G.

We quote a basic result on dynamics on $\mathrm{CAT}(0)$ space boundary by Ruane:

Lemma 2.2 (Ruane [12, Lemma 4.1]) *Let g be a hyperbolic isometry of a* $\mathrm{CAT}(0)$ *space X and let c be an axis of g. Let $z \in \partial X$, $z \neq g^{-\infty}$ and let $z_i = \overline{g}^i \cdot z$. If $w \in \partial X$ is an accumulation point of the sequence (z_i) in the cone topology, then $\angle(g^{-\infty}, w) + \angle(w, g^{\infty}) = \pi$, and $\angle(g^{-\infty}, z) = \angle(g^{-\infty}, w)$. If $w \neq g^{\infty}$, then $\mathrm{d_T}(g^{-\infty}, w) + \mathrm{d_T}(w, g^{\infty}) = \pi$. In this case c and a ray from $c(0)$ to w span a flat half plane, and $\mathrm{d_T}(g^{-\infty}, z) = \mathrm{d_T}(g^{-\infty}, w)$.*

Recall that a hyperbolic isometry is of rank one if none of its axes bounds a flat half plane, and it is of higher rank otherwise.

Lemma 2.3 *There is no rank-one isometry in G.*

Proof Take any hyperbolic $g \in G$. Assume without loss of generality that $g^{\infty} \in \partial X \setminus C_2$. Then for any point $y \in C_2$, $\overline{g}^n \cdot y$ cannot accumulate at g^{∞} since C_2 is closed in ∂X. Any accumulation point of $\overline{g}^n \cdot y$ will form a boundary of a half plane with $g^{\pm\infty}$ by Lemma 2.2. So g is not rank one. □

We note also that no finite subset of points on the boundary is stabilized by G, which readily follows from a result by Ruane, quoted in a paper by Papasoglu and Swenson, and the fact that our ∂X is not a suspension.

Lemma 2.4 (Ruane, Papasoglu–Swenson [11, Lemma 26]) *If G virtually stabilizes a finite subset A of ∂X, then G virtually has $\mathbb{Z}$ as a direct factor. In this case ∂X is a suspension.*

3 Endpoints of a hyperbolic element

We will show that there is no hyperbolic element of G with one of its endpoints in C_1 but not the other one. We will proceed by contradiction, using as a key result the following theorem by Papasoglu and Swenson to ∂X, itself a strengthening of a previous result by Ballmann and Buyalo [2]. This theorem is applicable to our ∂X in light of the previous lemmas.

Theorem 3.1 (Papasoglu and Swenson [11, Theorem 22]) *If the Tits diameter of ∂X is bigger than $\frac{3\pi}{2}$ then G contains a rank 1 hyperbolic element. In particular: If G does not fix a point of ∂X and does not have rank 1, and I is a (minimal) closed invariant set for the action of G on ∂X, then for any $x \in \partial X$, $\mathrm{d_T}(x, I) \leq \frac{\pi}{2}$.*

We put the word minimal in parentheses as it is not a necessary condition, for if $I \subset \partial X$ is a closed invariant set, then it contains a minimal closed invariant set I', and so for any $x \in \partial X$, $\mathrm{d_T}(x, I) \leq \mathrm{d_T}(x, I') \leq \frac{\pi}{2}$.

Note that the above theorem implies that ∂X has finite Tits diameter, and hence the CAT(1) space $\partial_{\mathrm{T}} X$ is connected.

Now assume that $g \in G$ is hyperbolic such that $g^{\infty} \in C_1$ and $g^{-\infty} \in \partial X \setminus C_1$.

Lemma 3.2 $\mathrm{Fix}(\overline{g})$ *contains boundary of a 2–flat.*

Proof Since $\overline{g}$ acts on $\partial_{\mathrm{T}} X$ by isometry and $\partial_{\mathrm{T}} X$ is connected, if $g^{-\infty} \in C_2$, then the arc between g^{∞} and $g^{-\infty}$ is fixed by $\overline{g}$; otherwise $g^{-\infty} \notin C_1 \cup C_2$, then $g^{-\infty}$ lies on an open arc joining a point in C_1 to a point in C_2, so this arc is fixed by $\overline{g}$. Hence in both cases there is an arc contained in $\partial\mathrm{Min}(g)$. Then by [12, Theorems 3.2 and 3.3], $\mathrm{Min}(g) = Y \times \mathbb{R}$, $\partial\mathrm{Min}(g) = \mathrm{Fix}(\overline{g})$ and is the suspension of ∂Y, and $Z_g/\langle g\rangle$ acts on the CAT(0) space Y geometrically. Here we have $\partial Y \neq \varnothing$, for otherwise $\partial\mathrm{Min}(g)$ would consist of only two points. Since Y has nonempty boundary, so by Swenson [14, Theorem 11] there is a hyperbolic element in $Z_g/\langle g\rangle$ which has an axis in Y with two endpoints on ∂Y. Thus there is a 2–flat in $\mathrm{Min}(g)$. □

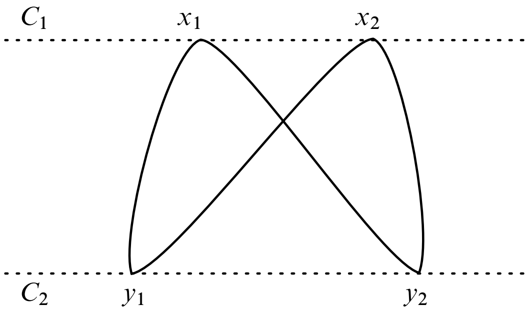

Figure 1: Boundary of a 2–flat in $\mathrm{Min}(h)$

Denote this 2–flat by F, and let z be a point in $\partial F \cap C_1$ other than g^∞.

