CAT(0) spaces with boundary
the join of two Cantor sets

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We will show that if a proper complete CAT(0) space $X$ has a visual boundary homeomorphic to the join of two Cantor sets, and $X$ admits a geometric group action by a group containing a subgroup isomorphic to $\mathbb{Z}^2$, then its Tits boundary is the spherical join of two uncountable discrete sets. If $X$ is geodesically complete, then $X$ is a product, and the group has a finite index subgroup isomorphic to a lattice in the product of two isometry groups of bounded valence bushy trees.

1 Introduction

CAT(0) spaces with homeomorphic visual boundaries can have very different Tits boundaries. However, if $X$ admits a proper and cocompact group action by isometries, or a geometric group action in short, then this places a restriction on the possible Tits boundaries for a given visual boundary. (We follow the definition of a proper group action in Bridson–Haefliger [3, Chapter I.8]; some use the term “properly discontinuous” for this.) Kim Ruane has showed in [13] that for a CAT(0) space $X$ with boundary $\partial X$ homeomorphic to the suspension of a Cantor set, if it admits a geometric group action, then the Tits boundary $\partial_T X$ is isometric to the suspension of an uncountable discrete set. In this paper we will show the following.

**Theorem 1.1** If a CAT(0) space $X$ has a boundary $\partial X$ homeomorphic to the join of two Cantor sets $C_1$ and $C_2$ and if $X$ admits a geometric group action by a group containing a subgroup isomorphic to $\mathbb{Z}^2$, then its Tits boundary $\partial_T X$ is isometric to the spherical join of two uncountable discrete sets. So if $X$ is geodesically complete, then $X = X_1 \times X_2$ with $\partial X_i$ homeomorphic to $C_i$, $i = 1, 2$.

As for the group acting on $X$, we will prove the following.

**Theorem 1.2** Let $X$ be a geodesically complete CAT(0) space such that $\partial X$ is homeomorphic to the join of two Cantor sets. Then for a group $G < \text{Isom}(X)$ acting
geometrically on $X$ and containing a subgroup isomorphic to $\mathbb{Z}^2$, either $G$ or a subgroup of $G$ of index 2 is a uniform lattice in $\text{Isom}(X_1) \times \text{Isom}(X_2)$. Furthermore, a finite index subgroup of $G$ is a lattice in $\text{Isom}(T_1) \times \text{Isom}(T_2)$, where $T_i$ is a bounded valence bushy tree quasi-isometric to $X_i$, $i = 1, 2$.

**Remark** The assumption that $G$ contains a subgroup isomorphic to $\mathbb{Z}^2$ is only used to obtain a hyperbolic element in $G$ with endpoints in $\partial X \setminus (C_1 \cup C_2)$, which we use in Section 4 to prove Theorem 1.1. It is conjectured that a CAT(0) group is either Gromov hyperbolic or it contains a subgroup isomorphic to $\mathbb{Z}^2$. Without using the assumption on $G$, we can show that $G$ cannot be hyperbolic, which follows from Lemma 2.3 and the flat plane theorem [3, Theorem III.H.1.5]. Thus if the conjecture is shown to be true for general CAT(0) groups, the assumption on $G$ will not be necessary. The conjecture has been proved for some classes of CAT(0) groups; see Kapovich–Klein [8] and Caprace–Haglund [5] for examples.

If $X_i$ are proper geodesically complete, one might hope that they are trees, so $G$ will be a uniform lattice in the product of two isometry groups of trees. Surprisingly, this may not be the case. Ontaneda constructed a 2–complex $Z$ which is non-positively curved and geodesically complete with free group $F_n$ as its fundamental group (see Ontaneda [10, Proposition 1]). Its universal cover is quasi-isometric to $F_n$, so it is a Gromov hyperbolic space with Cantor set boundary, while being also a CAT(0) space. Under an additional condition that the isotropy subgroup of Isom($X_i$) of every boundary point of $X_i$ acts cocompactly on $X_i$, then $X_i$ is a tree (see Caprace–Monod [6, Theorem 1.3]).

There are irreducible lattices in a product of two trees, so $G$ may not have a finite index subgroup which splits as a product. See Burger–Mozes [4] for a detailed investigation.

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## 2 Preliminaries

First we fix the notations. For a CAT(0) space $X$, its (visual) boundary with the cone topology is $\partial X$. For a subset $H \subset X$, we denote by $\partial H := \overline{H} \cap \partial X$, where the closure $\overline{H}$ is taken in $\overline{X} := X \cup \partial X$. The angular and the Tits metrics on the boundary are denoted as $\angle(\cdot, \cdot)$ and $d_T(\cdot, \cdot)$ respectively. We denote the boundary with the Tits metric by $\partial_T X$. The identity map from the Tits boundary $\partial_T X$ to the...
visual boundary $\partial X$ is continuous but usually not a homeomorphism (see Bridson–Haefliger [3, Proposition II.9.7]). If $g$ is a group element acting on $X$ by isometry, we denote by $\overline{g}$ the action of $g$ extended to $\partial X$ by homeomorphism. If $g$ acts on $X$ by a hyperbolic isometry, the two endpoints of its axes on $\partial X$ are denoted by $g^\pm\infty$. We refer to [3] for details on basic facts about CAT(0) spaces.

Let $X$ be a complete CAT(0) space with $\partial X$ homeomorphic to the join of two Cantor sets $C_1$ and $C_2$, and $G < \text{Isom}(X)$ be a group acting on $X$ geometrically. We will not assume that $G$ contains a subgroup isomorphic to $\mathbb{Z}^2$ until Section 4. By the following lemma, we can assume that $G$ stabilizes $C_1$ and $C_2$.

