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Modification rule of monodromies in an R_2–move

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An R_2–move is a homotopy of wrinkled fibrations which deforms images of indefinite fold singularities like the Reidemeister move of type II. Variants of this move are contained in several important deformations of wrinkled fibrations. In this paper, we first investigate how monodromies are changed by this move. For a given fibration and its vanishing cycles, we then give an algorithm to obtain vanishing cycles in a single reference fiber of a fibration obtained by flip and slip, which is a sequence of homotopies increasing fiber genera. As an application of this algorithm, we give several examples of diagrams which were introduced by Williams to describe smooth 4–manifolds by a finite sequence of simple closed curves in a closed surface.

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1 Introduction

Over the last few years, several new fibrations on 4–manifolds were introduced and studied by means of various tools: singularity theory, mapping class groups and so on. These studies originated from the work of Auroux, Donaldson and Katzarkov [2] in which they generalized the results of Donaldson [7] and Gompf [15] on relations between symplectic manifolds and Lefschetz fibrations to those on relations between near-symplectic 4–manifolds and corresponding fibrations, called *broken Lefschetz fibrations*. After their study, Perutz [19; 20] defined the Lagrangian matching invariant for near-symplectic 4–manifolds as a generalization of the standard surface counting of Donaldson and Smith [8] for symplectic 4–manifolds using broken Lefschetz fibrations. Although this invariant is a candidate for geometric interpretation of the Seiberg–Witten invariant, even smooth invariance of this invariant is not verified so far. We need to understand deformations (in the space of more general fibrations) between two broken Lefschetz fibrations in order to prove the invariance. There are several results on this matter (see Lekili [18], Williams [22; 21], and Gay and Kirby [13; 14], for example).

On the other hand, broken Lefschetz fibrations themselves have been studied in terms of mapping class groups using vanishing cycles. For example, the classification problem of fibrations with particular properties was solved by means of this combinatorial method (see Baykur and Kamada [5] and the author [16; 17]). It turns out that every closed

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oriented 4–manifold admits a broken Lefschetz fibration (see Akbulut and Karakurt [1], Baykur [3] and [18]). It is therefore natural to expect that broken Lefschetz fibrations enable us to deal with a broader range of 4–manifolds in a combinatorial way, as we dealt with symplectic 4–manifolds using Lefschetz fibrations. For developing topology of smooth 4–manifolds by means of mapping class groups, it is necessary to understand the relation between several deformations which appeared in the previous paragraph and vanishing cycles of fibrations.

In this paper, we will pay attention to a specific deformation of fibrations, called an R_2*–move*. In this move, the image of indefinite fold singularities is changed by the Reidemeister move of type II (we will define this move in Section 3; see Figure 5). In particular, the region with the highest genus fibers was cut off in this deformation. Furthermore, monodromies in this region might be changed by this move. This move appears in a lot of important deformations of fibrations. For example, *flip and slip*, which was first introduced by Baykur [3], is an application of flip twice followed by a variant of an R_2–move. Another variant of an R_2–move played a key role in the work of Williams [21], which gave a purely combinatorial description of 4–manifolds (which we will mention in Section 6).

The main purpose of this paper is to understand how monodromies are changed by an R_2–move. We will prove that modifications of monodromies in an R_2–move can be controlled by an intersection of kernels of some homomorphisms (see Theorem 3.9). We will also give an algorithm to obtain vanishing cycles in a reference fiber of a fibration obtained by flip and slip in terms of the mapping class group (see Theorems 4.1, 4.3, 5.1, 5.2, 6.5 and 6.7). Note that it is *not* easy to determine vanishing cycles in a single reference fiber of the fibration obtained by applying flip and slip. Indeed, in this modification, two regions with the highest genus fibers are connected by a variant of the R_2–move. It is easy to obtain vanishing cycles in fibers in the respective components since flip is a local deformation. However, we need to deal with a certain monodromy derived from a variant of the R_2–move to understand how these fibers are identified (see also Remark 2.3).

In Section 2, we will give several definitions and notation which we will use in this paper. Sections 3, 4 and 5 are the main parts of this paper. In Section 3, we will examine how monodromies are changed in R_2–moves. The results obtained in this section will play a key role in the following sections. In Sections 4 and 5, we will give an algorithm to obtain vanishing cycles of a fibration modified by flip and slip. We will first deal with fibrations with large fiber genera in Section 4, and then turn our attention to fibrations with small fiber genera in Section 5. In Section 6, we will give a modification rule of a diagram Williams introduced, which is called a *surface diagram*, when the corresponding fibration is changed by flip and slip. We will then construct surface diagrams of some standard 4–manifolds, S^4, $S^1 \times S^3$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ and so on. Note that, as far as the author knows, these are the first nontrivial examples of surface diagrams.

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2 Preliminaries

2.1 Wrinkled fibrations

We first define several singularities to which we will pay attention in this paper.

Definition 2.1 Let M and B be smooth manifolds of dimension 4 and 2, respectively. For a smooth map $f\colon M \to B$, we denote by $\mathcal{S}_f \subset M$ the set of critical points of f.

(1) An element $p \in \mathcal{S}_f$ is called an *indefinite fold singularity* of f if there exist real coordinates (t,x,y,z) (resp. (s,w)) around p (resp. $f(p)$) such that f is locally written in these coordinates as

$$f\colon (t,x,y,z) \mapsto (s,w) = (t, x^2 + y^2 - z^2).$$

(2) An element $p \in \mathcal{S}_f$ is called an *indefinite cusp singularity* of f if there exist real coordinates (t,x,y,z) (resp. (s,w)) around p (resp. $f(p)$) such that f is locally written in these coordinates as

$$f\colon (t,x,y,z) \mapsto (s,w) = (t, x^3 - 3tx + y^2 - z^2).$$

(3) We further assume that the manifolds M and B are oriented. An element $p \in \mathcal{S}_f$ is called a *Lefschetz singularity* of f if there exists complex coordinates (z,w) (resp. ξ) around p (resp. $f(p)$) compatible with orientation of the manifold M (resp. B) such that f is locally written in these coordinates as

$$f\colon (z,w) \mapsto \xi = zw.$$

We can also define definite fold singularities and definite cusp singularities. However, these singularities will not appear in this paper. We call an indefinite fold (resp. cusp) singularity a *fold* (resp. *cusp*) for simplicity.

Definition 2.2 Let M and B be oriented, compact, smooth manifolds of dimension 4 and 2, respectively. A smooth map $f\colon M \to B$ is called a *wrinkled fibration* if it satisfies the following conditions:

(1) $f^{-1}(\partial B) = \partial M$

(2) The set of singularities $\mathcal{S}_f$ consists of folds, cusps and Lefschetz singularities.

A wrinkled fibration f is called a *purely wrinkled fibration* if f has no Lefschetz singularities.

2.2 Mapping class groups and a homomorphism Φ_c

Let Σ_g be a closed, oriented, connected surface of genus g. We take subsets $A_i, B_j \subset \Sigma_g$. We define a group $\mathrm{Mod}\,(\Sigma_g; A_1, \dots, A_n)(B_1, \dots, B_m)$ by

$$\begin{aligned}\mathrm{Mod}\,(\Sigma_g; A_1, \dots, A_n)&(B_1, \dots, B_m)\\ &= \{[T] \in \pi_0(\mathrm{Diff}^+\,(\Sigma_g; A_1, \dots, A_n), \mathrm{id}) \mid T(B_j) = B_j \text{ for all } j\},\end{aligned}$$

where $\mathrm{Diff}^+\,(\Sigma_g; A_1, \dots, A_n)$ is defined as

$$\mathrm{Diff}^+\,(\Sigma_g; A_1, \dots, A_n) = \{\text{diffeomorphisms } T\colon \Sigma_g \to \Sigma_g \mid T(A_i) = A_i \text{ for all } i\}.$$

In this paper, we define a group structure on the above group by multiplication *reverse to the composition*, that is, for elements $T_1, T_2 \in \mathrm{Diff}^+\,(\Sigma_g; A_1, \dots, A_n)$, we define the product $T_1 \cdot T_2$ as

$$T_1 \cdot T_2 = T_2 \circ T_1.$$

We define a group structure of $\mathrm{Mod}\,(\Sigma_g; A_1, \dots, A_n)(B_1, \dots, B_m)$ in the same way. For simplicity, we denote by $\mathcal{M}_g$ the group $\mathrm{Mod}\,(\Sigma_g)$.

Let $c \subset \Sigma_g$ be a simple closed curve. For a given element $\psi \in \mathrm{Mod}\,(\Sigma_g)(c)$, we take a representative $T\colon \Sigma_g \to \Sigma_g \in \mathrm{Diff}^+\,(\Sigma_g)$ preserving the curve c setwise. The restriction $T|_{\Sigma_g \setminus c}\colon \Sigma_g \setminus c \to \Sigma_g \setminus c$ is also a diffeomorphism. Let S_c be the surface obtained by attaching two disks with marked points at the origin to $\Sigma_g \setminus c$ along c. The surface S_c is diffeomorphic to Σ_{g-1} with two marked points if c is nonseparating, or S_c is a disjoint union of Σ_{g_1} with a marked point and Σ_{g_2} with a marked point for some g_1, g_2 if c is separating. The diffeomorphism $T|_{\Sigma_g \setminus c}$ can be extended to a diffeomorphism $\widetilde{T}\colon S_c \to S_c$. We define an element $\Phi_c^*([T])$ as an isotopy class of $\widetilde{T}$, which is contained in the group $\mathrm{Mod}\,(S_c; \{v_1, v_2\})$, where v_1, v_2 are the marked points. The following map is a well-defined homomorphism:

$$\Phi_c^*\colon \mathrm{Mod}\,(\Sigma_g)(c) \to \mathrm{Mod}\,(S_c; \{v_1, v_2\})$$

Furthermore, we define a homomorphism Φ_c on $\mathrm{Mod}(\Sigma_g)(c)$ as the composition $F_{v_1,v_2} \circ \Phi_c^*$, where $F_{v_1,v_2}\colon \mathrm{Mod}(S_c; \{v_1, v_2\}) \to \mathrm{Mod}(S_c)$ is the forgetful map. The range of this map is $\mathcal{M}_{g-1}$ if c is nonseparating, $\mathcal{M}_{g_1} \times \mathcal{M}_{g_2}$ if c is separating and $g_1 \neq g_2$, and $(\mathcal{M}_{g_1} \times \mathcal{M}_{g_2}) \rtimes \mathbb{Z}/2\mathbb{Z}$ if c is separating and $g_1 = g_2$.

2.3 Several homotopies of fibrations

In this subsection, we will give a quick review of some deformations of smooth maps from 4–manifolds to surfaces which we will use in this paper. For details about this, see [18] or [22], for example.

2.3.1 Sink and unsink Lekili [18] introduced a homotopy which removes a Lefschetz singularity near a fold locus, and gives rise to a cusp singularity as in Figure 1.

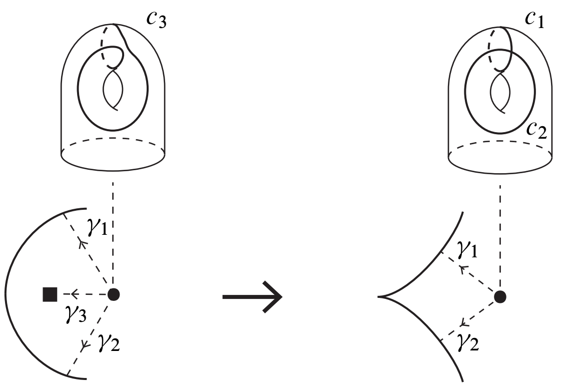

Figure 1: How a base diagram is changed in a sink: the image of a Lefschetz singularity is described by a square, while base points are described by dots.

This deformation is called a *sink* and the inverse move is called an *unsink*. We can always change a cusp into a Lefschetz singularity by an unsink. However, we can apply a sink only when c_3 corresponds to the curve $t_{c_1}(c_2)$ and c_1 intersects c_2 in a single point transversely, where c_i is a vanishing cycle determined by γ_i, which is a reference path in the base space described in Figure 1.

2.3.2 Flip and "flip and slip" The homotopy called a *flip* is locally written as

$$f_s\colon \mathbb{R}^4 \ni (t, x, y, z) \mapsto (t, x^4 - x^2 s + xt + y^2 - z^2) \in \mathbb{R}^2.$$

The set of singularities $\mathcal{S}_{f_s} \subset \mathbb{R}^4$ is equal to $\{(t, x, 0, 0) \in \mathbb{R}^4 \mid 4x^3 - 2sx + t = 0\}$. For $s < 0$, this set consists of indefinite folds. For $s > 0$, this set contains two cusps as in the right side of Figure 2.

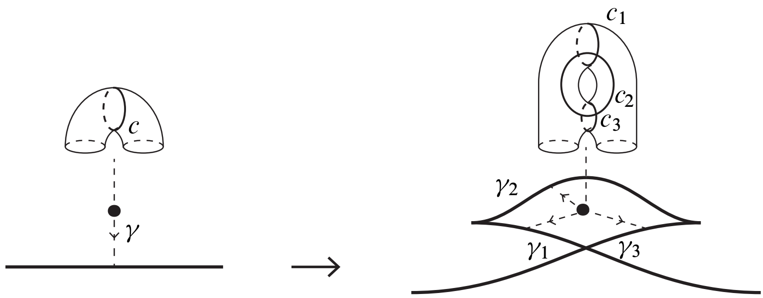

Figure 2: Left: the image of singularities for $s < 0$; right: the image of singularities for $s > 0$; c_i describes a vanishing cycle determined by the reference path γ_i; as is described, c_1 is disjoint from c_3

Baykur [3] and Lekili [18] introduced a certain global homotopy, which is called a *flip and slip*. This modification adds four cusps to the set of critical points of the fibration (see Figure 3). If a lower genus regular fiber of the original fibration (ie a regular fiber on the inside of the singular circle on the far left of Figure 3) is disconnected, then this fiber becomes connected after the modification. If a lower genus regular fiber is connected, this fiber becomes the higher genus fiber and its genus is increased by 2.

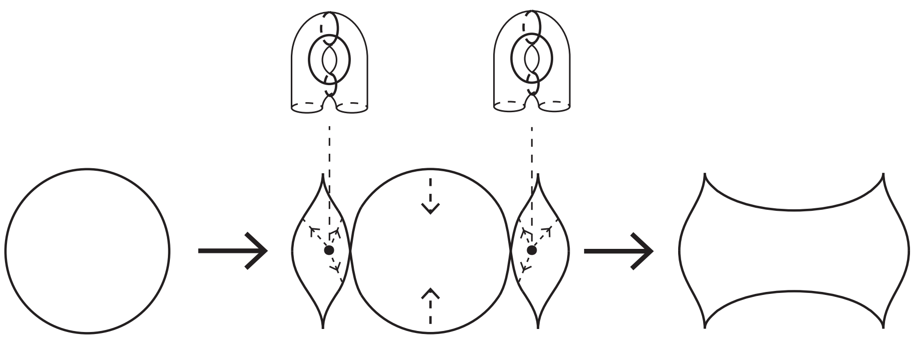

Figure 3: The circle in the far left figure describes the image of singularities of the original fibration. After applying a flip twice, we change the fibration by a homotopy which makes the singular image a circle without self-intersection.

Remark 2.3 It is *not* straightforward to deduce how all the vanishing cycles appear in a single reference fiber after performing a flip and slip. Indeed, to find the vanishing cycles, we need to know how to identify the two regular fibers in the regions with the highest genus fiber in the center of Figure 3. As we will show in the following sections, this identification depends on the choice of homotopies, especially the choice of a "slip" (from the middle figure to the right one in Figure 3).

We remark that such a modification can be also applied when the set of singularities of the original fibration contains cusps. We first apply a flip twice between two consecutive cusps. We then apply a slip in the same way as in the case that the original fibration contains no cusps (see Figure 4). We also call this deformation a *flip and slip*.

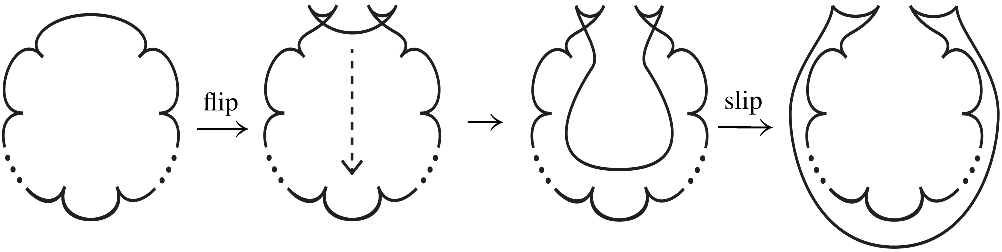

Figure 4: Base loci in a flip and slip when the original fibration has cusps

3 A fibration over the annulus with two components of indefinite folds

Let N be a 3–manifold obtained by a 1–handle attachment to $\Sigma_g \times I$ followed by a 2–handle attachment whose attaching circle is nonseparating and is disjoint from the belt circle of the 1–handle. The manifold N has a Morse function $h\colon N \to I$ with two critical points: one is the origin of the 1–handle $p_1 \in N$ whose index is 1, and the other is the origin of the 2–handle $p_2 \in N$ whose index is 2. We assume that the value of p_1 under h is $\frac{4}{9}$, and the value of p_2 under h is $\frac{5}{9}$. We put $M = N \times S^1$ and we define $f = h \times \mathrm{id}_{S^1}\colon M \to I \times S^1$. We denote by $Z_1 \subset M$ (resp. $Z_2 \subset M$) the component of indefinite folds of f satisfying $f(Z_1) = \{\frac{4}{9}\} \times S^1$ (resp. $f(Z_2) = \{\frac{5}{9}\} \times S^1$).

We identify S^1 with $[0,1]/\{0,1\}$. By construction of N, we can identify $f^{-1}(\{\frac{1}{2}\}\times\{0\})$ with the closed surface Σ_{g+1}. Moreover, this identification is unique up to Dehn twist t_c, where $c \subset \Sigma_{g+1}$ is the belt sphere of the 1–handle. We denote by $d \subset \Sigma_{g+1}$ the attaching circle of the 2–handle. In this section, we look at a monodromy of the fibration f, especially how a monodromy along the curve $\gamma = \{\frac{1}{2}\} \times S^1$ is changed by a certain homotopy of f. We remark that the number of connected components of the complement $\Sigma_{g+1} \setminus (c \cup d)$ is at most 2 since both c and d are nonseparating. We call (c, d) a *bounding pair of genus* g_1 if the complement $\Sigma_{g+1} \setminus (c \cup d)$ consists of two twice punctured surfaces of genus g_1 and $g_2 = g - g_1$.