Lemma 3.3 *If F_0 is a 2–flat whose boundary is contained in* $\mathrm{Fix}(\overline{h}) = \partial\mathrm{Min}(h)$ *for some hyperbolic $h \in G$, then ∂F_0 intersects each of C_1 and C_2 at exactly 2 points.*

Proof Suppose not, then denote the points at which ∂F_0 alternatively intersects C_1, C_2 by $x_1, y_1, x_2, y_2, \dots, x_n, y_n$. Consider the segment joining x_1 and y_2. We may assume that not both of x_1, y_2 are endpoints of h. (If not, choose y_1 and x_3 instead.) From the assumption on ∂F_0, this segment is not part of ∂F_0. Its two endpoints are fixed, but the arc joining them is not in $\mathrm{Fix}(\overline{h})$ because $\mathrm{Fix}(\overline{h})$ is a suspension with suspension points $h^{\pm\infty}$. However, this arc is stabilized by $\overline{h}$ because of the cone topology of ∂X. Take a point p in the open arc between x_1 and y_2. Since $\partial_{\mathrm{T}} X$ is connected there exists a Tits segment in this arc from p to one of x_1 and y_2, say x_1. The action of G on $\partial_{\mathrm{T}} X$ is by isometries. Choose a new point on this segment as p if necessary, we can assume $d_{\mathrm{T}}(p, x_1) < d_{\mathrm{T}}(y_2, x_1)$. Now $d_{\mathrm{T}}(\overline{h}\cdot p, \overline{h}\cdot x_1) = d_{\mathrm{T}}(\overline{h}\cdot p, x_1)$ and $\overline{h}\cdot p$ is also on the arc. $\overline{h}\cdot p$ cannot be on the open segment between p and x_1. If $\overline{h}\cdot p$ were on the open segment between p and y_2, the Tits geodesic from $\overline{h}\cdot p$ to x_1 would go through p or y_2, both would contradict $d_{\mathrm{T}}(\overline{h}\cdot p, x_1) = d_{\mathrm{T}}(p, x_1)$. So $\overline{h}\cdot p = p$. Then $p \in \partial\mathrm{Min}(h)$ and lies on a path in $\partial\mathrm{Min}(h)$ joining $h^{\pm\infty}$, forcing the arc to be in $\partial\mathrm{Min}(h)$, which contradicts the previous assertion. □

We describe our strategy for proving the main result about endpoints of hyperbolic elements in this section: Denote the segment in ∂X from g^∞ to z passing through $g^{-\infty}$ by β. Let y be the point where β intersects C_2. The essence of the following arguments is to look for a point in $\partial_{\mathrm{T}} X$ that is over $\pi/2$ away from C_1 or C_2, which are closed G–invariant subsets, so obtaining a contradiction to Theorem 3.1.

Lemma 3.4 *$g^{-\infty}$ cannot be on the closed segment in β from g^∞ to y.*

Proof Suppose $g^{-\infty}$ is on that segment. Since $d_T(g^\infty, g^{-\infty}) = \pi$, the Tits length of this segment from g^∞ to y is at least π. Let $0 < \delta < \pi/2$ be such that $2\delta \le d_T(y, C_1)$. Take a point p on this segment so that $d_T(p, g^\infty) = \pi/2+\delta$. Then $d_T(p, y) \ge \pi/2-\delta$. Now for any point $x \in C_1$ other than g^∞, if the Tits geodesic segment from p to x passes through y, then

$$d_T(p,x) \ge d_T(p,y) + d_T(y,C_1) \ge (\pi/2-\delta) + 2\delta = \pi/2+\delta;$$

while if it passes through g^∞, then obviously $d_T(p,x) > d_T(p,g^\infty) = \pi/2+\delta$. So $d_T(p,C_1) \ge \pi/2+\delta$, which contradicts Theorem 3.1. □

Now we deal with the case that $g^{-\infty}$ is in the open segment in β from y to z. We state a lemma first which will also be used in later arguments.

Lemma 3.5 *Suppose $h \in G$ is a hyperbolic element such that $F_0 \subset \mathrm{Min}(h)$ whose boundary intersects C_1 and C_2 alternatively at x_1, y_1, x_2, y_2. Assume that the endpoint $h^{-\infty}$ is on some open arc, say the open arc between x_i and y_j, while another endpoint h^∞ is not contained in the closed arc between x_i and y_j. Then for any point $x \in C_1$ other than x_1 and x_2, the sequence $\overline{h}^n \cdot x$ can only accumulate at x_1 or x_2. Similarly, for any point $y \in C_2$ other than y_1 and y_2, the sequence $\overline{h}^n \cdot x$ can only accumulate at y_1 or y_2.*

