Modification rule of monodromies in an $R_2$–move

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An $R_2$–move is a homotopy of wrinkled fibrations which deforms images of indefinite fold singularities like the Reidemeister move of type II. Variants of this move are contained in several important deformations of wrinkled fibrations. In this paper, we first investigate how monodromies are changed by this move. For a given fibration and its vanishing cycles, we then give an algorithm to obtain vanishing cycles in a single reference fiber of a fibration obtained by flip and slip, which is a sequence of homotopies increasing fiber genera. As an application of this algorithm, we give several examples of diagrams which were introduced by Williams to describe smooth 4–manifolds by a finite sequence of simple closed curves in a closed surface.

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1 Introduction

Over the last few years, several new fibrations on 4–manifolds were introduced and studied by means of various tools: singularity theory, mapping class groups and so on. These studies originated from the work of Auroux, Donaldson and Katzarkov [2] in which they generalized the results of Donaldson [7] and Gompf [15] on relations between symplectic manifolds and Lefschetz fibrations to those on relations between near-symplectic 4–manifolds and corresponding fibrations, called broken Lefschetz fibrations. After their study, Perutz [19; 20] defined the Lagrangian matching invariant for near-symplectic 4–manifolds as a generalization of the standard surface counting of Donaldson and Smith [8] for symplectic 4–manifolds using broken Lefschetz fibrations. Although this invariant is a candidate for geometric interpretation of the Seiberg–Witten invariant, even smooth invariance of this invariant is not verified so far. We need to understand deformations (in the space of more general fibrations) between two broken Lefschetz fibrations in order to prove the invariance. There are several results on this matter (see Lekili [18], Williams [22; 21], and Gay and Kirby [13; 14], for example).

On the other hand, broken Lefschetz fibrations themselves have been studied in terms of mapping class groups using vanishing cycles. For example, the classification problem of fibrations with particular properties was solved by means of this combinatorial method (see Baykur and Kamada [5] and the author [16; 17]). It turns out that every closed
oriented 4–manifold admits a broken Lefschetz fibration (see Akbulut and Karakurt [1], Baykur [3] and [18]). It is therefore natural to expect that broken Lefschetz fibrations enable us to deal with a broader range of 4–manifolds in a combinatorial way, as we dealt with symplectic 4–manifolds using Lefschetz fibrations. For developing topology of smooth 4–manifolds by means of mapping class groups, it is necessary to understand the relation between several deformations which appeared in the previous paragraph and vanishing cycles of fibrations.

In this paper, we will pay attention to a specific deformation of fibrations, called an $R_2$–move. In this move, the image of indefinite fold singularities is changed by the Reidemeister move of type II (we will define this move in Section 3; see Figure 5). In particular, the region with the highest genus fibers was cut off in this deformation. Furthermore, monodromies in this region might be changed by this move. This move appears in a lot of important deformations of fibrations. For example, flip and slip, which was first introduced by Baykur [3], is an application of flip twice followed by a variant of an $R_2$–move. Another variant of an $R_2$–move played a key role in the work of Williams [21], which gave a purely combinatorial description of 4–manifolds (which we will mention in Section 6).

The main purpose of this paper is to understand how monodromies are changed by an $R_2$–move. We will prove that modifications of monodromies in an $R_2$–move can be controlled by an intersection of kernels of some homomorphisms (see Theorem 3.9). We will also give an algorithm to obtain vanishing cycles in a reference fiber of a fibration obtained by flip and slip in terms of the mapping class group (see Theorems 4.1, 4.3, 5.1, 5.2, 6.5 and 6.7). Note that it is not easy to determine vanishing cycles in a single reference fiber of the fibration obtained by applying flip and slip. Indeed, in this modification, two regions with the highest genus fibers are connected by a variant of the $R_2$–move. It is easy to obtain vanishing cycles in fibers in the respective components since flip is a local deformation. However, we need to deal with a certain monodromy derived from a variant of the $R_2$–move to understand how these fibers are identified (see also Remark 2.3).

In Section 2, we will give several definitions and notation which we will use in this paper. Sections 3, 4 and 5 are the main parts of this paper. In Section 3, we will examine how monodromies are changed in $R_2$–moves. The results obtained in this section will play a key role in the following sections. In Sections 4 and 5, we will give an algorithm to obtain vanishing cycles of a fibration modified by flip and slip. We will first deal with fibrations with large fiber genera in Section 4, and then turn our attention to fibrations with small fiber genera in Section 5. In Section 6, we will give a modification rule of a diagram Williams introduced, which is called a surface diagram, when the corresponding fibration is changed by flip and slip. We will then construct
surface diagrams of some standard 4–manifolds, $S^4$, $S^1 \times S^3$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ and so on. Note that, as far as the author knows, these are the first nontrivial examples of surface diagrams.

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## 2 Preliminaries

### 2.1 Wrinkled fibrations

We first define several singularities to which we will pay attention in this paper.

**Definition 2.1** Let $M$ and $B$ be smooth manifolds of dimension 4 and 2, respectively. For a smooth map $f: M \to B$, we denote by $S_f \subset M$ the set of critical points of $f$.

1. An element $p \in S_f$ is called an **indefinite fold singularity** of $f$ if there exist real coordinates $(t, x, y, z)$ (resp. $(s, w)$) around $p$ (resp. $f(p)$) such that $f$ is locally written in these coordinates as
   \[ f: (t, x, y, z) \mapsto (s, w) = (t, x^2 + y^2 - z^2). \]

2. An element $p \in S_f$ is called an **indefinite cusp singularity** of $f$ if there exist real coordinates $(t, x, y, z)$ (resp. $(s, w)$) around $p$ (resp. $f(p)$) such that $f$ is locally written in these coordinates as
   \[ f: (t, x, y, z) \mapsto (s, w) = (t, x^3 - 3tx + y^2 - z^2). \]

3. We further assume that the manifolds $M$ and $B$ are oriented. An element $p \in S_f$ is called a **Lefschetz singularity** of $f$ if there exists complex coordinates $(z, w)$ (resp. $\xi$) around $p$ (resp. $f(p)$) compatible with orientation of the manifold $M$ (resp. $B$) such that $f$ is locally written in these coordinates as
   \[ f: (z, w) \mapsto \xi = zw. \]

We can also define definite fold singularities and definite cusp singularities. However, these singularities will not appear in this paper. We call an indefinite fold (resp. cusp) singularity a **fold** (resp. **cusp**) for simplicity.
Definition 2.2 Let $M$ and $B$ be oriented, compact, smooth manifolds of dimension 4 and 2, respectively. A smooth map $f: M \to B$ is called a wrinkled fibration if it satisfies the following conditions:

1. $f^{-1}(\partial B) = \partial M$
2. The set of singularities $S_f$ consists of folds, cusps and Lefschetz singularities.

A wrinkled fibration $f$ is called a purely wrinkled fibration if $f$ has no Lefschetz singularities.

2.2 Mapping class groups and a homomorphism $\Phi_c$

Let $\Sigma_g$ be a closed, oriented, connected surface of genus $g$. We take subsets $A_i, B_j \subset \Sigma_g$. We define a group $\text{Mod} (\Sigma_g; A_1, \ldots, A_n)(B_1, \ldots, B_m)$ by

$$\text{Mod} (\Sigma_g; A_1, \ldots, A_n)(B_1, \ldots, B_m) = \{ [T] \in \pi_0(\text{Diff}^+(\Sigma_g; A_1, \ldots, A_n), \text{id}) \mid T(B_j) = B_j \text{ for all } j \},$$

where $\text{Diff}^+(\Sigma_g; A_1, \ldots, A_n)$ is defined as

$$\text{Diff}^+(\Sigma_g; A_1, \ldots, A_n) = \{ \text{diffeomorphisms } T: \Sigma_g \to \Sigma_g \mid T(A_i) = A_i \text{ for all } i \}.$$

In this paper, we define a group structure on the above group by multiplication reverse to the composition, that is, for elements $T_1, T_2 \in \text{Diff}^+(\Sigma_g; A_1, \ldots, A_n)$, we define the product $T_1 \cdot T_2$ as

$$T_1 \cdot T_2 = T_2 \circ T_1.$$

We define a group structure of $\text{Mod} (\Sigma_g; A_1, \ldots, A_n)(B_1, \ldots, B_m)$ in the same way. For simplicity, we denote by $\mathcal{M}_g$ the group $\text{Mod} (\Sigma_g)$.

Let $c \subset \Sigma_g$ be a simple closed curve. For a given element $\psi \in \text{Mod} (\Sigma_g)(c)$, we take a representative $T: \Sigma_g \to \Sigma_g \in \text{Diff}^+ (\Sigma_g)$ preserving the curve $c$ setwise. The restriction $T|_{\Sigma_g \setminus c}: \Sigma_g \setminus c \to \Sigma_g \setminus c$ is also a diffeomorphism. Let $S_c$ be the surface obtained by attaching two disks with marked points at the origin to $\Sigma_g \setminus c$ along $c$. The surface $S_c$ is diffeomorphic to $\Sigma_{g-1}$ with two marked points if $c$ is nonseparating, or $S_c$ is a disjoint union of $\Sigma_{g_1}$ with a marked point and $\Sigma_{g_2}$ with a marked point for some $g_1, g_2$ if $c$ is separating. The diffeomorphism $T|_{\Sigma_g \setminus c}$ can be extended to a diffeomorphism $\widetilde{T}: S_c \to S_c$. We define an element $\Phi_c^*([T])$ as an isotopy class of $\widetilde{T}$, which is contained in the group $\text{Mod} (S_c; \{v_1, v_2\})$, where $v_1, v_2$ are the marked points. The following map is a well-defined homomorphism:

$$\Phi_c^*: \text{Mod} (\Sigma_g)(c) \to \text{Mod} (S_c; \{v_1, v_2\})$$
Furthermore, we define a homomorphism $\Phi_c$ on $\text{Mod}(\Sigma_g)(c)$ as the composition $F_{v_1,v_2} \circ \Phi_c^*$, where $F_{v_1,v_2}: \text{Mod}(S_c;\{v_1, v_2\}) \to \text{Mod}(S_c)$ is the forgetful map. The range of this map is $\mathcal{M}_{g-1}$ if $c$ is nonseparating, $\mathcal{M}_{g_1} \times \mathcal{M}_{g_2}$ if $c$ is separating and $g_1 \neq g_2$, and $(\mathcal{M}_{g_1} \times \mathcal{M}_{g_2}) \times \mathbb{Z}/2\mathbb{Z}$ if $c$ is separating and $g_1 = g_2$.

### 2.3 Several homotopies of fibrations

In this subsection, we will give a quick review of some deformations of smooth maps from 4–manifolds to surfaces which we will use in this paper. For details about this, see [18] or [22], for example.

#### 2.3.1 Sink and unsink

Lekili [18] introduced a homotopy which removes a Lefschetz singularity near a fold locus, and gives rise to a cusp singularity as in Figure 1.

This deformation is called a sink and the inverse move is called an unsink. We can always change a cusp into a Lefschetz singularity by an unsink. However, we can apply a sink only when $c_3$ corresponds to the curve $t_{c_1}(c_2)$ and $c_1$ intersects $c_2$ in a single point transversely, where $c_i$ is a vanishing cycle determined by $\gamma_i$, which is a reference path in the base space described in Figure 1.

#### 2.3.2 Flip and “flip and slip”

The homotopy called a flip is locally written as $f_s: \mathbb{R}^4 \ni (t, x, y, z) \mapsto (t, x^4 - x^2s + xt + y^2 - z^2) \in \mathbb{R}^2$.

The set of singularities $\mathcal{S}_{f_s} \subset \mathbb{R}^4$ is equal to $\{(t, x, 0, 0) \in \mathbb{R}^4 \mid 4x^3 - 2sx + t = 0\}$. For $s < 0$, this set consists of indefinite folds. For $s > 0$, this set contains two cusps as in the right side of Figure 2.
Baykur [3] and Lekili [18] introduced a certain global homotopy, which is called a *flip and slip*. This modification adds four cusps to the set of critical points of the fibration (see Figure 3). If a lower genus regular fiber of the original fibration (ie a regular fiber on the inside of the singular circle on the far left of Figure 3) is disconnected, then this fiber becomes connected after the modification. If a lower genus regular fiber is connected, this fiber becomes the higher genus fiber and its genus is increased by 2.

**Remark 2.3** It is not straightforward to deduce how all the vanishing cycles appear in a single reference fiber after performing a flip and slip. Indeed, to find the vanishing cycles, we need to know how to identify the two regular fibers in the regions with the highest genus fiber in the center of Figure 3. As we will show in the following sections, this identification depends on the choice of homotopies, especially the choice of a “slip” (from the middle figure to the right one in Figure 3).
We remark that such a modification can be also applied when the set of singularities of the original fibration contains cusps. We first apply a flip twice between two consecutive cusps. We then apply a slip in the same way as in the case that the original fibration contains no cusps (see Figure 4). We also call this deformation a flip and slip.

![Figure 4: Base loci in a flip and slip when the original fibration has cusps](image)

### 3 A fibration over the annulus with two components of indefinite folds

Let $N$ be a 3–manifold obtained by a 1–handle attachment to $\Sigma_g \times I$ followed by a 2–handle attachment whose attaching circle is nonseparating and is disjoint from the belt circle of the 1–handle. The manifold $N$ has a Morse function $h: N \to I$ with two critical points: one is the origin of the 1–handle $p_1 \in N$ whose index is 1, and the other is the origin of the 2–handle $p_2 \in N$ whose index is 2. We assume that the value of $p_1$ under $h$ is $\frac{4}{9}$, and the value of $p_2$ under $h$ is $\frac{5}{9}$. We put $M = N \times S^1$ and we define $f = h \times \text{id}_{S^1}: M \to I \times S^1$. We denote by $Z_1 \subset M$ (resp. $Z_2 \subset M$) the component of indefinite folds of $f$ satisfying $f(Z_1) = \{\frac{4}{9}\} \times S^1$ (resp. $f(Z_2) = \{\frac{5}{9}\} \times S^1$).

We identify $S^1$ with $[0,1]/\{0,1\}$. By construction of $N$, we can identify $f^{-1}(\{\frac{1}{2}\} \times \{0\})$ with the closed surface $\Sigma_{g+1}$. Moreover, this identification is unique up to Dehn twist $\tau_c$, where $c \subset \Sigma_{g+1}$ is the belt sphere of the 1–handle. We denote by $d \subset \Sigma_{g+1}$ the attaching circle of the 2–handle. In this section, we look at a monodromy of the fibration $f$, especially how a monodromy along the curve $\gamma = \{\frac{1}{2}\} \times S^1$ is changed by a certain homotopy of $f$. We remark that the number of connected components of the complement $\Sigma_{g+1} \setminus (c \cup d)$ is at most 2 since both $c$ and $d$ are nonseparating. We call $(c,d)$ a bounding pair of genus $g_1$ if the complement $\Sigma_{g+1} \setminus (c \cup d)$ consists of two twice punctured surfaces of genus $g_1$ and $g_2 = g - g_1$. 

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Let \( c, d \subset \Sigma_{g+1} \) be mutually disjoint nonseparating simple closed curves. We look at details of the homomorphisms

\[
\Phi_c: \text{Mod} (\Sigma_{g+1})(c, d) \to \text{Mod} (\Sigma_g)(d), \\
\Phi_d: \text{Mod} (\Sigma_{g+1})(c, d) \to \text{Mod} (\Sigma_g)(c).
\]

We first consider the case that \((c, d)\) is not a bounding pair. In this case, the union \(c \cup d\) is nonseparating in \(\Sigma_g\). As we mentioned in Section 2, for a nonseparating simple closed curve \(c \subset \Sigma_g\), the homomorphism \(\hat{\Phi}_c\) is defined by \(F_{v_1, v_2} \circ \Phi_c^*\). It is proved by Farb and Margalit in [12] that the kernel of the homomorphism \(\hat{\Phi}_c^*\) is generated by the Dehn twist \(t_c\).