**Lemma 2.1** Either $G$ or a subgroup of $G$ of index 2 stabilizes each of $C_1$ and $C_2$.

**Proof** Consider $\partial X$ as a complete bipartite graph with $C_1$, $C_2$ as the two sets of vertices. For any $g \in G$, if $\overline{g} \cdot x_1 \in C_i$ for some $x_1 \in C_1$, then $\overline{g} \cdot C_i = C_i$, $i = 1, 2$; otherwise $\overline{g} \cdot C_1 = C_2$ and $\overline{g} \cdot C_2 = C_1$. So the homomorphism from $G$ to symmetric group on two elements is well-defined and its kernel is the subgroup of $G$ which stabilizes each of $C_1$ and $C_2$. □

By an arc we specifically mean a segment from a point in $C_1$ to a point in $C_2$ which does not pass through any other point of $C_1$ or $C_2$, and by open (closed) segment a segment on the boundary excluding (including) its two endpoints. We will investigate the positions of the endpoints of hyperbolic elements in $G$.

We quote a basic result on dynamics on CAT(0) space boundary by Ruane:

**Lemma 2.2** (Ruane [12, Lemma 4.1]) Let $g$ be a hyperbolic isometry of a CAT(0) space $X$ and let $c$ be an axis of $g$. Let $z \in \partial X$, $z \neq g^{-\infty}$ and let $z_i = \overline{g}^i \cdot z$. If $w \in \partial X$ is an accumulation point of the sequence $(z_i)$ in the cone topology, then $\angle(g^{-\infty}, w) + \angle(w, g^\infty) = \pi$, and $\angle(g^{-\infty}, z) = \angle(g^{-\infty}, w)$. If $w \neq g^\infty$, then $d_T(g^{-\infty}, w) + d_T(w, g^\infty) = \pi$. In this case $c$ and a ray from $c(0)$ to $w$ span a flat half plane, and $d_T(g^{-\infty}, z) = d_T(g^{-\infty}, w)$.

Recall that a hyperbolic isometry is of rank one if none of its axes bounds a flat half plane, and it is of higher rank otherwise.

**Lemma 2.3** There is no rank-one isometry in $G$.

**Proof** Take any hyperbolic $g \in G$. Assume without loss of generality that $g^\infty \in \partial X \setminus C_2$. Then for any point $y \in C_2$, $\overline{g}^n \cdot y$ cannot accumulate at $g^\infty$ since $C_2$ is closed in $\partial X$. Any accumulation point of $\overline{g}^n \cdot y$ will form a boundary of a half plane with $g^\pm\infty$ by Lemma 2.2. So $g$ is not rank one. □
We note also that no finite subset of points on the boundary is stabilized by \( G \), which readily follows from a result by Ruane, quoted in a paper by Papasoglu and Swenson, and the fact that our \( \partial X \) is not a suspension.

**Lemma 2.4** (Ruane, Papasoglu–Swenson [11, Lemma 26])  
If \( G \) virtually stabilizes a finite subset \( A \) of \( \partial X \), then \( G \) virtually has \( \mathbb{Z} \) as a direct factor. In this case \( \partial X \) is a suspension.

### 3 Endpoints of a hyperbolic element

We will show that there is no hyperbolic element of \( G \) with one of its endpoints in \( C_1 \) but not the other one. We will proceed by contradiction, using as a key result the following theorem by Papasoglu and Swenson to \( \partial X \), itself a strengthening of a previous result by Ballmann and Buyalo [2]. This theorem is applicable to our \( \partial X \) in light of the previous lemmas.

**Theorem 3.1** (Papasoglu and Swenson [11, Theorem 22])  
If the Tits diameter of \( \partial X \) is bigger than \( \frac{3\pi}{2} \) then \( G \) contains a rank 1 hyperbolic element. In particular: If \( G \) does not fix a point of \( \partial X \) and does not have rank 1, and \( I \) is a (minimal) closed invariant set for the action of \( G \) on \( \partial X \), then for any \( x \in \partial X \), \( d_T(x, I) \leq \frac{\pi}{2} \).

We put the word minimal in parentheses as it is not a necessary condition, for if \( I \subset \partial X \) is a closed invariant set, then it contains a minimal closed invariant set \( I' \), and so for any \( x \in \partial X \), \( d_T(x, I) \leq d_T(x, I') \leq \frac{\pi}{2} \).

Note that the above theorem implies that \( \partial X \) has finite Tits diameter, and hence the CAT(1) space \( \partial_T X \) is connected.

Now assume that \( g \in G \) is hyperbolic such that \( g^\infty \in C_1 \) and \( g^{-\infty} \in \partial X \setminus C_1 \).

**Lemma 3.2**  
\( \text{Fix} (\tilde{g}) \) contains boundary of a 2–flat.

**Proof**  
Since \( \tilde{g} \) acts on \( \partial_T X \) by isometry and \( \partial_T X \) is connected, if \( g^{-\infty} \in C_2 \), then the arc between \( g^\infty \) and \( g^{-\infty} \) is fixed by \( \tilde{g} \); otherwise \( g^{-\infty} \notin C_1 \cup C_2 \), then \( g^{-\infty} \) lies on an open arc joining a point in \( C_1 \) to a point in \( C_2 \), so this arc is fixed by \( \tilde{g} \). Hence in both cases there is an arc contained in \( \partial\text{Min}(g) \). Then by [12, Theorems 3.2 and 3.3], \( \text{Min}(g) = Y \times \mathbb{R} \), \( \partial\text{Min}(g) = \text{Fix}(\tilde{g}) \) and is the suspension of \( \partial Y \), and \( Z_g/\langle g \rangle \) acts on the CAT(0) space \( Y \) geometrically. Here we have \( \partial Y \neq \emptyset \), for otherwise \( \partial\text{Min}(g) \) would consist of only two points. Since \( Y \) has nonempty boundary, so by Swenson [14, Theorem 11] there is a hyperbolic element in \( Z_g/\langle g \rangle \) which has an axis in \( Y \) with two endpoints on \( \partial Y \). Thus there is a 2–flat in \( \text{Min}(g) \). □

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Denote this 2–flat by $F$, and let $z$ be a point in $\partial F \cap C_1$ other than $g^\infty$.

**Lemma 3.3** If $F_0$ is a 2–flat whose boundary is contained in $\text{Fix}(\bar{h}) = \partial \text{Min}(h)$ for some hyperbolic $h \in G$, then $\partial F_0$ intersects each of $C_1$ and $C_2$ at exactly 2 points.

