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On connective KO–theory of elementary abelian 2–groups

GEOFFREY POWELL

A general notion of detection is introduced and used in the study of the cohomology of elementary abelian 2–groups with respect to the spectra in the Postnikov tower of orthogonal K–theory. This recovers and extends results of Bruner and Greenlees and is related to calculations of the (co)homology of the spaces of the associated Ω–spectra by Stong and by Cowen Morton.

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1 Introduction

The orthogonal K–theory of elementary abelian 2–groups possesses a rich structure and the spectra of the Postnikov tower of KO leads to interesting related functors $V \mapsto \mathrm{KO}\langle n\rangle^*(\mathrm{BV})$. The study of these is, for example, a first step towards a systematic analysis of $\mathrm{KO}\langle n\rangle^*(\mathrm{BG})$ for finite groups G. Bott periodicity reduces us to the consideration of $\mathrm{ko} = \mathrm{KO}\langle 0\rangle$, $\mathrm{ko}\langle 1\rangle$, $\mathrm{ko}\langle 2\rangle$ and $\mathrm{ko}\langle 4\rangle$, of which the case ko has been studied extensively (but nonfunctorially) by Bruner and Greenlees [5], based on their earlier work on the complex case [4]. A key property is that $\mathrm{ko}^*(\mathrm{BV})$ is *detected* by the periodic theory $\mathrm{KO}^*(\mathrm{BV})$ together with integral cohomology $H\mathbb{Z}^*(\mathrm{BV})$, via the zero layer $\mathrm{ko} \to H\mathbb{Z}$ of the Postnikov tower. The main result of this paper (Theorem 9.1) establishes the analogous property for the spectra $\mathrm{KO}\langle n\rangle$ using the Postnikov layers $\mathrm{KO}\langle n\rangle \to \Sigma^n H(\mathrm{KO}_n)$; this leads to a description of $\mathrm{KO}\langle n\rangle^*(\mathrm{BV})$ (see Corollary 9.2). This recovers, in particular, the results of Bruner and Greenlees [5] for ko.

The functorial structure gives information on the spaces of the associated Ω–spectra: Lannes' theory (cf Henn, Lannes and Schwartz [6] and Schwartz [12]) implies that $V \mapsto \mathrm{ko}\langle n\rangle^d(\mathrm{BV})$ determines (up to F–isomorphism) the mod-2 cohomology of the d^{th} space of the Ω–spectrum associated to $\mathrm{ko}\langle n\rangle$. This establishes a relation with results in the literature: the mod-2 cohomology rings of the connective covers of the classifying space BO of the infinite orthogonal group were determined by Stong [13], and the Hopf ring for ko and the Hopf module structures of the spectra $\mathrm{ko}\langle n\rangle$ over this Hopf ring were calculated by Cowen Morton [9]. Both these results establish

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an analogue of the detection property. Hence, the detection property for $\mathrm{ko}^*(\mathrm{BV})$ is related to the unstable origin of the fact that $\mathrm{ko} \to H\mathbb{Z}$ induces a monomorphism in homology $H\mathbb{F}_*(\mathrm{ko}) \hookrightarrow H\mathbb{F}_* H\mathbb{Z}$; a similar statement holds for $\mathrm{KO}\langle n\rangle$.

The main results of the paper (see Section 9) give a description of the functors $\mathrm{KO}\langle n\rangle^*(\mathrm{BV})$, based in part on the author's previous work [11] on the case of complex connective K–theory, which revisited the earlier work of Bruner and Greenlees [4] from a functorial viewpoint using new techniques. The abstract treatment of the detection property (given in Section 2) leads to an explicit relationship between the part of the theory which is detected in the periodic theory and the torsion part (see Theorem 2.10). These methods also apply to the study of $\mathrm{KO}\langle n\rangle_*(\mathrm{BV})$ for all n; this leads to a conceptual understanding of the relationship between cohomology and homology via the local cohomology spectral, generalising the results of [11] for ku. This will be explained elsewhere.

The proof requires an understanding of the homology of a complex which arises from the primary k–invariants of the Postnikov tower of KO, taking the cohomology of the classifying spaces BV (see Section 4); the complex is derived from an exact complex $\mathscr{E}_\bullet$ of $\mathcal{A}(1)$–modules (recall that $\mathcal{A}(1)$ is the subalgebra of the mod-2 Steenrod algebra $\mathcal{A}$ generated by Sq^1 and Sq^2), related to the exact complex of Toda [14]. The restriction to the category of $\mathcal{A}(1)$–modules provides the tools for calculating the homology of the cochain complex $\mathrm{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, H\mathbb{F}^*(\mathrm{BV}))$ (see Sections 5 and 6), based on ideas of Ossa [10] developed in the thesis of Cherng-Yih Yu [15] and by Bruner [3].

The first step towards establishing detection is to treat the case of ko (see Section 8). Much of the argument can be carried out using detection in periodic complex K–theory and the known structure of $\mathrm{ku}^*(\mathrm{BV})$. However, this is not sufficient to treat the classes which are divisible by η and which are detected in KO–cohomology; for these a general argument (cf Proposition A.2) related to the η–Bockstein spectral sequence is used, for which Proposition 7.4 is the crucial calculational input.

This leads to the determination of the functor $\mathrm{ko}^*(\mathrm{BV})$ (see Corollary 8.3); from this, detection is deduced for $\mathrm{ko}\langle n\rangle^*(\mathrm{BV})$ in general (Theorem 9.1), whence the functorial description given in Corollary 9.2.

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2 Abstract detection

Consider a tower $E_\bullet$ over F in the stable homotopy category:

$$\begin{array}{ccccccc}
\cdots \longrightarrow E_n & \xrightarrow{e_n} & E_{n-1} & \xrightarrow{e_{n-1}} & E_{n-2} & \longrightarrow \cdots \\
 & \searrow^{f_n} & & \searrow^{f_{n-1}} & \downarrow{\scriptstyle f_{n-2}} & \\
 & & & & F &
\end{array}$$

This has associated cofibre sequences

$$E_n \xrightarrow{e_n} E_{n-1} \xrightarrow{c_{n-1}} C_{n-1} \xrightarrow{\delta_{n-1}} \Sigma E_n$$

with composite morphism θ_n playing the role of a primary k–invariant:

$$C_n \xrightarrow{\delta_n} \Sigma E_{n+1} \xrightarrow{\Sigma c_{n+1}} \Sigma C_{n+1}, \qquad \theta_n = \Sigma c_{n+1} \circ \delta_n \colon C_n \to \Sigma C_{n+1}$$

These fit into the following commutative diagram, in which the horizontal sequence is the cofibre sequence:

(1)
$$\begin{array}{ccccccc}
\Sigma^{-1} C_{n-1} & & & & & & \\
\downarrow{\scriptstyle \Sigma^{-1}\delta_{n-1}} & \searrow^{\Sigma^{-1}\theta_{n-1}} & & & & & \\
E_n & \xrightarrow{c_n} & C_n & \xrightarrow{\delta_n} & \Sigma E_{n+1} & \longrightarrow & \Sigma E_n \\
 & & & \searrow_{\theta_n} & \downarrow{\scriptstyle \Sigma c_{n+1}} & & \\
 & & & & \Sigma C_{n+1} & &
\end{array}$$

Remark 2.1 The composite $\theta_n \circ \Sigma^{-1}\theta_{n-1}$ is trivial.

Detection is defined with respect to a fixed object X by considering the behaviour of $[X, E_n]^* \to [X, E_{n-1}]^*$ and $[X, E_n]^* \to [X, F]^*$, where $[-,-]^*$ denotes the $\mathbb{Z}$–graded morphism groups.

Definition 2.2 For $n \in \mathbb{Z}$ and a spectrum X, the tower satisfies:

(1) Level n detection if $(f_n, c_n)\colon [X, E_n]^* \to [X, F]^* \oplus [X, C_n]^*$ is a monomorphism.

(2) Weak level n detection if $(e_n, c_n)\colon [X, E_n]^* \to [X, E_{n-1}]^* \oplus [X, C_n]^*$ is a monomorphism.

Remark 2.3 One can also consider a family of spectra and define detection pointwise; this reduces to the single object case by taking the coproduct of the family.

Example 2.4 The case of interest here is the family $\Sigma^\infty \mathrm{BV}$, as V ranges over a skeleton of the category of finite-rank elementary abelian 2–groups.

Lemma 2.5 *For $n \in \mathbb{Z}$:*

(1) *Level n detection $\Rightarrow$ weak level n detection.*

(2) *Level $n-1$ detection and weak level n detection $\Rightarrow$ level n detection.*

Proof The proof is straightforward. $\square$

From the construction, it is clear that $c_{n-1}\colon E_{n-1} \to C_{n-1}$ induces a morphism

$$[X, E_{n-1}]^* \to \mathrm{Ker}\{[X, C_{n-1}]^* \xrightarrow{\theta_{n-1}} [X, \Sigma C_n]^*\}.$$

The following result gives an alternative formulation of weak detection.

Lemma 2.6 *For $n \in \mathbb{Z}$, the following conditions are equivalent:*

(1) *Weak level n detection holds.*

(2) *c_{n-1} induces a surjection $[X, E_{n-1}]^* \twoheadrightarrow \mathrm{Ker}(\theta_{n-1})$.*

Proof (2) $\Rightarrow$ (1) Suppose that $x \in [X, E_n]^*$ lies in the kernel of (e_n, c_n); since x is in the kernel of e_n, it is the image of some $\tilde{x} \in [X, \Sigma^{-1} C_{n-1}]^*$ and, moreover, $\tilde{x}$ lies in the kernel of $\Sigma^{-1}\theta_{n-1}$. Hence, by hypothesis (2), $\tilde{x}$ is the image of an element of $[X, \Sigma^{-1} E_{n-1}]^*$. This implies that $\tilde{x} \mapsto 0$ in $[X, E_n]^*$, so that $x = 0$, thus weak level n detection holds.

(1) $\Rightarrow$ (2) Consider an element $y \in [X, \Sigma^{-1} C_{n-1}]^*$ which lies in the kernel of $\Sigma^{-1}\theta_{n-1}$ and set $\bar{y} := \Sigma^{-1}\delta_{n-1} y \in [X, E_n]^*$. Since $e_n \Sigma^{-1}\delta_{n-1} = 0$, $e_n \bar{y} = 0$ and the hypothesis on y implies that $c_n \bar{y} = \Sigma^{-1}\theta_{n-1} y = 0$. Hence, weak detection implies that $\bar{y} \in [X, E_n]^*$ is zero; by exactness, y is the image of a class in $[X, \Sigma^{-1} E_{n-1}]^*$, as required. $\square$

Notation 2.7 For $n \in \mathbb{Z}$, write $\Phi_n [X, F]^*$ for the image of $[X, E_n]^* \xrightarrow{f_n} [X, F]^*$.