Let \(\text{Mod} (\Sigma_g)(c^{\text{ori}})\) be the subgroup of \(\text{Mod} (\Sigma_g)(c)\) whose element is represented by a diffeomorphism preserving an orientation of \(c\). We can define the homomorphism \(\Phi_c^{\text{ori}}\) \(: \text{Mod} (\Sigma_g)(c^{\text{ori}}) \to \mathcal{M}_{g-1}\) as we define \(\Phi_c\). Furthermore, we can decompose this map as

\[
\Phi_c^{\text{ori}}: \text{Mod} (\Sigma_g)(c^{\text{ori}}) \xrightarrow{\Phi_c^{\text{ori}}} \text{Mod} (\Sigma_{g-1}; v_1, v_2) \xrightarrow{F_{v_1, v_2}} \mathcal{M}_{g-1}.
\]

It is known that the following sequence is exact (see Birman [6]):

\[
\pi_1(\text{Diff}^+ (\Sigma_{g-1})) \xrightarrow{} \pi_1 (\Sigma_{g-1} \times \Sigma_{g-1} \setminus \Delta \Sigma_{g-1}) \xrightarrow{} \text{Mod}(\Sigma_{g-1}; v_1, v_2) \xrightarrow{} \mathcal{M}_{g-1} \xrightarrow{} \mathbb{1},
\]

where

\[\Delta \Sigma_{g-1} \subset \Sigma_{g-1} \times \Sigma_{g-1}\]

is the diagonal set. Let \(\text{Diff}_0^+ (\Sigma_{g-1})\) be the connected component of \(\text{Diff}^+ (\Sigma_{g-1})\) which contains the identity map. Since \(\text{Diff}_0^+ (\Sigma_{g-1})\) is contractible if \(g\) is greater than or equal to 3 (see Earle and Eells [9]), the kernel of the map

\[
F_{v_1, v_2}: \text{Mod}(\Sigma_{g-1}; v_1, v_2) \to \mathcal{M}_{g-1}
\]

is isomorphic to the fundamental group of \(\Sigma_{g-1} \times \Sigma_{g-1} \setminus \Delta \Sigma_{g-1}\).

Define \(\text{Mod} (\Sigma_{g+1})(c^{\text{ori}}, d)\), \(\text{Mod} (\Sigma_{g+1})(c, d^{\text{ori}})\) and \(\text{Mod} (\Sigma_{g+1})(c^{\text{ori}}, d^{\text{ori}})\) (subgroups of the group \(\text{Mod} (\Sigma_{g+1})(c, d)\)) as we define the group \(\text{Mod} (\Sigma_g)(c^{\text{ori}})\). From the construction of the maps, we obtain the following commutative diagram:
The map $\Phi_c \colon \text{Mod}(\Sigma_{g+1}; c, d) \to \text{Mod}(\Sigma_g; c, d)$ sends the group $\ker \Phi_c \cap \ker \Phi_d$ to the group $\ker F_{v_1, v_2} \cap \ker F_{w_1, w_2} \subset \text{Mod}(\Sigma_{g-1}; v_1, v_2, w_1, w_2)$, which is contained in $\ker (F_{v_1, v_2, w_1, w_2} \colon \text{Mod}(\Sigma_{g-1}; v_1, v_2, w_1, w_2) \to M_{g-1})$.

**Lemma 3.1** The following restrictions are isomorphisms:

\[
\Phi^*_{d, \text{ori}} \circ \Phi^*_{c, \text{ori}}|_{\ker \Phi_c \cap \ker \Phi_d} : \ker \Phi_c \cap \ker \Phi_d \to \ker F_{v_1, v_2} \cap \ker F_{w_1, w_2}
\]

\[
\Phi^*_{d, \text{ori}}|_{\ker \Phi_c \cap \ker \Phi_d} : \ker \Phi_c \cap \ker \Phi_d \to \ker \Phi_d \cap \ker F_{w_1, w_2}
\]

\[
\Phi^*_{c, \text{ori}}|_{\ker \Phi_c \cap \ker \Phi_d} : \ker \Phi_c \cap \ker \Phi_d \to \ker \Phi_c \cap \ker F_{v_1, v_2}
\]

**Proof of Lemma 3.1** We only prove that the first map is an isomorphism (we can prove the other maps are isomorphisms similarly). In this proof, we denote the map $\Phi^*_{d, \text{ori}} \circ \Phi^*_{c, \text{ori}}|_{\ker \Phi_c \cap \ker \Phi_d}$ by $\Phi$ for simplicity. We first prove that $\Phi$ is injective. We
take an element $\psi \in \text{Ker} \Phi$. Since the kernel of $\Phi_{\text{c,ori}}$ (resp. $\Phi_{\text{d,ori}}^*$) is generated by $t_c$ (resp. $t_d$), $\psi$ is equal to $t_c^m \cdot t_d^n$, for some $m, n \in \mathbb{Z}$. Since $\psi$ is contained in $\text{Ker} \Phi_c$, we have $\Phi_c(\psi) = t_d^n = 1$. Thus, we obtain $n = 0$. Similarly, we can obtain $m = 0$ and this completes the proof of injectivity of $\Phi$.

We next prove that $\Phi$ is surjective. For an element $\xi \in \text{Ker} F_{v_1, v_2} \cap \text{Ker} F_{w_1, w_2}$, we can take an element $\xi' \in \text{Mod} (\Sigma_{g-1})^4 (c^{\text{ori}}, d^{\text{ori}})$ which mapped to $\xi$ by the map $\Phi_{\text{d,ori}}^* \circ \Phi_{\text{c,ori}}^*$ since both of the maps $\Phi_{\text{d,ori}}^*$ and $\Phi_{\text{c,ori}}^*$ are surjective. By the commutative diagram (1), $\Phi_{\text{d,ori}}^*(\xi')$ is contained in the kernel of $\Phi_{\text{c,ori}}^*$. Thus, we obtain $\Phi_{\text{d,ori}}^*(\xi') = t_c^n \cdot t_d^m$, for some $m, n \in \mathbb{Z}$. Similarly, we obtain $\Phi_{\text{c,ori}}^*(\xi') = t_d^{-n} \cdot t_c^{-m}$ is contained in the group $\text{Ker} \Phi_c \cap \text{Ker} \Phi_d$ and mapped to $\xi$ by the map $\Phi$. This completes the proof of surjectivity of $\Phi$. 

Let $\epsilon: \text{Diff}^+ (\Sigma_{g-1}) \to (\Sigma_{g-1})^4 \setminus \tilde{\Delta}$ be the evaluation map at $v_1, v_2, w_1, w_2 \in \Sigma_{g-1}$, where $\tilde{\Delta}$ is the subset of $(\Sigma_{g-1})^4$ defined as 

$$
\tilde{\Delta} = \{(x_1, x_2, x_3, x_4) \in (\Sigma_{g-1})^4 \mid \text{there exist } i \text{ and } j, \ i \neq j, \text{ such that } x_i = x_j\}.
$$

The map $\epsilon$ is a locally trivial fibration with fiber $\text{Diff}^+ (\Sigma_{g-1}, v_1, v_2, w_1, w_2)$ (see [6]). Since $(\Sigma_{g-1})^4 \setminus \tilde{\Delta}$ is connected, we obtain the following exact sequence:

$$
(2) \quad \pi_1 (\text{Diff}^+ (\Sigma_{g-1}; v_1, v_2, w_1, w_2), \text{id}) \to \pi_1 (\text{Diff}^+ (\Sigma_{g-1}), \text{id}) \\
\to \pi_1 ((\Sigma_{g-1})^4 \setminus \tilde{\Delta}, (v_1, v_2, w_1, w_2)) \to \text{Mod} ((\Sigma_{g-1}; v_1, v_2, w_1, w_2) \\
\to \mathcal{M}_{g-1} \to 1
$$

Note that the map $\text{Mod} ((\Sigma_{g-1}; v_1, v_2, w_1, w_2) \to \mathcal{M}_{g-1}$ is $F_{v_1, v_2, w_1, w_2}$. The group $\text{Diff}^+ (\Sigma_{g-1})$ is contractible if $g \geq 3$ [9]. Therefore, if $g \geq 3$, the kernel of the map $F_{v_1, v_2, w_1, w_2}$ is isomorphic to the fundamental group of the configuration space $(\Sigma_{g-1})^4 \setminus \tilde{\Delta}$. Moreover, under the identification $\text{Ker} F_{v_1, v_2, w_1, w_2} \cong \pi_1 ((\Sigma_{g-1})^4 \setminus \tilde{\Delta}, (v_1, v_2, w_1, w_2))$, the kernel of the map $F_{w_1, w_2}$ corresponds to the homomorphism 

$$
p_{1,*}: \pi_1 ((\Sigma_{g-1})^4 \setminus \tilde{\Delta}, (v_1, v_2, w_1, w_2)) \to \pi_1 (\Sigma_{g-1} \times \Sigma_{g-1} \setminus \Delta \Sigma_{g-1}, (v_1, v_2)),
$$

where $p_1$ is the projection onto the first and second components. Similarly, the kernel of the map $F_{v_1, v_2}$ corresponds to the homomorphism 

$$
p_{2,*}: \pi_1 ((\Sigma_{g-1})^4 \setminus \tilde{\Delta}, (v_1, v_2, w_1, w_2)) \to \pi_1 (\Sigma_{g-1} \times \Sigma_{g-1} \setminus \Delta \Sigma_{g-1}, (w_1, w_2)),
$$

where $p_2$ is the projection onto the third and fourth components. Eventually, we obtain the isomorphism 

$$
\text{Ker} F_{v_1, v_2} \cap \text{Ker} F_{w_1, w_2} \cong \text{Ker} p_{1,*} \cap \text{Ker} p_{2,*}.
$$
For an oriented surface $S$ and points $x, y \in S$, we define $\Pi(S, x, y)$ as the set of embedded paths from $x$ to $y$. For an element $\eta \in \Pi(S, x, y)$, we denote by $L(\eta)\colon ([0, 1], \{0, 1\}) \to (S \setminus \{y\}, x)$ a loop in the neighborhood of $\eta$, which is injective on $[0, 1]$ and homotopic to a loop obtained by connecting $x$ to a sufficiently small counterclockwise circle around $y$ using $\eta$.

Lemma 3.2 For an element $\eta \in \Pi(\Sigma_{g-1} \setminus \{v_i, w_j\}, v_k, w_l)$ $(\{i, k\} = \{j, l\} = \{1, 2\})$, we denote by $l(\eta)$ the loop

$$
t \mapsto \begin{cases} (L(\eta)(t), v_2, w_1, w_2) & k = 1, \\ (v_1, L(\eta)(t), w_1, w_2) & k = 2, \end{cases}
$$

where $t \in [0, 1]$ and the right-hand side is in $\Sigma_g^4 \setminus \bar{\Delta}$. Then, the group $\text{Ker } p_{1,*} \cap \text{Ker } p_{2,*}$ is generated by the set

$$\{l(\eta) \in \pi_1((\Sigma_{g-1})^4 \setminus \bar{\Delta}, (v_1, v_2, w_1, w_2)) \mid \eta \in \Pi(\Sigma_{g-1} \setminus \{v_i, w_j\}, v_k, w_l), \{i, k\} = \{j, l\} = \{1, 2\}\}.$$

Proof of Lemma 3.2 When the space $S$ is obvious, we denote by $\Delta$ the diagonal subset of $S \times S$ for simplicity. It is obvious that an element $[l(\eta)]$ is contained in the group $\text{Ker } p_{1,*} \cap \text{Ker } p_{2,*}$ for any $\eta \in \Pi(\Sigma_{g-1} \setminus \{v_i, w_j\}, v_k, w_l)$. We prove that any element of $\text{Ker } p_{1,*} \cap \text{Ker } p_{2,*}$ can be represented by the product $[l(\eta_1) \cdots l(\eta_m)]$, for some $\eta_p \in \Pi(\Sigma_{g-1} \setminus \{v_{i_p}, w_{j_p}\}, v_{k_p}, w_{l_p})$. To prove this, we need the following theorem.

Theorem 3.3 (Fadell and Neuwirth [11, Theorem 3]) The projection

$$p_2\colon (\Sigma_{g-1})^4 \setminus \bar{\Delta} \to \Sigma_{g-1}^2 \setminus \Delta$$

is a locally trivial fibration with fiber $(\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta$.

By Theorem 3.3, we obtain the following homotopy exact sequence:

$$\pi_2((\Sigma_{g-1})^2 \setminus \Delta, (w_1, w_2)) \to \pi_1((\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta, (v_1, v_2)) \to \pi_1((\Sigma_{g-1})^4 \setminus \bar{\Delta}, (v_1, v_2, w_1, w_2)) \xrightarrow{p_{2,*}} \pi_1((\Sigma_{g-1})^2 \setminus \Delta, (w_1, w_2)) \to \pi_0((\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta, (v_1, v_2))$$

Since the space $(\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta$ is connected and the space $(\Sigma_{g-1})^2 \setminus \Delta$ is aspherical (cf [11, Corollary 2.2]), the inclusion map $i\colon (\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta \to (\Sigma_{g-1})^4 \setminus \bar{\Delta}$ gives the isomorphism

$$i_*\colon \pi_1((\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta, (v_1, v_2)) \to \text{Ker } p_{2,*}.$$
Let $i': (\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta \to (\Sigma_{g-1})^2 \setminus \Delta$ be the inclusion map. The group $\text{Ker } p_{1, *} \cap \text{Ker } p_{2, *}$ is isomorphic to the group $\text{Ker } i'_*$ since the following diagram commutes:

\[
\begin{array}{ccc}
(\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta & \xrightarrow{i} & (\Sigma_{g-1})^4 \setminus \tilde{\Delta} \\
\downarrow{i'} & & \downarrow{p_1} \\
(\Sigma_{g-1})^2 \setminus \Delta
\end{array}
\]

Thus, it is sufficient to prove that any element of $\text{Ker } i'_*$ can be represented by the product $[l'(\eta_1) \cdots l'(\eta_m)]$ for some $\eta_p \in \Pi(\Sigma_{g-1} \setminus \{v_{i_p}, w_{j_p}, v_{k_p}, w_{l_p}\})$, where $l'(\eta_p)$ is the loop defined by

\[
[0, 1] \ni t \mapsto \begin{cases} (L(\eta_p)(t), v_2) & k_p = 1, \\ (v_1, L(\eta_p)(t)) & k_p = 2, \end{cases}
\]

where $t \in [0, 1]$ and the right-hand side is in $(\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta$. We take an element $[\xi] \in \text{Ker } i'_*$, where $\xi: (S^1, 1) \to ((\Sigma_{g-1} \setminus \{w_1, w_2\})^2 \setminus \Delta, (v_1, v_2))$ is a loop $(1 \in S^1 \subset \mathbb{C})$. We can assume that $\xi$ is an embedding. Since $i'(\xi)$ is nullhomotopic in the space $\Sigma_{g-1}^2 \setminus \Delta$, we can take a map $\overline{\xi}: D^2 \to (\Sigma_{g-1})^2 \setminus \Delta$ satisfying the following conditions:

(a) The restriction $\overline{\xi}|_{\partial D^2}$ is equal to $i'(\xi)$.

(b) $\overline{\xi}$ is an immersion, that is, $\overline{\xi}$ satisfies:

- $\overline{\xi}^{-1}(p)$ is at most 2 for each $p \in \overline{\xi}(D^2)$.
- For any point $p \in \overline{\xi}(D^2)$ such that $\#\overline{\xi}^{-1}(p) = 2$, there exists a disk neighborhood $D_i \subset \Sigma_{g-1}^2 \setminus \Delta$ of the point $p_i \in \overline{\xi}^{-1}(p)$ such that $\overline{\xi}$ is an embedding over $D_i$, and that $\overline{\xi}(D_1)$ intersects $\overline{\xi}(D_2)$ at the unique point $p$ transversely, where $\{p_1, p_2\} = \overline{\xi}^{-1}(p)$.

(c) For each $i \in \{1, 2\}$, $\overline{\xi}^{-1}(\{w_i\} \times (\Sigma_{g-1} \setminus \{w_i\}))$ and $\overline{\xi}^{-1}((\Sigma_{g-1} \setminus \{w_i\}) \times \{w_i\})$ is a discrete set and is contained in $\text{Int } D^2 \cap \mathbb{R}$.

(d) The set $\overline{\xi}^{-1}(\{p \in \Sigma_{g-1}^2 \setminus \Delta \mid \#\overline{\xi}(p) = 2\})$ is contained in $\text{Int } D^2 \cap \mathbb{R}$.

(e) $\overline{\xi}(D^2)$ does not contain the point $(w_1, w_2)$ and $(w_2, w_1)$.