**Proof** Suppose not, then denote the points at which $\partial F_0$ alternatively intersects $C_1$, $C_2$ by $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$. Consider the segment joining $x_1$ and $y_2$. We may assume that not both of $x_1, y_2$ are endpoints of $h$. (If not, choose $y_1$ and $x_3$ instead.) From the assumption on $\partial F_0$, this segment is not part of $\partial F_0$. Its two endpoints are fixed, but the arc joining them is not in $\text{Fix}(\bar{h})$ because $\text{Fix}(\bar{h})$ is a suspension with suspension points $h^{\pm\infty}$. However, this arc is stabilized by $\bar{h}$ because of the cone topology of $\partial X$. Take a point $p$ in the open arc between $x_1$ and $y_2$. Since $\partial T X$ is connected there exists a Tits segment in this arc from $p$ to one of $x_1$ and $y_2$, say $x_1$. The action of $G$ on $\partial T X$ is by isometries. Choose a new point on this segment as $p$ if necessary, we can assume $d_T(p, x_1) < d_T(y_2, x_1)$. Now $d_T(\bar{h} \cdot p, \bar{h} \cdot x_1) = d_T(\bar{h} \cdot p, x_1)$ and $\bar{h} \cdot p$ is also on the arc. $\bar{h} \cdot p$ cannot be on the open segment between $p$ and $x_1$. If $\bar{h} \cdot p$ were on the open segment between $p$ and $y_2$, the Tits geodesic from $\bar{h} \cdot p$ to $x_1$ would go through $p$ or $y_2$, both would contradict $d_T(\bar{h} \cdot p, x_1) = d_T(p, x_1)$. So $\bar{h} \cdot p = p$. Then $p \in \partial \text{Min}(h)$ and lies on a path in $\partial \text{Min}(h)$ joining $h^{\pm\infty}$, forcing the arc to be in $\partial \text{Min}(h)$, which contradicts the previous assertion. \qed

We describe our strategy for proving the main result about endpoints of hyperbolic elements in this section: Denote the segment in $\partial X$ from $g^\infty$ to $z$ passing through $g^{-\infty}$ by $\beta$. Let $y$ be the point where $\beta$ intersects $C_2$. The essence of the following arguments is to look for a point in $\partial T X$ that is over $\pi/2$ away from $C_1$ or $C_2$, which are closed $G$–invariant subsets, so obtaining a contradiction to Theorem 3.1.

**Lemma 3.4** $g^{-\infty}$ cannot be on the closed segment in $\beta$ from $g^\infty$ to $y$. 

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**Proof** Suppose $g^{-\infty}$ is on that segment. Since $d_T(g^\infty, g^{-\infty}) = \pi$, the Tits length of this segment from $g^\infty$ to $y$ is at least $\pi$. Let $0 < \delta < \pi/2$ be such that $2\delta \leq d_T(y, C_1)$. Take a point $p$ on this segment so that $d_T(p, g^\infty) = \pi/2 + \delta$. Then $d_T(p, y) \geq \pi/2 - \delta$. Now for any point $x \in C_1$ other than $g^\infty$, if the Tits geodesic segment from $p$ to $x$ passes through $y$, then

$$d_T(p, x) \geq d_T(p, y) + d_T(y, C_1) \geq (\pi/2 - \delta) + 2\delta = \pi/2 + \delta;$$

while if it passes through $g^\infty$, then obviously $d_T(p, x) > d_T(p, g^\infty) = \pi/2 + \delta$. So $d_T(p, C_1) \geq \pi/2 + \delta$, which contradicts Theorem 3.1. \hfill \Box

Now we deal with the case that $g^{-\infty}$ is in the open segment in $\beta$ from $y$ to $z$. We state a lemma first which will also be used in later arguments.

**Lemma 3.5** Suppose $h \in G$ is a hyperbolic element such that $F_0 \subset \Min(h)$ whose boundary intersects $C_1$ and $C_2$ alternatively at $x_1, y_1, x_2, y_2$. Assume that the endpoint $h^{-\infty}$ is on some open arc, say the open arc between $x_i$ and $y_j$, while another endpoint $h^\infty$ is not contained in the closed arc between $x_i$ and $y_j$. Then for any point $x \in C_1$ other than $x_1$ and $x_2$, the sequence $\bar{h}^n \cdot x$ can only accumulate at $x_1$ or $x_2$. Similarly, for any point $y \in C_2$ other than $y_1$ and $y_2$, the sequence $\bar{h}^n \cdot x$ can only accumulate at $y_1$ or $y_2$.

**Proof** Suppose not, then the sequence has an accumulation point $x' \in C_1 \setminus \{x_1, x_2\}$. By Lemma 2.2, $x'$ forms boundary of a half flat plane with $h^\pm\infty$. This boundary goes from $h^\infty$ to $x'$, and then passes through $x_i$ or $y_j$ before ending at $h^{-\infty}$. If it passes through $x_i$, then the Tits length of segment on this boundary joining $h^\infty$ to $x_i$ is the total length of the half-plane boundary $\pi$ minus the length of the segment from $x_i$ to $h^{-\infty}$, thus it is equal to the length of the Tits geodesic segment on $\partial F_0$ joining these two points, so there are two geodesics for these two points. But this contradicts the uniqueness of Tits geodesic between two points less than $\pi$ apart. If the boundary of the half flat plane goes through $y_j$, apply the same argument to the points $h^\infty$ and $y_j$ and we have the same contradiction. For the case $y \in C_2 \setminus \{y_1, y_2\}$ use the same argument. \hfill \Box

**Lemma 3.6** $g^{-\infty}$ cannot be in the open segment from $y$ to $z$.

**Proof** Suppose $g^{-\infty}$ is on this segment. For any point $z' \in C_1$ other than $g^\infty$ and $z$, the sequence $\bar{g}^{-n} \cdot z'$ converges to $z$ by Lemma 3.5 and Lemma 2.2 which says that $\bar{g}^{-n} \cdot z'$ cannot accumulate at $g^\infty$.  

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The segment $\beta$ has Tits length larger than $\pi$, so there is a point $w \in \beta$ which is more than $\pi/2$ away from $g^\infty$ and from $z$.