This gives the decreasing filtration

$$\cdots \subset \Phi_n [X, F]^* \subset \Phi_{n-1}[X, F]^* \subset \cdots \subset [X, F]^*.$$

Lemma 2.8 *For $n \in \mathbb{Z}$, f_{n-1} induces a surjection*

$$\mathrm{Im}\{[X, E_n]^* \xrightarrow{e_n} [X, E_{n-1}]^*\} \twoheadrightarrow \Phi_n [X, F]^*.$$

If level $n-1$ detection holds, then this is an isomorphism.

Proof The first statement is clear, since $f_n = f_{n-1} \circ e_n$. The second statement is a consequence of the fact that the composite

$$[X, E_n]^* \xrightarrow{e_n} [X, E_{n-1}]^* \xrightarrow{c_{n-1}} [X, C_{n-1}]^*$$

is trivial, together with the hypothesis that level $n-1$ detection holds. □

Proposition 2.9 *For $n \in \mathbb{Z}$, there are natural morphisms*

$$\begin{array}{ccc} \mathrm{Ker}(\delta_n)/\,\mathrm{Im}(\Sigma^{-1}\theta_{n-1}) & \xhookrightarrow{\iota_n} & \mathrm{Ker}(\theta_n)/\,\mathrm{Im}(\Sigma^{-1}\theta_{n-1}) \\ \downarrow \cong & & \\ \mathrm{Im}(e_n)/\,\mathrm{Im}(e_n \circ e_{n+1}) & \underset{\sigma_n}{\twoheadrightarrow} & \Phi_n[X, F]^*/\Phi_{n+1}[X, F]^*. \end{array}$$

In particular, $\Phi_n[X, F]^*/\Phi_{n+1}[X, F]^*$ is a subquotient of $\mathrm{Ker}(\theta_n)/\,\mathrm{Im}(\Sigma^{-1}\theta_{n-1})$. Moreover:

(1) *Weak level $n+1$ detection holds if and only if ι_n is an isomorphism.*

(2) *If level $n-1$ detection holds, then σ_n is an isomorphism.*

If both the above conditions hold, then

$$\mathrm{Ker}(\theta_n)/\,\mathrm{Im}(\Sigma^{-1}\theta_{n-1}) \cong \Phi_n[X, F]^*/\Phi_{n+1}[X, F]^*.$$

Proof From diagram (1), there are inclusions $\mathrm{Im}(\Sigma^{-1}\theta_{n-1}) \subset \mathrm{Im}(c_n) = \mathrm{Ker}(\delta_n) \subset \mathrm{Ker}(\theta_n)$. The inclusion ι_n is induced by $\mathrm{Ker}(\delta_n) \subset \mathrm{Ker}(\theta_n)$ and the equivalence between weak level $n+1$ detection and ι_n being an isomorphism follows from Lemma 2.6.

The surjection σ_n is given by Lemma 2.8, using the argument outlined in its proof to show that it is an isomorphism under the hypothesis of level $n-1$ detection.

Using the equality $\mathrm{Im}(c_n) = \mathrm{Ker}(\delta_n)$, the vertical morphism is induced by e_n, which gives a well-defined surjection

$$\mathrm{Ker}(\delta_n) \twoheadrightarrow \mathrm{Im}(e_n)/\,\mathrm{Im}(e_n \circ e_{n+1}). \tag{2}$$

The cofibre sequence

$$\Sigma^{-1}C_{n-1} \xrightarrow{\Sigma^{-1}\delta_{n-1}} E_n \xrightarrow{e_n} E_{n-1}$$

induces an exact sequence

$$[X, \Sigma^{-1}C_{n-1}]^* \to [X, E_n]^* \to [X, E_{n-1}]^*,$$

and it is straightforward to deduce that the kernel of the surjection (2) is the image of $\Sigma^{-1}\theta_{n-1}$, as required. □

Theorem 2.10 *Suppose that detection holds for all $n \in \mathbb{Z}$. Then there are short exact sequences (natural in* $\mathrm{End}(X)$*)*

$$0 \to \mathrm{Im}(\Sigma^{-1}\theta_{n-1}) \to [X, E_n]^* \to \Phi_n[X, F]^* \to 0$$

which are formed by pullback along the natural surjection

$$\Phi_n[X, F]^* \twoheadrightarrow \Phi_n[X, F]^*/\Phi_{n+1}[X, F]^*$$

of the short exact sequence

$$0 \to \mathrm{Im}(\Sigma^{-1}\theta_{n-1}) \to \mathrm{Ker}(\theta_n) \to \Phi_n[X, F]^*/\Phi_{n+1}[X, F]^* \to 0.$$

Proof By definition, f_n induces a surjection $[X, E_n]^* \twoheadrightarrow \Phi_n[X, F]^*$. Since level $n-1$ detection holds, the kernel coincides with the kernel of $[X, E_n]^* \xrightarrow{e_n} [X, E_{n-1}]^*$ (as in the proof of Lemma 2.8) and hence identifies with the image of

$$[X, \Sigma^{-1}C_{n-1}]^* \xrightarrow{\Sigma^{-1}\delta_{n-1}} [X, E_n]^*.$$

By level n detection, this image is detected in $[X, C_n]^*$, where it identifies with the image of $\Sigma^{-1}\theta_{n-1}$, by definition of the latter.

Lemma 2.6, using the level $n+1$ detection hypothesis, implies that c_n induces a surjection $[X, E_n] \twoheadrightarrow \mathrm{Ker}(\theta_n)$. Combining this with Proposition 2.9 shows that there is a pullback square

$$\begin{array}{ccc} [X, E_n]^* & \twoheadrightarrow & \Phi_n[X, F]^* \\ \downarrow & & \downarrow \\ \mathrm{Ker}(\theta_n) & \twoheadrightarrow & \Phi_n[X, F]^*/\Phi_{n+1}[X, F]^*, \end{array}$$

level n detection ensuring that $[X, E_n]^*$ embeds into $\Phi_n[X, F]^* \oplus \mathrm{Ker}(\theta_n)$. This proves the final statement. □

3 Functors

This section introduces the categories of functors which feature in the paper and the objects which occur, using the notation of [11]. Let $\mathbb{F}$ denote the prime field with two elements and consider the category of functors from finite-dimensional $\mathbb{F}$–vector spaces to abelian groups; this contains the category $\mathscr{F}$ of functors from finite-dimensional $\mathbb{F}$–vector spaces to $\mathbb{F}$–vector spaces as a full subcategory. A functor is finite if it has a finite composition series and locally finite if it is the colimit of its finite subobjects.

In order to consider only covariant functors, vector space duality (denoted here by $V \mapsto V^\sharp$) is used where appropriate.

Example 3.1 A basic example is provided by the functor $V \mapsto H\mathbb{F}^*(BV^\sharp)$ of group cohomology with $\mathbb{F}$–coefficients (cohomology is always taken to be reduced; where necessary, a disjoint basepoint $(-)_+$ is added). In degree $n > 0$, this identifies with the n^{th} symmetric power functor S^n, which is finite.

Notation 3.2 Denote by:

(1) $\overline{P}_{\mathbb{Z}_2}$ the augmentation ideal of the $\mathbb{Z}_2$–group ring functor $\mathbb{Z}_2[V]$.

(2) $\overline{P}_{\mathbb{F}}$ the augmentation ideal of the $\mathbb{F}$–group ring functor $\mathbb{F}[V]$.

(3) $\overline{P}^n_{\mathbb{Z}_2}$ (respectively $\overline{P}^n_{\mathbb{F}}$) the n^{th} power of the augmentation ideal $\overline{P}_{\mathbb{Z}_2}$ (resp. $\overline{P}_{\mathbb{F}}$), which is understood as $\overline{P}_{\mathbb{Z}_2}$ (resp. $\overline{P}_{\mathbb{F}}$) for $0 \geq n \in \mathbb{Z}$.

(4) $\overline{I}_{\mathbb{F}}$ the subfunctor of $V \mapsto \mathbb{F}^{V^\sharp}$ of maps which send 0 to zero.

(5) $p_n\overline{I}_{\mathbb{F}} \subset \overline{I}_{\mathbb{F}}$ the largest subfunctor of $\overline{I}_{\mathbb{F}}$ of polynomial degree n.

Remark 3.3 (1) The functor $\overline{I}_{\mathbb{F}}$ is locally finite and uniserial; explicitly, $\overline{I}_{\mathbb{F}} = \lim_{\rightarrow} p_n\overline{I}_{\mathbb{F}}$ and $p_n\overline{I}_{\mathbb{F}}$ is finite, uniserial with composition factors $\Lambda^1, \dots, \Lambda^n$, where Λ^j is the j^{th} exterior power functor, which is an object of $\mathscr{F}$ and is simple.

(2) The functor $\overline{P}_{\mathbb{F}}$ is dual to $\overline{I}_{\mathbb{F}}$ and hence is uniserial and *not* locally finite (for duality, see Kuhn [7] and the author [11]); the filtration by powers of the augmentation ideal induces short exact sequences $0 \to \overline{P}^{n+1}_{\mathbb{F}} \to \overline{P}^n_{\mathbb{F}} \to \Lambda^n \to 0$, for $0 < n \in \mathbb{Z}$.

Notation 3.4 Let F, G be finite functors.

(1) Write $[F]$ for the element of the Grothendieck group of finite functors corresponding to F, so that $[F] = \sum_\lambda a_\lambda[S_\lambda]$, where $a_\lambda \in \mathbb{N}$ is the multiplicity of the simple S_λ in F; the function $a_{(-)}$ has finite support and the graded associated to a composition series of F is $\mathrm{gr}(F) \cong \bigoplus_\lambda S_\lambda^{\oplus a_\lambda}$.

(2) Write $[F] \leq [G]$ if $\mathrm{gr}(F)$ is a direct summand of $\mathrm{gr}(G)$. (This can be interpreted as an inequality of multiplicities of composition factors.)

Example 3.5 For $t \in \mathbb{N}$, there are equalities in the Grothendieck group:

(1) $[p_t\overline{I}_{\mathbb{F}}] = \sum_{j=1}^{t} [\Lambda^j]$.