We define a discrete set $B \subset \text{Int } D^2 \cap \mathbb{R}$ as

\[
B = \bigcup_{i=1}^{2} \overline{\xi}^{-1}(\{w_i\} \times (\Sigma_{g-1} \setminus \{w_i\})) \bigcup_{j=1}^{2} \overline{\xi}^{-1}(\Sigma_{g-1} \setminus \{w_j\}) \times \{w_j\}) \\
\cup \overline{\xi}^{-1}(\{p \in \Sigma_{g-1}^2 \setminus \Delta \mid \#\overline{\xi}(p) = 2\}).
\]
We next consider the case where $B = \{q_1, \ldots, q_n\} \subset D^2 \cap \mathbb{R}$. We assume that $q_1 < \cdots < q_n$. Denote by $S_i$ the upper semicircle centered at $(1 + q_i)/2$ whose ends are 1 and $q_i$. We also denote by $\xi_i$ a loop obtained by connecting a small counterclockwise circle around $q_i$ to the point $1 \in S^1$ using $S_i$. Since $\tilde{\xi}$ is an embedding over $S_i$, the image $\tilde{\xi}(S_i)$ is an embedded path, which we denote by $(\eta_1(S_i), \eta_2(S_i)) \subset \Sigma_{g-1} \setminus \Delta$. The loop $\tilde{\xi}(\xi_i)$ is homotopic to one of the following loops:

$$\tilde{\xi}(\xi_i) \simeq \begin{cases} l'(\eta_1(S_i)) & \text{if } \tilde{\xi}(q_i) \text{ is contained in } \{w_i\} \times (\Sigma_{g-1} \setminus \{w_i\}) \\ l'(\eta_2(S_i)) & \text{if } \tilde{\xi}(q_i) \text{ is contained in } (\Sigma_{g-1} \setminus \{w_j\}) \times \{w_j\} \\ \text{trivial loop} & \text{otherwise} \end{cases}$$

The loop $\tilde{\xi}$ is homotopic to $\tilde{\xi}|_{\xi_1 \cdots \xi_n}$, thus completing the proof of Lemma 3.2. □

We eventually obtain the following theorem.

**Theorem 3.4** For an element $\eta \in \Pi(\Sigma_{g-1} \setminus \{v_j, w_l\}) (\{i, k\} = \{j, l\} = \{1, 2\})$, we denote by $\delta(\eta) \subset \Sigma_{g-1}$ the boundary of a regular neighborhood of $\eta$. This is a simple closed curve in $\Sigma_{g-1} \setminus \{v_1, v_2, w_1, w_2\}$ and we can take a lift of this curve to $\tilde{\delta}(\eta) \subset \Sigma_{g+1} \setminus (c \cup d)$ by using the identification $\Sigma_{g-1} \setminus \{v_1, v_2, w_1, w_2\} \cong \Sigma_{g+1} \setminus (c \cup d)$. If $g$ is greater than 2, then the group $\text{Ker } \Phi_c \cap \text{Ker } \Phi_d$ is generated by the set

$$\{t_{\delta}(\eta) \cdot t_c^{-1} \cdot t_d^{-1} \in \text{Mod } (\Sigma_{g+1})(c, d) \mid \eta \in \Pi(\Sigma_{g-1} \setminus \{v_j, w_l\}, v_k, w_l) \quad (\{i, k\} = \{j, l\} = \{1, 2\})\}.$$ 

We next consider the case where $(c, d)$ is a separating curve. We put $g_2 = g - g_1$. By the same argument as in Lemma 3.1, we can prove the following lemma.

**Lemma 3.5** The following restrictions are isomorphisms:

$$\Phi_d^* \circ \Phi_c^* |_{\text{Ker } \Phi_c \cap \text{Ker } \Phi_d}: \text{Ker } \Phi_c \cap \text{Ker } \Phi_d \to \text{Ker } F_{v_1, v_2} \cap \text{Ker } F_{w_1, w_2}$$

$$\Phi_d^* |_{\text{Ker } \Phi_c \cap \text{Ker } \Phi_d}: \text{Ker } \Phi_c \cap \text{Ker } \Phi_d \to \text{Ker } \Phi_c \cap \text{Ker } F_{w_1, w_2}$$

$$\Phi_c^* |_{\text{Ker } \Phi_c \cap \text{Ker } \Phi_d}: \text{Ker } \Phi_c \cap \text{Ker } \Phi_d \to \text{Ker } F_{v_1, v_2} \cap \text{Ker } \Phi_d$$

The group $\text{Ker } F_{v_1, v_2}$ (resp. $\text{Ker } F_{w_1, w_2}$) is isomorphic to the group $\text{Ker } F_{v_1} \times \text{Ker } F_{v_2}$ (resp. $\text{Ker } F_{w_1} \times \text{Ker } F_{w_2}$). Thus, we obtain

$$\text{Ker } F_{v_1, v_2} \cap \text{Ker } F_{w_1, w_2} = (\text{Ker } F_{v_1} \cap \text{Ker } F_{w_1}) \times (\text{Ker } F_{v_2} \cap \text{Ker } F_{w_2}).$$
Furthermore, the group \( \text{Ker} \, F_{v_i} \cap \text{Ker} \, F_{w_i} \) is contained in the kernel of the homomorphism
\[
F_{v_i, w_i} : \text{Mod} \left( \Sigma g_i, v_i, w_i \right) \to M_{g_i}.
\]
This group is isomorphic to the group \( \pi_1((\Sigma g_i)^2 \setminus \Delta, (v_i, w_i)) \) if \( g_i \geq 2 \). Under this identification, it is easy to prove that \( \text{Ker} \, F_{v_i} \cap \text{Ker} \, F_{w_i} \) corresponds to the group \( \text{Ker} \, p_{1, \ast} \cap \text{Ker} \, p_{2, \ast} \), where we denote by \( p_j : (\Sigma g_i)^2 \setminus \Delta \to \Sigma g_i \) the projection onto the \( j \)-th component. Since \( p_2 \) is a locally trivial fibration with fiber \( \Sigma g_i \setminus \{ w_i \} \) (cf [11]), we can prove the following lemma by using van Kampen’s Theorem.

**Lemma 3.6** For an element \( \eta \in \Pi(\Sigma g_i, v_i, w_i) \), we denote by \( l(\eta) \) the loop
\[
[0, 1) \ni t \mapsto (L(\eta)(t), w_i) \in (\Sigma g_i)^2 \setminus \Delta.
\]
Then, the group \( \text{Ker} \, p_{1, \ast} \cap \text{Ker} \, p_{2, \ast} \) is generated by the set
\[
\{ [l(\eta)] \in \pi_1((\Sigma g_i)^2 \setminus \Delta, (v_i, w_i)) \mid \eta \in \Pi(\Sigma g_i, v_i, w_i) \}.
\]
As the case \( (c, d) \) is not a bounding pair, we eventually obtain the following theorem.

**Theorem 3.7** For an element \( \eta \in \Pi(\Sigma g_i, v_i, w_i) \), we denote by \( \delta(\eta) \subset \Sigma g_i \) the boundary of a regular neighborhood of \( \eta \). This is a simple closed curve in \( \Sigma g_i \setminus \{ v_i, w_i \} \) and we can take a lift of this curve to \( \widetilde{\delta}(\eta) \subset \Sigma_g g_{1+g_{2+1}} \setminus (c \cup d) \) by using the identification \( \Sigma g_1 \setminus \{ v_1, w_1 \} \sqcup \Sigma g_2 \setminus \{ v_2, w_2 \} \cong \Sigma g_{1+g_{2+1}} \setminus (c \cup d) \). If both of the numbers \( g_1 \) and \( g_2 \) are greater than or equal to 2, then the group \( \text{Ker} \, \Phi_c \cap \text{Ker} \, \Phi_d \) is generated by the set
\[
\{ t_{\delta(\eta)}^{-1} \cdot t_c^{-1} \cdot t_d^{-1} \in \text{Mod} (\Sigma g_{i+1})(c, d) \mid \eta \in \Pi(\Sigma g_i, v_i, w_i), i \in \{1, 2\} \}.
\]
We are now ready to discuss the fibration \( f : M \to I \times S^1 \) which we defined in the beginning of this section. Let \( N(p_i) \subset N \) be an open neighborhood of \( p_i \) in \( N \). We take a diffeomorphism \( \theta_i : B_1/\sqrt{3} \to N(p_i) \), where \( B_1/\sqrt{3} \subset \mathbb{R}^3 \) is a 3-ball with radius \( 1/\sqrt{3} \), so that \( h \circ \theta_i \) is described as
\[
h \circ \theta_1 : B_1/\sqrt{3} \to I, \quad (x, y, z) \mapsto x^2 + y^2 - z^2 + \frac{4}{9},
\]
\[
h \circ \theta_2 : B_1/\sqrt{3} \to I, \quad (x, y, z) \mapsto x^2 - y^2 - z^2 + \frac{5}{9}.
\]
We take a metric \( g \) of \( N \) so that the pullback \( \theta_i^* g \) is the standard metric on \( B_1/\sqrt{3} \). The metric \( g \) determines a rank 1 horizontal distribution \( \mathcal{H}_h = (\text{Ker} \, dh)^\perp \) of \( h|_{N \setminus \{ p_1, p_2 \}} \).
For each \( p \in N \setminus \{p_1, p_2\} \), we denote by \( c_p(t) \) a horizontal lift of the curve \( t \mapsto h(p) + t \) which satisfies \( c_p(0) = p \). We define submanifolds \( D^H_l(p_i) \) and \( D^H_u(p_i) \) as

\[
D^H_l(p_i) = \{p_i\} \cup \left\{ p \in N \mid h(p) < \frac{3+i}{9}, \lim_{t \to (3+i/9) - h(p)} c_p(t) = p_i \right\},
\]

\[
D^H_u(p_i) = \{p_i\} \cup \left\{ p \in N \mid h(p) > \frac{3+i}{9}, \lim_{t \to (3+i/9) - h(p)} c_p(t) = p_i \right\}.
\]

Note that \( D^H_l(p_1) \) and \( D^H_u(p_2) \) are diffeomorphic to the unit interval \( I \), but \( D^H_u(p_1) \) and \( D^H_l(p_2) \) are diffeomorphic to the 2–disk \( D^2 \). We take a homotopy \( h_t: N \to I \) with \( h_0 = h \) \((t \in I)\) satisfying the following conditions:

(a) The support of the homotopy is contained in \( N(p_1) \).
(b) For any \( t \in I \), \( h_t \) has two critical points \( p_1 \) and \( p_2 \).
(c) For any \( t \in I \), the critical point \( p_1 \) of \( h_t \) is nondegenerate and its index is 1.
(d) The function \( t \mapsto h_t(p_1) \) is monotone increasing.
(e) \( h_1(p_1) = \frac{2}{3} \)

This homotopy changes the order of critical points. We can take such a homotopy since \( c \) and \( d \) are disjoint. We take a smooth function \( \rho: I \to I \) satisfying the following properties:

- \( \rho \equiv 0 \) on \([0, \frac{1}{6}] \cup [\frac{5}{6}, 1]\)
- \( \rho \equiv 1 \) on \([\frac{1}{3}, \frac{2}{3}]\)
- \( \rho \) is monotone increasing on \([\frac{1}{6}, \frac{1}{3}]\)
- \( \rho(1 - s) = \rho(s) \) for any \( s \in [0, 1] \)

By using \( h_t \) and \( \rho \), we define a homotopy \( f_t: M = N \times S^1 \to I \times S^1 \) by

\[
f_t: M = N \times S^1 \to I \times S^1,
\]

\[
(x, s) \mapsto (h_t \rho(s)(x), s).
\]

Since \( N \) is obtained by attaching the 1–handle and the 2–handle to \( \Sigma_g \times I \), \( \partial N \) contains the surface \( \Sigma_g \times \{0\} \), which we denote by \( \Sigma \) for simplicity. Moreover, \( \Sigma \) intersects \( D^H_l(p_1) \) at two points \( v_1, v_2 \in \Sigma \), and \( \Sigma \) intersects \( D^H_u(p_2) \) at a simple closed curve \( d \subset \Sigma \). Let \( \Pi(\Sigma, v_i, d) \) be the set of embedded paths in \( \Sigma \) from the point \( v_i \) to a point in \( d \). For \( \eta \in \Pi(\Sigma, v_i, d) \), \( L(\eta): ([0, 1], \{0, 1\}) \to (\Sigma \setminus d, v_i) \) denotes a loop in the neighborhood of \( \eta \cup d \), which is injective on \([0, 1]\) and homotopic to a loop obtained by connecting \( v_i \) to \( d \) using \( \eta \). For an element \( \eta \in \Pi(\Sigma, v_i, d) \), we take a homotopy of horizontal distributions \( \{H^\eta_t\} \) \((t \in [0, 1])\) of \( h_1|_{N \setminus \{p_1, p_2\}} \) with \( H^\eta_0 = H_{h_1} \) which satisfies the following conditions:
As mentioned in the beginning of this section, we can identify the identifications boundary of a regular neighborhood of along the closed surface called in [21], we call this kind of move an of singular loci are changed like Reidemeister move of type II (cf Figure 5). As is the original fibration by the homotopies , the image of singular loci are changed like Reidemeister move of type II (cf Figure 5). As is called in [21], we call this kind of move an R2–move.

As mentioned in the beginning of this section, we can identify with the closed surface . Thus, we can take the monodromy of along . Since is contained in the group and an identification is unique up to Dehn twist , the mapping class is independent of an identification .

Lemma 3.8 Let be the simple closed curve corresponding to the boundary of a regular neighborhood of , for , under the identifications , then .
**Proof of Lemma 3.8** Since both sides of the boundary $\partial M$ are trivial surface bundles, $\varphi_\gamma$ is contained in the group $\text{Ker } \Phi_c \cap \text{Ker } \Phi_d$. We consider the element $\Phi_c^*(\varphi_\gamma) \in \text{Mod} (\Sigma_g; v_1, v_2)(d)$. This element can be realized as the monodromy of a certain fibration in the following way. We first take a sufficiently small neighborhood of the following subset of $M$:

$$\bigcup_{s \in [0, \frac{1}{3}]} (\mathcal{H}_{\nu_p(s)}(p_1) \cup D_u^{\mathcal{H}_{\nu_p(s)}(p_1)}) \times \{s\} \times \bigcup_{s \in [\frac{1}{3}, \frac{2}{3}]} (\mathcal{H}_{\nu_p(s)}^n(p_1) \cup D_u^{\mathcal{H}_{\nu_p(s)}^n(p_1)}) \times \{s\}$$

We denote this neighborhood by $U \subset M$. The restriction $\tilde{f}_0|_{M \setminus U}$ is a fibration with a connected singular locus $Z_2$. We take a suitable $U$ so that we can take a horizontal distribution $\tilde{\mathcal{H}}$ of $\tilde{f}_0|_{M \setminus (U \cup Z_2)}$ satisfying the following conditions:

- $\tilde{\mathcal{H}}$ is tangent to $\partial U$.
- $\tilde{\mathcal{H}}$ equals $\bigcup_{s \in S^1} (\mathcal{H}_{\nu_p(s)} \oplus T_s S^1)$ on a small neighborhood of $\partial N \times S^1 \subset M$ and $\tilde{f}_0^{-1}(I \times ([0, \frac{1}{6}] \cup [\frac{5}{6}, 1])) \subset M$.

This distribution gives a monodromy of $\tilde{f}_0|_{M \setminus \widetilde{U}}$ along $\gamma$. Identify $\Sigma = \Sigma_g \times \{0\} \subset \partial N$ with $\Sigma_g$. The fiber $\tilde{f}_0^{-1}(\{\frac{1}{2}\} \times \{0\}) \setminus \widetilde{U}$ is canonically identified with $\Sigma_g \setminus \{v_1, v_2\}$. By the condition (k) on the family of homotopies $\{h_{t, s}\}$, this monodromy is $\Phi_c^*(\varphi_\gamma)$.