By lower semi-continuity of the Tits metric,

$$d_T(w, z') = \lim_{n \to \infty} d_T(g^{-n} \cdot w, \bar{g}^{-n} \cdot z') \geq d_T(\lim_{n \to \infty} g^{-n} \cdot w, \lim_{n \to \infty} \bar{g}^{-n} \cdot z') = d_T(w, z).$$

So $d_T(w, C_1) > \pi/2$, a contradiction to Theorem 3.1. □

![Figure 2: $\partial F$ in Lemma 3.6](image)

We see from these lemmas that the endpoints of a hyperbolic element must be both in $C_1$, or both in $C_2$, or none is in $C_1 \cup C_2$.

If $g$ is a hyperbolic element of $G$ with endpoints not in $C_1 \cup C_2$, we have the following results.

**Lemma 3.7** $\partial \text{Min}(g)$ is the boundary of a 2–flat.

**Proof** Since $\partial \text{Min}(g)$ is a suspension, so it can only be a circle or a set of two points. However, as $\bar{g}$ is an isometry of $\partial_T X$, we see that $\bar{g}$ must fix the arc on which $g^\infty$ lies. So $\partial \text{Min}(g) = \text{Fix}(\bar{g})$ can only be a circle. Then by the same reason as in Lemma 3.2 $\text{Min}(g)$ contains a 2–flat, whose boundary is the circle. □

Suppose for convenience that $g^\infty$ is on the open arc from $x_1 \in C_1$ to $y_1 \in C_2$, and $x_2 \in C_1$, $y_2 \in C_2$ are the two other points on the boundary $\partial F$.

**Lemma 3.8** For $g$ as above, $g^{-\infty}$ can only be on the open arc from $x_2$ to $y_2$. 
We start with a lemma about the orbits of the group action, then we will prove Theorem 1.1.

Lemma 4.1 Suppose \( g^{-\infty} \) were not on this arc. Without loss of generality let \( g^{-\infty} \) be on the arc joining \( y_1 \) and \( x_2 \). Now the segment from \( x_1 \) to \( x_2 \) through \( y_1 \) has Tits length larger than \( \pi \), so we can choose a point \( p \) on this segment so that \( p \) is at distance more than \( \pi/2 \) away from \( x_1 \) and \( x_2 \). By Lemma 3.5, for any other point \( x' \in C_1 \), \( \overline{g^n} \cdot x' \) cannot have an accumulation point other than \( x_1 \) and \( x_2 \). Passing to a subsequence \( \overline{g^n} \cdot x' \rightarrow x_i, i = 1 \) or \( 2 \), we have

\[
d_T(p, x') = \lim_{n_k \to \infty} d_T(\overline{g^{n_k}} \cdot p, \overline{g^{n_k}} \cdot x') \\
\geq d_T\left( \lim_{n_k \to \infty} \overline{g^{n_k}} \cdot p, \lim_{n_k \to \infty} \overline{g^{n_k}} \cdot x' \right) = d_T(p, x_i),
\]

then \( d_T(p, C_1) > \pi/2 \), contradicting Theorem 3.1.

4 Main result

Now we add the assumption that \( G \) contains a subgroup isomorphic to \( \mathbb{Z}^2 \), then the flat torus theorem [3, Theorem II.7.1] implies that there exist two commuting hyperbolic elements \( g_1, g_2 \in G \), such that Min\((g_1)\), formed by the axes of \( g_1 \), contains axes of \( g_2 \) not parallel to those of \( g_1 \). Then an axis of \( g_1 \) and an axis of \( g_2 \) span a 2–flat in Min\((g_1)\), and elements \( g_1^m g_2^n \) are also hyperbolic and have axes in this 2–flat with endpoints dense on the boundary of this 2–flat. So we can choose some hyperbolic element \( g \) so that its endpoints are not in \( C_1 \cup C_2 \).

We start with a lemma about the orbits of the group action, then we will prove Theorem 1.1.

Lemma 4.1 For any two distinct points \( w_1, w_2 \in \partial X \), there is a sequence \((g_i)_{i=0}^\infty \subset G\) such that the points \( \overline{g_i} \cdot w_j \), where \( 0 \leq i < \infty \) and \( j \in \{1, 2\} \), are distinct.

Proof From Lemma 2.4 we know that every \( w \in \partial X \) has an infinite orbit \( G \cdot w \). So let \((h_i)_{i=0}^\infty \subset G\) be a sequence such that \( \overline{h_i} \cdot w_1 \) are distinct. We will construct the sequence \((g_i)\) inductively. First set \( g_0 = e \).

Suppose that for \( n \geq 0 \) we have \( g_0, \ldots, g_n \) such that \( \overline{g_i} \cdot w_j \), where \( 0 \leq i \leq n \) and \( j \in \{1, 2\} \), are distinct. Let \( S_n := \{ \overline{g_m} \cdot w_1, \overline{g_m} \cdot w_2 : 0 \leq m \leq n \} \). Pass to a subsequence of \((h_i)\) so that \( \overline{h_i} \cdot w_1 \notin S_n \). (We will keep denoting any subsequence by \((h_i)\).) If there exists some \( h_j \) such that \( \overline{h_j} \cdot w_2 \notin S_n \), then set \( g_{n+1} = h_j \). Otherwise, there exists some \( \overline{g_m} \cdot w_k \in S_n \) such that \( \overline{h_i} \cdot w_2 = \overline{g_m} \cdot w_k \) for infinitely many \( h_i \). Pass to this subsequence. Since the orbit of \( \overline{g_m} \cdot w_k \) is infinite, there exists \( h' \in G \) such that \( \overline{h'}(\overline{g_m} \cdot w_k) \notin S_n \), so \( \overline{h'h_i} \cdot w_2 \notin S_n \). Now \( \overline{h'h_i} \cdot w_1 \notin S_n \) for infinitely many \( h_i \). Set \( g_{n+1} = h'h_i \) for one such \( h_i \). Hence we get the desired sequence \((g_i)\). 

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Remark The only condition required on the group action is that every orbit is infinite. This proof can be used to show a similar result for any finite set \( \{w_1, \ldots, w_n\} \).