Proof Suppose not, then the sequence has an accumulation point $x' \in C_1 \setminus \{x_1, x_2\}$. By Lemma 2.2, x' forms boundary of a half flat plane with $h^{\pm\infty}$. This boundary goes from h^∞ to x', and then passes through x_i or y_j before ending at $h^{-\infty}$. If it passes through x_i, then the Tits length of segment on this boundary joining h^∞ to x_i is the total length of the half-plane boundary π minus the length of the segment from x_i to $h^{-\infty}$, thus it is equal to the length of the Tits geodesic segment on ∂F_0 joining these two points, so there are two geodesics for these two points. But this contradicts the uniqueness of Tits geodesic between two points less than π apart. If the boundary of the half flat plane goes through y_j, apply the same argument to the points h^∞ and y_j and we have the same contradiction. For the case $y \in C_2 \setminus \{y_1, y_2\}$ use the same argument. □

Lemma 3.6 *$g^{-\infty}$ cannot be in the open segment from y to z.*

Proof Suppose $g^{-\infty}$ is on this segment. For any point $z' \in C_1$ other than g^∞ and z, the sequence $\overline{g}^{-n} \cdot z'$ converges to z by Lemma 3.5 and Lemma 2.2 which says that $\overline{g}^{-n} \cdot z'$ cannot accumulate at g^∞.

The segment β has Tits length larger than π, so there is a point $w \in \beta$ which is more than $\pi/2$ away from g^∞ and from z.

By lower semi-continuity of the Tits metric,

$$\begin{aligned} d_T(w, z') &= \lim_{n\to\infty} d_T(\overline{g}^{-n}\cdot w, \overline{g}^{-n}\cdot z') \\ &\geq d_T(\lim_{n\to\infty} \overline{g}^{-n}\cdot w, \lim_{n\to\infty} \overline{g}^{-n}\cdot z') = d_T(w, z). \end{aligned}$$

So $d_T(w, C_1) > \pi/2$, a contradiction to Theorem 3.1. □

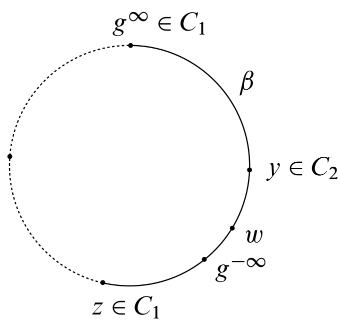

Figure 2: ∂F in Lemma 3.6

We see from these lemmas that the endpoints of a hyperbolic element must be both in C_1, or both in C_2, or none is in $C_1 \cup C_2$.

If g is a hyperbolic element of G with endpoints not in $C_1 \cup C_2$, we have the following results.

Lemma 3.7 *$\partial\mathrm{Min}(g)$ is the boundary of a 2–flat.*

Proof Since $\partial\mathrm{Min}(g)$ is a suspension, so it can only be a circle or a set of two points. However, as $\overline{g}$ is an isometry of $\partial_T X$, we see that $\overline{g}$ must fix the arc on which g^∞ lies. So $\partial\mathrm{Min}(g) = \mathrm{Fix}(\overline{g})$ can only be a circle. Then by the same reason as in Lemma 3.2 $\mathrm{Min}(g)$ contains a 2–flat, whose boundary is the circle. □

Suppose for convenience that g^∞ is on the open arc from $x_1 \in C_1$ to $y_1 \in C_2$, and $x_2 \in C_1$, $y_2 \in C_2$ are the two other points on the boundary ∂F.

Lemma 3.8 *For g as above, $g^{-\infty}$ can only be on the open arc from x_2 to y_2.*

Proof Suppose $g^{-\infty}$ were not on this arc. Without loss of generality let $g^{-\infty}$ be on the arc joining y_1 and x_2. Now the segment from x_1 to x_2 through y_1 has Tits length larger than π, so we can choose a point p on this segment so that p is at distance more than $\pi/2$ away from x_1 and x_2. By Lemma 3.5, for any other point $x' \in C_1$, $\bar{g}^n \cdot x'$ cannot have an accumulation point other than x_1 and x_2. Passing to a subsequence $\bar{g}^{n_k} \cdot x' \to x_i$, $i = 1$ or 2, we have

$$\begin{aligned} d_T(p, x') &= \lim_{n_k \to \infty} d_T(\bar{g}^{n_k} \cdot p, \bar{g}^{n_k} \cdot x') \\ &\geq d_T\Big(\lim_{n_k \to \infty} \bar{g}^{n_k} \cdot p, \lim_{n_k \to \infty} \bar{g}^{n_k} \cdot x'\Big) = d_T(p, x_i), \end{aligned}$$

then $d_T(p, C_1) > \pi/2$, contradicting Theorem 3.1. □

4 Main result

Now we add the assumption that G contains a subgroup isomorphic to $\mathbb{Z}^2$, then the flat torus theorem [3, Theorem II.7.1] implies that there exist two commuting hyperbolic elements $g_1, g_2 \in G$, such that $\mathrm{Min}(g_1)$, formed by the axes of g_1, contains axes of g_2 not parallel to those of g_1. Then an axis of g_1 and an axis of g_2 span a 2–flat in $\mathrm{Min}(g_1)$, and elements $g_1^n g_2^m$ are also hyperbolic and have axes in this 2–flat with endpoints dense on the boundary of this 2–flat. So we can choose some hyperbolic element g so that its endpoints are not in $C_1 \cup C_2$.