Since the region $[0, \frac{1}{2}] \times S^1$ does not contain any singular values of the fibration $\tilde{f}_0|_{M \setminus U}$, $\Phi_c^*(\varphi_\gamma)$ corresponds to the monodromy of $\tilde{f}_0|_{M \setminus U}$ along the loop.
We can take the following diffeomorphism by using the horizontal distribution
\[ \widetilde{\gamma} : \begin{cases} 
(0, t) & t \in [0, \frac{1}{3}], \\
\left(\frac{9}{2}(t - \frac{1}{3}), \frac{1}{3}\right) & t \in \left[\frac{1}{3}, \frac{4}{9}\right], \\
\left(\frac{1}{2}, 3(t - \frac{1}{3})\right) & t \in \left[\frac{4}{9}, \frac{5}{9}\right], \\
\left(\frac{9}{2}(\frac{2}{3} - t), \frac{2}{3}\right) & t \in \left[\frac{5}{9}, \frac{2}{3}\right], \\
(0, t) & t \in \left[\frac{2}{3}, 1\right], 
\end{cases} \]
where \( t \in I \) and the right-hand side is in \( I \times S^1 \). We denote by \( \psi_t : f_0^{-1}(\widetilde{\gamma}(0)) \cong \Sigma_g \rightarrow f_0^{-1}(\widetilde{\gamma}(t)) \) the diffeomorphism obtained by using the distribution \( \widetilde{\mathcal{H}} \) and the path \( \widetilde{\gamma}|_{[0,t]} \). Note that we can canonically identify \( f_0^{-1}(\widetilde{\gamma}(t)) \) with \( \Sigma_g \) for \( t \in [0, \frac{1}{3}] \cup \left[\frac{4}{9}, 1\right] \). Moreover, under the identification, \( \psi_t \) is the identity for \( t \in [0, \frac{1}{3}] \), and \( \psi_t = \psi_1 \) for \( t \in \left[\frac{2}{3}, 1\right] \) since \( \widetilde{\mathcal{H}} \) equals \( \bigsqcup_{s \in S^1}(\mathcal{H}_{h,\rho(s)} \oplus T_s S^1) \) on \( \partial N \times S^1 \).

We can take the following diffeomorphism by using the horizontal distribution \( \widetilde{\mathcal{H}} \) of \( f_0|M\setminus Z_1 \cup Z_2 \) together with its horizontal lifts of \( t \mapsto (t, s) \in I \times S^1 \):
\[ \widetilde{\psi}_s : \Sigma_g \cong f_0^{-1}((0, s)) \rightarrow f_0^{-1}((\frac{1}{2}, s))(s \in \left[\frac{1}{3}, \frac{2}{3}\right]) \]

By the definitions of \( \psi_t \) and \( \widetilde{\psi}_s \), we obtain the equalities
\[ \widetilde{\psi}^{-1}_{1/3} \circ \psi_{\frac{1}{3}} = \operatorname{id}_{\Sigma_g}, \]
\[ \widetilde{\psi}^{-1}_{2/3} \circ \psi_{\frac{2}{3}} = \psi_1, \]
\[ \widetilde{\psi}^{-1}_{3(t-(1/3))} \circ \psi_t(v_i) = L(\eta)(9t-4) \quad \text{for} \quad t \in \left[\frac{4}{9}, \frac{5}{9}\right]. \]

This means that the path \( [0, 1] \ni t \mapsto \widetilde{\psi}_{1/3}(t+1) \circ \psi_{1/9}(t+4) \in \text{Diff}^{+}(\Sigma_g; v_j) \) is the lift of the loop \( L(\eta) \) in \( \Sigma_g \setminus \{v_j\} \) under the locally trivial fibration
\[ \text{Diff}^{+}(\Sigma_g; v_i, v_j) \leftrightarrow \text{Diff}^{+}(\Sigma_g; v_j) \rightarrow \Sigma_g \setminus \{v_j\}, \]
where \( \varepsilon \) is the evaluation map. Thus, we obtain
\[ \Phi^*_c(\varphi_\gamma) = [\psi_1] = \text{Push}(L(\eta)) = t_{\delta(\eta)}^{-1} \cdot t_{d}^{-1} \in \text{Mod}(\Sigma_g; v_1, v_2)(d), \]
where \( \text{Push}(L(\eta)) \) is the pushing map along \( L(\eta) \). By Lemma 3.1 or 3.5, we have that \( \Phi^*_c|_{\ker \Phi_c \cap \ker \Phi_d} \) is an isomorphism. We therefore obtain
\[ \varphi_\gamma = \Phi^*_{c,1} \circ \Phi^*_c(\varphi_\gamma) \]
\[ = \Phi^*_{c,1} \circ (t_{\delta(\eta)}^{-1} \cdot t_{d}^{-1}) = t_{\delta(\eta)}^{-1} \cdot t_{c}^{-1} \cdot t_{d}^{-1}. \]

This completes the proof of Lemma 3.8.

Combining Theorems 3.4 and 3.7, we obtain the following theorem.
Theorem 3.9 Let \( f: M \to I \times S^1 \) and \( \gamma \subset I \times S^1 \) be as in the beginning of this section. Assume that \( g \) is greater than or equal to 3 when \((c,d)\) is not a bounding pair, and that both of the numbers \( g_1 \) and \( g_2 = g - g_1 \) are greater than or equal to 2 when \((c,d)\) is a bounding pair of genus \( g_1 \). For any \( \varphi \in \text{Ker} \Phi_c \cap \text{Ker} \Phi_d \), we can change \( f \) by successive application of \( R_2 \)-moves so that the monodromy of \( f|_{M \setminus (f^{-1}(Z_1) \cup f^{-1}(Z_2))} \) along \( \gamma \) corresponds to the element \( \varphi \).

4 Relation between vanishing cycles and flip and slip moves

Let \( f: M \to D^2 \) be a purely wrinkled fibration satisfying the following conditions:

1. The set of critical points \( S_f \) of \( f \) is an embedded circle in \( \text{Int} M \).
2. The restriction \( f|_{S_f} \) is an embedding.
3. Either of the following conditions on regular fibers holds:
   - A regular fiber on the outside of \( f(S_f) \) is connected, while that on the inside of \( f(S_f) \) is disconnected.
   - Every regular fiber is connected and the genus of a regular fiber on the outside of \( f(S_f) \) is higher than that on the inside of \( f(S_f) \).

We fix a point \( p_0 \in \partial D^2 \) and an identification \( f^{-1}(p_0) \cong \Sigma_g \). Let \( \varphi_0 \in \mathcal{M}_g \) be the monodromy along \( \partial D^2 \) oriented counterclockwise around the origin of \( D^2 \) with base point \( p_0 \). In this section, we will give an algorithm to obtain vanishing cycles in a single higher genus regular fiber of a fibration obtained by applying flip and slip to \( f \).

We first consider the simplest case, that is, assume that \( f \) has no cusps. We take a reference path \( \gamma_0 \) in \( \partial D^2 \) connecting \( p_0 \) to a point in the image of indefinite folds so that it satisfies \( \text{Int} \gamma_0 \cap f(S_f) = \emptyset \). This determines a vanishing cycle \( c \subset \Sigma_g \) of indefinite folds. Then, it is easy to prove that \( \varphi_0 \) is contained in the group \( \text{Ker} \Phi_c \). To give an algorithm precisely, we prepare several conditions. The first condition is on an embedded path \( \alpha \subset \Sigma_g \).

**Condition C\(_1\)(c)** A path \( \alpha \subset \Sigma_g \) intersects \( c \) at the unique point \( q \in c \) transversely.

We take a path \( \alpha \subset \Sigma_g \) so that \( \alpha \) satisfies the condition \( C_1(c) \). We put \( \partial \alpha = \{ w_1, w_2 \} \). The second condition is on a simple closed curve \( d \subset \Sigma_{g+1} \) and a diffeomorphism \( j: \Sigma_g \setminus \{ w_1, w_2 \} \to \Sigma_{g+1} \setminus d \).

**Condition C\(_2\)(c, \alpha)** The closure of \( j(\text{Int} \alpha) \) in \( \Sigma_{g+1} \) is a simple closed curve.

We take a simple closed curve \( d \subset \Sigma_{g+1} \) and a diffeomorphism \( j: \Sigma_g \setminus \{ w_1, w_2 \} \to \Sigma_{g+1} \setminus d \) so that they satisfy the condition \( C_2(c, \alpha) \). We put \( \tilde{c} = j(c) \).
The last condition is on an element \( \varphi \in \text{Mod}(\Sigma_{g+1})(\tilde{c}, d) \).

**Condition C\(_3\)(c, \alpha, d, j, \varphi_0)**  We have \( \Phi_{\tilde{c}}(\varphi) = 1 \) in \( \text{Mod}(\Sigma_g)(d) \) and \( \Phi_d(\varphi) = \varphi_0^{-1} \) in \( \text{Mod}(\Sigma_g)(c) \).

For the sake of simplicity, we will call the above conditions \( C_1, C_2 \) and \( C_3 \) if elements \( c, \alpha, d, j \) and \( \varphi_0 \) are obvious. Examples of \( \alpha, d, j \) are described in Figure 6.

![Diagram](image.png)

**Figure 6:** Examples of \( \alpha, d, j \)

**Theorem 4.1** Let \( f : M \to D^2 \) be a purely wrinkled fibration we took in the beginning of this section. We assume that \( f \) has no cusps.

1. Let \( \tilde{f} \) be a fibration obtained by applying flip and slip to \( f \). We take a point \( q_0 \) in the inside of \( f(S_{\tilde{f}}) \), and reference paths \( \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3 \) and \( \tilde{\gamma}_4 \) in \( D^2 \) connecting \( q_0 \) to a point on the respective fold arcs between cusps so that these paths appear in this order when we go around \( q_0 \) counterclockwise. Denote by \( e_i \subset \tilde{f}^{-1}(q_0) \) a vanishing cycle determined by the path \( \tilde{\gamma}_i \). Then, there exist an identification \( \tilde{f}^{-1}(q_0) \cong \Sigma_{g+1} \) and elements \( \alpha, d, j \) and \( \varphi \) satisfying the conditions \( C_1, C_2 \) and \( C_3 \) such that the following equality holds up to cyclic permutation:

\[
(e_1, e_2, e_3, e_4) = (\tilde{c}, \alpha', d, \tilde{\alpha}),
\]

where \( \tilde{\alpha} \) is the closure of \( j(\text{Int}\alpha) \) in \( \Sigma_{g+1} \) and \( \alpha' = \varphi^{-1}(\tilde{\alpha}) \) (see Figure 6).

2. Let \( \alpha, d, j \) and \( \varphi \) be elements satisfying the conditions \( C_1, C_2 \) and \( C_3 \). We take simple closed curves \( \tilde{c}, \tilde{\alpha} \) and \( \alpha' \) as in (1). Suppose that the genus of a higher genus fiber \( g \) of \( f \) is greater than or equal to 3 when \((\tilde{c}, d)\) is not a bounding pair, and that both of the genera \( g_1 \) and \( g_2 \) are greater than or equal to 2 when \((\tilde{c}, d)\) is a bounding pair of genus \( g_1 \), where we put \( g_2 = g - g_1 \). Then, there exists a fibration \( \tilde{f} \) obtained by applying flip and slip to \( f \) such that, for reference paths \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_4 \) as in (1), the corresponding vanishing cycles \( e_1, \ldots, e_4 \) satisfy the following equality up to cyclic permutation:

\[
(e_1, e_2, e_3, e_4) = (\tilde{c}, \alpha', d, \tilde{\alpha})
\]
Proof of Theorem 4.1(1) As in Figure 7, we take points $q_0, q_0', q_0'', q_1, q_1', q_1'' \in D^2$ and paths $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2, \delta_0, \delta_1 \subset D^2$.

Figure 7: The points $q_0, q_1$ are in the region with the highest genus fibers, while the points $q_0', q_0'', q_1', q_1''$ are on the set of critical values. The path $\tilde{\gamma}_0$ connects $p_0$ to $q_0''$ and the path $\delta_0$ connects $q_0$ to a point in the set of critical values. We take $\tilde{\gamma}_1$ and $\delta_1$ similarly. The path $\tilde{\gamma}_2$ connects $q_0$ to $q_1$. Note that we regard $D^2$ as a subset of $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and that $\infty \in D^2$.

We take an identification of the region $\Omega \subset D^2$ described in Figure 8 with the rectangle $I \times I$ so that the paths $\tilde{\gamma}_0, \tilde{\gamma}_1$ are contained in the side edges of the rectangle, that the path $\tilde{\gamma}_2$ corresponds to the middle horizontal line, and the set of critical values corresponds to the upper and the lower horizontal lines (see the right side of Figure 8). For each $x \in \tilde{\gamma}_2$, we denote by $u_x$ (resp. $l_x$) the vertical path which connects $x$ to the upper (resp. lower) singular image as in the right side of Figure 8.

Figure 8: The shaded region in the left figure is the region $\Omega$. The horizontal line with arrow in the right figure describes the path $\tilde{\gamma}_2$, while the horizontal dotted lines describe images of the singular loci.

We take a horizontal distribution $\mathcal{H}$ of $\tilde{f}|_{M \setminus S_{\tilde{f}}}$ so that it satisfies the following conditions.

(1) Let $w_1^{(i)}, w_2^{(i)}$ be points in $\tilde{f}^{-1}(p_0)$ which converge to an indefinite fold when $\tilde{f}^{-1}(p_0)$ approaches the singular fiber $\tilde{f}^{-1}(q_i')$ along $\tilde{\gamma}_i$ using $\mathcal{H}$. The set $\{w_1^{(0)}, w_2^{(0)}\}$ equals the set $\{w_1^{(1)}, w_2^{(1)}\}$. 
(2) Let $d^{(i)}$ (resp. $\tilde{c}^{(i)}$) be simple closed curves in $\tilde{f}^{-1}(q_i)$ which converge to an indefinite fold when $\tilde{f}^{-1}(q_i)$ approaches the singular fiber $\tilde{f}^{-1}(q_i)$ (resp. $\tilde{f}^{-1}(q_i')$) along $\tilde{y}_i$ using $\mathcal{H}$. For each $i = 0, 1$, $d^{(i)}$ is disjoint from $\tilde{c}^{(i)}$.

(3) We obtain a diffeomorphism $j_i$: $\tilde{f}^{-1}(p_0) \setminus \{w_1, w_2\} \to \tilde{f}^{-1}(q_i) \setminus d^{(i)}$ by using a horizontal lift of the curve $\tilde{y}_i$. By condition (2), $j_i^{-1}(\tilde{c}^{(i)})$ is a simple closed curve in $\tilde{f}^{-1}(p_0)$. $j_0^{-1}(\tilde{c}^{(0)}) = j_1^{-1}(\tilde{c}^{(1)}) = c$.

(4) Let $\tilde{a}^{(i)}$ be a simple closed curve in $\tilde{f}^{-1}(q_i)$ which converges to an indefinite fold when $\tilde{f}^{-1}(p_0)$ approaches a singular fiber along $\delta_i$ using $\mathcal{H}$. $\tilde{a}^{(i)}$ intersects both of the curves $\tilde{c}^{(i)}$ and $d^{(i)}$ transversely.

(5) We have $\#(\tilde{a}^{(i)} \cap d^{(i)}) = \#(\tilde{c}^{(i)} \cap \tilde{a}^{(i)}) = 1$.

(6) By conditions (4) and (5), the closure of $j_1^{-1}(\tilde{a}^{(i)} \setminus d^{(i)})$ is a segment between $w_1^{(i)}$ and $w_2^{(i)}$. The closure of $j_0^{-1}(\tilde{a}^{(0)} \setminus d^{(0)})$ equals the closure of $j_1^{-1}(\tilde{a}^{(1)} \setminus d^{(1)})$.

(7) Since the path $\tilde{y}_2$ does not contain any critical values of $\tilde{f}$, this path, together with $\mathcal{H}$, gives a diffeomorphism from $\tilde{f}^{-1}(q_0)$ to $\tilde{f}^{-1}(x)$ for each $x \in \tilde{y}_2$. This diffeomorphism sends the curve $d^{(0)}$ (resp. $\tilde{c}^{(0)}$) to the curve $d_x$ (resp. $\tilde{c}_x$), where $d_x$ (resp. $\tilde{c}_x$) is a simple closed curve in $\tilde{f}^{-1}(x)$ which converges to an indefinite fold when $\tilde{f}^{-1}(x)$ approaches a singular fiber along $u_x$ (resp. $l_x$) using $\mathcal{H}$.