(2) $[\overline{P}_{\mathbb{F}}/\overline{P}^{t+1}_{\mathbb{F}}] = [p_t\overline{I}_{\mathbb{F}}]$.

The following is clear:

Lemma 3.6 *If F is a subquotient of a finite functor G, then $[F] \leq [G]$.*

The following result gives information on the filtration by powers of the augmentation ideal of $\overline{P}_{\mathbb{Z}_2}$.

Proposition 3.7 [11] *For $n \in \mathbb{N}$, the canonical inclusion $\overline{P}^{n+1}_{\mathbb{Z}_2} \hookrightarrow \overline{P}^{n}_{\mathbb{Z}_2}$ induces a short exact sequence*

$$0 \to \overline{P}^{n+1}_{\mathbb{Z}_2} \hookrightarrow \overline{P}^{n}_{\mathbb{Z}_2} \to p_n \overline{I}_{\mathbb{F}} \to 0.$$

In particular, the cokernel of the inclusion $\overline{P}^{n+1}_{\mathbb{Z}_2} \hookrightarrow \overline{P}_{\mathbb{Z}_2}$ is a finite functor and $[\overline{P}_{\mathbb{Z}_2}/\overline{P}^{n+1}_{\mathbb{Z}_2}] = \sum_{j=1}^{n} [p_j \overline{I}_{\mathbb{F}}]$.

The 2–adic filtration of $\overline{P}_{\mathbb{Z}_2}$ and its relationship with the filtration by powers of the augmentation ideal is of importance; there is a short exact sequence

$$0 \to \overline{P}_{\mathbb{Z}_2} \xrightarrow{2} \overline{P}_{\mathbb{Z}_2} \to \overline{P}_{\mathbb{F}} \to 0$$

which restricts (for $n > 0$) to the short exact sequence

$$0 \to \overline{P}^{n}_{\mathbb{Z}_2} \xrightarrow{2} \overline{P}^{n+1}_{\mathbb{Z}_2} \to \overline{P}^{n+1}_{\mathbb{F}} \to 0.$$

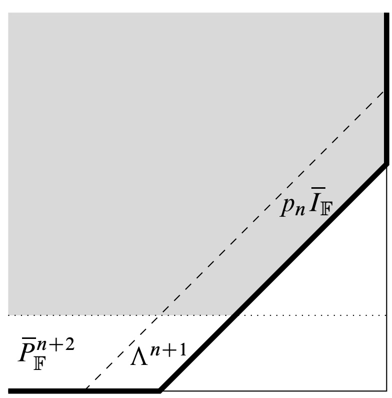

Figure 1: A representation of the subfunctors $\overline{P}^{n}_{\mathbb{Z}_2} \subset \overline{P}^{n+1}_{\mathbb{Z}_2} \subset \overline{P}_{\mathbb{Z}_2}$

This is illustrated by Figure 1, in which the bounding square represents $\overline{P}_{\mathbb{Z}_2}$, $\overline{P}^{n+1}_{\mathbb{Z}_2}$ is bounded by the heavy line and the shaded region indicates $2\overline{P}^{n}_{\mathbb{Z}_2} \subset \overline{P}^{n+1}_{\mathbb{Z}_2}$, which is isomorphic to $\overline{P}^{n}_{\mathbb{Z}_2}$. The region above the dotted line represents the inclusion

$2\bar{P}_{\mathbb{Z}_2} \subset \bar{P}_{\mathbb{Z}_2}$, whereas the region above the dashed line represents $\bar{P}^{n+2}_{\mathbb{Z}_2} \subset \bar{P}^{n+1}_{\mathbb{Z}_2}$, which restricts in the shaded region to the inclusion $\bar{P}^{n+1}_{\mathbb{Z}_2} \subset \bar{P}^{n}_{\mathbb{Z}_2}$. The indicated functors represent the subquotients corresponding to the respective areas. Hence the bottom row corresponds to the exact sequence $0 \to \bar{P}^{n+2}_{\mathbb{F}} \to \bar{P}^{n+1}_{\mathbb{F}} \to \Lambda^{n+1} \to 0$ and the diagonal to $0 \to p_n \bar{I}_{\mathbb{F}} \to p_{n+1} \bar{I}_{\mathbb{F}} \to \Lambda^{n+1} \to 0$.

Definition 3.8 For $F \in \mathrm{Ob}\,\mathscr{F}$ taking finite-dimensional values, the Poincaré series p_F is

$$p_F(t) := \sum_{i \geq 0} \dim F(\mathbb{F}^i) t^i.$$

The following general result concerning functors of $\mathscr{F}$ (taking values in $\mathbb{F}$–vector spaces) is used in Section 6 to deduce functorial information from Poincaré series.

Lemma 3.9 *Let* $F \in \mathrm{Ob}\,\mathscr{F}$ *be finite and suppose that* $p_F(t) = \sum_{i=0}^{\infty} \varepsilon_i \binom{t}{i}$, *with* $\varepsilon_i \in \{0, 1\}$. *Then* ε_i *has finite support and* $[F] = \sum_{i=0}^{\infty} \varepsilon_i [\Lambda^i]$.

Proof The Poincaré series p_F only depends upon $[F]$, hence the result is a consequence of the fact that, for each natural number n, there is a unique simple functor S in $\mathscr{F}$ such that $S(\mathbb{F}^i)$ is trivial for $i < n$ and $\dim S(\mathbb{F}^n) = 1$, namely the exterior power functor Λ^n, together with the fact that $\dim \Lambda^n(\mathbb{F}^d) = \binom{n}{d}$. The finiteness hypothesis on F clearly implies that ε_i has finite support. □

4 Background on the spectra associated to K–theory

4.1 The tower associated to KU–theory

As usual, ku is written for $\mathrm{KU}\langle 0 \rangle$ and Bott periodicity gives the isomorphisms $\mathrm{KU}\langle 2n \rangle \cong \Sigma^{2n}\mathrm{ku}$ and $\mathrm{KU}\langle 2n+1 \rangle \cong \mathrm{KU}\langle 2n+2 \rangle$, for $n \in \mathbb{Z}$, so that the associated cofibre sequences (as in Section 2) are determined by

$$\Sigma^2 \mathrm{ku} \xrightarrow{v} \mathrm{ku} \to H\mathbb{Z} \to \Sigma^3 \mathrm{ku},$$

where v is multiplication by the Bott element, where $\mathrm{KU}_* \cong \mathbb{Z}[v^{\pm 1}]$.

The functorial description given in [11] is a consequence of the fact that detection holds in the Postnikov tower of KU: the morphisms $\mathrm{ku} \to \mathrm{KU}$ and $\mathrm{ku} \to H\mathbb{Z}$ induce a monomorphism $\mathrm{ku}^*(BV^\sharp) \hookrightarrow H\mathbb{Z}^*(BV^\sharp) \oplus \mathrm{KU}^*(BV^\sharp)$. (This property was observed by Bruner and Greenlees in [4].) Integral cohomology $H\mathbb{Z}^*(BV^\sharp)$ embeds in $H\mathbb{F}^*(BV^\sharp)$ as the kernel of the Bockstein, hence there is a monomorphism $\mathrm{ku}^*(BV^\sharp) \hookrightarrow H\mathbb{F}^*(BV^\sharp) \oplus \mathrm{KU}^*(BV^\sharp)$. The structure of these functors can be described explicitly.

Notation 4.1 (cf [5]) Let Q^* (respectively TU^*) denote the image (resp. kernel) of $\mathrm{ku}^*(BV^\sharp) \to \mathrm{KU}^*(BV^\sharp)$.

Recall that the Milnor derivations Q_0, Q_1 are given by $Q_0 = \mathrm{Sq}^1$, $Q_1 = [\mathrm{Sq}^2, \mathrm{Sq}^1]$.

Theorem 4.2 [11] *Detection holds for the Postnikov tower of* KU *at all levels. In particular, there is a natural short exact sequence*

$$0 \to TU^* \to \mathrm{ku}^*(BV^\sharp) \to Q^* \to 0,$$

where

$$Q^n \cong \begin{cases} 0 & n \text{ odd}, \\ \bar{P}^d_{\mathbb{Z}_2} & n = 2d \geq 0, \\ \bar{P}_{\mathbb{Z}_2} & n = 2d \leq 0, \end{cases}$$

and TU^* *identifies with the image of* $Q_0 Q_1 \colon H\mathbb{F}^{*-4}(BV^\sharp) \to H\mathbb{F}^*(BV^\sharp)$.

4.2 The tower associated to KO–theory

Recall that

$$\mathrm{KO}_* \cong \mathbb{Z}[\eta, \alpha, \beta^{\pm 1}]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta),$$

where $|\eta| = 1$, $|\alpha| = 4$ and β is the Bott element, with $|\beta| = 8$. Bott periodicity gives $\mathrm{KO}\langle n+8\rangle \cong \Sigma^8 \mathrm{KO}\langle n\rangle$ for $n \in \mathbb{Z}$; the spectrum $\mathrm{KO}\langle 0\rangle$ is denoted ko.

The Postnikov tower for KO can be deduced by Bott periodicity from

(3)
$$\begin{array}{ccccc}
\Sigma^8\mathrm{ko} & \xrightarrow{\simeq} & \mathrm{ko}\langle 8\rangle & \longrightarrow & \Sigma^8 H\mathbb{Z} \\
 & & \downarrow & \nwarrow & \uparrow \mathrm{Sq}^5 \\
 & & \mathrm{ko}\langle 4\rangle & \longrightarrow & \Sigma^4 H\mathbb{Z} \\
 & & \downarrow & \nwarrow & \uparrow \mathrm{Sq}^3 \\
 & & \mathrm{ko}\langle 2\rangle & \longrightarrow & \Sigma^2 H\mathbb{F} \\
 & & \downarrow & \nwarrow & \uparrow \mathrm{Sq}^2 \\
 & & \mathrm{ko}\langle 1\rangle & \longrightarrow & \Sigma^1 H\mathbb{F} \\
 & & \downarrow & \nwarrow & \uparrow \mathrm{Sq}^2 \\
 & & \mathrm{ko} & \longrightarrow & H\mathbb{Z}.
\end{array}$$

(The dashed and curved arrows have the usual degree shift.) The curved arrows are the associated k–invariants; the cohomology operations are interpreted as in [5, Section A.5] (see Remark 4.3 below).