Let $c, d \subset \Sigma_{g+1}$ be mutually disjoint nonseparating simple closed curves. We look at details of the homomorphisms

$$\Phi_c\colon \mathrm{Mod}\,(\Sigma_{g+1})(c,d) \to \mathrm{Mod}\,(\Sigma_g)(d),$$
$$\Phi_d\colon \mathrm{Mod}\,(\Sigma_{g+1})(c,d) \to \mathrm{Mod}\,(\Sigma_g)(c).$$

We first consider the case that (c,d) is not a bounding pair. In this case, the union $c \sqcup d$ is nonseparating in Σ_g. As we mentioned in Section 2, for a nonseparating simple closed curve $c \subset \Sigma_g$, the homomorphism Φ_c is defined by $F_{v_1,v_2} \circ \Phi_c^*$. It is proved by Farb and Margalit in [12] that the kernel of the homomorphism Φ_c^* is generated by the Dehn twist t_c.

Let $\mathrm{Mod}\,(\Sigma_g)(c^{\mathrm{ori}})$ be the subgroup of $\mathrm{Mod}\,(\Sigma_g)(c)$ whose element is represented by a diffeomorphism preserving an orientation of c. We can define the homomorphism $\Phi_c^{\mathrm{ori}}\colon \mathrm{Mod}\,(\Sigma_g)(c^{\mathrm{ori}}) \to \mathcal{M}_{g-1}$ as we define Φ_c. Furthermore, we can decompose this map as

$$\Phi_c^{\mathrm{ori}}\colon \mathrm{Mod}\,(\Sigma_g)(c^{\mathrm{ori}}) \xrightarrow{\Phi_c^{*,\mathrm{ori}}} \mathrm{Mod}\,(\Sigma_{g-1}; v_1, v_2) \xrightarrow{F_{v_1,v_2}} \mathcal{M}_{g-1}.$$

It is known that the following sequence is exact (see Birman [6]):

$$\pi_1(\mathrm{Diff}^+(\Sigma_{g-1})) \to \pi_1(\Sigma_{g-1} \times \Sigma_{g-1} \setminus \Delta\Sigma_{g-1}) \to \mathrm{Mod}(\Sigma_{g-1}; v_1, v_2) \to \mathcal{M}_{g-1} \to 1,$$

where

$$\Delta\Sigma_{g-1} \subset \Sigma_{g-1} \times \Sigma_{g-1}$$

is the diagonal set. Let $\mathrm{Diff}_0^+(\Sigma_{g-1})$ be the connected component of $\mathrm{Diff}^+(\Sigma_{g-1})$ which contains the identity map. Since $\mathrm{Diff}_0^+(\Sigma_{g-1})$ is contractible if g is greater than or equal to 3 (see Earle and Eells [9]), the kernel of the map

$$F_{v_1,v_2}\colon \mathrm{Mod}\,(\Sigma_{g-1}; v_1, v_2) \to \mathcal{M}_{g-1}$$

is isomorphic to the fundamental group of $\Sigma_{g-1} \times \Sigma_{g-1} \setminus \Delta\Sigma_{g-1}$.

Define $\mathrm{Mod}\,(\Sigma_{g+1})(c^{\mathrm{ori}}, d)$, $\mathrm{Mod}\,(\Sigma_{g+1})(c, d^{\mathrm{ori}})$ and $\mathrm{Mod}\,(\Sigma_{g+1})(c^{\mathrm{ori}}, d^{\mathrm{ori}})$ (subgroups of the group $\mathrm{Mod}\,(\Sigma_{g+1})(c,d)$) as we define the group $\mathrm{Mod}\,(\Sigma_g)(c^{\mathrm{ori}})$. From the construction of the maps, we obtain the following commutative diagram:

$$
\begin{tikzcd}
 & & & \operatorname{Mod}(\Sigma_{g-1}; v_1, v_2, w_1, w_2) \arrow[ddddll, bend right, "F_{v_1,v_2}"'] \arrow[ddddrr, bend left, "F_{w_1,w_2}"] & & & \\
 & & \operatorname{Mod}(\Sigma_g; v_1, v_2)(d^{\mathrm{ori}}) \arrow[ur, "\Phi_d^{*,\mathrm{ori}}"] \arrow[dd, bend right, "F_{v_1,v_2}"'] & & \operatorname{Mod}(\Sigma_g; w_1, w_2)(c^{\mathrm{ori}}) \arrow[ul, "\Phi_c^{*,\mathrm{ori}}"'] \arrow[dd, bend left, "F_{w_1,w_2}"] & & \\
 & & & \operatorname{Mod}(\Sigma_{g+1})(c^{\mathrm{ori}}, d^{\mathrm{ori}}) \arrow[ul, "\Phi_c^{*,\mathrm{ori}}"] \arrow[ur, "\Phi_d^{*,\mathrm{ori}}"'] \arrow[dl, "\Phi_c^{\mathrm{ori}}"'] \arrow[dr, "\Phi_d^{\mathrm{ori}}"] & & & \\
 & & \operatorname{Mod}(\Sigma_g)(d^{\mathrm{ori}}) \arrow[dl, "\Phi_d^{*,\mathrm{ori}}"'] \arrow[dr, "\Phi_d^{\mathrm{ori}}"] & & \operatorname{Mod}(\Sigma_g)(c^{\mathrm{ori}}) \arrow[dl, "\Phi_c^{\mathrm{ori}}"'] \arrow[dr, "\Phi_c^{*,\mathrm{ori}}"] & & \\
 & \operatorname{Mod}(\Sigma_{g-1}; w_1, w_2) \arrow[rr, "F_{w_1,w_2}"] & & \mathcal{M}_{g-1} & & \operatorname{Mod}(\Sigma_{g-1}; v_1, v_2) \arrow[ll, "F_{v_1,v_2}"'] &
\end{tikzcd}
\tag{1}
$$

Since c is disjoint from d, the kernel of $\Phi_c\colon \operatorname{Mod}(\Sigma_{g+1})(c, d) \to \operatorname{Mod}(\Sigma_g)(d)$ is contained in the group $\operatorname{Mod}(\Sigma_{g+1})(c, d^{\mathrm{ori}})$. Similarly, $\operatorname{Ker}\Phi_d$ is contained in $\operatorname{Mod}(\Sigma_{g+1})(c^{\mathrm{ori}}, d)$. Thus, we obtain

$$
\begin{aligned}
\operatorname{Ker}\Phi_c \cap \operatorname{Ker}\Phi_d &\subset \operatorname{Mod}(\Sigma_{g+1})(c^{\mathrm{ori}}, d^{\mathrm{ori}}),\\
\operatorname{Ker}\Phi_c \cap \operatorname{Ker}\Phi_d &= \operatorname{Ker}\Phi_c^{\mathrm{ori}} \cap \operatorname{Ker}\Phi_d^{\mathrm{ori}}.
\end{aligned}
$$

The map $\Phi_d^{*,\mathrm{ori}} \circ \Phi_c^{*,\mathrm{ori}} = \Phi_c^{*,\mathrm{ori}} \circ \Phi_d^{*,\mathrm{ori}}$ sends the group $\operatorname{Ker}\Phi_c^{\mathrm{ori}} \cap \operatorname{Ker}\Phi_d^{\mathrm{ori}}$ to the group $\operatorname{Ker} F_{v_1,v_2} \cap \operatorname{Ker} F_{w_1,w_2} \subset \operatorname{Mod}(\Sigma_{g-1}; v_1, v_2, w_1, w_2)$, which is contained in

$$\operatorname{Ker}(F_{v_1,v_2,w_1,w_2}\colon \operatorname{Mod}(\Sigma_{g-1}, v_1, v_2, w_1, w_2) \to \mathcal{M}_{g-1}).$$

Lemma 3.1 *The following restrictions are isomorphisms:*

$$
\begin{aligned}
\Phi_d^{*,\mathrm{ori}} \circ \Phi_c^{*,\mathrm{ori}}|_{\operatorname{Ker}\Phi_c \cap \operatorname{Ker}\Phi_d}&\colon \operatorname{Ker}\Phi_c \cap \operatorname{Ker}\Phi_d \to \operatorname{Ker} F_{v_1,v_2} \cap \operatorname{Ker} F_{w_1,w_2}\\
\Phi_d^{*,\mathrm{ori}}|_{\operatorname{Ker}\Phi_c \cap \operatorname{Ker}\Phi_d}&\colon \operatorname{Ker}\Phi_c \cap \operatorname{Ker}\Phi_d \to \operatorname{Ker}\Phi_c^{\mathrm{ori}} \cap \operatorname{Ker} F_{w_1,w_2}\\
\Phi_c^{*,\mathrm{ori}}|_{\operatorname{Ker}\Phi_c \cap \operatorname{Ker}\Phi_d}&\colon \operatorname{Ker}\Phi_c \cap \operatorname{Ker}\Phi_d \to \operatorname{Ker} F_{v_1,v_2} \cap \operatorname{Ker}\Phi_d^{\mathrm{ori}}
\end{aligned}
$$

Proof of Lemma 3.1 We only prove that the first map is an isomorphism (we can prove the other maps are isomorphisms similarly). In this proof, we denote the map $\Phi_d^{*,\mathrm{ori}} \circ \Phi_c^{*,\mathrm{ori}}|_{\operatorname{Ker}\Phi_c \cap \operatorname{Ker}\Phi_d}$ by Φ for simplicity. We first prove that Φ is injective. We take an element $\psi \in \operatorname{Ker}\Phi$. Since the kernel of $\Phi_c^{*,\mathrm{ori}}$ (resp. $\Phi_d^{*,\mathrm{ori}}$) is generated by t_c (resp. t_d), ψ is equal to $t_c^m \cdot t_d^n$, for some $m,n \in \mathbb{Z}$. Since ψ is contained in $\operatorname{Ker}\Phi_c$, we have $\Phi_c(\psi) = t_d^n = 1$. Thus, we obtain $n = 0$. Similarly, we can obtain $m = 0$ and this completes the proof of injectivity of Φ.

We next prove that Φ is surjective. For an element $\xi \in \operatorname{Ker} F_{v_1,v_2} \cap \operatorname{Ker} F_{w_1,w_2}$, we can take an element $\overline{\xi} \in \operatorname{Mod}(\Sigma_{g+1})(c^{\mathrm{ori}}, d^{\mathrm{ori}})$ which mapped to ξ by the map $\Phi_d^{*,\mathrm{ori}} \circ \Phi_c^{*,\mathrm{ori}}$ since both of the maps $\Phi_d^{*,\mathrm{ori}}$ and $\Phi_c^{*,\mathrm{ori}}$ are surjective. By the commutative diagram (1), $\Phi_d^{\mathrm{ori}}(\overline{\xi})$ is contained in the kernel of $\Phi_c^{*,\mathrm{ori}}$. Thus, we obtain $\Phi_d^{\mathrm{ori}}(\overline{\xi}) = {t_c}^n$, for some $n \in \mathbb{Z}$. Similarly, we obtain $\Phi_c^{\mathrm{ori}}(\overline{\xi}) = {t_d}^m$, for some $m \in \mathbb{Z}$. Therefore, $\overline{\xi} \cdot {t_c}^{-n} \cdot {t_d}^{-m}$ is contained in the group $\operatorname{Ker}\Phi_c \cap \operatorname{Ker}\Phi_d$ and mapped to ξ by the map Φ. This completes the proof of surjectivity of Φ. □

Let $\varepsilon\colon \operatorname{Diff}^+(\Sigma_{g-1}) \to (\Sigma_{g-1})^4 \setminus \widetilde{\Delta}$ be the evaluation map at $v_1, v_2, w_1, w_2 \in \Sigma_{g-1}$, where $\widetilde{\Delta}$ is the subset of $(\Sigma_{g-1})^4$ defined as

$$\widetilde{\Delta} = \{(x_1,x_2,x_3,x_4) \in (\Sigma_{g-1})^4 \mid \text{there exist } i \text{ and } j,\, i \neq j, \text{ such that } x_i = x_j\}.$$

The map ε is a locally trivial fibration with fiber $\operatorname{Diff}^+(\Sigma_{g-1}, v_1, v_2, w_1, w_2)$ (see [6]). Since $(\Sigma_{g-1})^4 \setminus \widetilde{\Delta}$ is connected, we obtain the following exact sequence:

$$\begin{aligned}(2)\quad \pi_1(\operatorname{Diff}^+(\Sigma_{g-1}; v_1,v_2,w_1,w_2), \mathrm{id}) &\to \pi_1(\operatorname{Diff}^+(\Sigma_{g-1}), \mathrm{id}) \\ &\xrightarrow{\varepsilon_*} \pi_1((\Sigma_{g-1})^4 \setminus \widetilde{\Delta}, (v_1,v_2,w_1,w_2)) \to \operatorname{Mod}(\Sigma_{g-1}; v_1,v_2,w_1,w_2) \\ &\to \mathcal{M}_{g-1} \to 1\end{aligned}$$

Note that the map $\operatorname{Mod}(\Sigma_{g-1}; v_1,v_2,w_1,w_2) \to \mathcal{M}_{g-1}$ is F_{v_1,v_2,w_1,w_2}. The group $\operatorname{Diff}_0^+(\Sigma_{g-1})$ is contractible if $g \geq 3$ [9]. Therefore, if $g \geq 3$, the kernel of the map F_{v_1,v_2,w_1,w_2} is isomorphic to the fundamental group of the configuration space $(\Sigma_{g-1})^4 \setminus \widetilde{\Delta}$. Moreover, under the identification $\operatorname{Ker} F_{v_1,v_2,w_1,w_2} \cong \pi_1((\Sigma_{g-1})^4 \setminus \widetilde{\Delta}, (v_1,v_2,w_1,w_2))$, the kernel of the map F_{w_1,w_2} corresponds to the homomorphism

$$p_{1,*}\colon \pi_1((\Sigma_{g-1})^4 \setminus \widetilde{\Delta}, (v_1,v_2,w_1,w_2)) \to \pi_1(\Sigma_{g-1} \times \Sigma_{g-1} \setminus \Delta\Sigma_{g-1}, (v_1,v_2)),$$

where p_1 is the projection onto the first and second components. Similarly, the kernel of the map F_{v_1,v_2} corresponds to the homomorphism

$$p_{2,*}\colon \pi_1((\Sigma_{g-1})^4 \setminus \widetilde{\Delta}, (v_1,v_2,w_1,w_2)) \to \pi_1(\Sigma_{g-1} \times \Sigma_{g-1} \setminus \Delta\Sigma_{g-1}, (w_1,w_2)),$$

where p_2 is the projection onto the third and fourth components. Eventually, we obtain the isomorphism

$$\operatorname{Ker} F_{v_1,v_2} \cap \operatorname{Ker} F_{w_1,w_2} \cong \operatorname{Ker} p_{1,*} \cap \operatorname{Ker} p_{2,*}.$$

For an oriented surface S and points $x, y \in S$, we define $\Pi(S, x, y)$ as the set of embedded paths from x to y. For an element $\eta \in \Pi(S, x, y)$, we denote by $L(\eta)\colon ([0,1], \{0,1\}) \to (S \setminus \{y\}, x)$ a loop in the neighborhood of η, which is injective on $[0,1)$ and homotopic to a loop obtained by connecting x to a sufficiently small counterclockwise circle around y using η.

Lemma 3.2 *For an element $\eta \in \Pi(\Sigma_{g-1} \setminus \{v_i, w_j\}, v_k, w_l)$ ($\{i,k\} = \{j,l\} = \{1,2\}$), we denote by $l(\eta)$ the loop*

$$t \mapsto \begin{cases} (L(\eta)(t), v_2, w_1, w_2) & k = 1, \\ (v_1, L(\eta)(t), w_1, w_2) & k = 2, \end{cases}$$

where $t \in [0,1]$ and the right-hand side is in $\Sigma_g^4 \setminus \widetilde{\Delta}$. Then, the group $\operatorname{Ker} p_{1,*} \cap \operatorname{Ker} p_{2,*}$ *is generated by the set*

$$\big\{[l(\eta)] \in \pi_1((\Sigma_{g-1})^4 \setminus \widetilde{\Delta}, (v_1, v_2, w_1, w_2)) \\ \mid \eta \in \Pi(\Sigma_{g-1} \setminus \{v_i, w_j\}, v_k, w_l),\ \{i,k\} = \{j,l\} = \{1,2\}\big\}.$$

Proof of Lemma 3.2 When the space S is obvious, we denote by Δ the diagonal subset of $S \times S$ for simplicity. It is obvious that an element $[l(\eta)]$ is contained in the group $\operatorname{Ker} p_{1,*} \cap \operatorname{Ker} p_{2,*}$ for any $\eta \in \Pi(\Sigma_{g-1} \setminus \{v_i, w_j\}, v_k, w_l)$. We prove that any element of $\operatorname{Ker} p_{1,*} \cap \operatorname{Ker} p_{2,*}$ can be represented by the product $[l(\eta_1) \cdots l(\eta_m)]$, for some $\eta_p \in \Pi(\Sigma_{g-1} \setminus \{v_{i_p}, w_{j_p}\}, v_{k_p}, w_{l_p})$. To prove this, we need the following theorem.

Theorem 3.3 (Fadell and Neuwirth [11, Theorem 3]) *The projection*

$$p_2\colon (\Sigma_{g-1})^4 \setminus \widetilde{\Delta} \to (\Sigma_{g-1})^2 \setminus \Delta$$

is a locally trivial fibration with fiber $(\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta$.

By Theorem 3.3, we obtain the following homotopy exact sequence:

$$\pi_2((\Sigma_{g-1})^2 \setminus \Delta, (w_1, w_2)) \to \pi_1((\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta, (v_1, v_2)) \\ \to \pi_1((\Sigma_{g-1})^4 \setminus \widetilde{\Delta}, (v_1, v_2, w_1, w_2)) \xrightarrow{p_{2,*}} \pi_1((\Sigma_{g-1})^2 \setminus \Delta, (w_1, w_2)) \\ \to \pi_0((\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta, (v_1, v_2))$$

Since the space $(\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta$ is connected and the space $(\Sigma_{g-1})^2 \setminus \Delta$ is aspherical (cf [11, Corollary 2.2]), the inclusion map $i\colon (\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta \to (\Sigma_{g-1})^4 \setminus \widetilde{\Delta}$ gives the isomorphism

$$i_*\colon \pi_1((\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta, (v_1, v_2)) \to \operatorname{Ker} p_{2,*}.$$

Let $i'\colon (\Sigma_{g-1}\setminus\{w_1,w_2\})^2\setminus\Delta \to (\Sigma_{g-1})^2\setminus\Delta$ be the inclusion map. The group $\operatorname{Ker} p_{1,*}\cap\operatorname{Ker} p_{2,*}$ is isomorphic to the group $\operatorname{Ker} i'_*$ since the following diagram commutes:

$$\begin{tikzcd}
(\Sigma_{g-1}\setminus\{w_1,w_2\})^2\setminus\Delta \arrow[r, "i"] \arrow[d, "i'"'] & (\Sigma_{g-1})^4\setminus\widetilde{\Delta} \arrow[dl, "p_1"] \\
(\Sigma_{g-1})^2\setminus\Delta &
\end{tikzcd}$$

Thus, it is sufficient to prove that any element of $\operatorname{Ker} i'_*$ can be represented by the product $[l'(\eta_1)\cdots l'(\eta_m)]$ for some $\eta_p\in\Pi(\Sigma_{g-1}\setminus\{v_{i_p},w_{j_p}\},v_{k_p},w_{l_p})$, where $l'(\eta_p)$ is the loop defined by

$$[0,1]\ni t\mapsto\begin{cases}(L(\eta_p)(t),v_2) & k_p=1,\\ (v_1,L(\eta_p)(t)) & k_p=2,\end{cases}$$

where $t\in[0,1]$ and the right-hand side is in $(\Sigma_{g-1}\setminus\{w_1,w_2\})^2\setminus\Delta$. We take an element $[\xi]\in\operatorname{Ker} i'_*$, where $\xi\colon (S^1,1)\to((\Sigma_{g-1}\setminus\{w_1,w_2\})^2\setminus\Delta,(v_1,v_2))$ is a loop ($1\in S^1\subset\mathbb{C}$). We can assume that ξ is an embedding. Since $i'(\xi)$ is nullhomotopic in the space $\Sigma_{g-1}^2\setminus\Delta$, we can take a map $\overline{\xi}\colon D^2\to(\Sigma_{g-1})^2\setminus\Delta$ satisfying the following conditions:

(a) The restriction $\overline{\xi}|_{\partial D^2}$ is equal to $i'(\xi)$.