We choose indices of $w_1^{(i)}$ and $w_2^{(i)}$ so that $w_1^{(0)}$ corresponds to $w_1^{(1)}$ and we put $w_i = w_i^{(0)} = w_i^{(1)}$. We denote by $\alpha$ the closure of $j_0^{-1}(\tilde{a}^{(0)} \setminus d^{(0)})$ (which corresponds to the closure of $j_1^{-1}(\tilde{a}^{(1)} \setminus d^{(1)})$). Since we fixed an identification $\tilde{f}^{-1}(p_0) \cong \Sigma_g$, we can regard $w_1, w_2$ as points in $\Sigma_g$. We can also regard $\alpha$ as a segment in $\Sigma_g$ between $w_1$ and $w_2$. We choose an identification $\Sigma_g \setminus \{w_1, w_2\} \cong \Sigma_{g+1} \setminus d$, where $d \subset \Sigma_{g+1}$ is a nonseparating simple closed curve, so that the induced identification between $\Sigma_{g+1} \setminus d$ and $\tilde{f}^{-1}(q_i) \setminus d^{(i)}$ can be extended to an identification between $\Sigma_{g+1}$ and $\tilde{f}^{-1}(q_i)$ (to take such an identification, we modify $\mathcal{H}$ if necessary). By using this identification, we can regard $\tilde{c}^{(i)}$ as a curve in $\Sigma_{g+1}$, which we denote by $\tilde{c}$. We denote the identification between $\Sigma_{g+1}$ and $\tilde{f}^{-1}(q_i)$ as

$$\theta_i: \Sigma_{g+1} \cong \tilde{f}^{-1}(q_i), \quad i = 0, 1.$$ 

On the other hand, we obtain a diffeomorphism between $\tilde{f}^{-1}(q_0)$ and $\tilde{f}^{-1}(q_1)$ by taking horizontal lifts of $\tilde{y}_2$ using $\mathcal{H}$. We denote this diffeomorphism as

$$\theta_2: \tilde{f}^{-1}(q_0) \cong \tilde{f}^{-1}(q_1).$$

By condition (7), the diffeomorphism sends $d^{(0)}$ (resp. $\tilde{c}^{(0)}$) to the curve $d^{(1)}$ (resp. $\tilde{c}^{(1)}$). Thus, the isotopy class $[\theta_1^{-1} \circ \theta_2 \circ \theta_0]$ is contained in the subgroup.
Modification rule of monodromies in an $R_2$–move

Mod $(\Sigma_{g+1})(\bar{c}, d)$ of the mapping class group $\mathcal{M}_{g+1}$. We denote this class by $\varphi \in \text{Mod} (\Sigma_{g+1})(\bar{c}, d)$.

We denote by $\bar{\gamma}_2 \cdot \delta_1$ be the path in $D^2$, starting at the point $q_0$, obtained by connecting $\bar{\gamma}_2$ to $\delta_1$. This path gives the fiber $\bar{f}^{-1}(q_0)$ a vanishing cycle of $\bar{f}$. This vanishing cycle is equal to the curve $\theta_1^{-1}(\bar{\alpha}(0)) = \theta_1^{-1} \circ \theta_1(\bar{\alpha})$. This curve corresponds to the curve $\theta_0^{-1} \circ \theta_2^{-1} \circ \theta_1(\bar{\alpha}) = \varphi^{-1}(\bar{\alpha}) \subset \Sigma_{g+1}$ under the identification $\theta_0$. Thus, the proof is completed once we prove the following lemma.

**Lemma 4.2**  We have $\Phi_{\bar{c}}(\varphi) = 1$ and $\Phi_d(\varphi) = \varphi_0^{-1}$.

**Proof of Lemma 4.2**  The image $\Phi_d(\varphi)$ is equal to the monodromy along the curve $\delta_h$ described in the left side of Figure 9, which corresponds to $\varphi_0^{-1}$.

![Figure 9: Base diagrams of fibrations](image)

Thus, we have $\Phi_d(\varphi) = \varphi_0^{-1}$. To prove that $\Phi_{\bar{c}}(\varphi) = 1$, we consider the fibration obtained by applying an unsink to $\bar{f}$. We take the path $\bar{\gamma}_2'$ connecting $q_0$ to $q_1$ as in the right side of Figure 9. It is easy to see that the monodromy along this path corresponds to $(t_{td}(\bar{\alpha}) \cdot t_{t\bar{c}}(\bar{c})) \cdot \varphi \cdot (t_{td}(\bar{\alpha}) \cdot t_{t\bar{c}}(\bar{c}))^{-1}$. This preserves the curve $d$ and the image $\Phi_d((t_{td}(\bar{\alpha}) \cdot t_{t\bar{c}}(\bar{c})) \cdot \varphi \cdot (t_{td}(\bar{\alpha}) \cdot t_{t\bar{c}}(\bar{c}))^{-1})$ is trivial since this element is the monodromy along the curve obtained by pushing the curve $\bar{\gamma}_2'$ out of the region with higher genus fibers, which is nullhomotopic in the complement of the set of critical values. We can obtain the element $\Phi_{\bar{c}}(\varphi)$ by taking some conjugation of $\Phi_d((t_{td}(\bar{\alpha}) \cdot t_{t\bar{c}}(\bar{c})) \cdot \varphi \cdot (t_{td}(\bar{\alpha}) \cdot t_{t\bar{c}}(\bar{c}))^{-1})$. In particular, $\Phi_{\bar{c}}(\varphi)$ is also trivial and this completes the proof of Lemma 4.2. □

This completes the proof of Theorem 4.1(1).

**Proof of Theorem 4.1(2)**  In the proof of Theorem 4.1(1), we take a horizontal distribution of $\bar{f}|_{\mathcal{M}S_f}$ and an identification $\Sigma_g \setminus \{w_1, w_2\} \cong \Sigma_{g+1} \setminus d$. Once we take these auxiliary data, we can get vanishing cycles of $\bar{f}$ in a canonical way. We
first take a horizontal distribution of \( f|_M \setminus S_f \) so that the embedded path \( \alpha \subset \Sigma_g \) determined by the distribution corresponds to the given one. We next take an identification \( \Sigma_g \setminus \{w_1, w_2\} \cong \Sigma_{g+1} \setminus d \) by using the given \( d, j \). The element \([\theta_1^{-1} \circ \theta_2 \circ \theta_0]\), which appears in the proof of Theorem 4.1(1), is canonically determined by the chosen horizontal distribution of \( \tilde{f}|_M \setminus S_{\tilde{f}} \) of \( M \) and the chosen homotopy from \( f \).

Let \( \Omega \) be the region in \( D^2 \) as in Figure 8. We take an identification \( \Omega \cong I \times \gamma_2 \). We also take a diffeomorphism \( \Theta: \tilde{f}^{-1}(\gamma_0 \cap \Omega) \to \tilde{f}^{-1}(\gamma_1 \cap \Omega) \) so that it satisfies \( \tilde{f} \circ \Theta = i \circ \tilde{f} \), where \( i: I \times \{q_0\} \ni (t, q_0) \mapsto (t, q_1) \in I \times \{q_1\} \), and that the 4–manifold \( \tilde{f}^{-1}(\Omega)/\Theta \) is the trivial \( N \)–bundle over \( S^1 \), where \( N \) is a 3–manifold defined in Section 3. For any two elements \( \varphi_1, \varphi_2 \in \text{Mod}(\Sigma_{g+1}; \tilde{c}, d) \) satisfying the condition \( C_3 \), the element \( \varphi_1 \cdot \varphi_2^{-1} \) is contained in the group \( \text{Ker} \Phi_{\tilde{c}} \cap \text{Ker} \Phi_d \). Thus, Theorem 3.9 implies that we can change \( f \) into \( \tilde{f} \) by a flip and slip move so that the resulting element \([\theta_1^{-1} \circ \theta_2 \circ \theta_0]\) corresponds to \( \varphi^{-1} \in \text{Mod}(\Sigma_{g+1})(\tilde{c}, d) \) for the given \( \varphi \). This completes the proof of Theorem 4.1(2). \( \square \)

We next consider the case that \( f \) has cusps. We denote by \( \{s_1, \ldots, s_n\} \) the set of cusps of \( f \). We put \( u_i = f(s_i) \). The indices of \( s_i \) are chosen so that \( u_1, \ldots, u_n \) appear in this order when we travel the image \( f(S_f) \) clockwise around a point inside \( f(S_f) \). The points \( u_1, \ldots, u_n \) divide the image \( f(S_f) \) into \( n \) edges. We denote by \( l_i \subset f(S_f) \) the edge between \( u_i \) and \( u_{i+1} \), where we put \( u_{n+1} = u_1 \). For a point \( p_0 \in \partial D^2 \), we take reference paths \( \gamma_1, \ldots, \gamma_n \subset D^2 \) satisfying the following conditions (see also the left figure of Figure 10):

- \( \gamma_i \) connects \( p_0 \) to a point in \( \text{Int} \ l_i \).
- \( \gamma_i \cap \gamma_j = \{p_0\} \) for all \( i \neq j \)
- \( \text{Int} \ \gamma_i \cap f(S_f) = \emptyset \)
- \( \gamma_1, \ldots, \gamma_n \) appear in that order when we go around \( p_0 \) counterclockwise.

Let \( \gamma_{n+1} \) be a path obtained by connecting \( \partial D^2 \) oriented clockwise around the center of \( D^2 \) to \( \gamma_1 \). The paths give \( f^{-1}(p_0) \cong \Sigma_g \) vanishing cycles \( c_1, \ldots, c_{n+1} \). Note that, for each \( i \in \{1, \ldots, n\} \), \( c_i \) intersects \( c_{i+1} \) at a unique point transversely. In particular, every simple closed curve \( c_i \) is nonseparating. We also remark that \( c_{n+1} \) equals \( \varphi_0(c_1) \).

Let \( \hat{f}: M \to D^2 \) be the fibration obtained by changing all the cusp singularities of \( f \) into Lefschetz singularities by applying unsink to \( f \) \( n \) times. We take paths \( \varepsilon_1, \ldots, \varepsilon_n \) in \( D^2 \) satisfying the following conditions (see also the middle figure of Figure 10):

- \( \varepsilon_i \) connects \( p_0 \) to the image of the Lefschetz singularity derived from \( s_{i+1} \).
- \( \varepsilon_i \cap \varepsilon_j = \{p_0\} \) for all \( i \neq j \)
- \( \text{Int} \ \varepsilon_i \cap \hat{f}(S_{\hat{f}}) = \emptyset \)
\[ \gamma_1, \varepsilon_1, \gamma_2, \ldots, \gamma_n, \varepsilon_n, \gamma_{n+1} \] appear in that order when we go around \( p_0 \) counterclockwise.

The path \( \varepsilon_i \) gives a vanishing cycle of a Lefschetz critical point of \( \hat{f} \), the curve \( t_{c_i}(c_{i+1}) \).

Let \( \gamma_0 \) be a based loop in \( D^2 \setminus \hat{f}(S_f) \) with base point \( p_0 \) homotopic to the loop obtained by connecting \( p_0 \) to \( \hat{f}(S_f) \) oriented counterclockwise around a point inside \( f(S_f) \) using \( \gamma_1 \) (see the right figure of Figure 10). It is easy to see that the monodromy along \( \gamma_0 \) corresponds to the element

\[ \hat{\varphi}_0 = \varphi_0 \cdot (t_{c_1}(c_2) \cdots t_{c_n}(c_{n+1}))^{-1}. \]

This element preserves the curve \( c_1 \) and is contained in the kernel of the homomorphism \( \Phi_{c_1} \).

\[ \begin{array}{c}
\text{Figure 10: Left: the image of the critical locus of } \hat{f} \text{ and the reference paths } \gamma_1, \ldots, \gamma_{n+1}; \text{ middle: the image of the critical locus of } \hat{f} \text{ and the reference paths } \varepsilon_1, \ldots, \varepsilon_n; \text{ right: the loop } \gamma_0
\end{array} \]

Since application of flip and slip to \( f \) is equivalent to application of flip and slip to \( \hat{f} \) followed by application of sink \( n \) times, we can obtain vanishing cycles of a fibration obtained by applying flip and slip to \( f \) in the way quite similar to that in the case \( f \) has no cusps. In order to give the precise algorithm to obtain vanishing cycles, we prepare several conditions.

**Condition \( \widetilde{C}_1(c_1, \ldots, c_n) \)** A path \( \alpha \subset \Sigma_g \) intersects \( c_1 \) at the unique point \( q \in c_1 \) transversely. Furthermore, \( \partial \alpha \cap (c_1 \cup \cdots \cup c_{n+1}) = \emptyset \).

We take a path \( \alpha \subset \Sigma_g \) so that \( \alpha \) satisfies the condition \( \widetilde{C}_1(c_1, \ldots, c_n) \). We put \( \partial \alpha = \{w_1, w_2\} \). The second condition is on a simple closed curve \( d \subset \Sigma_{g+1} \) and a diffeomorphism \( j: \Sigma_g \setminus \{w_1, w_2\} \to \Sigma_{g+1} \setminus d \).

**Condition \( \widetilde{C}_2(c_1, \ldots, c_n, \alpha) \)** The closure of \( j(\operatorname{Int} \alpha) \) in \( \Sigma_{g+1} \) is a simple closed curve.

We take a simple closed curve \( d \subset \Sigma_{g+1} \) and a diffeomorphism \( j: \Sigma_g \setminus \{w_1, w_2\} \to \Sigma_{g+1} \setminus d \) so that they satisfy the condition \( \widetilde{C}_2(c_1, \ldots, c_n, \alpha) \). We put \( \tilde{c}_1 = j(c_1) \). The third condition is on an element \( \varphi \in \operatorname{Mod}(\Sigma_{g+1})(\tilde{c}_1, d) \).
Condition $\tilde{C}_3(c_1, \ldots, c_n, \alpha, d, j, \varphi_0)$: We have $\Phi_{\tilde{c}_1}(\varphi) = 1$ in $\text{Mod}(\Sigma_g)(d)$ and $\Phi_d(\varphi) = \hat{\varphi}_0^{-1}$ in $\text{Mod}(\Sigma_g)(c_1)$.

The last condition is on simple closed curves $\tilde{c}_2, \ldots, \tilde{c}_{n+1} \subset \Sigma_{g+1} \setminus d$.

Condition $\tilde{C}_4(c_1, \ldots, c_n, \alpha, d, j)$: For each $i \in \{2, \ldots, n+1\}$, $i(\tilde{c}_i)$ is isotopic to $c_i$ in $\Sigma_g$, where $i$ is an embedding defined by

$$i: \Sigma_{g+1} \setminus d \xrightarrow{j^{-1}} \Sigma_g \setminus \{w_1, w_2\} \hookrightarrow \Sigma_g.$$  

Furthermore, for each $i = 1, \ldots, n$, $\tilde{c}_i$ intersects $\tilde{c}_{i+1}$ at a unique point transversely.

As the case $f$ has no cusps, we will call the above conditions $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ and $\tilde{C}_4$ if elements $c_1, \ldots, c_n, \alpha, d, j$ and $\varphi_0$ are obvious. We can prove the following theorem by an argument similar to that in the proof of Theorem 4.1.

**Theorem 4.3**  Let $f: M \to D^2$ be a purely wrinkled fibration we took in the beginning of this section. Suppose that $f$ has $n > 0$ cusps. We take vanishing cycles $c_1, \ldots, c_{n+1}$ as above.