The associated diagram in mod-2 singular cohomology is well understood (cf [5, Section A.5] or Adams and Priddy [2]); in particular, $H\mathbb{F}_2^*(\mathrm{ko}\langle n\rangle)$ is a cyclic module over the mod-2 Steenrod algebra $\mathcal{A}$ and the morphism in cohomology induced by $\mathrm{ko}\langle n\rangle \to \Sigma^n H(\mathrm{KO}_n)$ is surjective. It follows that the curved arrows induce a periodic, exact sequence of $\mathcal{A}$–modules; this is the key exact sequence of Toda [14] used by Stong in [13].

Remark 4.3 The operation denoted Sq^5 in (3) is an integral lift of $\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2$. The equivalence of the two descriptions follows from the Adem relation $\mathrm{Sq}^5 = \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^4\mathrm{Sq}^1$, since $\mathrm{Sq}^4\mathrm{Sq}^1$ lifts trivially to integral cohomology.

Notation 4.4 Recall that $\mathcal{A}(1)$ (respectively $\mathcal{A}(0)$) is the finite sub-Hopf algebra of $\mathcal{A}$ generated by $\mathrm{Sq}^1, \mathrm{Sq}^2$ (respectively Sq^1) and $\mathcal{A}(1)/\!/\mathcal{A}(0)$ is the induced $\mathcal{A}(1)$–module $\mathcal{A}(1) \otimes_{\mathcal{A}(0)} \mathbb{F}$.

The Toda exact complex is induced from an exact complex $\mathscr{E}_\bullet$ of $\mathcal{A}(1)$–modules by applying the induction functor $\mathcal{A}\otimes_{\mathcal{A}(1)} -$. The complex $\mathscr{E}_\bullet$ is the periodic extension of

$$\begin{array}{ccccccc} \longleftarrow \Sigma^{-4}(\mathcal{A}(1)/\!/\mathcal{A}(0)) & & & & & & \\ {\scriptstyle \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2}\uparrow & & & & & & \\ \mathcal{A}(1)/\!/\mathcal{A}(0) & \xleftarrow{\mathrm{Sq}^2} & \Sigma^1\mathcal{A}(1) & \xleftarrow{\mathrm{Sq}^2} & \Sigma^2\mathcal{A}(1) & \xleftarrow{\mathrm{Sq}^3} & \Sigma^4(\mathcal{A}(1)/\!/\mathcal{A}(0)) \\ & & & & & & \uparrow{\scriptstyle \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2} \\ & & & & & & \Sigma^8(\mathcal{A}(1)/\!/\mathcal{A}(0)) \longleftarrow \end{array} \tag{4}$$

in which each morphism is of degree 1. On the level of objects $\mathscr{E}_0 = \mathcal{A}(1)/\!/\mathcal{A}(0)$, $\mathscr{E}_1 = \Sigma^1\mathcal{A}(1)$, $\mathscr{E}_2 = \Sigma^2\mathcal{A}(1)$ and $\mathscr{E}_3 = \Sigma^4(\mathcal{A}(1)/\!/\mathcal{A}(0))$; for $0 \le i \le 3$ and $k \in \mathbb{Z}$, $\mathscr{E}_{4k+i} = \Sigma^{8k}\mathscr{E}_i$.

For an $\mathcal{A}(1)$–module M, $\mathrm{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, M)$ is a periodic (up to suspension) cochain complex of $\mathbb{F}$–vector spaces, which is of the form

$$\begin{array}{ccccccc} \longrightarrow \Sigma^4\,\mathrm{Ann}_{(\mathrm{Sq}^1)} M & & & & & & \\ {\scriptstyle \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2}\downarrow & & & & & & \\ \mathrm{Ann}_{(\mathrm{Sq}^1)} M & \xrightarrow{\mathrm{Sq}^2} & \Sigma^{-1}M & \xrightarrow{\mathrm{Sq}^2} & \Sigma^{-2}M & \xrightarrow{\mathrm{Sq}^3} & \Sigma^{-4}\,\mathrm{Ann}_{(\mathrm{Sq}^1)} M, \\ & & & & & & \downarrow{\scriptstyle \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2} \\ & & & & & & \Sigma^{-8}\,\mathrm{Ann}_{(\mathrm{Sq}^1)} M \longrightarrow \end{array} \tag{5}$$

where the morphisms are of degree 1.

Example 4.5 The cochain complex $\mathrm{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, H\mathbb{F}^*(BV^\sharp))$ (V an elementary abelian 2–group) is isomorphic to the complex obtained by applying $[\Sigma^\infty BV^\sharp, -]$ to the sequence of curved arrows of diagram (3). Hence, by the techniques of Section 2, the homology of $\mathrm{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, H\mathbb{F}^*(BV^\sharp))$ is central to understanding $V \mapsto \mathrm{KO}\langle n\rangle^*(BV^\sharp)$.

In applying the methods of Section 2, it is natural to reindex in terms of the order of the spectra in the Postnikov tower, rather than connectivity:

Notation 4.6 For an integer $n = 4k + i$, ($i, k \in \mathbb{Z}$ such that $0 \le i \le 3$), write

$$\mathrm{KO}\{n\} := \Sigma^{8k}\mathrm{KO}\{i\},$$

where

$$\mathrm{KO}\{i\} = \begin{cases} \mathrm{ko}\langle i\rangle & 0 \le i \le 2, \\ \mathrm{ko}\langle 4\rangle & i = 3. \end{cases}$$

4.3 The complexification-realification sequences

Complex and orthogonal K–theories are related by the equivalence $\mathrm{KU} \simeq \mathrm{KO} \wedge C\eta$, which restricts to $\mathrm{ku} \simeq \mathrm{ko} \wedge C\eta$ (cf [5], for example). This yields the morphism between the associated complexification-realification cofibre sequences:

$$\begin{array}{ccccccc} \Sigma\mathrm{ko} & \xrightarrow{\eta} & \mathrm{ko} & \xrightarrow{c} & \mathrm{ku} & \xrightarrow{R} & \Sigma^2\mathrm{ko} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma\mathrm{KO} & \xrightarrow[\eta]{} & \mathrm{KO} & \xrightarrow[c]{} & \mathrm{KU} & \xrightarrow[R]{} & \Sigma^2\mathrm{KO}. \end{array} \tag{6}$$

Notation 4.7 (cf [5]) Let QO^* (respectively ST^*) denote the image (resp. kernel) of $\mathrm{ko}^*(BV^\sharp) \to \mathrm{KO}^*(BV^\sharp)$.

There are natural short exact sequences (recall the notation of Notation 4.1):

$$0 \longrightarrow ST^* \longrightarrow \mathrm{ko}^*(BV^\sharp) \longrightarrow QO^* \longrightarrow 0,$$
$$0 \longrightarrow TU^* \longrightarrow \mathrm{ku}^*(BV^\sharp) \longrightarrow Q^* \longrightarrow 0.$$

Hence, diagram (6) induces a short exact sequence of complexes

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & ST^{*+1} & \xrightarrow{\eta} & ST^* & \xrightarrow{c} & TU^* & \xrightarrow{R} & ST^{*+2} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{ko}^{*+1}(BV^\sharp) & \xrightarrow{\eta} & \mathrm{ko}^*(BV^\sharp) & \xrightarrow{c} & \mathrm{ku}^*(BV^\sharp) & \xrightarrow{R} & \mathrm{ko}^{*+2}(BV^\sharp) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & QO^{*+1} & \xrightarrow[\eta]{} & QO^* & \xrightarrow[c]{} & Q^* & \xrightarrow[R]{} & QO^{*+2} & \longrightarrow & \cdots \end{array} \tag{7}$$

in which the middle complex is exact.

The top row can be considered as an exact couple, as in the Appendix; in particular, there is an associated Bockstein operator: $\mathfrak{B}^*\colon TU^* \to TU^{*+2}$. By Theorem 4.2, TU^* identifies explicitly as a subfunctor of $H\mathbb{F}^*(BV^\sharp)$.

Proposition 4.8 [5] *There is a natural commutative diagram*

$$\begin{array}{ccc} TU^* & \xrightarrow{\mathfrak{B}^*} & TU^{*+2} \\ \downarrow & & \downarrow \\ H\mathbb{F}^*(BV^\sharp) & \xrightarrow{\mathrm{Sq}^2} & H\mathbb{F}^{*+2}(BV^\sharp). \end{array}$$

5 Cohomology of elementary abelian 2–groups

The results of this section are formulated in the category of bounded-below $\mathcal{A}(1)$–modules of finite type, which is abelian, closed under tensor products and has projective covers.

Notation 5.1 For M an $\mathcal{A}(1)$–module, let ΩM denote the first syzygy of M, namely the kernel of the surjection $P_M \twoheadrightarrow M$ from the projective cover of M. By convention, $\Omega^0 M = M$ and, for $n \in \mathbb{N}$, $\Omega^n M$ is defined by iteration.

Notation 5.2 (1) Let P denote the reduced $\mathbb{F}$–cohomology of $B\mathbb{Z}/2$, which identifies with the augmentation ideal $\overline{\mathbb{F}_2[u]}$ of $H\mathbb{F}^*(B\mathbb{Z}/2_+) \cong \mathbb{F}[u]$.

(2) Let R denote the $\mathcal{A}(1)$–module defined by the nonsplit extension $0 \to P \to R \to \Sigma^{-1}\mathbb{F} \to 0$ (cf [5, Section A.9]).

(3) Let P_0 denote the $\mathcal{A}(1)$–module defined by the nonsplit extension $0 \to \mathbb{F} \to P_0 \to R \to 0$.

Remark 5.3 (1) There is an isomorphism $P \cong \Sigma^{-1}\Omega P_0$ [3].

(2) The module P is Q_0–acyclic and R is Q_1–acyclic.

(3) The modules P_0 and ΣR are stably idempotent [3].

Proposition 5.4 [3] *For $n \in \mathbb{N}$, there is an isomorphism in the category of $\mathcal{A}(1)$–modules: $P^{\otimes n+1} \cong F_n \oplus \Sigma^{-n}\Omega^n P$, where F_n is a free $\mathcal{A}(1)$–module.*

Proof (Indications) The proof is by induction upon n, starting with the case $n = 0$. It is clear that $P^{\otimes n}$ is Q_0–acyclic; hence, by the criterion for $\mathcal{A}(1)$–freeness in terms of vanishing of Margolis homology (see Adams and Margolis [1]), $P^{\otimes n} \otimes R$ is $\mathcal{A}(1)$–free. The result follows by considering the short exact sequence

$$P^{\otimes n} \otimes (P \to R \to \Sigma^{-1}\mathbb{F}).$$ □

Corollary 5.5 *For an elementary abelian 2–group V of finite rank, there is a (non-functorial) isomorphism of $\mathcal{A}(1)$–modules*

$$H\mathbb{F}^*(BV^\sharp) \cong F_V \oplus \bigoplus_{i \geq 1} (\Lambda^i(V) \otimes \Sigma^{-i}\Omega^i P),$$

where $\Lambda^i(V)$ is concentrated in degree zero and F_V is a free $\mathcal{A}(1)$–module.