(b) $\overline{\xi}$ is a complete immersion, that is, $\overline{\xi}$ satisfies:
- $\overline{\xi}$ is an immersion.
- $\sharp\overline{\xi}^{-1}(p)$ is at most 2 for each $p\in\overline{\xi}(D^2)$.
- For any point $p\in\overline{\xi}(D^2)$ such that $\sharp\overline{\xi}^{-1}(p)=2$, there exists a disk neighborhood $D_i\subset\Sigma_{g-1}^2\setminus\Delta$ of the point $p_i\in\overline{\xi}^{-1}(p)$ such that $\overline{\xi}$ is an embedding over D_i, and that $\overline{\xi}(D_1)$ intersects $\overline{\xi}(D_2)$ at the unique point p transversely, where $\{p_1,p_2\}=\overline{\xi}^{-1}(p)$.

(c) For each $i\in\{1,2\}$, $\overline{\xi}^{-1}(\{w_i\}\times(\Sigma_{g-1}\setminus\{w_i\}))$ and $\overline{\xi}^{-1}((\Sigma_{g-1}\setminus\{w_i\})\times\{w_i\})$ is a discrete set and is contained in $\operatorname{Int} D^2\cap\mathbb{R}$.

(d) The set $\overline{\xi}^{-1}(\{p\in\Sigma_{g-1}^2\setminus\Delta\mid\sharp\overline{\xi}(p)=2\})$ is contained in $\operatorname{Int} D^2\cap\mathbb{R}$.

(e) $\overline{\xi}(D^2)$ does not contain the point (w_1,w_2) and (w_2,w_1).

We define a discrete set $B\subset\operatorname{Int} D^2\cap\mathbb{R}$ as

$$B=\coprod_{i=1}^{2}\overline{\xi}^{-1}(\{w_i\}\times(\Sigma_{g-1}\setminus\{w_i\})\coprod_{j=1}^{2}\overline{\xi}^{-1}(\Sigma_{g-1}\setminus\{w_j\})\times\{w_j\}) \\ \cup\overline{\xi}^{-1}(\{p\in\Sigma_{g-1}^2\setminus\Delta\mid\sharp\overline{\xi}(p)=2\}).$$

We put $B=\{q_1,\dots,q_n\}\subset D^2\cap\mathbb{R}$. We assume that $q_1<\dots<q_n$. Denote by S_i the upper semicircle centered at $(1+q_i)/2$ whose ends are 1 and q_i. We also denote by ζ_i a loop obtained by connecting a small counterclockwise circle around q_i to the point $1\in S^1$ using S_i. Since $\overline{\xi}$ is an embedding over S_i, the image $\overline{\xi}(S_i)$ is an embedded path, which we denote by $(\eta_1(S_i),\eta_2(S_i))\subset\Sigma_{g-1}^2\setminus\Delta$. The loop $\overline{\xi}(\zeta_i)$ is homotopic to one of the following loops:

$$\overline{\xi}(\zeta_i)\simeq\begin{cases} l'(\eta_1(S_i)) & \text{if } \overline{\xi}(q_i) \text{ is contained in } \{w_i\}\times(\Sigma_{g-1}\setminus\{w_i\}) \\ l'(\eta_2(S_i)) & \text{if } \overline{\xi}(q_i) \text{ is contained in } (\Sigma_{g-1}\setminus\{w_j\})\times\{w_j\} \\ \text{trivial loop} & \text{otherwise}\end{cases}$$

The loop ξ is homotopic to $\overline{\xi}|_{\zeta_1\cdots\zeta_n}$, thus completing the proof of Lemma 3.2. □

We eventually obtain the following theorem.

Theorem 3.4 *For an element $\eta\in\Pi(\Sigma_{g-1}\setminus\{v_i,w_j\},v_k,w_l)$ $(\{i,k\}=\{j,l\}=\{1,2\})$, we denote by $\delta(\eta)\subset\Sigma_{g-1}$ the boundary of a regular neighborhood of η. This is a simple closed curve in $\Sigma_{g-1}\setminus\{v_1,v_2,w_1,w_2\}$ and we can take a lift of this curve to $\widetilde{\delta}(\eta)\subset\Sigma_{g+1}\setminus(c\cup d)$ by using the identification $\Sigma_{g-1}\setminus\{v_1,v_2,w_1,w_2\}\cong\Sigma_{g+1}\setminus(c\cup d)$. If g is greater than 2, then the group $\operatorname{Ker}\Phi_c\cap\operatorname{Ker}\Phi_d$ is generated by the set*

$$\bigl\{t_{\widetilde{\delta}(\eta)}\cdot t_c^{-1}\cdot t_d^{-1}\in\operatorname{Mod}(\Sigma_{g+1})(c,d)\mid\eta\in\Pi(\Sigma_{g-1}\setminus\{v_i,w_j\},v_k,w_l)$$
$$(\{i,k\}=\{j,l\}=\{1,2\})\bigr\}.$$

We next consider the case where (c,d) is a bounding pair of genus g_1. Then, $c\subset\Sigma_g$ is a separating curve. We put $g_2=g-g_1$. By the same argument as in Lemma 3.1, we can prove the following lemma.

Lemma 3.5 *The following restrictions are isomorphisms:*

$$\Phi_d^*\circ\Phi_c^*|_{\operatorname{Ker}\Phi_c\cap\operatorname{Ker}\Phi_d}\colon\operatorname{Ker}\Phi_c\cap\operatorname{Ker}\Phi_d\to\operatorname{Ker}F_{v_1,v_2}\cap\operatorname{Ker}F_{w_1,w_2}$$

$$\Phi_d^*|_{\operatorname{Ker}\Phi_c\cap\operatorname{Ker}\Phi_d}\colon\operatorname{Ker}\Phi_c\cap\operatorname{Ker}\Phi_d\to\operatorname{Ker}\Phi_c\cap\operatorname{Ker}F_{w_1,w_2}$$

$$\Phi_c^*|_{\operatorname{Ker}\Phi_c\cap\operatorname{Ker}\Phi_d}\colon\operatorname{Ker}\Phi_c\cap\operatorname{Ker}\Phi_d\to\operatorname{Ker}F_{v_1,v_2}\cap\operatorname{Ker}\Phi_d$$

The group $\operatorname{Ker}F_{v_1,v_2}$ (resp. $\operatorname{Ker}F_{w_1,w_2}$) is isomorphic to the group $\operatorname{Ker}F_{v_1}\times\operatorname{Ker}F_{v_2}$ (resp. $\operatorname{Ker}F_{w_1}\times\operatorname{Ker}F_{w_2}$). Thus, we obtain

$$\operatorname{Ker}F_{v_1,v_2}\cap\operatorname{Ker}F_{w_1,w_2}=(\operatorname{Ker}F_{v_1}\cap\operatorname{Ker}F_{w_1})\times(\operatorname{Ker}F_{v_2}\cap\operatorname{Ker}F_{w_2}).$$

Furthermore, the group $\operatorname{Ker} F_{v_i} \cap \operatorname{Ker} F_{w_i}$ is contained in the kernel of the homomorphism

$$F_{v_i,w_i}\colon \operatorname{Mod}(\Sigma_{g_i}; v_i, w_i) \to \mathcal{M}_{g_i}.$$

This group is isomorphic to the group $\pi_1((\Sigma_{g_i})^2 \setminus \Delta, (v_i, w_i))$ if $g_i \geq 2$. Under this identification, it is easy to prove that $\operatorname{Ker} F_{v_i} \cap \operatorname{Ker} F_{w_i}$ corresponds to the group $\operatorname{Ker} p_{1,*} \cap \operatorname{Ker} p_{2,*}$, where we denote by $p_j\colon (\Sigma_{g_i})^2 \setminus \Delta \to \Sigma_{g_i}$ the projection onto the j^{th} component. Since p_2 is a locally trivial fibration with fiber $\Sigma_{g_i} \setminus \{w_i\}$ (cf [11]), we can prove the following lemma by using van Kampen's Theorem.

Lemma 3.6 *For an element $\eta \in \Pi(\Sigma_{g_i}, v_i, w_i)$, we denote by $l(\eta)$ the loop*

$$[0,1] \ni t \mapsto (L(\eta)(t), w_i) \in (\Sigma_{g_i})^2 \setminus \Delta.$$

Then, the group $\operatorname{Ker} p_{1,*} \cap \operatorname{Ker} p_{2,*}$ is generated by the set

$$\{[l(\eta)] \in \pi_1((\Sigma_{g_i})^2 \setminus \Delta, (v_i, w_i)) \mid \eta \in \Pi(\Sigma_{g_i}, v_i, w_i)\}.$$

As the case (c, d) is not a bounding pair, we eventually obtain the following theorem.

Theorem 3.7 *For an element $\eta \in \Pi(\Sigma_{g_i}, v_i, w_i)$, we denote by $\delta(\eta) \subset \Sigma_{g_i}$ the boundary of a regular neighborhood of η. This is a simple closed curve in $\Sigma_{g_i} \setminus \{v_i, w_i\}$ and we can take a lift of this curve to $\widetilde{\delta}(\eta) \subset \Sigma_{g_1+g_2+1} \setminus (c \cup d)$ by using the identification $\Sigma_{g_1} \setminus \{v_1, w_1\} \amalg \Sigma_{g_2} \setminus \{v_2, w_2\} \cong \Sigma_{g_1+g_2+1} \setminus (c \cup d)$. If both of the numbers g_1 and g_2 are greater than or equal to 2, then the group $\operatorname{Ker} \Phi_c \cap \operatorname{Ker} \Phi_d$ is generated by the set*

$$\{t_{\widetilde{\delta}(\eta)} \cdot t_c^{-1} \cdot t_d^{-1} \in \operatorname{Mod}(\Sigma_{g+1})(c,d) \mid \eta \in \Pi(\Sigma_{g_i}, v_i, w_i), i \in \{1,2\}\}.$$

We are now ready to discuss the fibration $f\colon M \to I \times S^1$ which we defined in the beginning of this section. Let $N(p_i) \subset N$ be an open neighborhood of p_i in N. We take a diffeomorphism $\theta_i\colon B_{1/\sqrt{3}} \to N(p_i)$, where $B_{1/\sqrt{3}} \subset \mathbb{R}^3$ is a 3–ball with radius $1/\sqrt{3}$, so that $h \circ \theta_i$ is described as

$$h \circ \theta_1\colon B_{1/\sqrt{3}} \longrightarrow I, \quad (x,y,z) \longmapsto x^2 + y^2 - z^2 + \tfrac{4}{9},$$

$$h \circ \theta_2\colon B_{1/\sqrt{3}} \longrightarrow I, \quad (x,y,z) \longmapsto x^2 - y^2 - z^2 + \tfrac{5}{9}.$$

We take a metric g of N so that the pullback $\theta_i^* g$ is the standard metric on $B_{1/\sqrt{3}}$. The metric g determines a rank 1 horizontal distribution $\mathcal{H}_h = (\operatorname{Ker} dh)^{\perp}$ of $h|_{N \setminus \{p_1, p_2\}}$.

For each $p \in N \setminus \{p_1, p_2\}$, we denote by $c_p(t)$ a horizontal lift of the curve $t \mapsto h(p)+t$ which satisfies $c_p(0) = p$. We define submanifolds $D_l^{\mathcal{H}_h}(p_i)$ and $D_u^{\mathcal{H}_h}(p_i)$ as

$$D_l^{\mathcal{H}_h}(p_i) = \{p_i\} \cup \left\{ p \in N \;\middle|\; h(p) < \tfrac{3+i}{9}, \lim_{t \to ((3+i)/9)-h(p)} c_p(t) = p_i \right\},$$

$$D_u^{\mathcal{H}_h}(p_i) = \{p_i\} \cup \left\{ p \in N \;\middle|\; h(p) > \tfrac{3+i}{9}, \lim_{t \to ((3+i)/9)-h(p)} c_p(t) = p_i \right\}.$$

Note that $D_l^{\mathcal{H}_h}(p_1)$ and $D_u^{\mathcal{H}_h}(p_2)$ are diffeomorphic to the unit interval I, but $D_u^{\mathcal{H}_h}(p_1)$ and $D_l^{\mathcal{H}_h}(p_2)$ are diffeomorphic to the 2–disk D^2. We take a homotopy $h_t\colon N \to I$ with $h_0 = h$ $(t \in I)$ satisfying the following conditions:

(a) The support of the homotopy is contained in $N(p_1)$.

(b) For any $t \in I$, h_t has two critical points p_1 and p_2.

(c) For any $t \in I$, the critical point p_1 of h_t is nondegenerate and its index is 1.

(d) The function $t \mapsto h_t(p_1)$ is monotone increasing.

(e) $h_1(p_1) = \frac{2}{3}$

This homotopy changes the order of critical points. We can take such a homotopy since c and d are disjoint. We take a smooth function $\rho\colon I \to I$ satisfying the following properties:

- $\rho \equiv 0$ on $[0, \frac{1}{6}] \amalg [\frac{5}{6}, 1]$
- $\rho \equiv 1$ on $[\frac{1}{3}, \frac{2}{3}]$
- ρ is monotone increasing on $[\frac{1}{6}, \frac{1}{3}]$
- $\rho(1-s) = \rho(s)$ for any $s \in [0, 1]$

By using h_t and ρ, we define a homotopy $f_t\colon M = N \times S^1 \to I \times S^1$ by

$$\begin{aligned} f_t\colon M = N \times S^1 &\longrightarrow I \times S^1, \\ (x, s) &\longmapsto (h_{t\rho(s)}(x), s). \end{aligned}$$

Since N is obtained by attaching the 1–handle and the 2–handle to $\Sigma_g \times I$, ∂N contains the surface $\Sigma_g \times \{0\}$, which we denote by Σ for simplicity. Moreover, Σ intersects $D_l^{\mathcal{H}_h}(p_1)$ at two points $v_1, v_2 \in \Sigma$, and Σ intersects $D_l^{\mathcal{H}_h}(p_2)$ at a simple closed curve $d \subset \Sigma$. Let $\Pi(\Sigma, v_i, d)$ be the set of embedded paths in Σ from the point v_i to a point in d. For $\eta \in \Pi(\Sigma, v_i, d)$, $L(\eta)\colon ([0,1], \{0,1\}) \to (\Sigma \setminus d, v_i)$ denotes a loop in the neighborhood of $\eta \cup d$, which is injective on $[0, 1)$ and homotopic to a loop obtained by connecting v_i to d using η. For an element $\eta \in \Pi(\Sigma, v_i, d)$, we take a homotopy of horizontal distributions $\{\mathcal{H}_t^\eta\}$ $(t \in [0,1])$ of $h_1|_{N \setminus \{p_1, p_2\}}$ with $\mathcal{H}_0^\eta = \mathcal{H}_{h_1}$ which satisfies the following conditions:

(f) The support of the homotopy is contained in $h_1^{-1}([\frac{5}{9},\frac{2}{3}])$.

(g) $\mathcal{H}_0^\eta = \mathcal{H}_1^\eta$

(h) The arc $D_l^{\mathcal{H}_t^\eta}(p_1)$ intersects Σ at the points $L(\eta)(t), v_j \in \Sigma$, where $\{i,j\}=\{1,2\}$.

Such a homotopy exists because $L(\eta)$ bounds a disk on the surface obtained by applying a surgery to Σ along d.

Since the submanifold $D_l^{\mathcal{H}_s^\eta}(p_1)$ does not contain p_2 for any $s\in[0,1]$, we can take a 1–parameter family of homotopies $h_{t,s}\colon N\to I$ $(t,s\in I)$ with $h_{0,s}=h_{\rho(s)}$ which satisfies the following conditions:

(i) For any $s\in[0,\frac{1}{3}]\cup[\frac{2}{3},1]$, the homotopy $h_{t,s}$ equals $h_{\rho(s)(1-t)}$.

(j) For any $t,s\in I$, $h_{t,s}$ has two critical points p_1 and p_2.

(k) For any $s\in[\frac{1}{3},\frac{2}{3}]$, the support of the homotopy $h_{t,s}$ is contained in a small neighborhood of
$$D_l^{\mathcal{H}^\eta_{3s-1}}(p_1)\cup D_u^{\mathcal{H}^\eta_{3s-1}}(p_1).$$

(l) For any $s\in I$, the homotopy $h_{t,s}$ is supported away from a neighborhood of ∂N.

(m) For any $t,s\in I$, the critical point p_1 of $h_{t,s}$ is nondegenerate and its index is 1.

(n) For any $s\in I$, $h_{t,s}(p_1)$ is equal to $h_{\rho(s)(1-t)}(p_1)$.

By using this family of homotopies, we define a homotopy $\widetilde{f}_t\colon M\to I\times S^1$ by
$$\begin{aligned}\widetilde{f}_t\colon M=N\times S^1 &\longrightarrow I\times S^1,\\ (x,s)&\longmapsto (h_{t,s}(x),s).\end{aligned}$$

Eventually, we obtain a new fibration $\widetilde{f}_1$. By construction, $\widetilde{f}_1$ can be obtained from the original fibration f by the homotopies f_t and $\widetilde{f}_t$. In these homotopies, the image of singular loci are changed like Reidemeister move of type II (cf Figure 5). As is called in [21], we call this kind of move an *R_2–move*.

As mentioned in the beginning of this section, we can identify $f^{-1}(\{\frac{1}{2}\}\times\{0\})$ with the closed surface Σ_{g+1}. Thus, we can take the monodromy $\varphi_\gamma\in\mathcal{M}_{g+1}$ of $\widetilde{f}_1$ along γ. Since φ is contained in the group $\mathrm{Mod}(\Sigma_{g+1})(c,d)$ and an identification $f^{-1}(\{\frac{1}{2}\}\times\{0\})\cong\Sigma_{g+1}$ is unique up to Dehn twist t_c, the mapping class φ_γ is independent of an identification $f^{-1}(\{\frac{1}{2}\}\times\{0\})\cong\Sigma_{g+1}$.