1. Let $\tilde{f}$ be a fibration obtained by applying flip and slip to $f$. We take a point $q_0$ in the inside of $f(S_{\tilde{f}})$, and reference paths $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n+4}$ in $D^2$ connecting $q_0$ to a point on the respective fold arcs between cusps so that these paths appear in this order when we go around $q_0$ counterclockwise. We denote by $e_i \subset \tilde{f}^{-1}(q_0)$ a vanishing cycle determined by the path $\tilde{\gamma}_i$. Then, there exist an identification $\tilde{f}^{-1}(q_0) \cong \Sigma_{g+1}$ and elements $\alpha, d, j, \tilde{c}_2, \ldots, \tilde{c}_{n+1}$ and $\varphi$ satisfying the conditions $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ and $\tilde{C}_4$ such that the following equality holds up to cyclic permutation:

$$(e_1, \ldots, e_{n+4}) = (\tilde{c}_1, \ldots, \tilde{c}_{n+1}, \alpha', d, \tilde{\alpha}),$$

where $\tilde{c}_1 = j(c_1)$, $\tilde{\alpha}$ is the closure of $j(\text{Int} \alpha)$ in $\Sigma_{g+1}$, and $\alpha'$ is defined as

$$\alpha' = (\varphi^{-1} \cdot t_{\tilde{c}_1}(\tilde{c}_2) \cdots t_{\tilde{c}_n}(\tilde{c}_{n+1}))(\tilde{\alpha}).$$

2. Let $\alpha, d, j, \tilde{c}_2, \ldots, \tilde{c}_{n+1}$ and $\varphi$ be elements satisfying the conditions $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ and $\tilde{C}_4$. We take simple closed curves $\tilde{c}_1, \tilde{\alpha}$ and $\alpha'$ as in (1). Suppose that the genus of higher genus fibers $g$ of $f$ is greater than or equal to 3. Then, there exists a fibration $\tilde{f}$ obtained by applying flip and slip to $f$ such that, for reference paths $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n+4}$ as in (1), the corresponding vanishing cycles $e_1, \ldots, e_{n+4}$ satisfy the following equality up to cyclic permutation:

$$(e_1, \ldots, e_{n+4}) = (\tilde{c}_1, \ldots, \tilde{c}_{n+1}, \alpha', d, \tilde{\alpha})$$
5  Fibrations with small fiber genera

Although Theorem 4.1(1) holds for a fibration with an arbitrary fiber genera, Theorems 4.1(2) and 4.3 do not hold if genera of fibers are too small. The main reason of this is nontriviality of the group $\pi_1(\text{Diff}^+(\Sigma_{g-1}), \text{id})$ when $g < 3$. To deal with fibrations with small fiber genera, we need to look at additional data on sections of fibrations. Let $f: M \to D^2$ be a purely wrinkled fibration we took in the beginning of Section 4.

5.1 Case 1: Every fiber of $f$ is connected

In this subsection, we assume that every fiber of $f$ is connected. We first consider the case $f$ has no cusps. We take a point $p_0$, an identification $f_1\sim_{\bar{\phi}}$, a reference path $\gamma_0 \subset D^2$, a vanishing cycle $c \subset \Sigma_g$, and a monodromy $\varphi_0 \in \text{Mod}(\Sigma_g)(c)$ as we took in Section 4. It is easy to see that $f$ has a section. We take a section $x = \sigma(p_0)$, which is contained in the complement $\Sigma_g \setminus c$. This section gives a lift $\tilde{\phi} \in \text{Mod}(\Sigma_g;x)(c)$. It is easy to show that this element is contained in the kernel of the homomorphism

$$\Phi_c^x: \text{Mod}(\Sigma_g;x)(c) \to \text{Mod}(\Sigma_{g-1};x),$$

which is defined as we define $\Phi_c$.

As in Section 4, we give several conditions. The first condition is on an embedded path $\alpha \subset \Sigma_g \setminus \{x\}$.

**Condition $C'_1(c, \sigma)$** A path $\alpha \subset \Sigma_g \setminus \{x\}$ intersects $c$ at the unique point $q \in c$ transversely.

We take a path $\alpha \subset \Sigma_g \setminus \{x\}$ so that $\alpha$ satisfies the condition $C'_1(c, \sigma)$. We put $\partial\alpha = \{w_1, w_2\}$. The second condition is on a simple closed curve $d \subset \Sigma_{g+1}$ and a diffeomorphism $j: \Sigma_g \setminus \{w_1, w_2\} \to \Sigma_{g+1} \setminus d$.

**Condition $C'_2(c, \alpha, \sigma)$** The closure of $j(\text{Int}\alpha)$ in $\Sigma_{g+1}$ is a simple closed curve.

We take a simple closed curve $d \subset \Sigma_{g+1}$ and a diffeomorphism $j: \Sigma_g \setminus \{w_1, w_2\} \to \Sigma_{g+1} \setminus d$ so that they satisfy the condition $C'_2(c, \alpha, \sigma)$. We put $\tilde{c} = j(c)$ and $\tilde{x} = j(x)$. The last condition is on an element $\varphi \in \text{Mod}(\Sigma_{g+1};\tilde{x})(\tilde{c}, d)$.

**Condition $C'_3(c, \alpha, d, j, \varphi_0, \sigma)$** We have that $\Phi_{\tilde{c}}(\varphi) = 1$ in $\text{Mod}(\Sigma_g, \tilde{x})(d)$ and that $\Phi_{\tilde{d}}(\varphi) = \varphi_0^{-1}$ in $\text{Mod}(\Sigma_g, x)(c)$. 

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Theorem 5.1  Let $f: M \to D^2$ be a purely wrinkled fibration as above.

(1) Let $\tilde{f}$ be a fibration obtained by applying flip and slip to $f$. We take a point $q_0$, reference paths $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_4$ in $D^2$ and $e_i \subset \tilde{f}^{-1}(q_0)$ as Theorem 4.1(1). Then, there exist an identification $\tilde{f}^{-1}(q_0) \cong \Sigma_{g+1}$ and elements $\alpha, d, j$ and $\varphi$ satisfying the conditions $C'_1, C'_2$ and $C'_3$ such that the following equality holds up to cyclic permutation:

$$(e_1, e_2, e_3, e_4) = (\tilde{c}, \alpha', d, \tilde{\alpha}),$$

where $\tilde{c} = j(c), \tilde{\alpha}$ is the closure of $j(\text{Int} \alpha)$ in $\Sigma_{g+1}$, and $\alpha' = \varphi^{-1}(\tilde{\alpha})$.

(2) Let $\alpha, d, j$ and $\varphi$ be elements satisfying the conditions $C'_1, C'_2$ and $C'_3$. We take simple closed curves $\tilde{c}, \tilde{\alpha}$ and $\alpha'$ as in (1). Suppose that the genus $g$ is greater than or equal to 2. Then, there exists a fibration $\tilde{f}$ obtained by applying flip and slip to $f$ such that, for reference paths $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_4$ as in (1), the corresponding vanishing cycles $e_1, \ldots, e_4$ satisfy the following equality up to cyclic permutation:

$$(e_1, e_2, e_3, e_4) = (\tilde{c}, \alpha', d, \tilde{\alpha})$$

Proof of Theorem 5.1(1)  The proof of Theorem 5.1(1) is quite similar to that of Theorem 4.1(1). The only difference is the following point: instead of a horizontal distribution $\mathcal{H}$ of the fibration $\tilde{f}|_{M \setminus S\tilde{f}}$, we take a horizontal distribution $\mathcal{H}_\sigma$ of the fibration $\tilde{f}|_{M \setminus S\tilde{f}}$, which satisfies the same conditions as that on $\mathcal{H}$, so that it is tangent to the image of the section $\sigma$. By using such a horizontal distribution, we can apply all the arguments in the proof of Theorem 4.1 straightforwardly. We omit details of the proof. \hfill $\Box$

Proof of Theorem 5.1(2)  As the proof of (1), the proof of (2) is also similar to that of Theorem 4.1(2). By the same argument as in the proof of Theorem 4.1(2), all we have to prove is that we can take a homotopy from $f$ to $\tilde{f}$ so that the element $[\theta_1^{-1} \circ \theta_2 \circ \theta_0]$ corresponds to $\varphi^{-1}$ for given $\varphi$.

We take a sufficiently small disk neighborhood $D$ of $x$ in $\Sigma_{g-1}$. Denote by $\tilde{\Sigma}_{g-1}$ the closure of the complement $\Sigma_{g-1} \setminus D$. It is easy to prove that the mapping class group $\pi_0(\text{Diff}^+(\tilde{\Sigma}_{g-1}), \text{id})$ is isomorphic to $\text{Mod}(\Sigma_{g-1}; x)$, where $\text{Diff}^+(\tilde{\Sigma}_{g-1})$ is the set of orientation-preserving diffeomorphisms of $\tilde{\Sigma}_{g-1}$ (note that an element in this group fixes the boundary of $\partial \tilde{\Sigma}_{g-1}$ set wise, but need not to fix $\partial \tilde{\Sigma}_{g-1}$ point wise). Moreover, we can obtain similar isomorphisms even if we consider groups of diffeomorphisms with fixed points or sets. It is known that the group $\pi_1(\text{Diff}^+(\tilde{\Sigma}_{g-1}), \text{id})$ is trivial if $g$ is greater than or equal to 2 (cf Earle and Schatz [10]). With these
observations understood, we can prove by the argument similar to that in Section 3 that the group \( \ker \Phi \_c \cap \ker \Phi \_d \) is generated by the set

\[
\{ t_{\delta(\eta)} \cdot t_{\zeta}^{-1} \cdot t_{\xi}^{-1} \in \text{Mod} (\Sigma_{g+1}; \tilde{x})(\tilde{c}, d) \mid \eta \in \pi(\Sigma_{g-1} \setminus \{ \tilde{x}, v_i, w_j \}, v_k, w_l) \},
\]

where \( \pi(\Sigma_{g-1} \setminus \{ \tilde{x}, v_i, w_j \}, v_k, w_l) \) and \( \tilde{\delta}(\eta) \) are defined as in Section 3. Thus, by the similar argument to that in the proof of Theorem 3.9, we can change \( [\theta_1^{-1} \circ \theta_2 \circ \theta_0] \) into \( [\theta_1^{-1} \circ \theta_2 \circ \theta_0] \cdot \psi \) for any \( \psi \in \ker \Phi \_c \cap \ker \Phi \_d \) by modifying a flip and slip from \( f \) to \( \tilde{f} \). This completes the proof of the statement (2).

We can deal with a fibration with cusps similarly by using sink and unsink as in Section 4. Suppose that \( f \) has \( n > 0 \) cusps and we take vanishing cycles \( c_1, \ldots, c_{n+1} \subset \Sigma \_g \) as we took in Section 4. We also take a section \( \sigma: D^2 \to M \) of \( f \). We put \( x = \sigma(p_0) \), which is contained in the complement \( \Sigma_g \setminus (c_1 \cup \cdots \cup c_{n+1}) \). This gives a lift \( \tilde{\varphi}_0 \in \text{Mod} (\Sigma_g; x)(c_1) \) of \( \varphi_0 \). As in Section 4, we put \( \tilde{\varphi}_0 = \tilde{\varphi}_0 \cdot (t_{\zeta_1}(\tilde{c}_2) \cdots t_{\zeta_n}(\tilde{c}_{n+1}))^{-1} \), and we give several conditions on elements \( a, d, j, \varphi, \tilde{c}_2, \ldots, \tilde{c}_{n+1} \).

**Condition \( \tilde{C}_1'(c_1, \ldots, c_n, \sigma) \)**  A path \( \alpha \subset \Sigma_g \setminus \{ x \} \) intersects \( c_1 \) at the unique point \( q \in c_1 \) transversely. Furthermore, \( \partial \alpha \cap (c_1 \cup \cdots \cup c_{n+1}) = \emptyset \).

**Condition \( \tilde{C}_2'(c_1, \ldots, c_n, \alpha, \sigma) \)**  The closure of \( j(\text{Int} \alpha) \) in \( \Sigma_{g+1} \) is a simple closed curve.

**Condition \( \tilde{C}_3'(c_1, \ldots, c_n, \alpha, d, j, \varphi_0, \sigma) \)**  We have \( \Phi \_c (\varphi) = 1 \) in \( \text{Mod} (\Sigma_g; \tilde{x})(d) \) and \( \Phi \_d (\varphi) = \tilde{\varphi}_0^{-1} \) in \( \text{Mod} (\Sigma_g; x)(c_1) \), where we put \( \tilde{x} = j(x) \) and \( \tilde{c} = j(c) \).

**Condition \( \tilde{C}_4'(c_1, \ldots, c_n, \alpha, d, j, \sigma) \)**  For each \( i \in \{ 2, \ldots, n+1 \} \), \( i(\tilde{c}_i) \) is isotopic to \( c_i \) in \( \Sigma_g \setminus \{ x \} \), where \( i \) is an embedding defined by

\[
i: \Sigma_{g+1} \setminus d \xrightarrow{j^{-1}} \Sigma_g \setminus \{ w_1, w_2 \} \to \Sigma_g.
\]

Furthermore, for each \( i = 1, \ldots, n \), \( \tilde{c}_i \) intersects \( \tilde{c}_{i+1} \) at a unique point transversely.

The following theorem can be proved in a way quite similar to that of the proof of Theorem 5.1.

**Theorem 5.2** Let \( f: M \to D^2 \) be a purely wrinkled fibration as above.

1. Let \( \tilde{f} \) be a fibration obtained by applying flip and slip to \( f \). We take a point \( q_0 \), reference paths \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n+4} \) in \( D^2 \), vanishing cycles \( e_1, \ldots, e_{n+4} \subset \tilde{f}^{-1}(q_0) \) as we took in Theorem 4.3(1). Then, there is an identification \( \tilde{f}^{-1}(q_0) = \Sigma_{g+1} \) and elements \( \alpha, d, j, \tilde{c}_2, \ldots, \tilde{c}_{n+1} \) and \( \varphi \) satisfying the conditions \( \tilde{C}_1', \tilde{C}_2', \tilde{C}_3', \tilde{C}_4' \) such that the following equality holds up to cyclic permutation:

\[\text{Mod} \quad \tilde{C}_1', \tilde{C}_2', \tilde{C}_3', \tilde{C}_4'.\]
\[(e_1, \ldots, e_{n+4}) = (\tilde{c}_1, \ldots, \tilde{c}_{n+1}, \alpha', d, \tilde{\alpha}),\]

where \(\tilde{c}_1 = j(c_1)\), \(\tilde{\alpha}\) is the closure of \(j(\text{Int } \alpha)\) in \(\Sigma_{g+1}\), and \(\alpha'\) is defined as

\[
\alpha' = (\varphi^{-1} \cdot t_{\tilde{c}_1} \cdot (\tilde{c}_2) \cdots t_{\tilde{c}_{n+1}})(\tilde{\alpha})
\]

(2) Let \(\alpha, d, j, \tilde{c}_2, \ldots, \tilde{c}_{n+1}\) and \(\varphi\) be elements satisfying the conditions \(\tilde{C}_1', \tilde{C}_2', \tilde{C}_3'\) and \(\tilde{C}_4'\). We take simple closed curves \(\tilde{c}_1, \tilde{\alpha}\) and \(\alpha'\) as in (1). Suppose that the genus \(g\) is greater than or equal to 2. Then, there exists a fibration \(\tilde{f}\) obtained by applying flip and slip move to \(f\) such that, for a reference path \(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n+4}\) as in (1), the corresponding vanishing cycles \(e_1, \ldots, e_{n+4}\) satisfy the following equality up to cyclic permutation:

\[(e_1, \ldots, e_{n+4}) = (\tilde{c}_1, \ldots, \tilde{c}_{n+1}, \alpha', d, \tilde{\alpha}).\]

### 5.2 Case 2: \(f\) has disconnected fibers

We next consider the case \(f\) has disconnected fibers. In this case, \(f\) has no cusps. We take a point \(p_0 \in \partial D^2\), an identification \(f^{-1}(p_0) \cong \Sigma_g\), a reference path \(\gamma_0\), a vanishing cycle \(c \subset \Sigma_g\), and a monodromy \(\varphi_0 \in \text{Mod}(\Sigma_g)(c)\) as we took in Section 4.

We also take a disconnected fiber of \(f\) and denote this by \(S_1 \sqcup S_2\), where \(S_i\) is a connected component of the fiber. We take a section \(\sigma_i: D^2 \to M\) of \(f\) which intersects \(S_i\) for each \(i = 1, 2\). We put \(x_i = \sigma_i(p_0)\), which is contained in the complement \(\Sigma_g \setminus c\). The sections \(\sigma_1\) and \(\sigma_2\) gives a lift \(\tilde{\varphi}_0 \in \text{Mod}(\Sigma_g; x_1, x_2)(c^\text{ori})\), and this element is contained in the kernel of the homomorphism

\[
\Phi^{x_1, x_2}_c: \text{Mod}(\Sigma_g; x_1, x_2)(c^\text{ori}) \to \text{Mod}(\Sigma_{g_1}; x_1) \times \text{Mod}(\Sigma_{g_2}; x_2),
\]

where \(g_i\) is the genus of the closed surface \(S_i\).