Lemma 4.2 For any \( x \in C_1 \), \( y \in C_2 \) we have \( d_T(x, y) = \pi/2 \). Hence \( \partial T X \) is metrically a spherical join of \( C_1 \) and \( C_2 \).

Proof Consider some \( g \in G \) which is hyperbolic with endpoints not on \( C_1 \cup C_2 \). Let \( \partial \text{Min}(g) = \partial F \). We will first prove that for \( x_1, x_2 \in C_1 \cap \partial F \), \( y_1, y_2 \in C_2 \cap \partial F \), we have \( d_T(x_i, y_j) = \pi/2 \), where \( i, j = 1, 2 \). Take any of the four arcs making up \( \partial F \), say the arc joining \( x_1 \) and \( y_1 \).

The endpoints of hyperbolic elements in \( Z_g \) are dense on \( \partial F \), so we can pick a \( g' \in Z_g \) so that \( g'^{\rightarrow \infty} \) is as close to the midpoint of arc \( x_2 \) and \( y_2 \) as we want. Let \( 0 < \delta < \min(d_T(x_2, C_2), d_T(y_2, C_1)) \). Pick \( g' \) so that \( |d_T(g'^{\rightarrow \infty}, x_2) - d_T(g'^{\rightarrow \infty}, y_2)| < \delta \). For any point \( x \in C_1 \) other than \( x_2 \), if the Tits geodesic segment from \( g'^{\rightarrow \infty} \) to \( x \) passes through \( y_2 \), then

\[
d_T(g'^{\rightarrow \infty}, x) \geq d_T(g'^{\rightarrow \infty}, y_2) + d_T(y_2, C_1) \\
> d_T(g'^{\rightarrow \infty}, x_2) - \delta + d_T(y_2, C_1) > d_T(g'^{\rightarrow \infty}, x_2);
\]

while if it passes through \( x_2 \) then obviously \( d_T(g'^{\rightarrow \infty}, x) > d_T(g'^{\rightarrow \infty}, x_2) \). For any \( y \in C_2 \) other than \( y_2 \), by similar reasoning on the Tits geodesic segment from \( g'^{\rightarrow \infty} \) to \( y \), we have \( d_T(g'^{\rightarrow \infty}, y) > d_T(g'^{\rightarrow \infty}, y_2) \).

For any arc joining \( x \neq x_2 \in C_1 \) and \( y \neq y_2 \in C_2 \), since \( d_T(g'^{\rightarrow \infty}, x) > d_T(g'^{\rightarrow \infty}, x_2) \), the point \( x_2 \) cannot be an accumulation point of \( \overline{g^n} \cdot x \) by Lemma 3.2, then by Lemma 3.5, \( \overline{g^n} \cdot x \to x_1 \). Likewise, \( \overline{g^n} \cdot y \to y_1 \). So

\[
d_T(x, y) = \lim_{n \to \infty} d_T(\overline{g^n} \cdot x, \overline{g^n} \cdot y) \\
\geq d_T\left( \lim_{n \to \infty} \overline{g^n} \cdot x, \lim_{n \to \infty} \overline{g^n} \cdot y \right) \\
= d_T(x_1, y_1).
\]

(4-1)

For any other arc joining \( x_i \) to \( y_j \) in \( \partial F \), by Lemma 4.1 there exists \( h \in G \) such that \( h \cdot x_i \neq x_2 \) and \( h \cdot y_j \neq y_2 \), so from the inequality (4-1) we get

\[
d_T(x_i, y_j) = d_T(h \cdot x_i, h \cdot y_j) \geq d_T(x_1, y_1).
\]

Thus all arcs have equal length \( \pi/2 \). Now for any \( x \in C_1 \), \( y \in C_2 \), by Lemma 3.5 the sequence \( \overline{g^n} \cdot x \) can accumulate at \( x_1 \) or \( x_2 \), and \( \overline{g^n} \cdot y \) can accumulate at \( y_1 \) or \( y_2 \), so passing to some subsequence \( (\overline{g^{nk}}) \), we have convergence sequences \( \overline{g^{nk}} \cdot x \to x_i \)
and \( \bar{g}^{n_k} \cdot y \to y_j \). Then we have the inequality

\[
(4-2) \quad d_T(x, y) = \lim_{n_k \to \infty} d_T(\bar{g}^{n_k} \cdot x, \bar{g}^{n_k} \cdot y) \geq d_T(x_i, y_j) = \pi/2.
\]

Take a point \( p \) on the open arc joining \( x \) and \( y \). Without loss of generality assume that \( p \) and \( x \) are connected in \( \partial_T X \) by a segment in the arc. For any \( \epsilon > 0 \), we may choose a new point on the segment from \( p \) to \( x \) to replace \( p \) so that \( 0 < d_T(x, p) < \epsilon \). Consider the Tits geodesic from \( p \) to some point in \( C_2 \). If it passes through \( x \), then it consists of the segment from \( p \) to \( x \) and an arc from \( x \) to some point in \( C_2 \), so by the inequality \((4-2)\) its Tits length is at least \( \pi/2 + d_T(x, p) \). By Theorem 3.1 \( d_T(p, C_2) \leq \pi/2 \), so there must be a Tits geodesic from \( p \) to some point in \( C_2 \) that does not pass through \( x \), hence it passes through \( y \). Its length is at least \( d_T(p, y) \), so \( y \) is the closest point in \( C_2 \) to \( p \), so \( d_T(p, y) = d_T(p, C_2) \leq \pi/2 \). Then \( d_T(x, y) \leq d_T(x, p) + d_T(p, y) < \pi/2 + \epsilon \). Letting \( \epsilon \to 0 \) we have \( d_T(x, y) \leq \pi/2 \). Combining with the inequality \((4-2)\), \( d_T(x, y) = \pi/2 \). \( \square \)

**Theorem 4.3** If \( X \) is a CAT(0) space which admits a geometric group action by a group containing a subgroup isomorphic to \( \mathbb{Z}^2 \), and \( \partial X \) is homeomorphic to the join of two Cantor sets, then \( \partial_T X \) is the spherical join of two uncountable discrete sets. If \( X \) is geodesically complete, that is, every geodesic segment in \( X \) can be extended to a geodesic line, then \( X \) is a product of two CAT(0) space \( X_1, X_2 \) with \( \partial X_i \) homeomorphic to a Cantor set.