Proof This is a straightforward consequence of Proposition 5.4 and of the Künneth theorem applied to $BV^\sharp \simeq (B\mathbb{Z}/2)^{\times \operatorname{rank}(V)}$. □

Remark 5.6 The functoriality with respect to V can be analysed by introducing a filtration and considering the associated graded object. This is not required here, since Lemma 3.9 can be applied in the case of interest.

6 Functorial cohomology calculations

The abstract detection results of Section 2 are applied to prove Proposition 6.7, which gives a lower bound for the image of

$$\mathrm{KO}\langle n\rangle^*(BV^\sharp) \to \mathrm{KO}^*(BV^\sharp).$$

This relies upon calculating the cohomology of $\operatorname{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, H\mathbb{F}^*(BV^\sharp))$. Since projective $\mathcal{A}(1)$–modules are also injective (cf Margolis [8, Chapter 12, Section 2]), this reduces to the calculation of the cohomology of $\operatorname{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, \Sigma^{-n}\Omega^n P)$, for $n \in \mathbb{N}$, by Corollary 5.5. This can be reduced further to the calculation of the cohomology of $\operatorname{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, P)$, by the following result.

Proposition 6.1 *Let M be an $\mathcal{A}(1)$–module which is bounded below, of finite-type and Q_0–acyclic. Then there is a natural isomorphism*

$$H^n(\operatorname{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, \Sigma^{-1}\Omega M)) \cong H^{n-1}(\operatorname{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, M))$$

of graded vector spaces.

Proof Since M is Q_0–acyclic, applying the functor $\operatorname{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, -)$ to the short exact sequence $\Omega M \to P_M \to M$ yields an exact sequence of cochain complexes

$$0 \to \operatorname{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, \Omega M) \to \operatorname{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, P_M) \to \operatorname{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, M) \to 0,$$

as seen as follows. The only nonprojective terms of $\mathscr{E}_\bullet$ are suspensions of $\mathcal{A}(1)/\!/\mathcal{A}(0)$; since $\operatorname{Hom}_{\mathcal{A}(1)}(\mathcal{A}(1)/\!/\mathcal{A}(0), -)$ is naturally isomorphic to $\operatorname{Hom}_{\mathcal{A}(0)}(\mathbb{F}, -)$, the fact that $\Omega M \to P_M \to M$ restricted to $\mathcal{A}(0)$ splits (since M is Q_0–free, by hypothesis) implies the exactness.

The projective cover P_M is also injective as an $\mathcal{A}(1)$–module, thus the middle complex is acyclic and the associated long exact sequence in cohomology provides the stated isomorphism. The shift in degree corresponding to the Σ^{-1} arises from the degree of the morphisms in $\mathscr{E}_\bullet$. □

Lemma 6.2 *The cohomology of* $\mathrm{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, P)$ *has Poincaré series given by*

i	$H^{4k+i}\big(\mathrm{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, P)\big)$
0	$t^{-8k}\big(\frac{t^4}{1-t^4}\big)$
1	$t^{-8k}\big(\frac{1}{1-t^4}\big)$
2	$t^{-8k}\big(t^{-1} + \frac{1}{1-t^4}\big)$
3	$t^{-8k}\big(t^{-2} + \frac{1}{1-t^4}\big)$

for integers k, $0 \le i \le 3$. *In particular, in any given cohomological and internal bidegree, the cohomology is at most one-dimensional.*

Proof By periodicity (up to suspension) of $\mathscr{E}_\bullet$, it suffices to calculate the cohomology of the following cochain complex:

$$\begin{array}{ccccccccc}
\longrightarrow & \Sigma^4\overline{\mathbb{F}[u^2]} & & & & & & & \\
& \Big\downarrow{\scriptstyle \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2} & & & & & & & \\
& \overline{\mathbb{F}[u^2]} & \xrightarrow{\mathrm{Sq}^2} & \Sigma^{-1}\overline{\mathbb{F}[u]} & \xrightarrow{\mathrm{Sq}^2} & \Sigma^{-2}\overline{\mathbb{F}[u]} & \xrightarrow{\mathrm{Sq}^3} & \Sigma^{-4}\overline{\mathbb{F}[u^2]} & \\
& & & & & & & \Big\downarrow{\scriptstyle \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2} & \\
& & & & & & & \Sigma^{-8}\overline{\mathbb{F}[u^2]} & \longrightarrow
\end{array}$$

The behaviour of the Steenrod operations on u^n depends on the congruence class of n modulo 4; $\mathrm{Sq}^2(u^n)$ is nonzero if and only if $n \equiv 2, 3 \pmod 4$, $\mathrm{Sq}^3(u^n)$ is nonzero if and only if $n \equiv 3 \pmod 4$ and the operation $\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2$ is identically zero on $\mathbb{F}[u^2]$.

It follows that the cohomology of the middle row is given by the classes

cohomological degree	
0	$u^{4(k+1)}$
1	$\Sigma^{-1}u^{4k+1}$
2	$\Sigma^{-2}u,\ \Sigma^{-2}u^{4k+2}$
3	$\Sigma^{-4}u^2,\ \Sigma^{-4}u^{4(k+1)}$

where $k \in \mathbb{N}$. □

Remark 6.3 The identification given in Example 4.5 and the application of the detection arguments of Section 2 imply that the cohomological degree n above corresponds to classes from $\mathrm{KO}\{n\}$–cohomology; this notation is adopted below.

From this, the cohomology $V \mapsto H_*(\mathrm{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, H\mathbb{F}^*(BV^\sharp)))$ can be deduced. The calculation is summarised in Proposition 6.4 and illustrated in Table 1.

	KO{−3}	KO{−2}	KO{−1}	KO{0}	KO{1}	KO{2}	KO{3}	KO{4}
−4								$[p_1\bar{I}_{\mathbb{F}}]$
−3								
−2							$[\Lambda^1]$	$[\Lambda^2]$
−1						$[\Lambda^1]$	$[\Lambda^2]$	$[\Lambda^3]$
0					$[p_1\bar{I}_{\mathbb{F}}]$	$[p_2\bar{I}_{\mathbb{F}}]$	$[p_3\bar{I}_{\mathbb{F}}]$	$[p_4\bar{I}_{\mathbb{F}}]$
1								
2								
3								
4				$[p_1\bar{I}_{\mathbb{F}}]$	$[p_2\bar{I}_{\mathbb{F}}]$	$[p_3\bar{I}_{\mathbb{F}}]$	$[p_4\bar{I}_{\mathbb{F}}]$	$[p_5\bar{I}_{\mathbb{F}}]$
5								
6			$[\Lambda^1]$	$[\Lambda^2]$	$[\Lambda^3]$	$[\Lambda^4]$	$[\Lambda^5]$	$[\Lambda^6]$
7		$[\Lambda^1]$	$[\Lambda^2]$	$[\Lambda^3]$	$[\Lambda^4]$	$[\Lambda^5]$	$[\Lambda^6]$	$[\Lambda^7]$
8	$[p_1\bar{I}_{\mathbb{F}}]$	$[p_2\bar{I}_{\mathbb{F}}]$	$[p_3\bar{I}_{\mathbb{F}}]$	$[p_4\bar{I}_{\mathbb{F}}]$	$[p_5\bar{I}_{\mathbb{F}}]$	$[p_6\bar{I}_{\mathbb{F}}]$	$[p_7\bar{I}_{\mathbb{F}}]$	$[p_8\bar{I}_{\mathbb{F}}]$
etc	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

Table 1: The Grothendieck group interpretation of the cohomology of $\mathrm{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, H\mathbb{F}^*(BV^\sharp))$

Proposition 6.4 *The nonzero values in the Grothendieck group of the functor $V \mapsto H^*(\mathrm{Hom}_{\mathcal{A}(1)}(\mathscr{E}_\bullet, H\mathbb{F}^*(BV^\sharp)))$ are given in bidegree $\mathrm{KO}\{n\}^d$, for $l \in \mathbb{Z}$, by:*

d	$\mathrm{KO}\{n\}$
$8l$	$[p_{n+4l}\bar{I}_{\mathbb{F}}]$
$8l+4$	$[p_{n+4l+1}\bar{I}_{\mathbb{F}}]$
$8l+6$	$[\Lambda^{n+4l+1}]$
$8l+7$	$[\Lambda^{n+4l+2}]$

Proof The result follows from Lemma 6.2, Corollary 5.5 and Proposition 6.1. For instance, the occurrence of the composition factors Λ^1 is given by Lemma 6.2; the *décalage* provided by Proposition 6.1 then shows that each factor of Λ^1 gives rise to a factor of Λ^2 to the right in Table 1 and this pattern continues.

The proof that the result holds as a statement in the Grothendieck group is a straightforward application of Lemma 3.9. □

Definition 6.5 For $n \in \mathbb{Z}$, define graded functors

$$\begin{aligned} C\{n\}^*\colon V &\mapsto \mathrm{Coker}\{\mathrm{KO}\{n\}^*(BV^\sharp) \to \mathrm{KO}^*(BV^\sharp)\}, \\ QO\{n\}^*\colon V &\mapsto \mathrm{Im}\{\mathrm{KO}\{n\}^*(BV^\sharp) \to \mathrm{KO}^*(BV^\sharp)\}. \end{aligned}$$

In the notation of Proposition 2.9, $\Phi_n[BV^\sharp, \mathrm{KO}]^* = QO\{n\}^*$; also $QO\{0\}^* = QO^*$ of Notation 4.7.