By using this lift, we can apply all the argument in Case 1 straightforwardly, and we can obtain the theorem similar to Theorem 5.1 (we need the assumption \(g_1, g_2 \geq 1\)). We omit the details of arguments.

**Remark 5.3** The statements of Theorems 5.1(2) and 5.2(2) do not hold if \(g = 1\) since the group \(\pi_1(\text{Diff}^+(\tilde{\Sigma}_0), \text{id})\) is nontrivial (cf [10]). To apply the same argument as in the proof of Theorem 5.1(2) to the case \(g = 1\), we need to take three disjoint sections of \(f\). We take points \(x_1, x_2, x_3 \in S^2\) and a sufficiently small disk neighborhood \(D_i\) of \(x_i \in S^2\) \((i = 1, 2, 3)\). We denote by \(S^2_{(3)}\) the closure of the complement \(S^2 \setminus (D_1 \sqcup D_2 \sqcup D_3)\). Earle and Schatz [10] showed that the group \(\pi_1(\text{Diff}^+(S^2_{(3)}), \text{id})\) is trivial, the statement similar to that in Theorems 5.1 and 5.2 hold for a fibration \(f\) with \(g = 1\) (note that the group \(\pi_1(\text{Diff}^+(S^2_{(2)}), \text{id})\) is nontrivial, where \(S^2_{(2)}\) is the
Modification rule of monodromies in an $R_2$–move

closure of $S^2 \setminus (D_1 \cup D_2)$; see [10] for details.) Furthermore, we can deal with a
fibration with disconnected fibers which contain spheres as connected components by
taking three disjoint sections so that these sections go through the sphere components.
We omit, however, details of arguments about this case for simplicity of the paper.

6 Application: Examples of surface diagrams

Williams [21] defined a certain cyclically ordered sequence of nonseparating simple
closed curves in a closed surface which describes a 4–manifold. This sequence is
obtained by looking at vanishing cycles of a simplified purely wrinkled fibration, which
is defined below. In this section, we will look at relation between flip and slip and
sequences of simple closed curves Williams defined. We will then give some new
examples of this sequence.

Definition 6.1 A purely wrinkled fibration $\zeta; M^4 \to S^2$ is called a simplified purely
wrinkled fibration if it satisfies the following conditions:

(1) All the fiber of $\xi$ are connected.
(2) The set of singularities $S_\xi \subset M$ of $\zeta$ is connected and nonempty.
(3) The restriction $\zeta|_{S_\xi}$ is injective.

It is easy to see that $\zeta$ has two types of regular fibers: $\Sigma_g$ and $\Sigma_{g-1}$ for some $g \geq 1$. We call the genus $g$ of a higher genus regular fiber the genus of $\zeta$. In this paper, we
call a simplified purely wrinkled fibration an SPWF for simplicity.

Let $\zeta; M \to S^2$ be a genus-$g$ SPWF. We denote by $\{s_1, \ldots, s_n\}$ the set of cusps of
$f$. We put $u_i = f(s_i)$. We take a regular value $p_0$ of $\zeta$ so that the genus of the fiber $\zeta^{-1}(p_0)$ is equal to $g$. The indices of $s_i$ are chosen so that $u_1, \ldots, u_n$ appear in
this order when we travel the image $\zeta(S_f)$ counterclockwise around $p_0$. The points
$u_1, \ldots, u_n$ divides the image $\zeta(S_\xi)$ into $n$ edges. We denote by $l_i \subset \zeta(S_\xi)$ the edge
between $u_i$ and $u_{i+1}$ (we regard the indices as in $\mathbb{Z}/n\mathbb{Z}$. In particular, $u_{n+1} = u_1$).
We take paths $\gamma_1, \ldots, \gamma_n \subset S^2$ satisfying the following conditions:

- $\gamma_i$ connects $p_0$ to a point in $\text{Int} l_i$.
- $\text{Int} \gamma_i \cap f(S_\xi) = \emptyset$
- $\gamma_i \cap \gamma_j = \{p_0\}$ if $i \neq j$

We fix an identification $\zeta^{-1}(p_0) \cong \Sigma_g$. These paths give $\Sigma_g$ a sequence of vanishing
cycles of $\zeta$, which we denote by $(c_1, \ldots, c_n)$.  

Definition 6.2 [21] Let $\zeta: M \to S^2$ be an SPWF with genus $g \geq 3$. We denote by $(c_1, \ldots, c_n)$ a sequence of simple closed curves in $\Sigma_g$ obtained as above. We call this sequence a surface diagram of a 4–manifold $M$.

Remark 6.3 We can define a surface diagram of an SPWF in the obvious way. In this paper, we call both of the diagram, that of a 4–manifold and that of an SPWF, a surface diagram.

Remark 6.4 It is known that every smooth map $h: M^4 \to S^2$ from an oriented, closed, connected 4–manifold $M$ is homotopic to an SPWF with genus greater than 2 (see [22]). In particular, every closed oriented connected 4–manifold has a surface diagram. Moreover, the total space of an SPWF is uniquely determined by a sequence of vanishing cycles if the genus is greater than 2 since the group $\pi_1(Diff^+(\Sigma_{g-1}), \text{id})$ is trivial if $g \geq 3$. Thus, a 4–manifold is uniquely determined by a surface diagram. However, it is known that there exist infinitely many SPWFs which have same vanishing cycles (see [5; 16], for example).

Let $\zeta: M \to S^2$ be a genus-$g$ SPWF and $(c_1, \ldots, c_n)$ a surface diagram of $\zeta$. For a base point $p_0$, we take a disk $D$ in $S^2 \setminus \zeta(S_\zeta)$ satisfying the following conditions:

- $p_0 \in \partial D$
- $\gamma_i \cap D = \{p_0\}$, where $\gamma_i \subset S^2$ is a reference path from $p_0$ which gives a vanishing cycle $c_i$.
- $\gamma_1, \ldots, \gamma_n, D$ appear in that order when we go around $p_0$ counterclockwise.

We consider the restriction $\zeta|_{M \setminus \zeta^{-1}((\text{Int} D))}$. This is a purely wrinkled fibration and satisfies the conditions in the beginning of Section 4. Thus, we can apply arguments in Section 4 to $\zeta|_{M \setminus \zeta^{-1}((\text{Int} D))}$. In particular, we can describe an algorithm to obtain a surface diagram of a fibration obtained by applying flip and slip to $\zeta$. As in Section 4, we prepare several conditions to give an algorithm precisely. We first remark that we can assume that $\varphi_0$ is trivial in this case since $\zeta^{-1}(\partial D)$ is bounded by the trivial fibration. In particular, we obtain

$$\hat{\varphi} = (t_{c_1}(c_2) \cdots t_{c_n-1}(c_n) \cdot t_{c_n}(c_1))^{-1}.$$

The first condition is on an embedded path $\alpha \subset \Sigma_g$.

Condition $W_1(c_1, \ldots, c_n)$ A path $\alpha \subset \Sigma_g$ intersects $c_1$ at the unique point $q \in c_1$ transversely. Furthermore, $\partial \alpha \cap (c_1 \cup \cdots \cup c_n) = \emptyset$.

We take a path $\alpha \subset \Sigma_g$ so that $\alpha$ satisfies the condition $W_1(c_1, \ldots, c_n)$. We put $\partial \alpha = \{w_1, w_2\}$. The second condition is on a simple closed curve $d \subset \Sigma_{g+1}$ and a diffeomorphism $j: \Sigma_g \setminus \{w_1, w_2\} \to \Sigma_{g+1} \setminus d$.
**Condition \(W_2(c_1, \ldots, c_n, \alpha)\)** The closure of \(j(\text{Int} \alpha)\) in \(\Sigma_{g+1}\) is a simple closed curve.

We take a simple closed curve \(d \subset \Sigma_{g+1}\) and a diffeomorphism \(j: \Sigma_g \setminus \{w_1, w_2\} \to \Sigma_{g+1} \setminus d\) so that they satisfy the condition \(W_2(c_1, \ldots, c_n, \alpha)\). We put \(\tilde{c}_1 = j(c_1)\).

The third condition is on an element \(\varphi \in \text{Mod}(\Sigma_{g+1})/(\tilde{c}_1, d)\).

**Condition \(W_3(c_1, \ldots, c_n, \alpha, d, j)\)** We have \(\Phi_{\tilde{c}_1}(\varphi) = 1\) in \(\text{Mod}(\Sigma_g)(d)\) and \(\Phi_d(\varphi) = t_{c_1}(c_2) \cdots t_{c_{n-1}}(c_n) \cdot t_{c_n}(c_1)\) in \(\text{Mod}(\Sigma_g)(c_1)\).

The last condition is on simple closed curves \(\tilde{c}_2, \ldots, \tilde{c}_n \subset \Sigma_{g+1} \setminus d\).

**Condition \(W_4(c_1, \ldots, c_n, \alpha, d, j)\)** For each \(i \in \{2, \ldots, n\}\), \(i(\tilde{c}_i)\) is isotopic to \(c_i\) in \(\Sigma_g\), where \(i\) is an embedding defined by

\[
i: \Sigma_{g+1} \setminus d \xrightarrow{j^{-1}} \Sigma_g \setminus \{w_1, w_2\} \hookrightarrow \Sigma_g.
\]

Furthermore, \(\tilde{c}_i\) intersects \(\tilde{c}_{i+1}\) at a unique point transversely for each \(i \in \mathbb{Z}/n\mathbb{Z}\).

By Theorem 4.3, we immediately obtain the following theorem.

**Theorem 6.5** Let \(\xi: M \to S^2\) be a genus-\(g\) SPWF and \((c_1, \ldots, c_n)\) a surface diagram of \(\xi\).

1. Let \(\tilde{\xi}\) be a genus-(\(g+1\)) SPWF obtained by applying flip and slip to \(\xi\). Then, there exist elements \(\alpha, d, j, \tilde{c}_2, \ldots, \tilde{c}_n, \varphi\) satisfying the conditions \(W_1, W_2, W_3\) and \(W_4\) so the sequence \((\tilde{c}_1, \ldots, \tilde{c}_n, \tilde{c}_1, \alpha', d, \tilde{\alpha})\) gives a surface diagram of \(\tilde{\xi}\), where \(\tilde{c}_1 = j(c_1), \tilde{\alpha}\) is the closure of \(j(\text{Int} \alpha)\) in \(\Sigma_{g+1}\), and \(\alpha'\) is defined by

\[
\alpha' = (\varphi^{-1} \cdot t_{\tilde{c}_1}(\tilde{c}_2) \cdots t_{\tilde{c}_n}(\tilde{c}_1)) (\tilde{\alpha}).
\]

2. Let \(\alpha, d, j, \tilde{c}_2, \ldots, \tilde{c}_n\) and \(\varphi\) be elements satisfying the conditions \(W_1, W_2, W_3\) and \(W_4\). Suppose that \(g\) is greater than or equal to 3. We take simple closed curves \(\alpha', \tilde{\alpha}\) as in (1). Then, there exists a genus-(\(g+1\)) SPWF \(\tilde{\xi}\) obtained by applying flip and slip to \(\xi\) such that \((\tilde{c}_1, \ldots, \tilde{c}_n, \tilde{c}_1, \alpha', d, \tilde{\alpha})\) is a surface diagram of \(\tilde{\xi}\).

As in Section 5, we can deal with SPWFs with small genera by looking at additional data. Let \(\xi: M \to S^2\) be a genus-\(g\) SPWF with surface diagram \((c_1, \ldots, c_n)\). We take a disk \(D \subset S^2\) as above. We also take a section \(\sigma: S^2 \setminus \text{Int} D \to M \setminus \xi^{-1}(\text{Int} D)\) of the fibration \(\xi|_M|\xi^{-1}(\text{Int} D)\). We put \(x = \sigma(p_0)\). We take a trivialization \(\xi^{-1}(D) \cong D \times \Sigma_g\) so that it is compatible with the identification \(\xi^{-1}(p_0) \cong \Sigma_g\). Let \(\beta_x \in \pi_1(\Sigma_g, x)\) be an element which is represented by the loop

\[
p_2 \circ \sigma: (\partial D, p_0) \to (\Sigma_g \setminus (c_1 \cup \cdots \cup c_n), x),
\]
where $p_2: D \times \Sigma_g \to \Sigma_g$ is the projection onto the second component. It is easy to see that the monodromy along $\partial D$ (oriented as a boundary of $S^2 \setminus \text{Int } D$) corresponds to the pushing map $\text{Push}(\beta_x)^{-1}$. Thus, we can assume $\tilde{\varphi}_0 = \text{Push}(\beta_x)^{-1} \in \text{Mod}(\Sigma_g \setminus x)(c_1)$ in this case. We call the loop $\beta_x$ an attaching loop.

Remark 6.6 We can obtain a handle decomposition of the total space of an SPWF by changing it into a simplified broken Lefschetz fibration using unsink. Indeed, Baykur [4] gave a way to obtain a handle decomposition of the total spaces of simplified broken Lefschetz fibrations from monodromy representation (or equivalently, vanishing cycles of the fibrations). The loop $t \mapsto (t, \beta_x(t)) \in D \times \Sigma_g$ corresponds to the attaching circle of the 2-handle in the lower side of the fibration. This is because $\beta_x$ is called an attaching loop.

We consider the following conditions on elements $\alpha, d, j, \varphi, \tilde{c}_2, \ldots, \tilde{c}_n$ as in Section 5.

Condition $W'_1(c_1, \ldots, c_n, \sigma)$ A path $\alpha \subset \Sigma_g \setminus \{x\}$ intersects $c_1$ at the unique point $q \in c$ transversely. Furthermore, $\partial \alpha \cap (c_1 \cup \cdots \cup c_n) = \emptyset$.

Condition $W'_2(c_1, \ldots, c_n, \alpha, \sigma)$ The closure of $j(\text{Int } \alpha)$ in $\Sigma_{g+1}$ is a simple closed curve.

Condition $W'_3(c_1, \ldots, c_n, \alpha, d, j, \sigma)$ Here, we set $\tilde{c}_1 = j^{-1}(c_1)$ and $\tilde{c} = j(x)$, $\Phi_{\tilde{c}_1}(\varphi) = 1$ in $\text{Mod}(\Sigma_g \setminus x)(d)$ and $\Phi_{\tilde{c}}(\varphi) = t_{tc_1(c_2)} \cdots t_{tc_{n-1}(c_n)} \cdot t_{tc_n}(c_1) \cdot \text{Push}(\beta_x)$ in $\text{Mod}(\Sigma_g \setminus x)(c)$.

Condition $W'_4(c_1, \ldots, c_n, \alpha, d, j, \sigma)$ For each $i \in \{2, \ldots, n\}$, the curve $\tilde{c}_i \subset \Sigma_{g+1} \setminus \{x\}$ satisfies that $\varphi(\tilde{c}_i)$ is isotopic to $c_i$ in $\Sigma_g \setminus \{x\}$, where $\varphi$ is the embedding defined by

$$i: \Sigma_{g+1} \setminus d \xrightarrow{j^{-1}} \Sigma_g \setminus \partial \alpha \hookrightarrow \Sigma_g.$$

Then, we can obtain the following theorem through Theorem 5.2.

Theorem 6.7 Let $\xi: M \to S^2$ be a genus-$g$ SPWF and $(c_1, \ldots, c_n)$ a surface diagram of $\xi$. We take a disk $D \subset S^2$, $\sigma: S^2 \setminus \text{Int } D \to M \setminus \xi^{-1}(\text{Int } D)$, and an element $\beta_x \in \pi_1(\Sigma_g, x)$ as above.

1. Let $\tilde{\xi}$ be a genus-$(g + 1)$ SPWF obtained by applying flip and slip to $\xi$. Then, there exist elements $\alpha, d, j, \tilde{c}_2, \ldots, \tilde{c}_n, \varphi$ satisfying the conditions $W'_1$, $W'_2$, $W'_3$ and $W'_4$ such that the sequence $(\tilde{c}_1, \ldots, \tilde{c}_n, \tilde{c}_1, \alpha', d, \tilde{\alpha})$ gives a surface diagram $\tilde{\xi}$, where $\tilde{c}_1 = j^{-1}(c_1)$, $\tilde{\alpha}$ is the closure of $j^{-1}(\text{Int } \alpha)$ in $\Sigma_{g+1}$, and $\alpha'$ is defined by

$$\alpha' = (\varphi^{-1} \cdot t_{tc_1(\tilde{c}_2)} \cdots t_{tc_n(\tilde{c}_1)})(\tilde{\alpha}).$$

Theorem 5.2
Let \( \alpha, d, j, \bar{c}_2, \ldots, \bar{c}_n \) and \( \varphi \) be elements satisfying the conditions \( W'_1, W'_2, W'_3 \) and \( W'_4 \). Suppose that \( g \) is greater than or equal to 2. We take simple closed curves \( \alpha', \bar{\alpha} \) as in (1). Then, there exists a genus-(\( g + 1 \)) SPWF \( \tilde{\zeta} \) obtained by applying flip and slip to \( \zeta \) such that \((\bar{c}_1, \ldots, \bar{c}_n, \bar{c}_1, \alpha', d, \bar{\alpha}) \) is a surface diagram of \( \tilde{\zeta} \).