Lemma 3.8 *Let $\widetilde{\delta}(\eta)\subset\Sigma_{g+1}$ be the simple closed curve corresponding to the boundary of a regular neighborhood of $\eta\in\Pi(\Sigma_{g-1},v_i,w_i)$, for $i\in\{1,2\}$, under the identifications $\Sigma_{g-1}\setminus\{v_1,v_2,w_1,w_2\}\cong\Sigma\setminus(\{v_1,v_2\}\cup d)\cong\Sigma_{g+1}\setminus(c\cup d)$. Then $\varphi_\gamma=t_{\widetilde{\delta}(\eta)}\cdot t_c^{-1}\cdot t_d^{-1}$.*

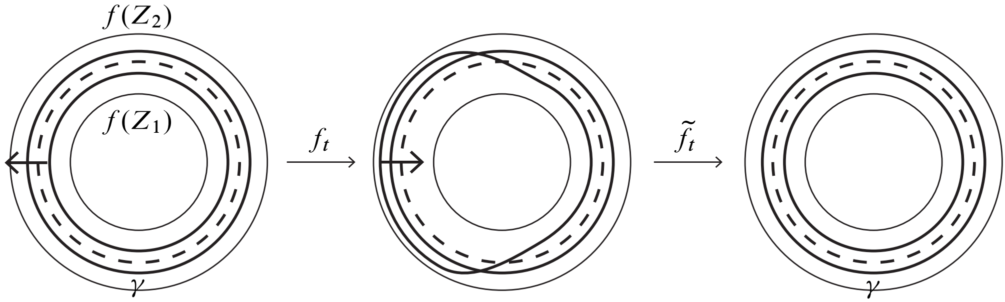

Figure 5: Left: the image of singular loci of $f = f_0$; the bold circles describe the image $f(Z_1) \amalg f(Z_2)$ and the bold dotted circle describes γ; center: the image of singular loci of $f_1 = \widetilde{f}_0$; right: the image of singular loci of $\widetilde{f}_1$, which corresponds to that of f

Proof of Lemma 3.8 Since both sides of the boundary ∂M are trivial surface bundles, φ_γ is contained in the group $\operatorname{Ker} \Phi_c \cap \operatorname{Ker} \Phi_d$. We consider the element $\Phi_c^*(\varphi_\gamma) \in \operatorname{Mod}(\Sigma_g; v_1, v_2)(d)$. This element can be realized as the monodromy of a certain fibration in the following way. We first take a sufficiently small neighborhood of the following subset of M:

$$\coprod_{s\in[0,\frac{1}{3}]\amalg[\frac{2}{3},1]} \Big((D_l^{\mathcal{H}_{h_{\rho(s)}}}(p_1) \cup D_u^{\mathcal{H}_{h_{\rho(s)}}}(p_1)) \times \{s\}\Big) \times \coprod_{s\in[\frac{1}{3},\frac{2}{3}]} \Big((D_l^{\mathcal{H}^\eta_{3s-1}}(p_1) \cup D_u^{\mathcal{H}^\eta_{3s-1}}(p_1)) \times \{s\}\Big)$$

We denote this neighborhood by $U \subset M$. The restriction $\widetilde{f}_0|_{M\setminus U}$ is a fibration with a connected singular locus Z_2. We take a suitable U so that we can take a horizontal distribution $\widetilde{\mathcal{H}}$ of $\widetilde{f}_0|_{M\setminus(U\cup Z_2)}$ satisfying the following conditions:

- $\widetilde{\mathcal{H}}$ is tangent to ∂U.
- $\widetilde{\mathcal{H}}$ equals $\coprod_{s\in S^1}(\mathcal{H}_{h_{\rho(s)}} \oplus T_s S^1)$ on a small neighborhood of
$$\partial N \times S^1 \subset M \quad \text{and} \quad \widetilde{f}_0^{-1}(I \times ([0, \tfrac{1}{6}] \cup [\tfrac{5}{6}, 1])) \subset M.$$

This distribution gives a monodromy of $\widetilde{f}_0|_{M\setminus\bar{U}}$ along γ. Identify $\Sigma = \Sigma_g \times \{0\} \subset \partial N$ with Σ_g. The fiber $\widetilde{f}_0^{-1}(\{\frac{1}{2}\} \times \{0\}) \setminus \bar{U}$ is canonically identified with $\Sigma_g \setminus \{v_1, v_2\}$. By the condition (k) on the family of homotopies $\{h_{t,s}\}$, this monodromy is $\Phi_c^*(\varphi_\gamma)$.

Since the region $[0, \frac{1}{2}] \times S^1$ does not contain any singular values of the fibration $\widetilde{f}_0|_{M\setminus U}$, $\Phi_c^*(\varphi_\gamma)$ corresponds to the monodromy of $\widetilde{f}_0|_{M\setminus U}$ along the loop

$$\widetilde{\gamma}\colon \mapsto \begin{cases} (0,t) & t\in[0,\frac{1}{3}], \\ (\frac{9}{2}(t-\frac{1}{3}),\frac{1}{3}) & t\in[\frac{1}{3},\frac{4}{9}], \\ (\frac{1}{2},3(t-\frac{1}{3})) & t\in[\frac{4}{9},\frac{5}{9}], \\ (\frac{9}{2}(\frac{2}{3}-t),\frac{2}{3}) & t\in[\frac{5}{9},\frac{2}{3}], \\ (0,t) & t\in[\frac{2}{3},1], \end{cases}$$

where $t\in I$ and the right-hand side is in $I\times S^1$. We denote by $\psi_t\colon \widetilde{f}_0^{-1}(\widetilde{\gamma}(0))\cong \Sigma_g\to \widetilde{f}_0^{-1}(\widetilde{\gamma}(t))$ the diffeomorphism obtained by using the distribution $\widetilde{\mathcal{H}}$ and the path $\widetilde{\gamma}|_{[0,t]}$. Note that we can canonically identify $\widetilde{f}_0^{-1}(\widetilde{\gamma}(t))$ with Σ_g for $t\in[0,\frac{1}{3}]\amalg[\frac{2}{3},1]$. Moreover, under the identification, ψ_t is the identity for $t\in[0,\frac{1}{3}]$, and $\psi_t=\psi_1$ for $t\in[\frac{2}{3},1]$ since $\widetilde{\mathcal{H}}$ equals $\coprod_{s\in S^1}(\mathcal{H}_{h_{\rho(s)}}\oplus T_sS^1)$ on $\partial N\times S^1$.

We can take the following diffeomorphism by using the horizontal distribution $\widetilde{\mathcal{H}}$ of $\widetilde{f}_0|_{M\setminus Z_1\cup Z_2}$ together with its horizontal lifts of $t\mapsto(t,s)\in I\times S^1$:

$$\widetilde{\psi}_s\colon \Sigma_g\cong \widetilde{f}_0^{-1}((0,s))\to \widetilde{f}_0^{-1}((\tfrac{1}{2},s))(s\in[\tfrac{1}{3},\tfrac{2}{3}])$$

By the definitions of ψ_t and $\widetilde{\psi}_s$, we obtain the equalities

$$\begin{aligned} \widetilde{\psi}_{1/3}^{-1}\circ\psi_{\frac{4}{9}} &= \mathrm{id}_{\Sigma_g}, \\ \widetilde{\psi}_{2/3}^{-1}\circ\psi_{\frac{5}{9}} &= \psi_1, \\ \widetilde{\psi}_{3(t-(1/3))}^{-1}\circ\psi_t(v_i) &= L(\eta)(9t-4) \quad \text{for } t\in[\tfrac{4}{9},\tfrac{5}{9}]. \end{aligned}$$

This means that the path $[0,1]\ni t\mapsto \widetilde{\psi}_{(1/3)(t+1)}^{-1}\circ\psi_{(1/9)(t+4)}\in \mathrm{Diff}^+(\Sigma_g;v_j)$ is the lift of the loop $L(\eta)$ in $\Sigma_g\setminus\{v_j\}$ under the locally trivial fibration

$$\mathrm{Diff}^+(\Sigma_g;v_i,v_j)\hookrightarrow \mathrm{Diff}^+(\Sigma_g;v_j)\xrightarrow{\varepsilon}\Sigma_g\setminus\{v_j\},$$

where ε is the evaluation map. Thus, we obtain

$$\Phi_c^*(\varphi_\gamma)=[\psi_1]=\mathrm{Push}(L(\eta))=t_{\widetilde{\delta}(\eta)}\cdot t_d^{-1}\in \mathrm{Mod}(\Sigma_g;v_1,v_2)(d),$$

where $\mathrm{Push}(L(\eta))$ is the pushing map along $L(\eta)$. By Lemma 3.1 or 3.5, we have that $\Phi_c^*|_{\mathrm{Ker}\,\Phi_c\cap\mathrm{Ker}\,\Phi_d}$ is an isomorphism. We therefore obtain

$$\begin{aligned} \varphi_\gamma &= \Phi_c^{*,-1}\circ\Phi_c^*(\varphi_\gamma) \\ &= \Phi_c^{*,-1}(t_{\widetilde{\delta}(\eta)}\cdot t_d^{-1}) = t_{\widetilde{\delta}(\eta)}\cdot t_c^{-1}\cdot t_d^{-1}. \end{aligned}$$

This completes the proof of Lemma 3.8. □

Combining Theorems 3.4 and 3.7, we obtain the following theorem.

Theorem 3.9 *Let $f\colon M \to I \times S^1$ and $\gamma \subset I \times S^1$ be as in the beginning of this section. Assume that g is greater than or equal to 3 when (c,d) is not a bounding pair, and that both of the numbers g_1 and $g_2 = g - g_1$ are greater than or equal to 2 when (c,d) is a bounding pair of genus g_1. For any $\varphi \in \operatorname{Ker}\Phi_c \cap \operatorname{Ker}\Phi_d$, we can change f by successive application of R_2–moves so that the monodromy of $f|_{M\setminus(f^{-1}(f(Z_1))\cup f^{-1}(Z_2))}$ along γ corresponds to the element φ.*

4 Relation between vanishing cycles and flip and slip moves

Let $f\colon M \to D^2$ be a purely wrinkled fibration satisfying the following conditions:

(1) The set of critical points $\mathcal{S}_f$ of f is an embedded circle in $\operatorname{Int} M$.

(2) The restriction $f|_{\mathcal{S}_f}$ is an embedding.

(3) Either of the following conditions on regular fibers holds:
- A regular fiber on the outside of $f(\mathcal{S}_f)$ is connected, while that on the inside of $f(\mathcal{S}_f)$ is disconnected.
- Every regular fiber is connected and the genus of a regular fiber on the outside of $f(\mathcal{S}_f)$ is higher than that on the inside of $f(\mathcal{S}_f)$.

We fix a point $p_0 \in \partial D^2$ and an identification $f^{-1}(p_0) \cong \Sigma_g$. Let $\varphi_0 \in \mathcal{M}_g$ be the monodromy along ∂D^2 oriented counterclockwise around the origin of D^2 with base point p_0. In this section, we will give an algorithm to obtain vanishing cycles in a single higher genus regular fiber of a fibration obtained by applying flip and slip to f.

We first consider the simplest case, that is, assume that f has no cusps. We take a reference path γ_0 in ∂D^2 connecting p_0 to a point in the image of indefinite folds so that it satisfies $\operatorname{Int}\gamma_0 \cap f(\mathcal{S}_f) = \varnothing$. This determines a vanishing cycle $c \subset \Sigma_g$ of indefinite folds. Then, it is easy to prove that φ_0 is contained in the group $\operatorname{Ker}\Phi_c$. To give an algorithm precisely, we prepare several conditions. The first condition is on an embedded path $\alpha \subset \Sigma_g$.

Condition $C_1(c)$ A path $\alpha \subset \Sigma_g$ intersects c at the unique point $q \in c$ transversely.

We take a path $\alpha \subset \Sigma_g$ so that α satisfies the condition $C_1(c)$. We put $\partial\alpha = \{w_1, w_2\}$. The second condition is on a simple closed curve $d \subset \Sigma_{g+1}$ and a diffeomorphism $j\colon \Sigma_g \setminus \{w_1, w_2\} \to \Sigma_{g+1} \setminus d$.

Condition $C_2(c,\alpha)$ The closure of $j(\operatorname{Int}\alpha)$ in Σ_{g+1} is a simple closed curve.

We take a simple closed curve $d \subset \Sigma_{g+1}$ and a diffeomorphism $j\colon \Sigma_g \setminus \{w_1, w_2\} \to \Sigma_{g+1} \setminus d$ so that they satisfy the condition $C_2(c,\alpha)$. We put $\widetilde{c} = j(c)$.

The last condition is on an element $\varphi \in \mathrm{Mod}(\Sigma_{g+1})(\widetilde{c}, d)$.

Condition $C_3(c, \alpha, d, j, \varphi_0)$ We have $\Phi_{\widetilde{c}}(\varphi) = 1$ in $\mathrm{Mod}(\Sigma_g)(d)$ and $\Phi_d(\varphi) = \varphi_0^{-1}$ in $\mathrm{Mod}(\Sigma_g)(c)$.

For the sake of simplicity, we will call the above conditions C_1, C_2 and C_3 if elements c, α, d, j and φ_0 are obvious. Examples of α, d, j are described in Figure 6.

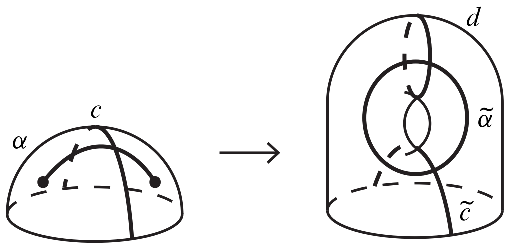

Figure 6: Examples of α, d, j

Theorem 4.1 *Let $f\colon M \to D^2$ be a purely wrinkled fibration we took in the beginning of this section. We assume that f has no cusps.*

(1) *Let $\widetilde{f}$ be a fibration obtained by applying flip and slip to f. We take a point q_0 in the inside of $f(\mathcal{S}_{\widetilde{f}})$, and reference paths $\widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\gamma}_3$ and $\widehat{\gamma}_4$ in D^2 connecting q_0 to a point on the respective fold arcs between cusps so that these paths appear in this order when we go around q_0 counterclockwise. Denote by $e_i \subset \widetilde{f}^{-1}(q_0)$ a vanishing cycle determined by the path $\widehat{\gamma}_i$. Then, there exist an identification $\widetilde{f}^{-1}(q_0) \cong \Sigma_{g+1}$ and elements α, d, j and φ satisfying the conditions C_1, C_2 and C_3 such that the following equality holds up to cyclic permutation:*

$$(e_1, e_2, e_3, e_4) = (\widetilde{c}, \alpha', d, \widetilde{\alpha}),$$

where $\widetilde{\alpha}$ is the closure of $j(\mathrm{Int}\,\alpha)$ in Σ_{g+1} and $\alpha' = \varphi^{-1}(\widetilde{\alpha})$ (see Figure 6).

(2) *Let α, d, j and φ be elements satisfying the conditions C_1, C_2 and C_3. We take simple closed curves $\widetilde{c}, \widetilde{\alpha}$ and α' as in (1). Suppose that the genus of a higher genus fiber g of f is greater than or equal to 3 when $(\widetilde{c}, d)$ is not a bounding pair, and that both of the genera g_1 and g_2 are greater than or equal to 2 when $(\widetilde{c}, d)$ is a bounding pair of genus g_1, where we put $g_2 = g - g_1$. Then, there exists a fibration $\widetilde{f}$ obtained by applying flip and slip to f such that, for reference paths $\widehat{\gamma}_1, \ldots, \widehat{\gamma}_4$ as in (1), the corresponding vanishing cycles $e_1, \ldots, e_4$ satisfy the following equality up to cyclic permutation:*

$$(e_1, e_2, e_3, e_4) = (\widetilde{c}, \alpha', d, \widetilde{\alpha})$$

Proof of Theorem 4.1(1) As in Figure 7, we take points $q_0, q_0', q_0'', q_1, q_1', q_1'' \in D^2$ and paths $\widetilde{\gamma}_0, \widetilde{\gamma}_1, \widetilde{\gamma}_2, \delta_0, \delta_1 \subset D^2$.

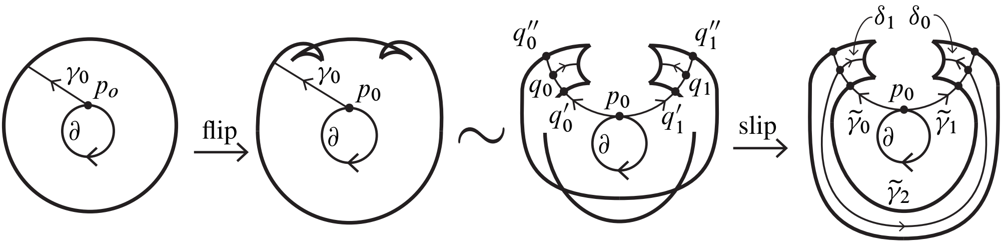

Figure 7: The points q_0, q_1 are in the region with the highest genus fibers, while the points q_0', q_0'', q_1', q_1'' are on the set of critical values. The path $\widetilde{\gamma}_0$ connects p_0 to q_0'' and the path δ_0 connects q_0 to a point in the set of critical values. We take $\widetilde{\gamma}_1$ and δ_1 similarly. The path $\widetilde{\gamma}_2$ connects q_0 to q_1. Note that we regard D^2 as a subset of $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and that $\infty \in D^2$.

We take an identification of the region $\Omega \subset D^2$ described in Figure 8 with the rectangle $I \times I$ so that the paths $\widetilde{\gamma}_0, \widetilde{\gamma}_1$ are contained in the side edges of the rectangle, that the path $\widetilde{\gamma}_2$ corresponds to the middle horizontal line, and the set of critical values corresponds to the upper and the lower horizontal lines (see the right side of Figure 8). For each $x \in \widetilde{\gamma}_2$, we denote by u_x (resp. l_x) the vertical path which connects x to the upper (resp. lower) singular image as in the right side of Figure 8.

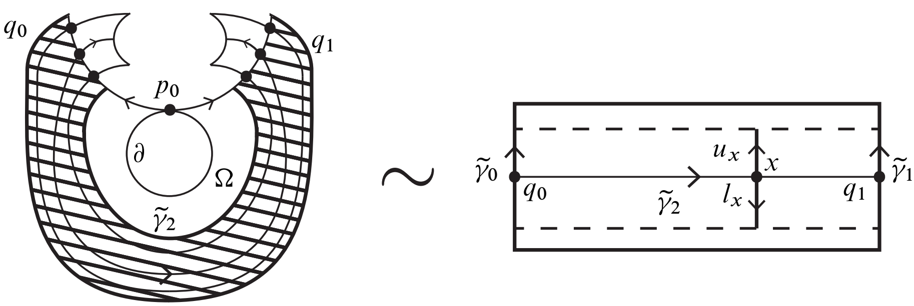

Figure 8: The shaded region in the left figure is the region Ω. The horizontal line with arrow in the right figure describes the path $\widetilde{\gamma}_2$, while the horizontal dotted lines describe images of the singular loci.