Lemma 6.6 *For $n \in \mathbb{Z}$, there is a natural short exact sequence*

$$0 \to QO\{n-1\}^*/QO\{n\}^* \to C\{n\}^* \to C\{n-1\}^* \to 0$$

and, in a fixed degree d, $C\{n\}^d$ admits a finite filtration with associated graded object

$$\bigoplus_{j<n} QO\{j\}^d/QO\{j+1\}^d.$$

Proof By definition, there is a short exact sequence of graded functors

$$0 \to QO\{n\}^* \to \mathrm{KO}^*(BV^\sharp) \to C\{n\}^* \to 0.$$

The inclusion $QO\{n\}^* \hookrightarrow QO^*\{n-1\}$ induces the stated short exact sequence. The second statement follows recursively, using the observation that, in a fixed degree d, $C\{n\}^d = 0$ for $n \ll 0$. □

Proposition 6.7 *For $n, d \in \mathbb{Z}$, $C\{n\}^d$ is a finite functor. Moreover, there are inequalities in the Grothendieck group*

$$[C\{n\}^d] \le \begin{cases} \sum_{s=1}^{4l+n-1} [p_s \bar{I}_{\mathbb{F}}] = \big[\bar{P}_{\mathbb{Z}_2}/\bar{P}_{\mathbb{Z}_2}^{4l+n}\big] & d = 8l, \\ \sum_{s=1}^{4l+n} [p_s \bar{I}_{\mathbb{F}}] = \big[\bar{P}_{\mathbb{Z}_2}/\bar{P}_{\mathbb{Z}_2}^{4l+n+1}\big] & d = 8l+4, \\ \sum_{s=1}^{4l+n+1} [\Lambda^s] = \big[\bar{P}_{\mathbb{F}}/\bar{P}_{\mathbb{F}}^{4l+n+2}\big] & d = 8l+6, \\ \sum_{s=1}^{4l+n+2} [\Lambda^s] = \big[\bar{P}_{\mathbb{F}}/\bar{P}_{\mathbb{F}}^{4l+n+3}\big] & d = 8l+7, \end{cases}$$

and, in the remaining cases, $C\{n\}^d = 0$.

In a fixed degree d, *equality holds if and only if, for all* $j < n$,

$$QO\{j\}^d/QO\{j+1\}^d \cong \operatorname{Ker}(\theta_j)^d/\operatorname{Im}(\Sigma^{-1}\theta_{j-1})^d.$$

Proof The stated equalities in the Grothendieck group follow from Proposition 3.7 and Example 3.5.

Lemma 6.6 gives $[C\{n\}^d] = \sum_{j<n}[QO\{j\}^d/QO\{j+1\}^d]$, hence, to prove the inequality, it suffices to give an upper bound for $[QO\{j\}^d/QO\{j+1\}^d]$; this is provided by Lemma 3.6.

Proposition 2.9 implies that $QO\{j\}^d/QO\{j+1\}^d$ is a subquotient of

$$\operatorname{Ker}(\theta_j)^d/\operatorname{Im}(\Sigma^{-1}\theta_{j-1})^d$$

and the value of the latter in the Grothendieck group is given by Proposition 6.4; this proves the inequalities.

Finally, since the functors involved are finite, equality holds in degree d if and only $QO\{j\}^d/QO\{j+1\}^d \cong \operatorname{Ker}(\theta_j)^d/\operatorname{Im}(\Sigma^{-1}\theta_{j-1})^d$ for all $j < n$. □

7 An Sq^2–homology calculation

Recall that TU^* identifies as the image of the iterated Milnor operation

$$Q_0Q_1\colon H\mathbb{F}^{*-4}(BV^\sharp) \to H\mathbb{F}^*(BV^\sharp).$$

Proposition 4.8 implies that the operation Sq^2 induces a complex

$$\cdots \to TU^{*-2} \xrightarrow{\mathrm{Sq}^2} TU^* \xrightarrow{\mathrm{Sq}^2} TU^{*+2} \to \cdots .$$

The work of Bruner and Greenlees [5] on the ko–(co)homology of elementary abelian 2–groups shows the importance of the calculation of the homology of this complex. In [5, Proposition 9.7.2], they calculate the homology and their result can be interpreted as a functorial calculation.

The purpose of this section is to show that the methods employed in Section 6 provide an alternative, direct proof. However, it is no longer possible to reduce to a calculation involving only P, since there is no analogue of Proposition 6.1 in this case. Thus further precision is required on the structure of the $\mathcal{A}(1)$–modules $\Sigma^{-n}\Omega^n P$.

Notation 7.1

(1) For $n \in \mathbb{N}$, let P_n denote the $\mathcal{A}(1)$–module $\Sigma^{-n}\Omega^n P_0$, so that $P_1 = P$.

(2) Let J denote the $\mathcal{A}(1)$–module $\Sigma^{-2}\mathcal{A}(1)/(\mathrm{Sq}^1\mathrm{Sq}^2)$.

The module J is an element of the stable Picard group of $\mathcal{A}(1)$–modules of order 2, namely $J \otimes J \cong \mathbb{F} \oplus F$, where F is free (cf [2]).

Theorem 7.2 [3; 15] *For $n \in \mathbb{N}$, there is an isomorphism of $\mathcal{A}(1)$–modules $P_{n+4} \cong \Sigma^8 P_n$.*

Remark 7.3 The periodicity can be seen by establishing that there is an isomorphism of $\mathcal{A}(1)$–modules $P_0 \otimes J \cong \Sigma^{-4} P_2 \oplus F'$, where F' is free. It follows also that $P \otimes J \cong \Sigma^{-4} P_3 \oplus F''$ for a free $\mathcal{A}(1)$–module F''.

Proposition 7.4 *For $n \in \mathbb{Z}$,*

$$\mathrm{Ker}\{\mathrm{Sq}^2\colon TU^n \to TU^{n+2}\}/\,\mathrm{Im}\{\mathrm{Sq}^2\colon TU^{n-2} \to TU^n\} \cong \begin{cases} \Lambda^{4k+2} & n = 8k+6, \\ \Lambda^{4k+3} & n = 8k+7, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof Consider the isomorphism $H\mathbb{F}^*(BV^\sharp) \cong F_V \oplus \bigoplus_{i\ge 1} \Lambda^i(V) \otimes P_i$ in $\mathcal{A}(1)$–modules, where F_V is a free $\mathcal{A}(1)$–module (bounded below, of finite type); the Sq^2–complex splits as a corresponding direct sum. Using the periodicity isomorphism for the P_i's given by Theorem 7.2, this reduces the calculation of the Sq^2–homology evaluated upon $V^\sharp$ to the calculation of the respective homologies for the $\mathcal{A}(1)$–modules: $\mathcal{A}(1)$, P_0, P_1, P_2, P_3.

(1) The image of Q_0Q_1 applied to $\mathcal{A}(1)$ has two classes, which are linked by the operation Sq^2, hence the free summand contributes nothing to the homology.

(2) By inspection, the operation Q_0Q_1 acts trivially upon P_0 and P_1, hence these contribute nothing to the Sq^2–homology.

(3) The structure of P_2, P_3 is described explicitly in [3; 15]. The relevant part of the structure can be understood using Remark 7.3; the nontrivial morphism $\mathbb{F} \hookrightarrow P_0$ induces an embedding $\Sigma^4 J \hookrightarrow P_2$ and the surjection $P \to \Sigma\mathbb{F}$ induces a surjection $P_3 \twoheadrightarrow \Sigma^5 J$. Upon restricting to the subalgebra $E(1) := \Lambda(Q_0, Q_1) \subset \mathcal{A}(1)$, this gives isomorphisms

$$P_2|_{E(1)} \cong \Sigma^2 E(1) \oplus L_2,$$

$$P_3|_{E(1)} \cong \Sigma^3 E(1) \oplus L_3,$$

where L_2, L_3 are indecomposable $E(1)$–modules on which Q_0Q_1 acts trivially.

Thus, for both P_2 and P_3, the image of $Q_0 Q_1$ is a single class and Sq^2 acts trivially on the associated complex. Explicitly, for P_2, the Sq^2–homology is 1–dimensional, concentrated in degree 6 and, for P_3, is 1–dimensional, concentrated in degree 7.

Lemma 3.9 implies that these classes correspond to the simple functors Λ^2 and Λ^3 in degrees 6 and 7 respectively. The general result follows, by using the periodicity given by Theorem 7.2. □

8 Detection for ko

This section determines the functorial structure of $\mathrm{ko}^*(BV^\sharp)$ as a first step towards the determination of $\mathrm{KO}\langle n\rangle^*(BV^\sharp)$; the arguments use the abstract detection result, Proposition 2.9, which depends upon understanding the image of $\mathrm{KO}\langle n\rangle^*(BV^\sharp) \to \mathrm{KO}^*(BV^\sharp)$.

In degrees which are multiples of four, a direct approach treating all the cases simultaneously is possible, using the fact that $\mathrm{KO}^*(BV^\sharp) \to \mathrm{KU}^*(BV^\sharp)$ is injective in these degrees, so that the known structure of $\mathrm{ku}^*(BV^\sharp)$ can be used to provide an upper bound for the image of $\mathrm{KO}\langle n\rangle^*(BV^\sharp)$, which can be played off against the lower bound provided by Proposition 6.7. In the remaining degrees in which $\mathrm{KO}^*(BV^\sharp)$ is nontrivial (those congruent to $6, 7 \pmod 8$), the map to $\mathrm{KU}^*(BV^\sharp)$ is zero, hence this strategy cannot be applied. Instead, a Bockstein argument derived from the complexification-realification cofibre sequence of Section 4.3 is used.

The KO–cohomology $\mathrm{KO}^*(BV^\sharp)$ can be deduced from the case of KU, which is concentrated in even degrees, where $\mathrm{KU}^{2d}(BV^\sharp) \cong \overline{P}_{\mathbb{Z}_2}(V)$ (see [11] and compare Theorem 4.2), by using the long exact sequence associated to

$$\Sigma\mathrm{KO} \xrightarrow{\eta} \mathrm{KO} \xrightarrow{c} \mathrm{KU} \xrightarrow{R} \Sigma^2\mathrm{KO}.$$

Proposition 8.1 (cf [5]) *There are isomorphisms*

$$\mathrm{KO}^{8k+l}(BV^\sharp) \cong \begin{cases} \overline{P}_{\mathbb{F}} & l = 7, \\ \overline{P}_{\mathbb{F}} & l = 6, \\ \overline{P}_{\mathbb{Z}_2} & l = 4, \\ \overline{P}_{\mathbb{Z}_2} & l = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(1) *Complexification* $c\colon \mathrm{KO}^{8k+l}(BV^\sharp) \to \mathrm{KU}^{8k+l}(BV^\sharp)$ *is zero unless we have* $l \equiv 0 \pmod 4$*; for* $l = 0$ *it is an isomorphism and, for* $l = 4$*,* $\overline{P}_{\mathbb{Z}_2} \overset{2}{\hookrightarrow} \overline{P}_{\mathbb{Z}_2}$.