We start with a lemma about the orbits of the group action, then we will prove Theorem 1.1.

Lemma 4.1 *For any two distinct points $w_1, w_2 \in \partial X$, there is a sequence $(g_i)_{i=0}^{\infty} \subset G$ such that the points $\bar{g}_i \cdot w_j$, where $0 \leq i < \infty$ and $j \in \{1, 2\}$, are distinct.*

Proof From Lemma 2.4 we know that every $w \in \partial X$ has an infinite orbit $G \cdot w$. So let $(h_i)_{i=0}^{\infty} \subset G$ be a sequence such that $\bar{h}_i \cdot w_1$ are distinct. We will construct the sequence (g_i) inductively. First set $g_0 = e$.

Suppose that for $n \geq 0$ we have $g_0, \dots, g_n$ such that $\bar{g}_i \cdot w_j$, where $0 \leq i \leq n$ and $j \in \{1, 2\}$, are distinct. Let $S_n := \{\bar{g}_m \cdot w_1, \bar{g}_m \cdot w_2 : 0 \leq m \leq n\}$. Pass to a subsequence of (h_i) so that $\bar{h}_i \cdot w_1 \notin S_n$. (We will keep denoting any subsequence by (h_i).) If there exists some h_j such that $\bar{h}_j \cdot w_2 \notin S_n$, then set $g_{n+1} = h_j$. Otherwise, there exists some $\bar{g}_m \cdot w_k \in S_n$ such that $\bar{h}_i \cdot w_2 = \bar{g}_m \cdot w_k$ for infinitely many h_i. Pass to this subsequence. Since the orbit of $\bar{g}_m \cdot w_k$ is infinite, there exists $h' \in G$ such that $\bar{h}' \cdot (\bar{g}_m \cdot w_k) \notin S_n$, so $\overline{h'h_i} \cdot w_2 \notin S_n$. Now $\overline{h'h_i} \cdot w_1 \notin S_n$ for infinitely many h_i. Set $g_{n+1} = h'h_i$ for one such h_i. Hence we get the desired sequence (g_i). □

Remark The only condition required on the group action is that every orbit is infinite. This proof can be used to show a similar result for any finite set $\{w_1, \dots w_n\}$.

Lemma 4.2 *For any $x \in C_1$, $y \in C_2$ we have $\mathrm{d_T}(x, y) = \pi/2$. Hence $\partial_{\mathrm{T}} X$ is metrically a spherical join of C_1 and C_2.*

Proof Consider some $g \in G$ which is hyperbolic with endpoints not on $C_1 \cup C_2$. Let $\partial\mathrm{Min}(g) = \partial F$. We will first prove that for $x_1, x_2 \in C_1 \cap \partial F$, $y_1, y_2 \in C_2 \cap \partial F$, we have $\mathrm{d_T}(x_i, y_j) = \pi/2$, where $i, j = 1, 2$. Take any of the four arcs making up ∂F, say the arc joining x_1 and y_1.

The endpoints of hyperbolic elements in Z_g are dense on ∂F, so we can pick a $g' \in Z_g$ so that $g'^{-\infty}$ is as close to the midpoint of arc x_2 and y_2 as we want. Let $0 < \delta < \min(\mathrm{d_T}(x_2, C_2), \mathrm{d_T}(y_2, C_1))$. Pick g' so that $|\mathrm{d_T}(g'^{-\infty}, x_2) - \mathrm{d_T}(g'^{-\infty}, y_2)| < \delta$. For any point $x \in C_1$ other than x_2, if the Tits geodesic segment from $g'^{-\infty}$ to x passes through y_2, then

$$\begin{aligned}\mathrm{d_T}(g'^{-\infty}, x) &\geq \mathrm{d_T}(g'^{-\infty}, y_2) + \mathrm{d_T}(y_2, C_1) \\ &> \mathrm{d_T}(g'^{-\infty}, x_2) - \delta + \mathrm{d_T}(y_2, C_1) > \mathrm{d_T}(g'^{-\infty}, x_2);\end{aligned}$$

while if it passes through x_2 then obviously $\mathrm{d_T}(g'^{-\infty}, x) > \mathrm{d_T}(g'^{-\infty}, x_2)$. For any $y \in C_2$ other than y_2, by similar reasoning on the Tits geodesic segment from $g'^{-\infty}$ to y, we have $\mathrm{d_T}(g'^{-\infty}, y) > \mathrm{d_T}(g'^{-\infty}, y_2)$.