**Proof** We have shown that for any \( x \in C_1, \ y \in C_2, \ d_T(x, y) = \pi/2 \) in Lemma 4.2, so every two distinct points in \( C_i \) has Tits distance at most \( \pi \) for \( i = 1, 2 \). Since the identity map \( \partial_T X \to \partial X \) is continuous, any Tits path joining two distinct points in \( C_i \) is also a path in the visual boundary \( \partial X \), and every such path in \( \partial X \) has to pass through some point in the other \( C_j \), thus the distance between the two point must be exactly \( \pi \), hence \( C_i \) with the Tits metric is an uncountable discrete set. Then \( \partial_T X \) is isomorphic to the spherical join of \( C_1 \) and \( C_2 \), giving the first result. So with the additional assumption that \( X \) is geodesically complete, it follows by Bridson–Haefliger [3, Theorem II.9.24] that \( X \) splits as a product \( X_1 \times X_2 \), with \( \partial X_i = C_i \) for \( i = 1, 2 \). \( \square \)

## 5 Some properties of the group

We will show Theorem 1.2 in this section. Assuming that \( X \) is geodesically complete, and hence reducible by Theorem 4.3, we have the following result for the group \( G \). We do not require that \( G \) stabilizes each of \( C_1 \) and \( C_2 \) in this section.
Theorem 5.1 Let $X$ be a CAT(0) space such that $\partial X$ is homeomorphic to the join of two Cantor sets and suppose $X$ is geodesically complete. For a group $G < \text{Isom}(X)$ containing $\mathbb{Z}^2$ and acting geometrically on $X$, either $G$ or a subgroup of it of index 2 is a uniform lattice in $\text{Isom}(X_1) \times \text{Isom}(X_2)$, where $X_1, X_2$ are given by Theorem 4.3.

Proof We know from Theorem 4.3 that $X = X_1 \times X_2$, so we only need to show that $G$ or a subgroup of it of index 2 preserves this decomposition.

By Lemma 2.1, either $G$ or a subgroup of it of index 2 stabilizes $C_1$ and $C_2$. Replacing $G$ by its subgroup if necessary, we assume $G$ stabilizes $C_1$ and $C_2$.

Denote by $\pi_i$ the projection of $X$ to $X_i$, $i = 1, 2$. Take any $p_1, p_2 \in X$ such that $\pi_2(p_1) = \pi_2(p_2)$. Extend $[p_1, p_2]$ to a geodesic line $\gamma$, its projection to each of $X_i$ is the image of a geodesic line. Since $X_1$ is totally geodesic, the geodesic segment $[p_1, p_2]$ projects to a single point $\pi_2(p_1)$ on $X_2$, that is, a degenerated geodesic segment, so $\pi_2(\gamma)$ is also a degenerated geodesic line. Thus the endpoints $\gamma(\pm \infty)$ are in $C_1$. Now $g \cdot \gamma$ is a geodesic line passing through $g \cdot p_1, g \cdot p_2$, and its endpoints $\bar{g} \cdot \gamma(\pm \infty) \in C_1$, so $\pi_2(g \cdot p_1) = \pi_2(g \cdot p_2)$. Similarly, for any $q_1, q_2 \in X$ such that $\pi_1(q_1) = \pi_1(q_2)$ we have $\pi_1(g \cdot q_1) = \pi_1(g \cdot q_2)$. So $G$ preserves the decomposition $X = X_1 \times X_2$, hence the result. \hfill \Box

We will show that $\text{Isom}(X_i)$ is isomorphic to a subgroup of $\text{Homeo}(C_i)$ by the following lemma.

Lemma 5.2 Suppose $X'$ is a proper complete CAT(0) space, and $G' < \text{Isom}(X')$ acts properly on $X'$ by isometries.

1. If $S \subset \partial X'$ is a set of points on the boundary such that the intersection

$$\bigcap_{w \in S} B_{\partial}(w, \pi/2)$$

is empty, then there exists a point $q \in X$ such that any $g \in \text{Isom}(X')$ that stabilizes all horospheres with centers in $S$ will fix $q$. In particular, such $g$ is elliptic.

2. Assume that $G'$ does not have parabolic isometries of positive translation lengths. If $\partial X'$ is not a suspension and the radius of $\partial X'$ is larger than $\pi/2$, then the map $G' \to \text{Homeo}(\partial X')$, defined by extending the action of $G'$ to the boundary $\partial X'$, has a finite kernel, that is, the subgroup of $G'$ that acts trivially on the boundary is finite. Moreover, assume the action of $G'$ is cocompact, then the kernel fixes a subspace of $X'$ with boundary $\partial X'$.
Proof  To prove (1), as any such $g$ stabilizes all horospheres by assumption, thus $g$ stabilizes all horoballs centered at every $w \in S$. Take an arbitrary point $q' \in X$ and choose for each $w$ a closed horoball $H_w$ centered at $w$ that contains $q'$. Their intersection $\bigcap_{w \in S} H_w$ is non-empty since it contains $q'$. By Caprace–Monod [6, Lemma 3.5] $\partial H_w = B_T(w, \pi/2)$, then $\partial(\bigcap_{w \in S} H_w) \subset \bigcap_{w \in S}(\partial H_w) = \emptyset$. So $\bigcap_{w \in S} H_w$ is bounded. Also as every $H_w$ is stabilized by $g$, so is $\bigcap_{w \in S} H_w$. As $\bigcap_{w \in S} H_w$ is convex and compact, it contains a unique circumcenter $q$. Then $g$ fixes $q$.

To prove (2), first we claim that if $g \in G'$ has zero translation length, and $g$ fixes a point $w \in \partial X$, then the horospheres centered at $w$ are stabilized by $g$. Let $\gamma$ be a geodesic ray with endpoint $w$, and $b_\gamma(\cdot)$ be the corresponding Busemann function. Since $g \cdot \gamma$ is asymptotic to $\gamma$, we have $b_\gamma(x) = b_{g \cdot \gamma}(g \cdot x) = b_\gamma(g \cdot x) + C$ for some constant $C$. Then as Busemann functions are $1$–Lipschitz, it follows that $|C| \leq d_X(x, g \cdot x)$. We have assumed that $|g| = \inf_x d_X(x, g \cdot x) = 0$, so $C = 0$, that is, $b_\gamma(x) = b_\gamma(g \cdot x)$, hence the claim.