(2) *Realification* $R\colon \mathrm{KU}^{8k+l-2}(BV^\sharp)\to \mathrm{KO}^{8k+l}(BV^\sharp)$ *is zero unless* $l\in\{0,4,6\}$*; for* $l=0$ *it is* $\overline{P}_{\mathbb{Z}_2}\overset{2}{\hookrightarrow}\overline{P}_{\mathbb{Z}_2}$*, for* $l=4$ *is an isomorphism and for* $l=6$*, it is the surjection* $\overline{P}_{\mathbb{Z}_2}\twoheadrightarrow \overline{P}_{\mathbb{F}}$*.*

(3) *Multiplication by* η*,* $\mathrm{KO}^*(BV^\sharp)\overset{\eta}{\to}\mathrm{KO}^{*-1}(BV^\sharp)$*, is zero except for*

$$\mathrm{KO}^{8(k+1)}(BV^\sharp)\cong \overline{P}_{\mathbb{Z}_2}\overset{\eta}{\twoheadrightarrow}\mathrm{KO}^{8k+7}(BV^\sharp)\cong \overline{P}_{\mathbb{F}}\underset{\cong}{\overset{\eta}{\longrightarrow}}\mathrm{KO}^{8k+6}(BV^\sharp)\cong \overline{P}_{\mathbb{F}}.$$

The key to the calculation of $\mathrm{ko}^*(BV^\sharp)$ is the short exact sequence of complexes (7) of Section 4.3. Recall the functors Q^*, QO^* of Notation 4.1 and 4.7.

Proposition 8.2 *For* $k\in\mathbb{Z}$*, there are isomorphisms*

$$QO^{8k+l}\cong\begin{cases}\overline{P}^{4k+3}_{\mathbb{F}} & l=7,\\ \overline{P}^{4k+2}_{\mathbb{F}} & l=6,\\ \overline{P}^{4k+1}_{\mathbb{Z}_2} & l=4,\\ \overline{P}^{4k}_{\mathbb{Z}_2} & l=0,\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, the complexification $QO^{8k+l}\overset{c}{\to}Q^{8k+l}$ *is zero unless* $l\equiv 0 \pmod 4$*, the map* $QO^{8k}\overset{c}{\to}Q^{8k}$ *is an isomorphism and the map* $QO^{8k+4}\overset{c}{\to}Q^{8k+4}$ *is the inclusion*

$$\overline{P}^{4k+1}_{\mathbb{Z}_2}\hookrightarrow \overline{P}^{4k+2}_{\mathbb{Z}_2}.$$

The complex $\cdots\to QO^{*+1}\overset{\eta}{\to}QO^*\overset{c}{\to}Q^*\overset{R}{\to}\cdots$ *is exact, except for the segments*

$$\begin{array}{ccccccccccc}
QO^{8k+7} & \overset{\eta}{\longrightarrow} & QO^{8k+6} & \overset{c}{\longrightarrow} & Q^{8k+6} & \overset{R}{\longrightarrow} & QO^{8(k+1)} & \overset{\eta}{\longrightarrow} & QO^{8k+7} & \overset{c}{\longrightarrow} & Q^{8k+7}\\
\| & & \| & & \| & & \| & & \| & & \|\\
\overline{P}^{4k+3}_{\mathbb{F}} & \hookrightarrow & \overline{P}^{4k+2}_{\mathbb{F}} & \underset{0}{\longrightarrow} & \overline{P}^{4k+3}_{\mathbb{Z}_2} & \hookrightarrow & \overline{P}^{4k+4}_{\mathbb{Z}_2} & \longrightarrow & \overline{P}^{4k+3}_{\mathbb{F}} & \underset{0}{\longrightarrow} & 0\\
 & & \vdots\downarrow & & & & & & \vdots\downarrow & & \\
0 & & \Lambda^{4k+2} & & 0 & & 0 & & \Lambda^{4k+3} & & 0,
\end{array}$$

where the homology is given by the bottom line, with the corresponding surjections indicated by the dotted arrows.

Proof The morphism $\mathrm{KO}^*(BV^\sharp)\overset{c}{\to}\mathrm{KU}^*(BV^\sharp)$ induces an inclusion

$$QO^{8k+4\varepsilon}\hookrightarrow QU^{8k+4\varepsilon}\cong \overline{P}^{4k+2\varepsilon}_{\mathbb{Z}_2}$$

for $\varepsilon \in \{0, 1\}$. This gives

$$QO^{8k} \subseteq \overline{P}^{4k}_{\mathbb{Z}_2},$$
$$QO^{8k+4} \subseteq \overline{P}^{4k+1}_{\mathbb{Z}_2} \cong (2\overline{P}_{\mathbb{Z}_2}) \cap \overline{P}^{4k+2}_{\mathbb{Z}_2},$$

as upper bounds and the inclusions $QO^{8k+4\varepsilon} \hookrightarrow \mathrm{KO}^{8k+4\varepsilon}(BV^{\sharp})$ correspond respectively to

$$QO^{8k} \hookrightarrow \overline{P}^{4k}_{\mathbb{Z}_2} \hookrightarrow \overline{P}_{\mathbb{Z}_2},$$
$$QO^{8k+4} \hookrightarrow \overline{P}^{4k+1}_{\mathbb{Z}_2} \hookrightarrow \overline{P}_{\mathbb{Z}_2}.$$

A comparison between the cokernels of $\overline{P}^{4k}_{\mathbb{Z}_2} \hookrightarrow \overline{P}_{\mathbb{Z}_2}$ (respectively $\overline{P}^{4k+1}_{\mathbb{Z}_2} \hookrightarrow \overline{P}_{\mathbb{Z}_2}$) and the bounds provided by Proposition 6.7 shows that the inequalities are isomorphisms, by Proposition 3.7.

In the remaining nontrivial cases, in degrees congruent to $6, 7 \pmod 8$, an upper bound is obtained by appealing to the general method of the Appendix, as follows.

Multiplication by η gives the commutative diagram

$$\begin{array}{ccccc} QO^{8(k+1)} & & \xrightarrow{\quad\eta\quad} & & QO^{8k+7} \\ \cong\big\downarrow & & & & \big\downarrow \\ \overline{P}^{4(k+1)}_{\mathbb{Z}_2} & \hookrightarrow & \overline{P}_{\mathbb{Z}_2} & \twoheadrightarrow & \overline{P}_{\mathbb{F}}, \end{array}$$

which identifies the image of $QO^{8(k+1)}$ in QO^{8k+7} as $\overline{P}^{4(k+1)}_{\mathbb{F}}$.

The complexes

$$QO^{8(k+1)} \xrightarrow{\eta} QO^{8k+7} \to Q^{8k+7} = 0,$$
$$QO^{8k+7} \xrightarrow{\eta} QO^{8k+6} \xrightarrow{0} Q^{8k+6},$$

(where the last morphism is zero, since Q^{8k+6} takes values in torsion-free abelian groups and QO^{8k+6} is torsion) have homology appearing as a subquotient of the simple functors Λ^{4k+3} and Λ^{4k+2} respectively, by Lemma A.1, using Proposition 7.4 and the shift in homological degrees associated to the short exact sequence of complexes (7) of Section 4.3. This provides the upper bounds

$$QO^{8k+7} \subseteq \overline{P}^{4k+3}_{\mathbb{F}},$$
$$QO^{8k+6} \subseteq \overline{P}^{4k+2}_{\mathbb{F}}, \tag{8}$$

where both are equalities if $QO^{8k+6} = \overline{P}^{4k+2}_{\mathbb{F}}$.

Realification

$$\begin{array}{ccc} \overline{P}^{4k+2}_{\mathbb{Z}_2} \cong Q^{8k+4} & \xrightarrow{R} & QO^{8k+6} \\ \big\downarrow & & \big\downarrow \\ \overline{P}_{\mathbb{Z}_2} \cong \mathrm{KU}^{8k+4}(BV^\sharp) & \twoheadrightarrow & \mathrm{KO}^{8k+6}(BV^\sharp) \cong \overline{P}_{\mathbb{F}} \end{array}$$

gives a lower bound of $\overline{P}^{4k+2}_{\mathbb{F}}$ for QO^{8k+6}, whence it follows that both the inequalities in (8) are equalities.

Finally, using the structure of the functors $\overline{P}^t_{\mathbb{F}}$ and $\overline{P}^t_{\mathbb{Z}_2}$ (as reviewed in Section 3), it is straightforward to calculate the homology of the complex

$$\cdots \to QO^{*+1} \xrightarrow{\eta} QO^* \xrightarrow{c} Q^* \xrightarrow{R} \cdots . \qquad \square$$

Corollary 8.3 *Detection holds for* ko*–cohomology of elementary abelian* 2*–groups: the morphisms* $\mathrm{ko} \to \mathrm{KO}$ *and* $\mathrm{ko} \to H\mathbb{Z}$ *induce a natural monomorphism*

$$\mathrm{ko}^*(BV^\sharp) \hookrightarrow H\mathbb{Z}^*(BV^\sharp) \oplus \mathrm{KO}^*(BV^\sharp).$$

The functor ST^* *is the image of* $\mathrm{Sq}^2\colon TU^{*-2} \to TU^*$ *and* $\eta\colon ST^{*+1} \to ST^*$ *is trivial.*

Proof By applying the long exact sequence in homology associated to the short exact sequence of complexes (7) of Section 4.3, Proposition 8.2 implies that the exact couple $\cdots \to ST^{*+1} \to ST^* \to TU^* \to \cdots$ has homology concentrated at the TU^* term, where it coincides with the Bockstein homology.

Therefore Proposition A.2 applies; it follows that $ST^* \to TU^*$ is a monomorphism and that ST^* is the image of the operator $TU^{*-2} \to TU^*$, which is induced by Sq^2, by Proposition 4.8.