We take a horizontal distribution $\mathcal{H}$ of $\widetilde{f}|_{M \setminus \mathcal{S}_{\widetilde{f}}}$ so that it satisfies the following conditions.

(1) Let $w_1^{(i)}, w_2^{(i)}$ be points in $\widetilde{f}^{-1}(p_0)$ which converge to an indefinite fold when $\widetilde{f}^{-1}(p_0)$ approaches the singular fiber $\widetilde{f}^{-1}(q_i')$ along $\widetilde{\gamma}_i$ using $\mathcal{H}$. The set $\{w_1^{(0)}, w_2^{(0)}\}$ equals the set $\{w_1^{(1)}, w_2^{(1)}\}$.

(2) Let $d^{(i)}$ (resp. $\tilde{c}^{(i)}$) be simple closed curves in $\tilde{f}^{-1}(q_i)$ which converge to an indefinite fold when $\tilde{f}^{-1}(q_i)$ approaches the singular fiber $\tilde{f}^{-1}(q_i')$ (resp. $\tilde{f}^{-1}(q_i'')$) along $\tilde{\gamma}_i$ using $\mathcal{H}$. For each $i=0,1$, $d^{(i)}$ is disjoint from $\tilde{c}^{(i)}$.

(3) We obtain a diffeomorphism $j_i\colon \tilde{f}^{-1}(p_0)\setminus\{w_1,w_2\}\to\tilde{f}^{-1}(q_i)\setminus d^{(i)}$ by using a horizontal lift of the curve $\tilde{\gamma}_i$. By condition (2), $j_i^{-1}(\tilde{c}^{(i)})$ is a simple closed curve in $\tilde{f}^{-1}(p_0)$. $j_0^{-1}(\tilde{c}^{(0)})=j_1^{-1}(\tilde{c}^{(1)})=c$.

(4) Let $\tilde{\alpha}^{(i)}$ be a simple closed curve in $\tilde{f}^{-1}(q_i)$ which converges to an indefinite fold when $\tilde{f}^{-1}(p_0)$ approaches a singular fiber along δ_i using $\mathcal{H}$. $\tilde{\alpha}^{(i)}$ intersects both of the curves $\tilde{c}^{(i)}$ and $d^{(i)}$ transversely.

(5) We have $\sharp(\tilde{\alpha}^{(i)}\cap d^{(i)})=\sharp(\tilde{c}^{(i)}\cap\tilde{\alpha}^{(i)})=1$.

(6) By conditions (4) and (5), the closure of $j_i^{-1}(\tilde{\alpha}^{(i)}\setminus d^{(i)})$ is a segment between $w_1^{(i)}$ and $w_2^{(i)}$. The closure of $j_0^{-1}(\tilde{\alpha}^{(0)}\setminus d^{(0)})$ equals the closure of $j_1^{-1}(\tilde{\alpha}^{(1)}\setminus d^{(1)})$.

(7) Since the path $\tilde{\gamma}_2$ does not contain any critical values of $\tilde{f}$, this path, together with $\mathcal{H}$, gives a diffeomorphism from $\tilde{f}^{-1}(q_0)$ to $\tilde{f}^{-1}(x)$ for each $x\in\tilde{\gamma}_2$. This diffeomorphism sends the curve $d^{(0)}$ (resp. $\tilde{c}^{(0)}$) to the curve d_x (resp. $\tilde{c}_x$), where d_x (resp. $\tilde{c}_x$) is a simple closed curve in $\tilde{f}^{-1}(x)$ which converges to an indefinite fold when $\tilde{f}^{-1}(x)$ approaches a singular fiber along u_x (resp. l_x) using $\mathcal{H}$.

We choose indices of $w_1^{(i)}$ and $w_2^{(i)}$ so that $w_1^{(0)}$ corresponds to $w_1^{(1)}$ and we put $w_i=w_i^{(0)}=w_i^{(1)}$. We denote by α the closure of $j_0^{-1}(\tilde{\alpha}^{(0)}\setminus d^{(0)})$ (which corresponds to the closure of $j_1^{-1}(\tilde{\alpha}^{(1)}\setminus d^{(1)})$). Since we fixed an identification $\tilde{f}^{-1}(p_0)\cong\Sigma_g$, we can regard w_1,w_2 as points in Σ_g. We can also regard α as a segment in Σ_g between w_1 and w_2. We choose an identification $\Sigma_g\setminus\{w_1,w_2\}\cong\Sigma_{g+1}\setminus d$, where $d\subset\Sigma_{g+1}$ is a nonseparating simple closed curve, so that the induced identification between $\Sigma_{g+1}\setminus d$ and $\tilde{f}^{-1}(q_i)\setminus d^{(i)}$ can be extended to an identification between Σ_{g+1} and $\tilde{f}^{-1}(q_i)$ (to take such an identification, we modify $\mathcal{H}$ if necessary). By using this identification, we can regard $\tilde{c}^{(i)}$ as a curve in Σ_{g+1}, which we denote by $\tilde{c}$. We denote the identification between Σ_{g+1} and $\tilde{f}^{-1}(q_i)$ as

$$\theta_i\colon \Sigma_{g+1}\xrightarrow{\cong}\tilde{f}^{-1}(q_i),\quad i=0,1.$$

On the other hand, we obtain a diffeomorphism between $\tilde{f}^{-1}(q_0)$ and $\tilde{f}^{-1}(q_1)$ by taking horizontal lifts of $\tilde{\gamma}_2$ using $\mathcal{H}$. We denote this diffeomorphism as

$$\theta_2\colon \hat{f}^{-1}(q_0)\xrightarrow{\cong}\hat{f}^{-1}(q_1).$$

By condition (7), the diffeomorphism sends $d^{(0)}$ (resp. $\tilde{c}^{(0)}$) to the curve $d^{(1)}$ (resp. $\tilde{c}^{(1)}$). Thus, the isotopy class $[\theta_1^{-1}\circ\theta_2\circ\theta_0]$ is contained in the subgroup

$\mathrm{Mod}\,(\Sigma_{g+1})(\widetilde{c},d)$ of the mapping class group $\mathcal{M}_{g+1}$. We denote this class by $\varphi \in \mathrm{Mod}\,(\Sigma_{g+1})(\widetilde{c},d)$.

We denote by $\widetilde{\gamma}_2 \cdot \delta_1$ be the path in D^2, starting at the point q_0, obtained by connecting $\widetilde{\gamma}_2$ to δ_1. This path gives the fiber $\widetilde{f}^{-1}(q_0)$ a vanishing cycle of $\widetilde{f}$. This vanishing cycle is equal to the curve $\theta_2^{-1}(\widetilde{\alpha}^{(1)}) = \theta_2^{-1} \circ \theta_1(\widetilde{\alpha})$. This curve corresponds to the curve $\theta_0^{-1} \circ \theta_2^{-1} \circ \theta_1(\widetilde{\alpha}) = \varphi^{-1}(\widetilde{\alpha}) \subset \Sigma_{g+1}$ under the identification θ_0. Thus, the proof is completed once we prove the following lemma.

Lemma 4.2 *We have $\Phi_{\widetilde{c}}(\varphi) = 1$ and $\Phi_d(\varphi) = \varphi_0^{-1}$.*

Proof of Lemma 4.2 The image $\Phi_d(\varphi)$ is equal to the monodromy along the curve δ_h described in the left side of Figure 9, which corresponds to φ_0^{-1}.

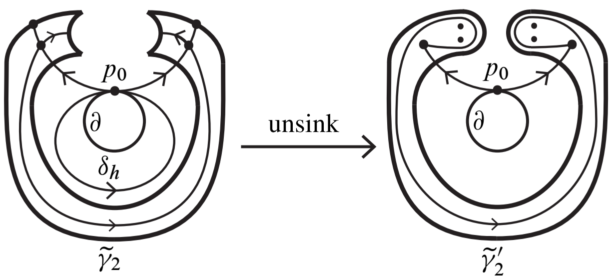

Figure 9: Base diagrams of fibrations

Thus, we have $\Phi_d(\varphi) = \varphi_0^{-1}$. To prove that $\Phi_{\widetilde{c}}(\varphi) = 1$, we consider the fibration obtained by applying an unsink to $\widetilde{f}$. We take the path $\widetilde{\gamma}_2'$ connecting q_0 to q_1 as in the right side of Figure 9. It is easy to see that the monodromy along this path corresponds to $(t_{t_d(\widetilde{\alpha})} \cdot t_{t_{\widetilde{\alpha}}(\widetilde{c})}) \cdot \varphi \cdot (t_{t_d(\widetilde{\alpha})} \cdot t_{t_{\widetilde{\alpha}}(\widetilde{c})})^{-1}$. This preserves the curve d and the image $\Phi_d((t_{t_d(\widetilde{\alpha})} \cdot t_{t_{\widetilde{\alpha}}(\widetilde{c})}) \cdot \varphi \cdot (t_{t_d(\widetilde{\alpha})} \cdot t_{t_{\widetilde{\alpha}}(\widetilde{c})})^{-1})$ is trivial since this element is the monodromy along the curve obtained by pushing the curve $\widetilde{\gamma}_2'$ out of the region with higher genus fibers, which is nullhomotopic in the complement of the set of critical values. We can obtain the element $\Phi_{\widetilde{c}}(\varphi)$ by taking some conjugation of $\Phi_d((t_{t_d(\widetilde{\alpha})} \cdot t_{t_{\widetilde{\alpha}}(\widetilde{c})}) \cdot \varphi \cdot (t_{t_d(\widetilde{\alpha})} \cdot t_{t_{\widetilde{\alpha}}(\widetilde{c})})^{-1})$. In particular, $\Phi_{\widetilde{c}}(\varphi)$ is also trivial and this completes the proof of Lemma 4.2. □

This completes the proof of Theorem 4.1(1). □

Proof of Theorem 4.1(2) In the proof of Theorem 4.1(1), we take a horizontal distribution of $\widetilde{f}|_{M \setminus \mathcal{S}_{\widetilde{f}}}$ and an identification $\Sigma_g \setminus \{w_1, w_2\} \cong \Sigma_{g+1} \setminus d$. Once we take these auxiliary data, we can get vanishing cycles of $\widetilde{f}$ in a canonical way. We first take a horizontal distribution of $\widetilde{f}|_{M\setminus\mathcal{S}_{\widetilde{f}}}$ so that the embedded path $\alpha\subset\Sigma_g$ determined by the distribution corresponds to the given one. We next take an identification $\Sigma_g\setminus\{w_1,w_2\}\cong\Sigma_{g+1}\setminus d$ by using the given d,j. The element $[\theta_1^{-1}\circ\theta_2\circ\theta_0]$, which appears in the proof of Theorem 4.1(1), is canonically determined by the chosen horizontal distribution of $\widetilde{f}|_{M\setminus\mathcal{S}_{\widetilde{f}}}$ of M and the chosen homotopy from f.

Let Ω be the region in D^2 as in Figure 8. We take an identification $\Omega\cong I\times\widetilde{\gamma}_2$. We also take a diffeomorphism $\Theta\colon \widetilde{f}^{-1}(\widetilde{\gamma}_0\cap\Omega)\to\widetilde{f}^{-1}(\widetilde{\gamma}_1\cap\Omega)$ so that it satisfies $\widetilde{f}\circ\Theta=i\circ\widetilde{f}$, where $i\colon I\times\{q_0\}\ni(t,q_0)\mapsto(t,q_1)\in I\times\{q_1\}$, and that the 4–manifold $\widetilde{f}^{-1}(\Omega)/\Theta$ is the trivial N–bundle over S^1, where N is a 3–manifold defined in Section 3. For any two elements $\varphi_1,\varphi_2\in\mathrm{Mod}\,(\Sigma_{g+1};\widetilde{c},d)$ satisfying the condition C_3, the element $\varphi_1\cdot\varphi_2^{-1}$ is contained in the group $\mathrm{Ker}\,\Phi_{\widetilde{c}}\cap\mathrm{Ker}\,\Phi_d$. Thus, Theorem 3.9 implies that we can change f into $\widetilde{f}$ by a flip and slip move so that the resulting element $[\theta_1^{-1}\circ\theta_2\circ\theta_0]$ corresponds to $\varphi^{-1}\in\mathrm{Mod}\,(\Sigma_{g+1})(\widetilde{c},d)$ for the given φ. This completes the proof of Theorem 4.1(2). □

We next consider the case that f has cusps. We denote by $\{s_1,\dots,s_n\}$ the set of cusps of f. We put $u_i=f(s_i)$. The indices of s_i are chosen so that $u_1,\dots,u_n$ appear in this order when we travel the image $f(\mathcal{S}_f)$ clockwise around a point inside $f(\mathcal{S}_f)$. The points $u_1,\dots,u_n$ divide the image $f(\mathcal{S}_f)$ into n edges. We denote by $l_i\subset f(\mathcal{S}_f)$ the edge between u_i and u_{i+1}, where we put $u_{n+1}=u_1$. For a point $p_0\in\partial D^2$, we take reference paths $\gamma_1,\dots,\gamma_n\subset D^2$ satisfying the following conditions (see also the left figure of Figure 10):

- γ_i connects p_0 to a point in $\mathrm{Int}\,l_i$.
- $\gamma_i\cap\gamma_j=\{p_0\}$ for all $i\neq j$
- $\mathrm{Int}\,\gamma_i\cap f(\mathcal{S}_f)=\varnothing$
- $\gamma_1,\dots,\gamma_n$ appear in that order when we go around p_0 counterclockwise.

Let γ_{n+1} be a path obtained by connecting ∂D^2 oriented clockwise around the center of D^2 to γ_1. The paths give $f^{-1}(p_0)\cong\Sigma_g$ vanishing cycles $c_1,\dots,c_{n+1}$. Note that, for each $i\in\{1,\dots,n\}$, c_i intersects c_{i+1} at a unique point transversely. In particular, every simple closed curve c_i is nonseparating. We also remark that c_{n+1} equals $\varphi_0(c_1)$.

Let $\widehat{f}\colon M\to D^2$ be the fibration obtained by changing all the cusp singularities of f into Lefschetz singularities by applying unsink to f n times. We take paths $\varepsilon_1,\dots,\varepsilon_n$ in D^2 satisfying the following conditions (see also the middle figure of Figure 10):

- ε_i connects p_0 to the image of the Lefschetz singularity derived from s_{i+1}.
- $\varepsilon_i\cap\varepsilon_j=\{p_0\}$ for all $i\neq j$
- $\mathrm{Int}\,\varepsilon_i\cap\widehat{f}(\mathcal{S}_{\widehat{f}})=\varnothing$

- $\gamma_1, \varepsilon_1, \gamma_2, \dots, \gamma_n, \varepsilon_n, \gamma_{n+1}$ appear in that order when we go around p_0 counterclockwise.

The path ε_i gives a vanishing cycle of a Lefschetz critical point of $\widehat{f}$, the curve $t_{c_i}(c_{i+1})$. Let γ_0 be a based loop in $D^2 \setminus \widehat{f}(\mathcal{S}_{\widehat{f}})$ with base point p_0 homotopic to the loop obtained by connecting p_0 to $\widehat{f}(\mathcal{S}_{\widehat{f}})$ oriented counterclockwise around a point inside $f(\mathcal{S}_f)$ using γ_1 (see the right figure of Figure 10). It is easy to see that the monodromy along γ_0 corresponds to the element

$$\widehat{\varphi}_0 = \varphi_0 \cdot \left(t_{t_{c_1}(c_2)} \cdots t_{t_{c_n}(c_{n+1})}\right)^{-1}.$$

This element preserves the curve c_1 and is contained in the kernel of the homomorphism Φ_{c_1}.

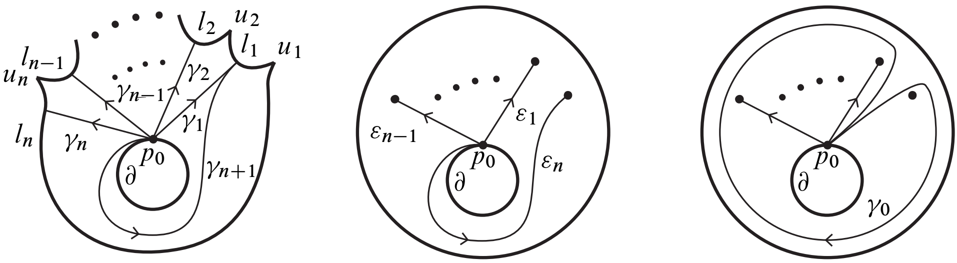

Figure 10: Left: the image of the critical locus of f and the reference paths $\gamma_1, \dots, \gamma_{n+1}$; middle: the image of the critical locus of $\widehat{f}$ and the reference paths $\varepsilon_1, \dots, \varepsilon_n$; right: the loop γ_0

Since application of flip and slip to f is equivalent to application of flip and slip to $\widehat{f}$ followed by application of sink n times, we can obtain vanishing cycles of a fibration obtained by applying flip and slip to f in the way quite similar to that in the case f has no cusps. In order to give the precise algorithm to obtain vanishing cycles, we prepare several conditions.

Condition $\widetilde{C}_1(c_1, \dots, c_n)$ A path $\alpha \subset \Sigma_g$ intersects c_1 at the unique point $q \in c_1$ transversely. Furthermore, $\partial\alpha \cap (c_1 \cup \cdots \cup c_{n+1}) = \varnothing$.

We take a path $\alpha \subset \Sigma_g$ so that α satisfies the condition $\widetilde{C}_1(c_1, \dots, c_n)$. We put $\partial\alpha = \{w_1, w_2\}$. The second condition is on a simple closed curve $d \subset \Sigma_{g+1}$ and a diffeomorphism $j\colon \Sigma_g \setminus \{w_1, w_2\} \to \Sigma_{g+1} \setminus d$.

Condition $\widetilde{C}_2(c_1, \dots, c_n, \alpha)$ The closure of $j(\operatorname{Int}\alpha)$ in Σ_{g+1} is a simple closed curve.

We take a simple closed curve $d \subset \Sigma_{g+1}$ and a diffeomorphism $j\colon \Sigma_g \setminus \{w_1, w_2\} \to \Sigma_{g+1} \setminus d$ so that they satisfy the condition $\widetilde{C}_2(c_1, \dots, c_n, \alpha)$. We put $\widetilde{c}_1 = j(c_1)$. The third condition is on an element $\varphi \in \operatorname{Mod}(\Sigma_{g+1})(\widetilde{c}_1, d)$.

Condition $\widetilde{C}_3(c_1, \dots, c_n, \alpha, d, j, \varphi_0)$ We have $\Phi_{\widetilde{c}_1}(\varphi) = 1$ in $\mathrm{Mod}(\Sigma_g)(d)$ and $\Phi_d(\varphi) = \widehat{\varphi}_0^{-1}$ in $\mathrm{Mod}(\Sigma_g)(c_1)$.