Now if $g \in G'$ acts by hyperbolic isometry, then $\partial \text{Min}(g) = \text{Fix}(\overline{g})$ is a suspension. Since we assumed $\partial X'$ is not a suspension, any $g$ acting trivially on the whole boundary $\partial X'$ is not hyperbolic, so by assumption $g$ is either elliptic or parabolic with zero translation length, thus by the previous claim $g$ stabilizes all the horospheres centered at any point on $\partial X'$. As $\partial_T X'$ has radius larger than $\pi/2$, for every $x \in \partial X'$ there is some $w \in \partial X'$ such that $d_T(x, w) > \pi/2$, so $x \notin B_T(w, \pi/2)$, hence $S = \partial X'$ satisfies the condition in (1). Now (1) implies that the kernel of $G' \to \text{Homeo}(\partial X')$ is a subgroup of the stabilizer of some point $q \in X'$. As the action of $G'$ is proper, the kernel is finite.

Let $K$ be the kernel. The set fixed by $K$ is closed and convex. For any point $q$ fixed by the kernel, as $g \cdot q$ is fixed by $gKg^{-1} = K$, then $G' \cdot q$ is fixed by $K$. If the action of $G'$ is cocompact, then $G' \cdot q$ has boundary $\partial X'$, and thus so is the set fixed by $K$. □

Remark  If we further assume that the space of directions at any point of $X'$ is compact (for instance, when $X'$ is geodetically complete), then it was proved by Fujiwara, Nagano and Shioya [7] that the fixed point set on $\partial_T X'$ of any parabolic isometry, possibly with positive translation length, has Tits radius $\leq \pi/2$. So in this case the assumption on parabolic isometries in (2) of the previous lemma is not needed.

Corollary 5.3  Let $X$ be a geodesically complete CAT(0) space such that $\partial X$ is homeomorphic to the join of two Cantor sets. Then for a group $G < \text{Isom}(X)$ containing $\mathbb{Z}^2$ and acting geometrically on $X$, either $G$ or a subgroup of it of index 2 is isomorphic to a subgroup of $\text{Homeo}(C_1) \times \text{Homeo}(C_2)$.

Proof  This follows from Theorem 5.1 and Lemma 5.2. □
We can still show this without the geodesic completeness assumption.

**Theorem 5.4** Let $X$ be a CAT(0) space such that $\partial X$ is homeomorphic to the join of two Cantor sets. Then for a group $G < \text{Isom}(X)$ containing $\mathbb{Z}^2$ and acting geometrically on $X$, a finite quotient of either $G$ or a subgroup of $G$ of index 2 is isomorphic to a subgroup in $\text{Homeo}(C_1) \times \text{Homeo}(C_2)$.

**Proof** Assume $G$ stabilizes each of $C_1$ and $C_2$ as in the proof of Theorem 5.1. Each $g \in G$ acts on $\partial X$ as a homeomorphism, so it acts on $C_i \subset \partial X$ also as a homeomorphism.

Suppose $\bar{g}$ acts trivially on $C_1$ and $C_2$, that is, $g$ is in the kernel of $G \to \text{Homeo}(C_1) \times \text{Homeo}(C_2)$. Then for any point $x \in \partial X$ outside $C_1 \cup C_2$, the arc on which $x$ lies is a Tits geodesic segment of length $\pi/2$ in $\partial_T X$. Since $\bar{g}$ acts on $\partial_T X$ by isometry and both endpoints of this Tits geodesic segment are fixed by $\bar{g}$, so $\bar{g}$ fixes the whole arc, thus $\bar{g} \cdot x = x$. Hence $\bar{g}$ acts trivially on $\partial X$. One can check that $\partial_T X$ has radius larger than $\pi/2$, so by Lemma 5.2 $G \to \text{Homeo}(\partial X)$ has finite kernel. Hence the result.

In the case when $X$ is geodesically complete, actually we can prove a stronger result, expressed in the last statement of Theorem 1.2. Observe that $X_i$ is a Gromov hyperbolic space by the flat plane theorem, which states that a proper cocompact CAT(0) space $Y$ is hyperbolic if and only if it does not contain a subspace isometric to $\mathbb{E}^2$. Recall that a cocompact space is defined as a space $Y$ which has a compact subset whose images under the action by $\text{Isom}(Y)$ cover $Y$. The (projected) action of $G$ on $X_i$ is cocompact, even though the image in $\text{Isom}(X_i)$ may not be discrete. As $\partial X_i$ does not contain $S^1$, the result follows.

We will show $X_i$ is quasi-isometric to a tree. This is equivalent to having the bottleneck property by a theorem of Manning, which he proved with an explicit construction:

**Theorem 5.5** (Manning [9, Theorem 4.6]) Let $Y$ be a geodesic metric space. The following are equivalent:

1. $Y$ is quasi-isometric to some simplicial tree $\Gamma$.
2. (Bottleneck property) There is some $\Delta > 0$ so that for all $x, y \in Y$ there is a midpoint $m = m(x, y)$ with $d(x, m) = d(y, m) = \frac{1}{2} d(x, y)$ and the property that any path from $x$ to $y$ must pass within less than $\Delta$ of the point $m$.

Pick a base point $p$ in $X_i$. There exists some $r > 0$ such that $G \cdot B(p, r)$ covers $X_i$.

**Lemma 5.6** There exists $R > 0$ such that for any $x, y$ in the same connected component of $X_i \setminus B(p, R)$, the geodesic segment $[x, y]$ does not intersect $B(p, r)$.
Suppose on the contrary that for $R_n$ increasing to infinity, we can find $x_n, y_n$ in the same connected component of $X_i \setminus B(p, R_n)$ and $[x_n, y_n]$ intersects $B(p, r)$. Since $\bar{X}_i$ is compact in the cone topology, passing to a subsequence we have $x_n \to \bar{x}$, $y_n \to \bar{y}$ for some $\bar{x}, \bar{y} \in \partial X_i$. By Bridson–Haefliger [3, Lemma II.9.22], there is a geodesic line from $\bar{x}$ to $\bar{y}$ intersecting $B(p, r)$. In particular, $\bar{x} \neq \bar{y}$.