For any arc joining $x \neq x_2 \in C_1$ and $y \neq y_2 \in C_2$, since $\mathrm{d_T}(g'^{-\infty}, x) > \mathrm{d_T}(g'^{-\infty}, x_2)$, the point x_2 cannot be an accumulation point of $\overline{g'}^n \cdot x$ by Lemma 2.2, then by Lemma 3.5, $\overline{g'}^n \cdot x \to x_1$. Likewise, $\overline{g'}^n \cdot y \to y_1$. So

$$\begin{aligned}\mathrm{d_T}(x, y) &= \lim_{n\to\infty} \mathrm{d_T}(\overline{g'}^n \cdot x, \overline{g'}^n \cdot y) \\ &\geq \mathrm{d_T}\Big(\lim_{n\to\infty} \overline{g'}^n \cdot x, \lim_{n\to\infty} \overline{g'}^n \cdot y\Big) \\ &= \mathrm{d_T}(x_1, y_1).\end{aligned} \tag{4-1}$$

For any other arc joining x_i to y_j in ∂F, by Lemma 4.1 there exists $h \in G$ such that $\overline{h} \cdot x_i \neq x_2$ and $\overline{h} \cdot y_j \neq y_2$, so from the inequality (4-1) we get

$$\mathrm{d_T}(x_i, y_j) = \mathrm{d_T}(\overline{h} \cdot x_i, \overline{h} \cdot y_j) \geq \mathrm{d_T}(x_1, y_1).$$

Thus all arcs have equal length $\pi/2$. Now for any $x \in C_1$, $y \in C_2$, by Lemma 3.5 the sequence $\overline{g}^n \cdot x$ can accumulate at x_1 or x_2, and $\overline{g}^n \cdot y$ can accumulate at y_1 or y_2, so passing to some subsequence $(\overline{g}^{n_k})$, we have convergence sequences $\overline{g}^{n_k} \cdot x \to x_i$

and $\overline{g}^{n_k} \cdot y \to y_j$. Then we have the inequality

$$\mathrm{d_T}(x,y) = \lim_{n_k \to \infty} \mathrm{d_T}(\overline{g}^{n_k} \cdot x, \overline{g}^{n_k} \cdot y) \geq \mathrm{d_T}(x_i, y_j) = \pi/2. \tag{4-2}$$

Take a point p on the open arc joining x and y. Without loss of generality assume that p and x are connected in $\partial_{\mathrm{T}} X$ by a segment in the arc. For any $\epsilon > 0$, we may choose a new point on the segment from p to x to replace p so that $0 < \mathrm{d_T}(x,p) < \epsilon$. Consider the Tits geodesic from p to some point in C_2. If it passes through x, then it consists of the segment from p to x and an arc from x to some point in C_2, so by the inequality (4-2) its Tits length is at least $\pi/2 + \mathrm{d_T}(x,p)$. By Theorem 3.1 $\mathrm{d_T}(p, C_2) \leq \pi/2$, so there must be a Tits geodesic from p to some point in C_2 that does not pass through x, hence it passes through y. Its length is at least $\mathrm{d_T}(p,y)$, so y is the closest point in C_2 to p, so $\mathrm{d_T}(p,y) = \mathrm{d_T}(p, C_2) \leq \pi/2$. Then $\mathrm{d_T}(x,y) \leq \mathrm{d_T}(x,p) + \mathrm{d_T}(p,y) < \pi/2 + \epsilon$. Letting $\epsilon \to 0$ we have $\mathrm{d_T}(x,y) \leq \pi/2$. Combining with the inequality (4-2), $\mathrm{d_T}(x,y) = \pi/2$. □

Theorem 4.3 *If X is a* CAT(0) *space which admits a geometric group action by a group containing a subgroup isomorphic to $\mathbb{Z}^2$, and ∂X is homeomorphic to the join of two Cantor sets, then $\partial_{\mathrm{T}} X$ is the spherical join of two uncountable discrete sets. If X is geodesically complete, that is, every geodesic segment in X can be extended to a geodesic line, then X is a product of two* CAT(0) *space X_1, X_2 with ∂X_i homeomorphic to a Cantor set.*

Proof We have shown that for any $x \in C_1$, $y \in C_2$, $\mathrm{d_T}(x,y) = \pi/2$ in Lemma 4.2, so every two distinct points in C_i has Tits distance at most π for $i = 1,2$. Since the identity map $\partial_{\mathrm{T}} X \to \partial X$ is continuous, any Tits path joining two distinct points in C_i is also a path in the visual boundary ∂X, and every such path in ∂X has to pass through some point in the other C_j, thus the distance between the two point must be exactly π, hence C_i with the Tits metric is an uncountable discrete set. Then $\partial_{\mathrm{T}} X$ is isomorphic to the spherical join of C_1 and C_2, giving the first result. So with the additional assumption that X is geodesically complete, it follows by Bridson–Haefliger [3, Theorem II.9.24] that X splits as a product $X_1 \times X_2$, with $\partial X_i = C_i$ for $i = 1,2$. □

5 Some properties of the group

We will show Theorem 1.2 in this section. Assuming that X is geodesically complete, and hence reducible by Theorem 4.3, we have the following result for the group G. We do not require that G stabilizes each of C_1 and C_2 in this section.

Theorem 5.1 *Let X be a $\mathrm{CAT}(0)$ space such that ∂X is homeomorphic to the join of two Cantor sets and suppose X is geodesically complete. For a group $G < \mathrm{Isom}(X)$ containing $\mathbb{Z}^2$ and acting geometrically on X, either G or a subgroup of it of index 2 is a uniform lattice in $\mathrm{Isom}(X_1) \times \mathrm{Isom}(X_2)$, where X_1, X_2 are given by Theorem 4.3.*

Proof We know from Theorem 4.3 that $X = X_1 \times X_2$, so we only need to show that G or a subgroup of it of index 2 preserves this decomposition.