The last condition is on simple closed curves $\widetilde{c}_2, \dots, \widetilde{c}_{n+1} \subset \Sigma_{g+1} \setminus d$.

Condition $\widetilde{C}_4(c_1, \dots, c_n, \alpha, d, j)$ For each $i \in \{2, \dots, n+1\}$, $i(\widetilde{c}_i)$ is isotopic to c_i in Σ_g, where i is an embedding defined by

$$i\colon \Sigma_{g+1} \setminus d \xrightarrow{j^{-1}} \Sigma_g \setminus \{w_1, w_2\} \hookrightarrow \Sigma_g.$$

Furthermore, for each $i = 1, \dots, n$, $\widetilde{c}_i$ intersects $\widetilde{c}_{i+1}$ at a unique point transversely.

As the case f has no cusps, we will call the above conditions $\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3$ and $\widetilde{C}_4$ if elements $c_1, \dots, c_n, \alpha, d, j$ and φ_0 are obvious. We can prove the following theorem by an argument similar to that in the proof of Theorem 4.1.

Theorem 4.3 *Let $f\colon M \to D^2$ be a purely wrinkled fibration we took in the beginning of this section. Suppose that f has $n > 0$ cusps. We take vanishing cycles $c_1, \dots, c_{n+1}$ as above.*

(1) *Let $\widetilde{f}$ be a fibration obtained by applying flip and slip to f. We take a point q_0 in the inside of $f(\mathcal{S}_{\widetilde{f}})$, and reference paths $\widehat{\gamma}_1, \dots, \widehat{\gamma}_{n+4}$ in D^2 connecting q_0 to a point on the respective fold arcs between cusps so that these paths appear in this order when we go around q_0 counterclockwise. We denote by $e_i \subset \widetilde{f}^{-1}(q_0)$ a vanishing cycle determined by the path $\widehat{\gamma}_i$. Then, there exist an identification $\widetilde{f}^{-1}(q_0) \cong \Sigma_{g+1}$ and elements $\alpha, d, j, \widetilde{c}_2, \dots, \widetilde{c}_{n+1}$ and φ satisfying the conditions $\widetilde{C}_1$, $\widetilde{C}_2$, $\widetilde{C}_3$ and $\widetilde{C}_4$ such that the following equality holds up to cyclic permutation:*

$$(e_1, \dots, e_{n+4}) = (\widetilde{c}_1, \dots, \widetilde{c}_{n+1}, \alpha', d, \widetilde{\alpha}),$$

where $\widetilde{c}_1 = j(c_1)$, $\widetilde{\alpha}$ is the closure of $j(\mathrm{Int}\,\alpha)$ in Σ_{g+1}, and α' is defined as

$$\alpha' = (\varphi^{-1} \cdot t_{t_{\widetilde{c}_1}(\widetilde{c}_2)} \cdots t_{t_{\widetilde{c}_n}(\widetilde{c}_{n+1})})(\widetilde{\alpha}).$$

(2) *Let $\alpha, d, j, \widetilde{c}_2, \dots, \widetilde{c}_{n+1}$ and φ be elements satisfying the conditions $\widetilde{C}_1$, $\widetilde{C}_2$, $\widetilde{C}_3$ and $\widetilde{C}_4$. We take simple closed curves $\widetilde{c}_1, \widetilde{\alpha}$ and α' as in (1). Suppose that the genus of higher genus fibers g of f is greater than or equal to 3. Then, there exists a fibration $\widetilde{f}$ obtained by applying flip and slip to f such that, for reference paths $\widehat{\gamma}_1, \dots, \widehat{\gamma}_{n+4}$ as in (1), the corresponding vanishing cycles $e_1, \dots, e_{n+4}$ satisfy the following equality up to cyclic permutation:*

$$(e_1, \dots, e_{n+4}) = (\widetilde{c}_1, \dots, \widetilde{c}_{n+1}, \alpha', d, \widetilde{\alpha})$$

5 Fibrations with small fiber genera

Although Theorem 4.1(1) holds for a fibration with an arbitrary fiber genera, Theorems 4.1(2) and 4.3 do not hold if genera of fibers are too small. The main reason of this is nontriviality of the group $\pi_1(\mathrm{Diff}^+(\Sigma_{g-1}), \mathrm{id})$ when $g < 3$. To deal with fibrations with small fiber genera, we need to look at additional data on sections of fibrations. Let $f\colon M \to D^2$ be a purely wrinkled fibration we took in the beginning of Section 4.

5.1 Case 1: Every fiber of f is connected

In this subsection, we assume that every fiber of f is connected. We first consider the case f has no cusps. We take a point p_0, an identification $f^{-1}(p_0) \cong \Sigma_g$, a reference path $\gamma_0 \subset D^2$, a vanishing cycle $c \subset \Sigma_g$, and a monodromy $\varphi_0 \in \mathrm{Mod}(\Sigma_g)(c)$ as we took in Section 4. It is easy to see that f has a section. We take a section $\sigma\colon D^2 \to M$ of f. We put $x = \sigma(p_0)$, which is contained in the complement $\Sigma_g \setminus c$. This section gives a lift $\widetilde{\varphi} \in \mathrm{Mod}(\Sigma_g; x)(c)$. It is easy to show that this element is contained in the kernel of the homomorphism

$$\Phi_c^x\colon \mathrm{Mod}(\Sigma_g; x)(c) \to \mathrm{Mod}(\Sigma_{g-1}; x),$$

which is defined as we define Φ_c.

As in Section 4, we give several conditions. The first condition is on an embedded path $\alpha \subset \Sigma_g \setminus \{x\}$.

Condition $C_1'(c, \sigma)$ A path $\alpha \subset \Sigma_g \setminus \{x\}$ intersects c at the unique point $q \in c$ transversely.

We take a path $\alpha \subset \Sigma_g \setminus \{x\}$ so that α satisfies the condition $C_1'(c, \sigma)$. We put $\partial\alpha = \{w_1, w_2\}$. The second condition is on a simple closed curve $d \subset \Sigma_{g+1}$ and a diffeomorphism $j\colon \Sigma_g \setminus \{w_1, w_2\} \to \Sigma_{g+1} \setminus d$.

Condition $C_2'(c, \alpha, \sigma)$ The closure of $j(\mathrm{Int}\,\alpha)$ in Σ_{g+1} is a simple closed curve.

We take a simple closed curve $d \subset \Sigma_{g+1}$ and a diffeomorphism $j\colon \Sigma_g \setminus \{w_1, w_2\} \to \Sigma_{g+1} \setminus d$ so that they satisfy the condition $C_2'(c, \alpha, \sigma)$. We put $\widetilde{c} = j(c)$ and $\widetilde{x} = j(x)$. The last condition is on an element $\varphi \in \mathrm{Mod}(\Sigma_{g+1}; \widetilde{x})(\widetilde{c}, d)$.

Condition $C_3'(c, \alpha, d, j, \varphi_0, \sigma)$ We have that $\Phi_{\widetilde{c}}^{\widetilde{x}}(\varphi) = 1$ in $\mathrm{Mod}(\Sigma_g, \widetilde{x})(d)$ and that $\Phi_d^{\widetilde{x}}(\varphi) = \widetilde{\varphi}_0^{-1}$ in $\mathrm{Mod}(\Sigma_g, x)(c)$.

Theorem 5.1 *Let $f\colon M \to D^2$ be a purely wrinkled fibration as above.*

(1) *Let $\widetilde{f}$ be a fibration obtained by applying flip and slip to f. We take a point q_0, reference paths $\widehat{\gamma}_1, \dots, \widehat{\gamma}_4$ in D^2 and $e_i \subset \widetilde{f}^{-1}(q_0)$ as Theorem 4.1(1). Then, there exist an identification $\widetilde{f}^{-1}(q_0) \cong \Sigma_{g+1}$ and elements α, d, j and φ satisfying the conditions C_1', C_2' and C_3' such that the following equality holds up to cyclic permutation:*

$$(e_1, e_2, e_3, e_4) = (\widetilde{c}, \alpha', d, \widetilde{\alpha}),$$

where $\widetilde{c} = j(c)$, $\widetilde{\alpha}$ is the closure of $j(\operatorname{Int}\alpha)$ in Σ_{g+1}, and $\alpha' = \varphi^{-1}(\widetilde{\alpha})$.

(2) *Let α, d, j and φ be elements satisfying the conditions C_1', C_2' and C_3'. We take simple closed curves $\widetilde{c}, \widetilde{\alpha}$ and α' as in (1). Suppose that the genus g is greater than or equal to 2. Then, there exists a fibration $\widetilde{f}$ obtained by applying flip and slip to f such that, for reference paths $\widehat{\gamma}_1, \dots, \widehat{\gamma}_4$ as in (1), the corresponding vanishing cycles $e_1, \dots, e_4$ satisfy the following equality up to cyclic permutation:*

$$(e_1, e_2, e_3, e_4) = (\widetilde{c}, \alpha', d, \widetilde{\alpha})$$

Proof of Theorem 5.1(1) The proof of Theorem 5.1(1) is quite similar to that of Theorem 4.1(1). The only difference is the following point: instead of a horizontal distribution $\mathcal{H}$ of the fibration $\widetilde{f}|_{M \setminus \mathcal{S}_{\widetilde{f}}}$, we take a horizontal distribution $\mathcal{H}_\sigma$ of the fibration $\widetilde{f}|_{M \setminus \mathcal{S}_{\widetilde{f}}}$, which satisfies the same conditions as that on $\mathcal{H}$, so that it is tangent to the image of the section σ. By using such a horizontal distribution, we can apply all the arguments in the proof of Theorem 4.1 straightforwardly. We omit details of the proof. □

Proof of Theorem 5.1(2) As the proof of (1), the proof of (2) is also similar to that of Theorem 4.1(2). By the same argument as in the proof of Theorem 4.1(2), all we have to prove is that we can take a homotopy from f to $\widetilde{f}$ so that the element $[\theta_1^{-1} \circ \theta_2 \circ \theta_0]$ corresponds to φ^{-1} for given φ.

We take a sufficiently small disk neighborhood D of x in Σ_{g-1}. Denote by $\widetilde{\Sigma}_{g-1}$ the closure of the complement $\Sigma_{g-1} \setminus D$. It is easy to prove that the mapping class group $\pi_0(\mathrm{Diff}^+(\widetilde{\Sigma}_{g-1}), \mathrm{id})$ is isomorphic to $\mathrm{Mod}(\Sigma_{g-1}; x)$, where $\mathrm{Diff}^+(\widetilde{\Sigma}_{g-1})$ is the set of orientation-preserving diffeomorphisms of $\widetilde{\Sigma}_{g-1}$ (note that an element in this group fixes the boundary of $\partial\widetilde{\Sigma}_{g-1}$ set wise, but need not to fix $\partial\widetilde{\Sigma}_{g-1}$ point wise). Moreover, we can obtain similar isomorphisms even if we consider groups of diffeomorphisms with fixed points or sets. It is known that the group $\pi_1(\mathrm{Diff}^+(\widetilde{\Sigma}_{g-1}), \mathrm{id})$ is trivial if g is greater than or equal to 2 (cf Earle and Schatz [10]). With these observations understood, we can prove by the argument similar to that in Section 3 that the group $\operatorname{Ker}\Phi_{\tilde{c}}^{\tilde{x}}\cap\operatorname{Ker}\Phi_{d}^{\tilde{x}}$ is generated by the set

$$\begin{aligned}\{t_{\tilde{\delta}(\eta)}\cdot t_{\tilde{c}}^{-1}\cdot t_d^{-1}\in\operatorname{Mod}(\Sigma_{g+1};\tilde{x})(\tilde{c},d)\mid \eta\in\Pi(\Sigma_{g-1}\setminus\{\tilde{x},v_i,w_j\},v_k,w_l),\\ \{i,k\}=\{j,l\}=\{1,2\}\},\end{aligned}$$

where $\Pi(\Sigma_{g-1}\setminus\{\tilde{x},v_i,w_j\},v_k,w_l)$ and $\tilde{\delta}(\eta)$ are defined as in Section 3. Thus, by the similar argument to that in the proof of Theorem 3.9, we can change $[\theta_1^{-1}\circ\theta_2\circ\theta_0]$ into $[\theta_1^{-1}\circ\theta_2\circ\theta_0]\cdot\psi$ for any $\psi\in\operatorname{Ker}\Phi_{\tilde{c}}^{\tilde{x}}\cap\operatorname{Ker}\Phi_{d}^{\tilde{x}}$ by modifying a flip and slip from f to $\tilde{f}$. This completes the proof of the statement (2). □

We can deal with a fibration with cusps similarly by using sink and unsink as in Section 4. Suppose that f has $n>0$ cusps and we take vanishing cycles $c_1,\dots,c_{n+1}\subset\Sigma_g$ as we took in Section 4. We also take a section $\sigma\colon D^2\to M$ of f. We put $x=\sigma(p_0)$, which is contained in the complement $\Sigma_g\setminus(c_1\cup\cdots\cup c_{n+1})$. This gives a lift $\tilde{\varphi}_0\in\operatorname{Mod}(\Sigma_g;x)(c_1)$ of φ_0. As in Section 4, we put $\hat{\varphi}_0=\tilde{\varphi}_0\cdot(t_{t_{\tilde{c}_1}(\tilde{c}_2)}\cdots t_{t_{\tilde{c}_n}(\tilde{c}_{n+1})})^{-1}$, and we give several conditions on elements $\alpha,d,j,\varphi,\tilde{c}_2,\dots,\tilde{c}_{n+1}$.

Condition $\tilde{C}_1'(c_1,\dots,c_n,\sigma)$ A path $\alpha\subset\Sigma_g\setminus\{x\}$ intersects c_1 at the unique point $q\in c_1$ transversely. Furthermore, $\partial\alpha\cap(c_1\cup\cdots\cup c_{n+1})=\varnothing$.

Condition $\tilde{C}_2'(c_1,\dots,c_n,\alpha,\sigma)$ The closure of $j(\operatorname{Int}\alpha)$ in Σ_{g+1} is a simple closed curve.

Condition $\tilde{C}_3'(c_1,\dots,c_n,\alpha,d,j,\varphi_0,\sigma)$ We have $\Phi_{\tilde{c}_1}^{\tilde{x}}(\varphi)=1$ in $\operatorname{Mod}(\Sigma_g;\tilde{x})(d)$ and $\Phi_d^{\tilde{x}}(\varphi)=\hat{\varphi}_0^{-1}$ in $\operatorname{Mod}(\Sigma_g;x)(c_1)$, where we put $\tilde{x}=j(x)$ and $\tilde{c}=j(c)$.

Condition $\tilde{C}_4'(c_1,\dots,c_n,\alpha,d,j,\sigma)$ For each $i\in\{2,\dots,n+1\}$, $i(\tilde{c}_i)$ is isotopic to c_i in $\Sigma_g\setminus\{x\}$, where i is an embedding defined by

$$i\colon \Sigma_{g+1}\setminus d\xrightarrow{j^{-1}}\Sigma_g\setminus\{w_1,w_2\}\hookrightarrow\Sigma_g.$$

Furthermore, for each $i=1,\dots,n$, $\tilde{c}_i$ intersects $\tilde{c}_{i+1}$ at a unique point transversely.

The following theorem can be proved in a way quite similar to that of the proof of Theorem 5.1.

Theorem 5.2 *Let $f\colon M\to D^2$ be a purely wrinkled fibration as above.*

(1) *Let $\tilde{f}$ be a fibration obtained by applying flip and slip to f. We take a point q_0, reference paths $\hat{\gamma}_1,\dots,\hat{\gamma}_{n+4}$ in D^2, vanishing cycles $e_1,\dots,e_{n+4}\subset\tilde{f}^{-1}(q_0)$ as we took in Theorem 4.3(1). Then, there is an identification $\tilde{f}^{-1}(q_0)\cong\Sigma_{g+1}$ and elements $\alpha,d,j,\tilde{c}_2,\dots,\tilde{c}_{n+1}$ and φ satisfying the conditions $\tilde{C}_1'$, $\tilde{C}_2'$, $\tilde{C}_3'$ and $\tilde{C}_4'$ such that the following equality holds up to cyclic permutation:*

$$(e_1, \dots, e_{n+4}) = (\widetilde{c}_1, \dots, \widetilde{c}_{n+1}, \alpha', d, \widetilde{\alpha}),$$

where $\widetilde{c}_1 = j(c_1)$, $\widetilde{\alpha}$ is the closure of $j(\mathrm{Int}\,\alpha)$ in Σ_{g+1}, and α' is defined as

$$\alpha' = (\varphi^{-1} \cdot t_{t_{\widetilde{c}_1}(\widetilde{c}_2)} \cdots t_{t_{\widetilde{c}_n}(\widetilde{c}_{n+1})})(\widetilde{\alpha})$$

(2) *Let $\alpha, d, j, \widetilde{c}_2, \dots, \widetilde{c}_{n+1}$ and φ be elements satisfying the conditions $\widetilde{C}'_1$, $\widetilde{C}'_2$, $\widetilde{C}'_3$ and $\widetilde{C}'_4$. We take simple closed curves $\widetilde{c}_1, \widetilde{\alpha}$ and α' as in (1). Suppose that the genus g is greater than or equal to 2. Then, there exists a fibration $\widetilde{f}$ obtained by applying flip and slip move to f such that, for a reference path $\widehat{\gamma}_1, \dots, \widehat{\gamma}_{n+4}$ as in (1), the corresponding vanishing cycles $e_1, \dots, e_{n+4}$ satisfy the following equality up to cyclic permutation:*

$$(e_1, \dots, e_{n+4}) = (\widetilde{c}_1, \dots, \widetilde{c}_{n+1}, \alpha', d, \widetilde{\alpha}).$$

5.2 Case 2: f has disconnected fibers

We next consider the case f has disconnected fibers. In this case, f has no cusps. We take a point $p_0 \in \partial D^2$, an identification $f^{-1}(p_0) \cong \Sigma_g$, a reference path γ_0, a vanishing cycle $c \subset \Sigma_g$, and a monodromy $\varphi_0 \in \mathrm{Mod}\,(\Sigma_g)(c)$ as we took in Section 4. We also take a disconnected fiber of f and denote this by $S_1 \amalg S_2$, where S_i is a connected component of the fiber. We take a section $\sigma_i \colon D^2 \to M$ of f which intersects S_i for each $i = 1, 2$. We put $x_i = \sigma_i(p_0)$, which is contained in the complement $\Sigma_g \setminus c$. The sections σ_1 and σ_2 gives a lift $\widehat{\varphi}_0 \in \mathrm{Mod}\,(\Sigma_g; x_1, x_2)(c^{\mathrm{ori}})$, and this element is contained in the kernel of the homomorphism

$$\Phi_c^{x_1, x_2} \colon \mathrm{Mod}\,(\Sigma_g; x_1, x_2)(c^{\mathrm{ori}}) \to \mathrm{Mod}\,(\Sigma_{g_1}; x_1) \times \mathrm{Mod}\,(\Sigma_{g_2}; x_2),$$

where g_i is the genus of the closed surface S_i.