Since different connected components in the boundary of a hyperbolic space correspond to different ends of the space (see Bridson–Haefliger [3, Exercise III.3.8]), and $\partial X_i$ is a Cantor set, so $\bar{x}$ and $\bar{y}$ are in different ends of $X_i$, which are separated by $B(p, R_n)$ for $R_n$ large enough. But then $x_n, y_n$ will be in different connected components of $X_i \setminus B(p, R_n)$, contradicting the assumption. Hence the result. □

**Lemma 5.7** $X_i$ has the bottleneck property.

**Proof** For any $x, y \in X_i$, we may translate by some $g \in G$ so that the midpoint $m$ of $[x, y]$ is in $B(p, r)$. We may assume that $d(x, y) > 2(R + r)$, then $x, y \in X_i \setminus B(p, R)$. By **Lemma 5.6**, $x, y$ are in different connected components of $X_i \setminus B(p, R)$, hence any path connecting $x$ to $y$ must intersect $B(p, R)$, so some point on this path is at a distance at most $R + r$ from $m$. Thus the bottleneck property is satisfied. □

**Lemma 5.8** $X_i$ is quasi-isometric to a bounded valence tree with no terminal vertex.

**Proof** First we describe briefly Manning’s construction in his proof of **Theorem 5.5**. Let $R’ = 20\Delta$. Start with a single point $\star$ in $Y$. Call the vertex set containing this point $V_0$, and let $\Gamma_0$ be a tree with only one vertex and no edge, and $\beta_0 : \Gamma_0 \to Y$ be the map sending the vertex to $\star$. Then for each $k \geq 1$, Let $N_{k-1}$ be the open $R$–neighborhood of $V_{k-1}$. Let $C_k$ be the set consists of path components of $Y \setminus N_k$. For each $C \in C_k$ pick some point $v$ at $C \cap \bar{N}_k$. There is a unique path component in $C_{k-1}$ containing $C$, corresponding to a terminal vertex $w \in V_{k-1}$. Connect $v$ to $w$ by a geodesic segment. Let $V_k$ be the union of $V_{k-1}$ and the set of new points from each of the path components in $C_k$. Add new vertices and edges to the tree $\Gamma_{k-1}$ accordingly to get the tree $\Gamma_k$. Extend $\beta_{k-1}$ to $\beta_k$ by mapping new vertices of $\Gamma_k$ to corresponding new vertices in $V_k$, and new edges to corresponding geodesic segments. The tree $\Gamma = \bigcup_{k \geq 0} \Gamma_k$, and $\beta : \Gamma \to Y$ is defined to be $\beta_k$ on $\Gamma_k$.

Apply the construction above to $X_i$. Since $X_i$ is geodesically complete, each terminal vertex in $V_{k-1}$ will be connected by at least one vertex in $V_k \setminus V_{k-1}$, and similarly so for terminal vertices of $\Gamma_{k-1}$. So the tree $\Gamma$ has no terminal vertex.

Manning proved that the length of each geodesic segment added in the construction is bounded above by $R’ + 6\Delta$. Consider $w \in V_{k-1}$ with corresponding path component
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Every path component \( C \in C_k \) such that \( C \subset C_w \) gives a new segment joining \( w \). Together with geodesic completeness of \( X_i \), this implies that such \( C \) will contain at least one path component of \( X_i \setminus B(w, R' + 6\Delta) \), and every path component of \( X_i \setminus B(w, R' + 6\Delta) \) is contained in at most one such \( C \). (Geodesic completeness is used to ensure that no such \( C \) will disappear when passing to \( X_i \setminus B(w, R' + 6\Delta) \).) Thus the number of new vertices in \( V_k \) joining \( w \) is bounded by the number of path components of \( X_i \setminus B(w, R' + 6\Delta) \). Call the vertex in \( \Gamma \) corresponding to \( w \) as \( p_w \). Since no more new segments will join \( w \) in subsequent steps, the degree of \( p_w \) in \( \Gamma \) equals one plus the number of new vertices in \( V_k \) joining \( w \). Translate \( X_i \) by some \( g \) so that \( g \cdot w \in B(p, r) \). The number of path components in \( X_i \setminus B(w, R' + 6\Delta) \) equals that in \( X_i \setminus B(g \cdot w, R' + 6\Delta) \), which is at most the number of path components in \( X_i \setminus B(p, r + R' + 6\Delta) \), as \( B(g \cdot w, R' + 6\Delta) \subset B(p, r + R' + 6\Delta) \). Hence we obtain a universal bound of the degree of \( p_w \) in \( \Gamma \), which means \( \Gamma \) has bounded valence. \( \square \)

A tree of bounded valence with no terminal vertex is quasi-isometric to the trivalent tree. Such tree is called a bounded valence bushy tree. Therefore we have shown the following:

**Theorem 5.9** If \( X_i \) is a proper cocompact and geodesically complete CAT(0) space whose boundary \( \partial X_i \) is homeomorphic to a Cantor set, then \( X_i \) is quasi-isometric to a bounded valence bushy tree.

Now each of \( X_1, X_2 \) is quasi-isometric to a bushy tree, thus \( X \) is quasi-isometric to the product of two bounded valence bushy trees, and so is \( G \). Therefore we can apply a theorem by Ahlin [1, Theorem 1] on quasi-isometric rigidity of lattices in products of trees to show that a finite index subgroup of \( G \) is a lattice in \( \text{Isom}(T_1 \times T_2) \) where \( T_i \) is a bounded valence bushy tree quasi-isometric to \( X_i, i = 1, 2 \). Notice that \( \text{Isom}(T_1) \times \text{Isom}(T_2) \) is isomorphic to a subgroup of \( \text{Isom}(T_1 \times T_2) \) of index 1 or 2 (which can be proved similarly as Lemma 2.1), we finally proved the last statement of Theorem 1.2.

**References**


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