By Lemma 2.1, either G or a subgroup of it of index 2 stabilizes C_1 and C_2. Replacing G by its subgroup if necessary, we assume G stabilizes C_1 and C_2.

Denote by π_i the projection of X to X_i, $i = 1, 2$. Take any $p_1, p_2 \in X$ such that $\pi_2(p_1) = \pi_2(p_2)$. Extend $[p_1, p_2]$ to a geodesic line γ, its projection to each of X_i is the image of a geodesic line. Since X_1 is totally geodesic, the geodesic segment $[p_1, p_2]$ projects to a single point $\pi_2(p_1)$ on X_2, that is, a degenerated geodesic segment, so $\pi_2(\gamma)$ is also a degenerated geodesic line. Thus the endpoints $\gamma(\pm\infty)$ are in C_1. Now $g \cdot \gamma$ is a geodesic line passing through $g \cdot p_1$, $g \cdot p_2$, and its endpoints $\overline{g} \cdot \gamma(\pm\infty) \in C_1$, so $\pi_2(g \cdot p_1) = \pi_2(g \cdot p_2)$. Similarly, for any $q_1, q_2 \in X$ such that $\pi_1(q_1) = \pi_1(q_2)$ we have $\pi_1(g \cdot q_1) = \pi_1(g \cdot q_2)$. So G preserves the decomposition $X = X_1 \times X_2$, hence the result. □

We will show that $\mathrm{Isom}(X_i)$ is isomorphic to a subgroup of $\mathrm{Homeo}(C_i)$ by the following lemma.

Lemma 5.2 *Suppose X' is a proper complete $\mathrm{CAT}(0)$ space, and $G' < \mathrm{Isom}(X')$ acts properly on X' by isometries.*

(1) *If $S \subset \partial X'$ is a set of points on the boundary such that the intersection*

$$\bigcap_{w \in S} \overline{\mathrm{B}_{\mathrm{T}}(w, \pi/2)}$$

is empty, then there exists a point $q \in X$ such that any $g \in \mathrm{Isom}(X')$ that stabilizes all horospheres with centers in S will fix q. In particular, such g is elliptic.

(2) *Assume that G' does not have parabolic isometries of positive translation lengths. If $\partial X'$ is not a suspension and the radius of $\partial_{\mathrm{T}} X'$ is larger than $\pi/2$, then the map $G' \to \mathrm{Homeo}(\partial X')$, defined by extending the action of G' to the boundary $\partial X'$, has a finite kernel, that is, the subgroup of G' that acts trivially on the boundary is finite. Moreover, assume the action of G' is cocompact, then the kernel fixes a subspace of X' with boundary $\partial X'$.*

Proof To prove (1), as any such g stabilizes all horospheres by assumption, thus g stabilizes all horoballs centered at every $w \in S$. Take an arbitrary point $q' \in X$ and choose for each w a closed horoball H_w centered at w that contains q'. Their intersection $\bigcap_{w\in S} H_w$ is non-empty since it contains q'. By Caprace–Monod [6, Lemma 3.5] $\partial H_w = \overline{\mathrm{B}_{\mathrm{T}}(w,\pi/2)}$, then $\partial(\bigcap_{w\in S} H_w) \subset \bigcap_{w\in S}(\partial H_w) = \varnothing$. So $\bigcap_{w\in S} H_w$ is bounded. Also as every H_w is stabilized by g, so is $\bigcap_{w\in S} H_w$. As $\bigcap_{w\in S} H_w$ is convex and compact, it contains a unique circumcenter q. Then g fixes q.

To prove (2), first we claim that if $g \in G'$ has zero translation length, and g fixes a point $w \in \partial X$, then the horospheres centered at w are stabilized by g. Let γ be a geodesic ray with endpoint w, and $b_\gamma(\cdot)$ be the corresponding Busemann function. Since $g\cdot\gamma$ is asymptotic to γ, we have $b_\gamma(x) = b_{g\cdot\gamma}(g\cdot x) = b_\gamma(g\cdot x) + C$ for some constant C. Then as Busemann functions are 1–Lipschitz, it follows that $|C| \le \mathrm{d}_X(x, g\cdot x)$. We have assumed that $|g| = \inf_x \mathrm{d}_X(x, g\cdot x) = 0$, so $C = 0$, that is, $b_\gamma(x) = b_\gamma(g\cdot x)$, hence the claim.

Now if $g \in G'$ acts by hyperbolic isometry, then $\partial\mathrm{Min}(g) = \mathrm{Fix}(\overline{g})$ is a suspension. Since we assumed $\partial X'$ is not a suspension, any g acting trivially on the whole boundary $\partial X'$ is not hyperbolic, so by assumption g is either elliptic or parabolic with zero translation length, thus by the previous claim g stabilizes all the horospheres centered at any point on $\partial X'$. As $\partial_{\mathrm{T}} X'$ has radius larger than $\pi/2$, for every $x \in \partial X'$ there is some $w \in \partial X'$ such that $\mathrm{d}_{\mathrm{T}}(x,w) > \pi/2$, so $x \notin \overline{\mathrm{B}_{\mathrm{T}}(w,\pi/2)}$, hence $S = \partial X'$ satisfies the condition in (1). Now (1) implies that the kernel of $G' \to \mathrm{Homeo}(\partial X')$ is a subgroup of the stabilizer of some point $q \in X'$. As the action of G' is proper, the kernel is finite.