By using this lift, we can apply all the argument in Case 1 straightforwardly, and we can obtain the theorem similar to Theorem 5.1 (we need the assumption $g_1, g_2 \geq 1$). We omit the details of arguments.

Remark 5.3 The statements of Theorems 5.1(2) and 5.2(2) do not hold if $g = 1$ since the group $\pi_1(\mathrm{Diff}^+\,(\widetilde{\Sigma}_0), \mathrm{id})$ is not trivial (cf [10]). To apply the same argument as in the proof of Theorem 5.1(2) to the case $g = 1$, we need to take three disjoint sections of f. We take points $x_1, x_2, x_3 \in S^2$ and a sufficiently small disk neighborhood D_i of $x_i \in S^2$ ($i = 1, 2, 3$). We denote by $S^2_{(3)}$ the closure of the complement $S^2 \setminus (D_1 \sqcup D_2 \sqcup D_3)$. Earle and Schatz [10] showed that the group $\pi_1(\mathrm{Diff}^+\,(S^2_{(3)}), \mathrm{id})$ is trivial, the statement similar to that in Theorems 5.1 and 5.2 hold for a fibration f with $g = 1$ (note that the group $\pi_1(\mathrm{Diff}^+\,(S^2_{(2)}), \mathrm{id})$ is nontrivial, where $S^2_{(2)}$ is the closure of $S^2 \setminus (D_1 \sqcup D_2)$; see [10] for details.) Furthermore, we can deal with a fibration with disconnected fibers which contain spheres as connected components by taking three disjoint sections so that these sections go through the sphere components. We omit, however, details of arguments about this case for simplicity of the paper.

6 Application: Examples of surface diagrams

Williams [21] defined a certain cyclically ordered sequence of nonseparating simple closed curves in a closed surface which describes a 4–manifold. This sequence is obtained by looking at vanishing cycles of a *simplified purely wrinkled fibration*, which is defined below. In this section, we will look at relation between flip and slip and sequences of simple closed curves Williams defined. We will then give some new examples of this sequence.

Definition 6.1 A purely wrinkled fibration $\zeta; M^4 \to S^2$ is called a *simplified purely wrinkled fibration* if it satisfies the following conditions:

(1) All the fiber of ξ are connected.

(2) The set of singularities $\mathcal{S}_\zeta \subset M$ of ζ is connected and nonempty.

(3) The restriction $\zeta|_{\mathcal{S}_\zeta}$ is injective.

It is easy to see that ζ has two types of regular fibers: Σ_g and Σ_{g-1} for some $g \geq 1$. We call the genus g of a higher genus regular fiber the *genus* of ζ. In this paper, we call a simplified purely wrinkled fibration an SPWF for simplicity.

Let ζ: $M \to S^2$ be a genus-g SPWF. We denote by $\{s_1, \ldots, s_n\}$ the set of cusps of f. We put $u_i = f(s_i)$. We take a regular value p_0 of ζ so that the genus of the fiber $\zeta^{-1}(p_0)$ is equal to g. The indices of s_i are chosen so that $u_1, \ldots, u_n$ appear in this order when we travel the image $\zeta(\mathcal{S}_f)$ counterclockwise around p_0. The points $u_1, \ldots, u_n$ divides the image $\zeta(\mathcal{S}_\zeta)$ into n edges. We denote by $l_i \subset \zeta(\mathcal{S}_\zeta)$ the edge between u_i and u_{i+1} (we regard the indices as in $\mathbb{Z}/n\mathbb{Z}$. In particular, $u_{n+1} = u_1$). We take paths $\gamma_1, \ldots, \gamma_n \subset S^2$ satisfying the following conditions:

- γ_i connects p_0 to a point in $\operatorname{Int} l_i$.
- $\operatorname{Int} \gamma_i \cap f(\mathcal{S}_\zeta) = \varnothing$
- $\gamma_i \cap \gamma_j = \{p_0\}$ if $i \neq j$

We fix an identification $\zeta^{-1}(p_0) \cong \Sigma_g$. These paths give Σ_g a sequence of vanishing cycles of ζ, which we denote by $(c_1, \ldots, c_n)$.

Definition 6.2 [21] Let $\zeta\colon M \to S^2$ be an SPWF with genus $g \geq 3$. We denote by $(c_1, \dots, c_n)$ a sequence of simple closed curves in Σ_g obtained as above. We call this sequence a *surface diagram* of a 4–manifold M.

Remark 6.3 We can define a surface diagram of an SPWF in the obvious way. In this paper, we call both of the diagram, that of a 4–manifold and that of an SPWF, a surface diagram.

Remark 6.4 It is known that every smooth map $h\colon M^4 \to S^2$ from an oriented, closed, connected 4–manifold M is homotopic to an SPWF with genus greater than 2 (see [22]). In particular, every closed oriented connected 4–manifold has a surface diagram. Moreover, the total space of an SPWF is uniquely determined by a sequence of vanishing cycles if the genus is greater than 2 since the group $\pi_1(\mathrm{Diff}^+(\Sigma_{g-1}), \mathrm{id})$ is trivial if $g \geq 3$. Thus, a 4–manifold is uniquely determined by a surface diagram. However, it is known that there exist infinitely many SPWFs which have same vanishing cycles (see [5; 16], for example).

Let $\zeta\colon M \to S^2$ be a genus-g SPWF and $(c_1, \dots, c_n)$ a surface diagram of ζ. For a base point p_0, we take a disk D in $S^2 \setminus \zeta(\mathcal{S}_\zeta)$ satisfying the following conditions:

- $p_0 \in \partial D$
- $\gamma_i \cap D = \{p_0\}$, where $\gamma_i \subset S^2$ is a reference path from p_0 which gives a vanishing cycle c_i.
- $\gamma_1, \dots, \gamma_n, D$ appear in that order when we go around p_0 counterclockwise.

We consider the restriction $\zeta|_{M \setminus \zeta^{-1}(\mathrm{Int}\, D)}$. This is a purely wrinkled fibration and satisfies the conditions in the beginning of Section 4. Thus, we can apply arguments in Section 4 to $\zeta|_{M \setminus \zeta^{-1}(\mathrm{Int}\, D)}$. In particular, we can describe an algorithm to obtain a surface diagram of a fibration obtained by applying flip and slip to ζ. As in Section 4, we prepare several conditions to give an algorithm precisely. We first remark that we can assume that φ_0 is trivial in this case since $\zeta^{-1}(\partial D)$ is bounded by the trivial fibration. In particular, we obtain

$$\widehat{\varphi} = (t_{t_{c_1}(c_2)} \cdots t_{t_{c_{n-1}}(c_n)} \cdot t_{t_{c_n}(c_1)})^{-1}.$$

The first condition is on an embedded path $\alpha \subset \Sigma_g$.

Condition $W_1(c_1, \dots, c_n)$ A path $\alpha \subset \Sigma_g$ intersects c_1 at the unique point $q \in c_1$ transversely. Furthermore, $\partial\alpha \cap (c_1 \cup \cdots \cup c_n) = \varnothing$.

We take a path $\alpha \subset \Sigma_g$ so that α satisfies the condition $W_1(c_1, \dots, c_n)$. We put $\partial\alpha = \{w_1, w_2\}$. The second condition is on a simple closed curve $d \subset \Sigma_{g+1}$ and a diffeomorphism $j\colon \Sigma_g \setminus \{w_1, w_2\} \to \Sigma_{g+1} \setminus d$.

Condition $W_2(c_1,\ldots,c_n,\alpha)$ The closure of $j(\operatorname{Int}\alpha)$ in Σ_{g+1} is a simple closed curve.

We take a simple closed curve $d \subset \Sigma_{g+1}$ and a diffeomorphism $j\colon \Sigma_g \setminus \{w_1,w_2\} \to \Sigma_{g+1}\setminus d$ so that they satisfy the condition $W_2(c_1,\ldots,c_n,\alpha)$. We put $\widetilde{c}_1 = j(c_1)$. The third condition is on an element $\varphi \in \operatorname{Mod}(\Sigma_{g+1})(\widetilde{c}_1, d)$.

Condition $W_3(c_1,\ldots,c_n,\alpha,d,j)$ We have $\Phi_{\widetilde{c}_1}(\varphi) = 1$ in $\operatorname{Mod}(\Sigma_g)(d)$ and $\Phi_d(\varphi) = t_{t_{c_1}(c_2)}\cdots t_{t_{c_{n-1}}(c_n)}\cdot t_{t_{c_n}(c_1)}$ in $\operatorname{Mod}(\Sigma_g)(c_1)$.

The last condition is on simple closed curves $\widetilde{c}_2,\ldots,\widetilde{c}_n \subset \Sigma_{g+1}\setminus d$.

Condition $W_4(c_1,\ldots,c_n,\alpha,d,j)$ For each $i \in \{2,\ldots,n\}$, $i(\widetilde{c}_i)$ is isotopic to c_i in Σ_g, where i is an embedding defined by

$$i\colon \Sigma_{g+1}\setminus d \xrightarrow{j^{-1}} \Sigma_g \setminus \{w_1,w_2\} \hookrightarrow \Sigma_g.$$

Furthermore, $\widetilde{c}_i$ intersects $\widetilde{c}_{i+1}$ at a unique point transversely for each $i \in \mathbb{Z}/n\mathbb{Z}$.

By Theorem 4.3, we immediately obtain the following theorem.

Theorem 6.5 *Let $\zeta\colon M \to S^2$ be a genus-g SPWF and $(c_1,\ldots,c_n)$ a surface diagram of ζ.*

(1) *Let $\widetilde{\zeta}$ be a genus-$(g+1)$ SPWF obtained by applying flip and slip to ζ. Then, there exist elements $\alpha, d, j, \widetilde{c}_2,\ldots,\widetilde{c}_n,\varphi$ satisfying the conditions W_1, W_2, W_3 and W_4 so the sequence $(\widetilde{c}_1,\ldots,\widetilde{c}_n,\widetilde{c}_1,\alpha',d,\widetilde{\alpha})$ gives a surface diagram of $\widetilde{\zeta}$, where $\widetilde{c}_1 = j(c_1)$, $\widetilde{\alpha}$ is the closure of $j(\operatorname{Int}\alpha)$ in Σ_{g+1}, and α' is defined by*

$$\alpha' = \left(\varphi^{-1}\cdot t_{t_{\widetilde{c}_1}(\widetilde{c}_2)}\cdots t_{t_{\widetilde{c}_n}(\widetilde{c}_1)}\right)(\widetilde{\alpha}).$$

(2) *Let $\alpha, d, j, \widetilde{c}_2,\ldots,\widetilde{c}_n$ and φ be elements satisfying the conditions W_1, W_2, W_3 and W_4. Suppose that g is greater than or equal to 3. We take simple closed curves $\alpha',\widetilde{\alpha}$ as in (1). Then, there exists a genus-$(g+1)$ SPWF $\widetilde{\zeta}$ obtained by applying flip and slip to ζ such that $(\widetilde{c}_1,\ldots,\widetilde{c}_n,\widetilde{c}_1,\alpha',d,\widetilde{\alpha})$ is a surface diagram of $\widetilde{\zeta}$.*

As in Section 5, we can deal with SPWFs with small genera by looking at additional data. Let $\zeta\colon M \to S^2$ be a genus-g SPWF with surface diagram $(c_1,\ldots,c_n)$. We take a disk $D \subset S^2$ as above. We also take a section $\sigma\colon S^2\setminus \operatorname{Int} D \to M\setminus\zeta^{-1}(\operatorname{Int} D)$ of the fibration $\zeta|_{M\setminus\zeta^{-1}(\operatorname{Int} D)}$. We put $x = \sigma(p_0)$. We take a trivialization $\zeta^{-1}(D)\cong D\times\Sigma_g$ so that it is compatible with the identification $\zeta^{-1}(p_0)\cong\Sigma_g$. Let $\beta_x \in \pi_1(\Sigma_g, x)$ be an element which is represented by the loop

$$p_2\circ\sigma\colon (\partial D, p_0) \to (\Sigma_g\setminus(c_1\cup\cdots\cup c_n), x),$$

where $p_2\colon D\times\Sigma_g\to\Sigma_g$ is the projection onto the second component. It is easy to see that the monodromy along ∂D (oriented as a boundary of $S^2\setminus\operatorname{Int} D$) corresponds to the pushing map $\operatorname{Push}(\beta_x)^{-1}$. Thus, we can assume $\widetilde{\varphi}_0=\operatorname{Push}(\beta_x)^{-1}\in\operatorname{Mod}(\Sigma_g;x)(c_1)$ in this case. We call the loop β_x an *attaching loop*.

Remark 6.6 We can obtain a handle decomposition of the total space of an SPWF by changing it into a simplified broken Lefschetz fibration using unsink. Indeed, Baykur [4] gave a way to obtain a handle decomposition of the total spaces of simplified broken Lefschetz fibrations from monodromy representation (or equivalently, vanishing cycles of the fibrations). The loop $t\mapsto(t,\beta_x(t))\in D\times\Sigma_g$ corresponds to the attaching circle of the 2–handle in the lower side of the fibration. This is because β_x is called an attaching loop.

We consider the following conditions on elements $\alpha, d, j, \varphi, \widetilde{c}_2,\dots,\widetilde{c}_n$ as in Section 5.

Condition $W_1'(c_1,\dots,c_n,\sigma)$ A path $\alpha\subset\Sigma_g\setminus\{x\}$ intersects c_1 at the unique point $q\in c$ transversely. Furthermore, $\partial\alpha\cap(c_1\cup\dots\cup c_n)=\varnothing$.

Condition $W_2'(c_1,\dots,c_n,\alpha,\sigma)$ The closure of $j(\operatorname{Int}\alpha)$ in Σ_{g+1} is a simple closed curve.

Condition $W_3'(c_1,\dots,c_n,\alpha,d,j,\sigma)$ Here, we set $\widetilde{c}_1=j^{-1}(c_1)$ and $\widetilde{x}=j(x)$, $\Phi^{\widetilde{x}}_{\widetilde{c}_1}(\varphi)=1$ in $\operatorname{Mod}(\Sigma_g;x)(d)$ and $\Phi^{\widetilde{x}}_{d}(\varphi)=t_{t_{c_1}(c_2)}\cdots t_{t_{c_{n-1}}(c_n)}\cdot t_{t_{c_n}(c_1)}\cdot\operatorname{Push}(\beta_x)$ in $\operatorname{Mod}(\Sigma_g;x)(c)$.

Condition $W_4'(c_1,\dots,c_n,\alpha,d,j,\sigma)$ For each $i\in\{2,\dots,n\}$, the curve $\widetilde{c}_i\subset\Sigma_{g+1}\setminus\{\widetilde{x}\}$ satisfies that $i(\widetilde{c}_i)$ is isotopic to c_i in $\Sigma_g\setminus\{x\}$, where i is the embedding defined by

$$i\colon\Sigma_{g+1}\setminus d\xrightarrow{j^{-1}}\Sigma_g\setminus\partial\alpha\hookrightarrow\Sigma_g.$$

Then, we can obtain the following theorem through Theorem 5.2.

Theorem 6.7 *Let $\zeta\colon M\to S^2$ be a genus-g SPWF and $(c_1,\dots,c_n)$ a surface diagram of ζ. We take a disk $D\subset S^2$, $\sigma\colon S^2\setminus\operatorname{Int}D\to M\setminus\zeta^{-1}(\operatorname{Int}D)$, and an element $\beta_x\in\pi_1(\Sigma_g,x)$ as above.*

(1) *Let $\widetilde{\zeta}$ be a genus-$(g+1)$ SPWF obtained by applying flip and slip to ζ. Then, there exist elements $\alpha,d,j,\widetilde{c}_2,\dots,\widetilde{c}_n,\varphi$ satisfying the conditions W_1', W_2', W_3' and W_4' such that the sequence $(\widetilde{c}_1,\dots,\widetilde{c}_n,\widetilde{c}_1,\alpha',d,\widetilde{\alpha})$ gives a surface diagram $\widetilde{\zeta}$, where $\widetilde{c}_1=j^{-1}(c_1)$, $\widetilde{\alpha}$ is the closure of $j^{-1}(\operatorname{Int}\alpha)$ in Σ_{g+1}, and α' is defined by*

$$\alpha'=\left(\varphi^{-1}\cdot t_{t_{\widetilde{c}_1}(\widetilde{c}_2)}\cdots t_{t_{\widetilde{c}_n}(\widetilde{c}_1)}\right)(\widetilde{\alpha}).$$

(2) *Let $\alpha, d, j, \widetilde{c}_2, \ldots, \widetilde{c}_n$ and φ be elements satisfying the conditions W_1', W_2', W_3' and W_4'. Suppose that g is greater than or equal to 2. We take simple closed curves $\alpha', \widetilde{\alpha}$ as in (1). Then, there exists a genus-$(g+1)$ SPWF $\widetilde{\zeta}$ obtained by applying flip and slip to ζ such that $(\widetilde{c}_1, \ldots, \widetilde{c}_n, \widetilde{c}_1, \alpha', d, \widetilde{\alpha})$ is a surface diagram of $\widetilde{\zeta}$.*

Example 6.8 Let $p_1 \colon S^2 \times \Sigma_k \to S^2$ be the projection onto the first component ($k \geq 0$). By applying a birth (for details about this move, see [18; 22], for example), we can change p_1 into a genus-$(k+1)$ SPWF with two cusps. We then apply a flip and slip move to this SPWF m times. As a result, we obtain a genus-$(k+m+1)$ SPWF on the manifold $S^2 \times \Sigma_k$. We denote this fibration by $\widetilde{p}_1^{(m)} \colon S^2 \times \Sigma_k \to S^2$.

Claim A surface diagram of $\widetilde{p}_1^{(m)}$ corresponds to

$$(d_0, d_1, \ldots, d_{2m}, d_{2m+1}, d_{2m}, \ldots, d_1),$$

where $d_i \subset \Sigma_{k+m+1}$ is a simple closed curve described in the left side of Figure 11.