Example 6.8 Let \( p_1: S^2 \times \Sigma_k \to S^2 \) be the projection onto the first component \((k \geq 0)\). By applying a birth (for details about this move, see [18; 22], for example), we can change \( p_1 \) into a genus-(\( k + 1 \)) SPWF with two cusps. We then apply a flip and slip move to this SPWF \( m \) times. As a result, we obtain a genus-(\( k + m + 1 \)) SPWF on the manifold \( S^2 \times \Sigma_k \). We denote this fibration by \( \tilde{p}_1^{(m)}: S^2 \times \Sigma_k \to S^2 \).

Claim A surface diagram of \( \tilde{p}_1^{(m)} \) corresponds to

\[
(d_0, d_1, \ldots, d_{2m}, d_{2m+1}, d_{2m}, \ldots, d_1),
\]

where \( d_i \subset \Sigma_{k+m+1} \) is a simple closed curve described in the left side of Figure 11.

![Figure 11: Simple closed curves in the genus-(\( k + m + 1 \)) closed surface \( \Sigma_{k+m+1} \)](image)

We prove this claim by induction on \( m \). The claim is obvious when \( m = 0 \). We assume that \( m > 0 \). For simplicity, we denote the Dehn twist along the curve \( d_i \) by \( i \) and its inverse by \( \bar{i} \). For an integer \( n > 0 \), let \( S_n \) be a regular neighborhood of the union \( d_0 \cup \cdots \cup d_n \). By direct calculation, we can prove the following relation in \( \text{Mod}(S_n; \partial S_n) \):

\[
(3) \quad t_{t_{d_0}(d_1)} \cdots t_{t_{d_{n-1}}(d_n)} \cdot t_{t_{d_n}(d_{n-1})} \cdots t_{t_{d_1}(d_0)} =
\begin{cases}
\tilde{d}^4 \cdot (01)^3 & n = 1 \\
\tilde{d}^{2n+2} \cdot (01 \cdots n)^{n+2} \cdot (\bar{z} \cdots \bar{n})^n & n \geq 2
\end{cases}
\]

By induction hypothesis, a sequence \((d_0, \ldots, d_{2m-2}, d_{2m-1}, d_{2m-2}, \ldots, d_1)\) is a surface diagram of \( \tilde{p}_1^{(m-1)} \). We will stabilize this diagram by using Theorem 6.5. We take a path \( \alpha \subset \Sigma_{k+(m-1)+1} \) as in the left side of Figure 11. Let \( j: \Sigma_{k+m} \setminus \partial \alpha \to \Sigma_{k+m+1} \setminus d \) be a diffeomorphism, where \( d \) is a nonseparating simple closed curve. By using \( j \), we regard \( d_i \) as a curve in \( \Sigma_{k+m+1} \). It is easy to see that the element

\[
t_{t_{d_0}(d_1)} \cdots t_{t_{d_{2m-2}}(d_{2m-2})} \cdot t_{t_{d_{2m-1}}(d_{2m-2})} \cdots t_{t_{d_1}(d_0)}
\]
is contained in the group \( \text{Mod} (\Sigma_{k+m+1})(d, d_0) \). Moreover, by the relation (3), we can calculate the image under \( \Phi_{d_0} \): \( \text{Mod} (\Sigma_{k+m+1})(d, d_0) \to \text{Mod} (\Sigma_{k+m})(d) \) as follows:

\[
\Phi_{d_0}(t_{d_0}(d_1) \cdots t_{d_{2m-2}}(d_{2m-2}) \cdots t_{d_1}(d_0)) = \begin{cases} 
\Phi_{d_0}(0^4 \cdot (01)^3) & m = 1 \\
\Phi_{d_0}(0^{4m+2} \cdot (01 \cdots 2m - 1)^{2m+1} \cdot (23 \cdots 2m - 1)^{2m-1}) & m \geq 2 \\
id & 
\end{cases}
\]

where the last equality is proved by the chain relation of the mapping class group. Note that this equality still holds in the group \( \text{Mod} (\Sigma; \partial \Sigma) \), where \( \Sigma \) is a regular neighborhood of the union \( d \cup d_0 \cup \cdots \cup d_{2m-1} \). We put

\[
\varphi = t_{d_0}(d_1) \cdots t_{d_{2m-2}}(d_{2m-2}) \cdots t_{d_1}(d_0) \in \text{Mod} (\Sigma_{k+m+1})(d, d_0).
\]

The elements \( \alpha, d, j, d_0, \ldots, d_{2m-1}, \varphi \) satisfy conditions \( W_1, W_2, W_3 \) and \( W_4 \). Therefore, by Theorem 6.5, the following sequence is a surface diagram of \( \widetilde{\varphi}^{(m)} \):

\[
(d_0, \ldots, d_{2m-2}, d_{2m-1}, d_{2m-2}, \ldots, d_1, d_0, \bar{\alpha}, d, \bar{\alpha})
\]

Note that this still holds when the genus of \( \widetilde{\varphi}^{(m-1)} \) is less than 3 since the above calculation of elements of mapping class groups can be done in regular neighborhoods of curves. This proves the claim on surface diagrams of \( S^2 \times \Sigma_k \).

**Remark 6.9** It is known that there is a genus-\( k \) SPWF \( q \): \( S^2 \times \Sigma_{k-1} \# S^1 \times S^3 \to S^2 \) without cusp singularities for \( k \geq 1 \). This was introduced in [4], and was called the step fibration. By the same argument as in Example 6.8, we can prove that \((d_0, d_1, \ldots, d_{2m-1}, d_2, d_{2m-1}, \ldots, d_1)\) is a surface diagram of the fibration obtained by applying flip and slip to \( q \) \( m \) times.

We can also prove the claims on surface diagrams of \( S^2 \times \Sigma_k \) and \( S^2 \times \Sigma_{k-1} \# S^1 \times S^3 \) by using Lemma 6.13.

**Example 6.10** We next construct a surface diagram of \( \# 2S^1 \times S^3 \), which will be used to construct a surface diagram of \( S^4 \). To do this, we first prove the following lemma.

**Lemma 6.11** \( \# 2S^1 \times S^3 \) admits a genus-2 SPWF \( \xi \) without cusps. Moreover, an attaching loop \( \beta_x \) of this fibration is described as in the left side of Figure 12, where \( e_0 \) is a vanishing cycle of indefinite fold singularity.
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\[ \beta_x \quad e_0 \quad \alpha \]

\[ j \]

\[ \delta_+ \quad \delta_- \quad d \]

Figure 12: We have that \( \tilde{\alpha} \) is the closure of \( j(\text{Int}\alpha) \) in \( \Sigma_3 \).

**Proof of Lemma 6.11** It is easy to show that there exists a genus-2 SPWF \( \zeta \) without cusps and whose attaching loop is \( \beta_x \) which is described in Figure 12. Furthermore, we can draw a Kirby diagram of the total space of \( \zeta \) as described in Figure 13. It can be easily shown by Kirby calculus that this manifold is diffeomorphic to \( \#2S^1 \times S^3 \).

We take a path \( \alpha \subset \Sigma_2 \) as in the left side of Figure 12. We also take a diffeomorphism \( j: \Sigma_2 \setminus \partial\alpha \to \Sigma_3 \setminus d \), where \( d \) is a nonseparating simple closed curve in \( \Sigma_3 \), so that the closure of \( j(\text{Int}\alpha) \) is a simple closed curve. Let \( \delta_+ , \delta_- \subset \Sigma_3 \) be simple closed curves described as in the right side of Figure 12. We define an element \( \varphi \in \text{Mod}(\Sigma_3 ; x)(d, e_0) \) as

\[ \varphi = \text{Push}(\beta_x) \cdot t_{\delta_+} \cdot t_{\delta_-}^{-1} . \]

It is easy to see that this element satisfies \( \Phi^x_d(\varphi) = \text{Push}(\beta_x) \) and \( \Phi^x_{e_0}(\varphi) = \text{id} \). Thus, the elements \( \alpha, d, j, e_0, \varphi \) satisfy the conditions \( W'_1, W'_2, W'_3 \) and \( W'_4 \). By Theorem 6.7, \( (e_0, \alpha', d, \tilde{\alpha}) \) is a surface diagram of the fibration obtained by applying flip and slip to \( \zeta \), where \( \alpha' = (\varphi^{-1})(\tilde{\alpha}) \) (see Figure 14).

**Remark 6.12** More generally, we can obtain a genus-\((m + 2)\) surface diagram of \( \#2S^1 \times S^3 \) by looking at vanishing cycles of a fibration obtained by applying flip and slip to \( \zeta \) \( m \) times.
Claim  Let $e_1, \ldots, e_{3m+1}$ be simple closed curves in $\Sigma_{m+2}$ described in Figure 15. The following sequence is the surface diagram of $\# 2S^1 \times S^3$:

$$(e_1, e_2, \ldots, e_{2m-1}, e_{2m}, e_{2m+1}, e_{2m+2}, e_{2m-1}, e_{2m+3}, e_{2m-3}, \ldots, e_{3m}, e_3, e_{3m+1})$$

Before looking at the next example, we prove the following lemma.
Lemma 6.13  Let $(c_1, \ldots, c_n)$ be a genus-$g$ surface diagram of an SPWF $\zeta: M \to S^2$. We take a simple closed curve $\gamma \subset \Sigma_g$ which intersects $c_{i_0}$ at a unique point transversely. Then there exists a genus-$g$ SPWF $\zeta_s: M_s \to S^2$ whose surface diagram is $(c_1, \ldots, c_{i_0-1}, c_{i_0}, \gamma, c_{i_0}, c_{i_0+1}, \ldots, c_n)$. Moreover, if $g$ is greater than or equal to 3, the manifold $M_s$ is obtained from $M$ by applying surgery along $\gamma$, where we regard $\gamma$ as in a regular fiber of $\zeta$.

Proof of Lemma 6.13  By applying cyclic permutation to the sequence $(c_1, \ldots, c_n)$ if necessary, we can assume that $i_0 = 1$. It is easy to see that the element $t_{tc_1(\gamma)} \cdot t_{tc_2(\gamma)} \cdot t_{tc_3(\gamma)} \cdots t_{tc_n(\gamma)}$ is contained in the kernel of $\Phi_{c_1}$. Thus, the product $t_{tc_1(\gamma)} \cdot t_{tc_2(\gamma)} \cdot t_{tc_3(\gamma)} \cdots t_{tc_n(\gamma)}$ is also contained in the kernel of $\Phi_{c_1}$. This implies existence of a genus-$g$ simplified broken Lefschetz fibration with vanishing cycles $(c_1, t_{c_1(\gamma)}, t_{c_2(\gamma)}, t_{c_3(\gamma)}, \ldots, t_{c_n(\gamma)})$. Such a fibration can be changed into a genus-$g$ SPWF $\zeta_s: M_s \to S^2$ with surface diagram $(c_1, \gamma, c_1, \ldots, c_n)$ by applying sink. To prove the statement on $M_s$, we look at the submanifold $S$ of $M$ satisfying the following conditions:

1. The image $f(S)$ is a disk and the intersection $f(S \cap S_f)$ forms a connected arc without cusps.
2. A vanishing cycle of indefinite folds in $f(S)$ is $c_1$.
3. The restriction $f|_{S \setminus f^{-1}(S_f)}: S \setminus S_f \to f(S) \setminus f(S_f)$ is a disjoint union of trivial fibrations.
4. The higher genus fiber of $f|_S: S \to f(S)$ is a regular neighborhood of the union $c_1 \cup \gamma$.

Figure 16: Left: a Kirby diagram of $S$; right: a Kirby diagram of $\tilde{S}$

We can easily draw a Kirby diagram of $S$ as in the left side of Figure 16. This diagram implies that $S$ is diffeomorphic to $S^1 \times D^3$, and that a generator of $\pi_1(S)$ corresponds to a simple closed curve $\gamma$. Let $\tilde{S}$ be a manifold which is described in the right side of Figure 16. This manifold admit a fibration to $D^2$ with connected indefinite fold,
which forms an arc, and two Lefschetz singularities. Furthermore, a regular fiber of the fibration is either a genus-1 surface with one boundary component or a disk. By Kirby calculus, we can prove that this manifold is diffeomorphic to $D^2 \times S^2$. By the construction of the fibration $\zeta$, the manifold $M_\delta$ can be obtained by removing $S$ from $M$, and then attaching $\widetilde{S}$ along the boundary. This completes the proof of Lemma 6.13. \hfill \Box

**Example 6.14** Let $e_1, e_2, e_3, e_4$ be simple closed curves in $\Sigma_3$ as described in Figure 17. As is shown, a sequence $(e_1, e_2, e_3, e_4)$ is a surface diagram of $\#2S^1 \times S^3$. We take a curve $\gamma_i$ ($i = 1, 2, 3, 4$) as shown in Figure 17. The curve $\gamma_1$ intersects $e_1$ at a unique point transversely. By Lemma 6.13, a sequence $(e_1, \gamma_1, e_1, e_2, e_3, e_4)$ is a surface diagram of some 4–manifold obtained by applying surgery to $\#2S^1 \times S^3$. Indeed, we can prove by Kirby calculus that this diagram represents the manifold $S^1 \times S^3$. In the same way, we can prove the following correspondence between surface diagrams and 4–manifolds:

<table>
<thead>
<tr>
<th>surface diagram</th>
<th>corresponding 4–manifold</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(e_1, \gamma_1, e_1, e_2, e_3, e_4)$</td>
<td>$S^4$</td>
</tr>
<tr>
<td>$(e_1, \gamma_1, e_1, e_2, e_3, \gamma_3, e_3, e_4)$</td>
<td>$S^1 \times S^3 # S^2 \times S^2$</td>
</tr>
<tr>
<td>$(e_1, \gamma_1, e_1, e_2, e_3, \gamma_4, e_3, \gamma_4, e_3, e_4)$</td>
<td>$S^2 \times S^2$</td>
</tr>
<tr>
<td>$(e_1, \gamma_1, \gamma_2, \gamma_1, e_1, e_2, e_3, \gamma_4, e_3, e_4)$</td>
<td>$\mathbb{C}P^2 # \mathbb{C}P^2$</td>
</tr>
</tbody>
</table>

Figure 17: Simple closed curves in $\Sigma_3$

In particular, we have obtained two genus-3 SPWFs on $S^2 \times S^2$ which is derived from the following two surface diagrams: the diagram $(d_0, d_1, d_2, d_3, d_4, d_5, d_4, d_3, d_2, d_1)$ in Example 6.8, and the diagram $(e_1, \gamma_1, e_1, e_2, e_3, \gamma_4, e_3, \gamma_4, e_3, e_4)$ as above. The SPWF which corresponds to the former diagram is homotopic to the projection $p_1: S^2 \times S^2 \to S^2$ onto the first projection. Indeed, this SPWF was constructed by applying birth and flip and slip to $p_1$. On the other hand, it is easy to prove (by Kirby calculus, for example) that a regular fiber of the SPWF corresponding to the latter diagram
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is nullhomologous in $S^2 \times S^2$. Thus, two genus-3 SPWFs above are not homotopic. In the same way, we can prove that two SPWFs on $S^1 \times S^3 \# S^2 \times S^2$ derived from the following two diagrams are not homotopic: the diagram $(d_0, d_1, d_2, d_3, d_4, d_3, d_2, d_1)$ which is obtained by applying flip and slip to the step fibration twice (see Remark 6.9), and the diagram $(e_1, \gamma_1, e_1, e_2, e_3, \gamma_3, e_3, e_4)$ as above.

References


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