Let K be the kernel. The set fixed by K is closed and convex. For any point q fixed by the kernel, as $g\cdot q$ is fixed by $gKg^{-1} = K$, then $G'\cdot q$ is fixed by K. If the action of G' is cocompact, then $G'\cdot q$ has boundary $\partial X'$, and thus so is the set fixed by K. □

Remark If we further assume that the space of directions at any point of X' is compact (for instance, when X' is geodesically complete), then it was proved by Fujiwara, Nagano and Shioya [7] that the fixed point set on $\partial_{\mathrm{T}} X'$ of any parabolic isometry, possibly with positive translation length, has Tits radius $\le \pi/2$. So in this case the assumption on parabolic isometries in (2) of the previous lemma is not needed.

Corollary 5.3 Let X be a geodesically complete CAT(0) space such that ∂X is homeomorphic to the join of two Cantor sets. Then for a group $G < \mathrm{Isom}(X)$ containing $\mathbb{Z}^2$ and acting geometrically on X, either G or a subgroup of it of index 2 is isomorphic to a subgroup of $\mathrm{Homeo}(C_1) \times \mathrm{Homeo}(C_2)$.

Proof This follows from Theorem 5.1 and Lemma 5.2. □

We can still show this without the geodesic completeness assumption.

Theorem 5.4 *Let X be a* CAT(0) *space such that ∂X is homeomorphic to the join of two Cantor sets. Then for a group $G < \mathrm{Isom}(X)$ containing $\mathbb{Z}^2$ and acting geometrically on X, a finite quotient of either G or a subgroup of G of index 2 is isomorphic to a subgroup in $\mathrm{Homeo}(C_1) \times \mathrm{Homeo}(C_2)$.*

Proof Assume G stabilizes each of C_1 and C_2 as in the proof of Theorem 5.1. Each $g \in G$ acts on ∂X as a homeomorphism, so it acts on $C_i \subset \partial X$ also as a homeomorphism.

Suppose $\overline{g}$ acts trivially on C_1 and C_2, that is, g is in the kernel of $G \to \mathrm{Homeo}(C_1) \times \mathrm{Homeo}(C_2)$. Then for any point $x \in \partial X$ outside $C_1 \cup C_2$, the arc on which x lies is a Tits geodesic segment of length $\pi/2$ in $\partial_{\mathrm{T}} X$. Since $\overline{g}$ acts on $\partial_{\mathrm{T}} X$ by isometry and both endpoints of this Tits geodesic segment are fixed by $\overline{g}$, so $\overline{g}$ fixes the whole arc, thus $\overline{g} \cdot x = x$. Hence $\overline{g}$ acts trivially on ∂X. One can check that $\partial_{\mathrm{T}} X$ has radius larger than $\pi/2$, so by Lemma 5.2 $G \to \mathrm{Homeo}(\partial X)$ has finite kernel. Hence the result. □

In the case when X is geodesically complete, actually we can prove a stronger result, expressed in the last statement of Theorem 1.2. Observe that X_i is a Gromov hyperbolic space by the flat plane theorem, which states that a proper cocompact CAT(0) space Y is hyperbolic if and only if it does not contain a subspace isometric to $\mathbb{E}^2$. Recall that a cocompact space is defined as a space Y which has a compact subset whose images under the action by $\mathrm{Isom}(Y)$ cover Y. The (projected) action of G on X_i is cocompact, even though the image in $\mathrm{Isom}(X_i)$ may not be discrete. As ∂X_i does not contain S^1, the result follows.

We will show X_i is quasi-isometric to a tree. This is equivalent to having the *bottleneck property* by a theorem of Manning, which he proved with an explicit construction:

Theorem 5.5 (Manning [9, Theorem 4.6]) *Let Y be a geodesic metric space. The following are equivalent:*

(1) *Y is quasi-isometric to some simplicial tree Γ.*
(2) *(Bottleneck property) There is some $\Delta > 0$ so that for all $x, y \in Y$ there is a midpoint $m = m(x, y)$ with $d(x,m) = d(y,m) = \frac{1}{2} d(x, y)$ and the property that any path from x to y must pass within less than Δ of the point m.*

Pick a base point p in X_i. There exists some $r > 0$ such that $G \cdot B(p, r)$ covers X_i.

Lemma 5.6 *There exists $R > 0$ such that for any x, y in the same connected component of $X_i \setminus B(p, R)$, the geodesic segment $[x, y]$ does not intersect $B(p, r)$.*

Proof Suppose on the contrary that for R_n increasing to infinity, we can find x_n, y_n in the same connected component of $X_i \setminus B(p, R_n)$ and $[x_n, y_n]$ intersects $B(p, r)$. Since $\overline{X}_i$ is compact in the cone topology, passing to a subsequence we have $x_n \to \overline{x}$, $y_n \to \overline{y}$ for some $\overline{x}, \overline{y} \in \partial X_i$. By Bridson–Haefliger [3, Lemma II.9.22], there is a geodesic line from $\overline{x}$ to $\overline{y}$ intersecting $B(p, r)$. In particular, $\overline{x} \neq \overline{y}$.