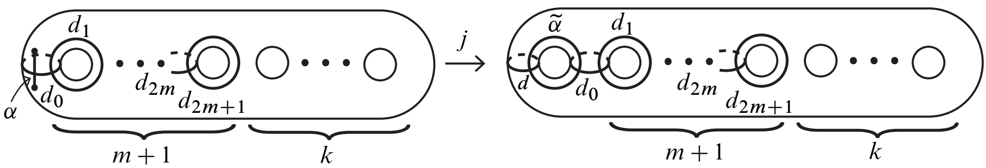

Figure 11: Simple closed curves in the genus-$(k+m+1)$ closed surface Σ_{k+m+1}

We prove this claim by induction on m. The claim is obvious when $m = 0$. We assume that $m > 0$. For simplicity, we denote the Dehn twist along the curve d_i by i and its inverse by $\overline{\imath}$. For an integer $n > 0$, let S_n be a regular neighborhood of the union $d_0 \cup \cdots \cup d_n$. By direct calculation, we can prove the following relation in $\mathrm{Mod}(\overline{S_n}; \partial \overline{S_n})$:

(3)
$$t_{t_{d_0}(d_1)} \cdots t_{t_{d_{n-1}}(d_n)} \cdot t_{t_{d_n}(d_{n-1})} \cdots t_{t_{d_1}(d_0)} = \begin{cases} \overline{0}^4 \cdot (01)^3 & n = 1 \\ \overline{0}^{2n+2} \cdot (01 \cdots n)^{n+2} \cdot (\overline{23} \cdots \overline{n})^n & n \geq 2 \end{cases}$$

By induction hypothesis, a sequence $(d_0, \ldots, d_{2m-2}, d_{2m-1}, d_{2m-2}, \ldots, d_1)$ is a surface diagram of $\widetilde{p}_1^{(m-1)}$. We will stabilize this diagram by using Theorem 6.5. We take a path $\alpha \subset \Sigma_{k+(m-1)+1}$ as in the left side of Figure 11. Let $j \colon \Sigma_{k+m} \setminus \partial\alpha \to \Sigma_{k+m+1} \setminus d$ be a diffeomorphism, where d is a nonseparating simple closed curve. By using j, we regard d_i as a curve in Σ_{k+m+1}. It is easy to see that the element

$$t_{t_{d_0}(d_1)} \cdots t_{t_{d_{2m-2}}(d_{2m-1})} \cdot t_{t_{d_{2m-1}}(d_{2m-2})} \cdots t_{t_{d_1}(d_0)}$$

is contained in the group $\mathrm{Mod}\,(\Sigma_{k+m+1})(d,d_0)$. Moreover, by the relation (3), we can calculate the image under $\Phi_{d_0}\colon \mathrm{Mod}\,(\Sigma_{k+m+1})(d,d_0) \to \mathrm{Mod}\,(\Sigma_{k+m})(d)$ as follows:

$$\begin{aligned}\Phi_{d_0}&(t_{t_{d_0}(d_1)}\cdots t_{t_{d_{2m-2}}(d_{2m-1})}\cdot t_{t_{d_{2m-1}}(d_{2m-2})}\cdots t_{t_{d_1}(d_0)})\\ &= \begin{cases}\Phi_{d_0}(\overline{0}^4\cdot(01)^3) & m=1\\ \Phi_{d_0}(\overline{0}^{4m+2}\cdot(01\cdots 2m-1)^{2m+1}\cdot(\overline{23}\cdots\overline{2m-1})^{2m-1}) & m\geq 2\end{cases}\\ &= \mathrm{id},\end{aligned}$$

where the last equality is proved by the chain relation of the mapping class group. Note that this equality still holds in the group $\mathrm{Mod}\,(\overline{S};\partial\overline{S})$, where S is a regular neighborhood of the union $d\cup d_0\cup\cdots\cup d_{2m-1}$. We put

$$\varphi = t_{t_{d_0}(d_1)}\cdots t_{t_{d_{2m-2}}(d_{2m-1})}\cdot t_{t_{d_{2m-1}}(d_{2m-2})}\cdots t_{t_{d_1}(d_0)} \in \mathrm{Mod}\,(\Sigma_{k+m+1})(d,d_0).$$

The elements $\alpha, d, j, d_0,\dots,d_{2m-1},\varphi$ satisfy conditions W_1, W_2, W_3 and W_4. Therefore, by Theorem 6.5, the following sequence is a surface diagram of $\widetilde{p}_1^{(m)}$:

$$(d_0,\dots,d_{2m-2},d_{2m-1},d_{2m-2},\dots,d_1,d_0,\widetilde{\alpha},d,\widetilde{\alpha})$$

Note that this still holds when the genus of $\widetilde{p}_1^{(m-1)}$ is less than 3 since the above calculation of elements of mapping class groups can be done in regular neighborhoods of curves. This proves the claim on surface diagrams of $S^2\times\Sigma_k$.

Remark 6.9 It is known that there is a genus-k SPWF $q\colon S^2\times\Sigma_{k-1}\# S^1\times S^3\to S^2$ without cusp singularities for $k\geq 1$. This was introduced in [4], and was called the *step fibration*. By the same argument as in Example 6.8, we can prove that $(d_0,d_1,\dots,d_{2m-1},d_{2m},d_{2m-1},\dots,d_1)$ is a surface diagram of the fibration obtained by applying flip and slip to q m times.

We can also prove the claims on surface diagrams of $S^2\times\Sigma_k$ and $S^2\times\Sigma_{k-1}\# S^1\times S^3$ by using Lemma 6.13.

Example 6.10 We next construct a surface diagram of $\#2S^1\times S^3$, which will be used to construct a surface diagram of S^4. To do this, we first prove the following lemma.

Lemma 6.11 *$\#2S^1\times S^3$ admits a genus-2 SPWF ζ without cusps. Moreover, an attaching loop β_x of this fibration is described as in the left side of Figure 12, where e_0 is a vanishing cycle of indefinite fold singularity.*

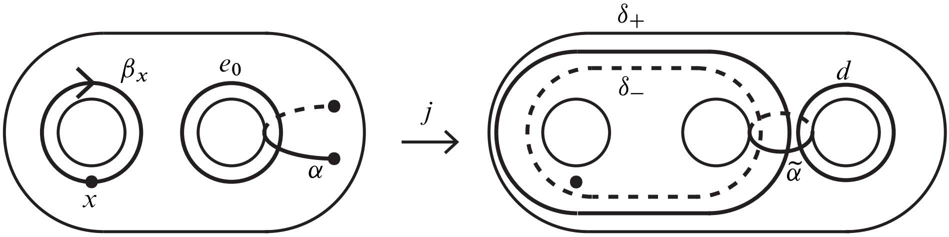

Figure 12: We have that $\widetilde{\alpha}$ is the closure of $j(\mathrm{Int}\,\alpha)$ in Σ_3.

Proof of Lemma 6.11 It is easy to show that there exists a genus-2 SPWF ζ without cusps and whose attaching loop is β_x which is described in Figure 12. Furthermore, we can draw a Kirby diagram of the total space of ζ as described in Figure 13. It can be easily shown by Kirby calculus that this manifold is diffeomorphic to $\#2S^1 \times S^3$. □

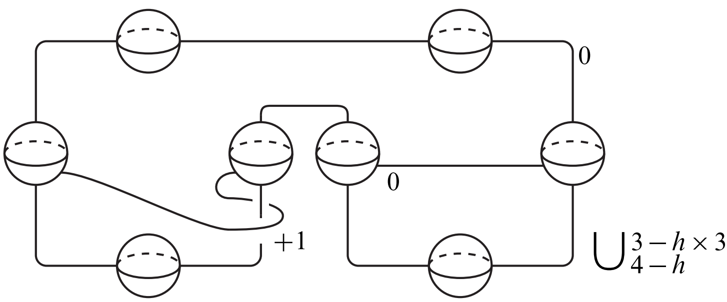

Figure 13: Kirby diagram of the fibration ζ

We take a path $\alpha \subset \Sigma_2$ as in the left side of Figure 12. We also take a diffeomorphism $j\colon \Sigma_2 \setminus \partial\alpha \to \Sigma_3 \setminus d$, where d is a nonseparating simple closed curve in Σ_3, so that the closure of $j(\mathrm{Int}\,\alpha)$ is a simple closed curve. Let $\delta_+, \delta_- \subset \Sigma_3$ be simple closed curves descried as in the right side of Figure 12. We define an element $\varphi \in \mathrm{Mod}\,(\Sigma_3; x)(d, e_0)$ as

$$\varphi = \mathrm{Push}(\beta_x)\cdot t_{\delta_+}\cdot t_{\delta_-}^{-1}.$$

It is easy to see that this element satisfies $\Phi^x_d(\varphi) = \mathrm{Push}(\beta_x)$ and $\Phi^x_{e_0}(\varphi) = \mathrm{id}$. Thus, the elements $\alpha, d, j, e_0, \varphi$ satisfy the conditions W_1', W_2', W_3' and W_4'. By Theorem 6.7, $(e_0, \alpha', d, \widetilde{\alpha})$ is a surface diagram of the fibration obtained by applying flip and slip to ζ, where $\alpha' = (\varphi^{-1})(\widetilde{\alpha})$ (see Figure 14).

Remark 6.12 More generally, we can obtain a genus-$(m+2)$ surface diagram of $\#2S^1 \times S^3$ by looking at vanishing cycles of a fibration obtained by applying flip and slip to ζ m times.

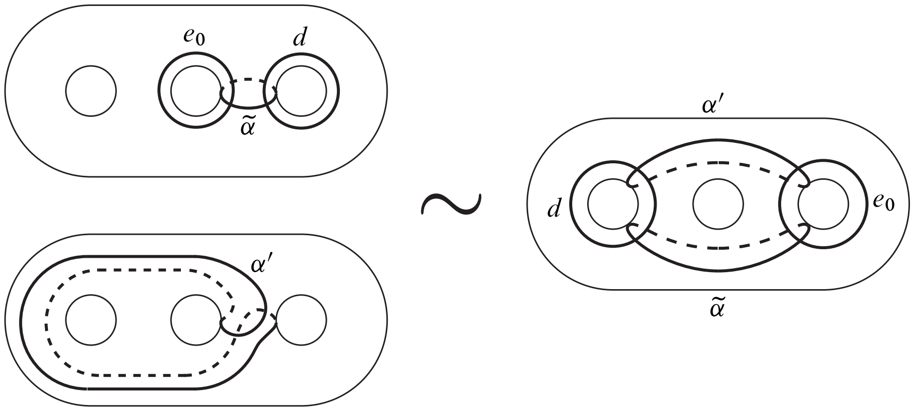

Figure 14: Simple closed curves contained in a surface diagram of $\#2S^1 \times S^3$

Claim Let $e_1, \dots, e_{3m+1}$ be simple closed curves in Σ_{m+2} described in Figure 15. The following sequence is the surface diagram of $\#2S^1 \times S^3$:

$$(e_1, e_2, \dots, e_{2m-1}, e_{2m}, e_{2m+1}, e_{2m+2}, e_{2m-1}, e_{2m+3}, e_{2m-3}, \dots, e_{3m}, e_3, e_{3m+1})$$

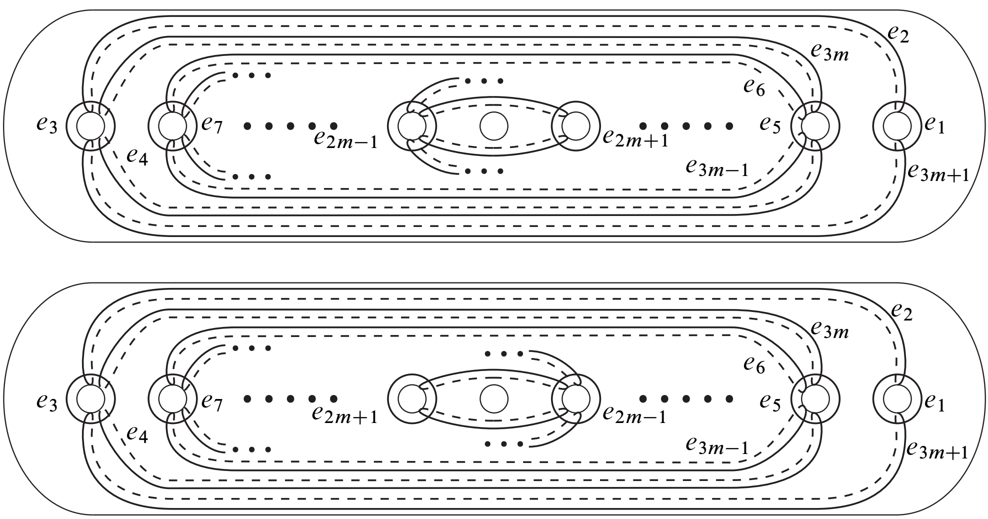

Figure 15: The upper figure describes simple closed curves $e_1, \dots, e_{3m+1}$ in the case m is even, while the lower figure describes simple closed curves $e_1, \dots, e_{3m+1}$ in the case m is odd.

Before looking at the next example, we prove the following lemma.

Lemma 6.13 *Let $(c_1,\dots,c_n)$ be a genus-g surface diagram of an SPWF $\zeta\colon M\to S^2$. We take a simple closed curve $\gamma\subset\Sigma_g$ which intersects c_{i_0} at a unique point transversely. Then there exists a genus-g SPWF $\zeta_s\colon M_s\to S^2$ whose surface diagram is $(c_1,\dots,c_{i_0-1},c_{i_0},\gamma,c_{i_0},c_{i_0+1},\dots,c_n)$. Moreover, if g is greater than or equal to 3, the manifold M_s is obtained from M by applying surgery along γ, where we regard γ as in a regular fiber of ζ.*

Proof of Lemma 6.13 By applying cyclic permutation to the sequence $(c_1,\dots,c_n)$ if necessary, we can assume that $i_0=1$. It is easy to see that the element $t_{t_{c_1}(\gamma)}\cdot t_{t_\gamma(c_1)}$ is contained in the kernel of Φ_{c_1}. Thus, the product $t_{t_{c_1}(\gamma)}\cdot t_{t_\gamma(c_1)}\cdot t_{t_{c_1}(c_2)}\cdots t_{t_{c_n}(c_1)}$ is also contained in the kernel of Φ_{c_1}. This implies existence of a genus-g simplified broken Lefschetz fibration with vanishing cycles $(c_1,t_{c_1}(\gamma),t_\gamma(c_1),t_{c_1}(c_2),\dots,t_{c_n}(c_1))$. Such a fibration can be changed into a genus-g SPWF $\zeta_s\colon M_s\to S^2$ with surface diagram $(c_1,\gamma,c_1,\dots,c_n)$ by applying sink. To prove the statement on M_s, we look at the submanifold S of M satisfying the following conditions:

(1) The image $f(S)$ is a disk and the intersection $f(S\cap\mathcal{S}_f)$ forms a connected arc without cusps.

(2) A vanishing cycle of indefinite folds in $f(S)$ is c_1.

(3) The restriction $f|_{S\setminus f^{-1}(\mathcal{S}_f)}\colon S\setminus\mathcal{S}_f\to f(S)\setminus f(\mathcal{S}_f)$ is a disjoint union of trivial fibrations.

(4) The higher genus fiber of $f|_S\colon S\to f(S)$ is a regular neighborhood of the union $c_1\cup\gamma$.

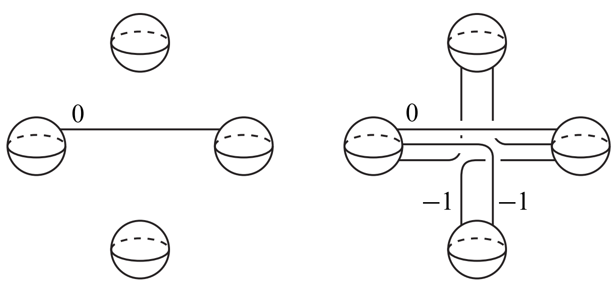

Figure 16: Left: a Kirby diagram of S; right: a Kirby diagram of $\overline{S}$

We can easily draw a Kirby diagram of S as in the left side of Figure 16. This diagram implies that S is diffeomorphic to $S^1\times D^3$, and that a generator of $\pi_1(S)$ corresponds to a simple closed curve γ. Let $\overline{S}$ be a manifold which is described in the right side of Figure 16. This manifold admit a fibration to D^2 with connected indefinite fold, which forms an arc, and two Lefschetz singularities. Furthermore, a regular fiber of the fibration is either a genus-1 surface with one boundary component or a disk. By Kirby calculus, we can prove that this manifold is diffeomorphic to $D^2 \times S^2$. By the construction of the fibration ζ_s, the manifold M_s can be obtained by removing S from M, and then attaching $\overline{S}$ along the boundary. This completes the proof of Lemma 6.13. □

Example 6.14 Let e_1, e_2, e_3, e_4 be simple closed curves in Σ_3 as described in Figure 17. As is shown, a sequence (e_1, e_2, e_3, e_4) is a surface diagram of $\#2S^1 \times S^3$. We take a curve γ_i $(i = 1, 2, 3, 4)$ as shown in Figure 17. The curve γ_1 intersects e_1 at a unique point transversely. By Lemma 6.13, a sequence $(e_1, \gamma_1, e_1, e_2, e_3, e_4)$ is a surface diagram of some 4–manifold obtained by applying surgery to $\#2S^1 \times S^3$. Indeed, we can prove by Kirby calculus that this diagram represents the manifold $S^1 \times S^3$. In the same way, we can prove the following correspondence between surface diagrams and 4–manifolds:

surface diagram	corresponding 4–manifold
$(e_1, \gamma_1, e_1, e_2, e_3, \gamma_4, e_3, e_4)$	S^4
$(e_1, \gamma_1, e_1, e_2, e_3, \gamma_3, e_3, e_4)$	$S^1 \times S^3 \# S^2 \times S^2$
$(e_1, \gamma_1, e_1, e_2, e_3, \gamma_4, e_3, \gamma_4, e_3, e_4)$	$S^2 \times S^2$
$(e_1, \gamma_1, \gamma_2, \gamma_1, e_1, e_2, e_3, \gamma_4, e_3, e_4)$	$\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$

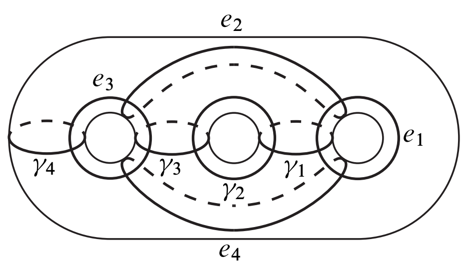

Figure 17: Simple closed curves in Σ_3

In particular, we have obtained two genus-3 SPWFs on $S^2 \times S^2$ which is derived from the following two surface diagrams: the diagram $(d_0, d_1, d_2, d_3, d_4, d_5, d_4, d_3, d_2, d_1)$ in Example 6.8, and the diagram $(e_1, \gamma_1, e_1, e_2, e_3, \gamma_4, e_3, \gamma_4, e_3, e_4)$ as above. The SPWF which corresponds to the former diagram is homotopic to the projection $p_1\colon S^2 \times S^2 \to S^2$ onto the first projection. Indeed, this SPWF was constructed by applying birth and flip and slip to p_1. On the other hand, it is easy to prove (by Kirby calculus, for example) that a regular fiber of the SPWF corresponding to the latter diagram is nullhomologous in $S^2 \times S^2$. Thus, two genus-3 SPWFs above are not homotopic. In the same way, we can prove that two SPWFs on $S^1 \times S^3 \# S^2 \times S^2$ derived from the following two diagrams are not homotopic: the diagram $(d_0, d_1, d_2, d_3, d_4, d_3, d_2, d_1)$ which is obtained by applying flip and slip to the step fibration twice (see Remark 6.9), and the diagram $(e_1, \gamma_1, e_1, e_2, e_3, \gamma_3, e_3, e_4)$ as above.

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