Since different connected components in the boundary of a hyperbolic space correspond to different ends of the space (see Bridson–Haefliger [3, Exercise III.H.3.8]), and ∂X_i is a Cantor set, so $\overline{x}$ and $\overline{y}$ are in different ends of X_i, which are separated by $B(p, R_n)$ for R_n large enough. But then x_n, y_n will be in different connected components of $X_i \setminus B(p, R_n)$, contradicting the assumption. Hence the result. □

Lemma 5.7 *X_i has the bottleneck property.*

Proof For any $x, y \in X_i$, we may translate by some $g \in G$ so that the midpoint m of $[x, y]$ is in $B(p, r)$. We may assume that $d(x, y) > 2(R+r)$, then $x, y \in X_i \setminus B(p, R)$. By Lemma 5.6, x, y are in different connected components of $X_i \setminus B(p, R)$, hence any path connecting x to y must intersect $B(p, R)$, so some point on this path is at a distance at most $R + r$ from m. Thus the bottleneck property is satisfied. □

Lemma 5.8 *X_i is quasi-isometric to a bounded valence tree with no terminal vertex.*

Proof First we describe briefly Manning's construction in his proof of Theorem 5.5. Let $R' = 20\Delta$. Start with a single point $\star$ in Y. Call the vertex set containing this point V_0, and let Γ_0 be a tree with only one vertex and no edge, and $\beta_0 : \Gamma_0 \to Y$ be the map sending the vertex to $\star$. Then for each $k \geq 1$, Let N_{k-1} be the open R–neighborhood of V_{k-1}. Let $\mathcal{C}_k$ be the set consists of path components of $Y \setminus N_{k-1}$. For each $C \in \mathcal{C}_k$ pick some point v at $C \cap \overline{N}_k$. There is a unique path component in $\mathcal{C}_{k-1}$ containing C, corresponding to a terminal vertex $w \in V_{k-1}$. Connect v to w by a geodesic segment. Let V_k be the union of V_{k-1} and the set of new points from each of the path components in $\mathcal{C}_k$. Add new vertices and edges to the tree Γ_{k-1} accordingly to get the tree Γ_k. Extend β_{k-1} to β_k by mapping new vertices of Γ_k to corresponding new vertices in V_k, and new edges to corresponding geodesic segments. The tree $\Gamma = \bigcup_{k \geq 0} \Gamma_k$, and $\beta : \Gamma \to Y$ is defined to be β_k on Γ_k.

Apply the construction above to X_i. Since X_i is geodesically complete, each terminal vertex in V_{k-1} will be connected by at least one vertex in $V_k \setminus V_{k-1}$, and similarly so for terminal vertices of Γ_{k-1}. So the tree Γ has no terminal vertex.

Manning proved that the length of each geodesic segment added in the construction is bounded above by $R' + 6\Delta$. Consider $w \in V_{k-1}$ with corresponding path component

$C_w \in \mathcal{C}_{k-1}$. Every path component $C \in \mathcal{C}_k$ such that $C \subset C_w$ gives a new segment joining w. Together with geodesic completeness of X_i, this implies that such C will contain at least one path component of $X_i \setminus B(w, R'+6\Delta)$, and every path component of $X_i \setminus B(w, R'+6\Delta)$ is contained in at most one such C. (Geodesic completeness is used to ensure that no such C will disappear when passing to $X_i \setminus B(w, R'+6\Delta)$.) Thus the number of new vertices in V_k joining w is bounded by the number of path components of $X_i \setminus B(w, R'+6\Delta)$. Call the vertex in Γ corresponding to w as p_w. Since no more new segments will join w in subsequent steps, the degree of p_w in Γ equals one plus the number of new vertices in V_k joining w. Translate X_i by some g so that $g \cdot w \in B(p, r)$. The number of path components in $X_i \setminus B(w, R'+6\Delta)$ equals that in $X_i \setminus B(g \cdot w, R'+6\Delta)$, which is at most the number of path components in $X_i \setminus B(p, r+R'+6\Delta)$, as $B(g \cdot w, R'+6\Delta) \subset B(p, r+R'+6\Delta)$. Hence we obtain a universal bound of the degree of p_w in Γ, which means Γ has bounded valence. □

A tree of bounded valence with no terminal vertex is quasi-isometric to the trivalent tree. Such tree is called a bounded valence *bushy* tree. Therefore we have shown the following:

Theorem 5.9 *If X_i is a proper cocompact and geodesically complete* CAT(0) *space whose boundary ∂X_i is homeomorphic to a Cantor set, then X_i is quasi-isometric to a bounded valence bushy tree.*

Now each of X_1, X_2 is quasi-isometric to a bushy tree, thus X is quasi-isometric to the product of two bounded valence bushy trees, and so is G. Therefore we can apply a theorem by Ahlin [1, Theorem 1] on quasi-isometric rigidity of lattices in products of trees to show that a finite index subgroup of G is a lattice in $\mathrm{Isom}(T_1 \times T_2)$ where T_i is a bounded valence bushy tree quasi-isometric to X_i, $i = 1, 2$. Notice that $\mathrm{Isom}(T_1) \times \mathrm{Isom}(T_2)$ is isomorphic to a subgroup of $\mathrm{Isom}(T_1 \times T_2)$ of index 1 or 2 (which can be proved similarly as Lemma 2.1), we finally proved the last statement of Theorem 1.2.

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