To show detection for ko, it suffices to show that ST^* maps monomorphically to $H\mathbb{Z}^*(BV^\sharp)$. By the above, $\mathrm{ko}^*(BV^\sharp) \to \mathrm{ku}^*(BV^\sharp)$ induces an injection $ST^* \hookrightarrow TU^*$, and the composite $TU^* \to \mathrm{ku}^*(BV^\sharp) \to H\mathbb{Z}^*(BV^\sharp)$ is a monomorphism, by detection for ku (Theorem 4.2), hence the result follows. $\square$

9 Detection for $\mathrm{KO}\langle n\rangle$

Throughout this section, the reindexing of the spectra $\mathrm{KO}\langle n\rangle$ introduced in Notation 4.6 is used; for example, as in Definition 6.5, $QO\{n\}$ is the image of $\mathrm{KO}\{n\}(BV^\sharp)$ in $\mathrm{KO}(BV^\sharp)$. Similarly, θ_n denotes the stable cohomology operation derived from the Postnikov tower of KO, as in Section 2.

Theorem 9.1 *For each $n \in \mathbb{Z}$, detection of level n with respect to the family of spectra $\{\Sigma^\infty B(\mathbb{Z}/2)^{\oplus d} \mid 1 \le d \in \mathbb{Z}\}$ holds for the Postnikov tower $\mathrm{KO}\{n\}$.*

Proof The result follows from the general result on detection, Proposition 2.9. Using the notation of the proposition, the functorial homology $\mathrm{Ker}(\theta_n)/\,\mathrm{Im}(\Sigma^{-1}\theta_{n-1})$ is a finite functor in each degree, hence to prove weak detection at each level, it is sufficient to show that the filtration quotient $\Phi_n[BV^\sharp, \mathrm{KO}]^*/\Phi_{n-1}[BV^\sharp, \mathrm{KO}]^*$ is abstractly isomorphic to $\mathrm{Ker}(\theta_n)/\,\mathrm{Im}(\Sigma^{-1}\theta_{n-1})$, for each n. Here, by definition $\Phi_n[BV^\sharp, \mathrm{KO}]^*$ is the graded functor $QO\{n\}$.

Proposition 8.2 establishes that the inequalities of Proposition 6.7 are equalities for $n = 4m$, $m \in \mathbb{Z}$. To conclude, one argues as in Proposition 6.7: Lemma 6.6 provides the equality

$$(9)\qquad [\ker\{C\{4(m+1)\}^* \twoheadrightarrow C\{4m\}^*\}] = \sum_{j=4m}^{4m+3} [QO\{j\}^*/QO\{j+1\}^*] \le \sum_{j=4m}^{4m+3} [\mathrm{Ker}(\theta_j)^*/\,\mathrm{Im}(\Sigma^{-1}\theta_{j-1})^*]$$

and Proposition 2.9 provides the inequality. As explained above, the left-hand side is determined by Proposition 6.7 and coincides with the lower term (9), by Proposition 6.4; the functors occurring are finite in each degree, so the inequality is in fact an equality. Hence, the final statement of Proposition 6.7 provides the required isomorphism, thus proving weak detection.

Finally, Lemma 2.5 establishes detection at each level, since detection has been proved for ko by Corollary 8.3, hence holds by Bott periodicity for all of the theories $\mathrm{KO}\{4s\}$, $s \in \mathbb{Z}$. □

From this one derives the explicit description of the functors $\mathrm{KO}\{n\}^*(BV^\sharp)$, in particular recovering the results of [5] for ko.

Corollary 9.2 *For $n \in \mathbb{Z}$, there is a natural short exact sequence*

$$0 \to \mathrm{Im}(\Sigma^{-1}\theta_{n-1}) \to \mathrm{KO}\{n\}^*(BV^\sharp) \to QO\{n\} \to 0,$$

which is determined as a pullback of the short exact sequence associated to the quotient $\mathrm{Ker}(\theta_n)/\,\mathrm{Im}(\Sigma^{-1}\theta_{n-1})$, by Theorem 2.10.

The nonzero functors $QO\{i\}^{8k+l}$, for $0 \le i \le 3$ and $0 \le l \le 7$, are given by

i	$8k$	$8k+4$	$8k+6$	$8k+7$
0	$\overline{P}^{4k}_{\mathbb{Z}_2}$	$\overline{P}^{4k+1}_{\mathbb{Z}_2}$	$\overline{P}^{4k+2}_{\mathbb{F}}$	$\overline{P}^{4k+3}_{\mathbb{F}}$
1	$\overline{P}^{4k+1}_{\mathbb{Z}_2}$	$\overline{P}^{4k+2}_{\mathbb{Z}_2}$	$\overline{P}^{4k+3}_{\mathbb{F}}$	$\overline{P}^{4(k+1)}_{\mathbb{F}}$
2	$\overline{P}^{4k+2}_{\mathbb{Z}_2}$	$\overline{P}^{4k+3}_{\mathbb{Z}_2}$	$\overline{P}^{4(k+1)}_{\mathbb{F}}$	$\overline{P}^{4(k+1)+1}_{\mathbb{F}}$
3	$\overline{P}^{4k+3}_{\mathbb{Z}_2}$	$\overline{P}^{4(k+1)}_{\mathbb{Z}_2}$	$\overline{P}^{4(k+1)+1}_{\mathbb{F}}$	$\overline{P}^{4(k+1)+2}_{\mathbb{F}}$

which determines the functors $QO\{n\}$, for all $n \in \mathbb{Z}$, by Bott periodicity.

The subfunctors $\mathrm{Im}(\Sigma^{-1}\theta_n)$ *are given for* $0 \le n \le 3$ *by*

n	$\mathrm{Im}(\Sigma^{-1}\theta_n)$
0	$\mathrm{Im}\{H\mathbb{Z}^{*-5}(BV^\sharp) \xrightarrow{\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2} H\mathbb{Z}^*(BV^\sharp)\} \cong \mathrm{Im}\{H\mathbb{F}^{*-6}(BV^\sharp) \xrightarrow{\mathrm{Sq}^2\mathrm{Sq}^2\mathrm{Sq}^2} H\mathbb{F}^*(BV^\sharp)\}$
1	$\mathrm{Im}\{H\mathbb{Z}^{*-1}(BV^\sharp) \xrightarrow{\mathrm{Sq}^2} H\mathbb{Z}^{*+1}(BV^\sharp)\} \cong \mathrm{Im}\{H\mathbb{F}^{*-2}(BV^\sharp) \xrightarrow{\mathrm{Sq}^2\mathrm{Sq}^1} H\mathbb{F}^{*+1}(BV^\sharp)\}$
2	$\mathrm{Im}\{H\mathbb{F}^{*}(BV^\sharp) \xrightarrow{\mathrm{Sq}^2} H\mathbb{F}^{*+2}(BV^\sharp)\}$
3	$\mathrm{Im}\{H\mathbb{F}^{*+1}(BV^\sharp) \xrightarrow{\mathrm{Sq}^3} H\mathbb{F}^{*+4}(BV^\sharp)\}$

which extends to all integers n by Bott periodicity.

Proof The short exact sequence is provided by Proposition 2.9 and Theorem 2.10, as a consequence of detection established in Theorem 9.1.

The identification of the functors $QO\{i\}$ is a straightforward consequence of the equalities derived from Proposition 6.7 in the proof of Theorem 9.1 above, using the structure of the functors $\overline{P}^t_{\mathbb{Z}_2}$ reviewed in Section 3. □

Appendix: General Bockstein results

Fix an exact couple in an abelian category, considered as a complex of the form

$$\cdots \to D^{n+1} \xrightarrow{i^{n+1}} D^n \xrightarrow{q^n} E^n \xrightarrow{\partial^n} D^{n+2} \to \cdots .$$

The associated Bockstein-type operator (the differential associated to the exact couple) is $\mathfrak{B}^n\colon E^n \to E^{n+2}$, defined by $\mathfrak{B}^n := q^{n+2} \circ \partial^n$.

The following is clear:

Lemma A.1 *For $n \in \mathbb{Z}$,*

$$\mathrm{Im}(\mathfrak{B}^{n-2}) \subseteq \mathrm{Im}(q^n) \subseteq \mathrm{Ker}(\partial^n) \subseteq \mathrm{Ker}(\mathfrak{B}^n),$$

hence $H^n := \mathrm{Ker}(\partial^n)/\,\mathrm{Im}(q^n)$ *is a subquotient of* $H^n_{\mathfrak{B}} := \mathrm{Ker}(\mathfrak{B}^n)/\,\mathrm{Im}(\mathfrak{B}^{n-2})$.

Moreover if $H^n_{\mathfrak{B}}$ *has a finite composition series, then* $H^n \cong H^n_{\mathfrak{B}}$ *if and only if* $\mathrm{Im}(\mathfrak{B}^{n-2}) = \mathrm{Im}(q^n)$ *and* $\mathrm{Ker}(\mathfrak{B}^n) = \mathrm{Ker}(\partial^n)$.

This is applied in the following basic result.

Proposition A.2 *Suppose that the exact couple* $D^{*+1} \xrightarrow{i} D^* \xrightarrow{q} E^* \xrightarrow{\partial} D^{*+2}$ *satisfies the following hypotheses:*

(1) $D^n = 0$ *for* $n \ll 0$.

(2) *The complex is exact except at the terms* E^n, *where the homology* H^n *coincides with* $H^n_{\mathfrak{B}}$.

(3) $H^n_{\mathfrak{B}}$ *has a finite composition series, for all* $n \in \mathbb{Z}$.

Then $i^n = 0$ *for all* $n \in \mathbb{Z}$ *and the complex decomposes as complexes of the form*

$$D^n \hookrightarrow E^n \twoheadrightarrow D^{n+2}.$$

In particular, D^n *identifies with the image of the operator* $\mathfrak{B}^{n-2}$.

Proof The result follows by an increasing induction upon n, using the hypothesis $D^n = 0$ for $n \ll 0$ for the initial step.

Suppose that $i^n\colon D^n \to D^{n-1}$ is zero. Exactness of $E^{n-2} \xrightarrow{\partial^{n-2}} D^n \xrightarrow{i^n} D^{n-1}$ implies that ∂^{n-2} is an epimorphism; the hypothesis $H^n_{\mathfrak{B}} = H^n$ gives $\mathrm{Ker}(\mathfrak{B}^{n-2}) = \mathrm{Ker}(\partial^{n-2})$, by Lemma A.1.

Using this fact, inspection of

$$\begin{array}{ccccc} & & E^{n-2} & & \\ & & {\scriptstyle\partial^{n-2}}\downarrow & \searrow{\scriptstyle\mathfrak{B}^{n-2}} & \\ D^{n+1} & \xrightarrow[i^{n+1}]{} & D^n & \xrightarrow[q^n]{} & E^n \end{array}$$

shows that q^n is a monomorphism and exactness of $D^{n+1} \xrightarrow{i^{n+1}} D^n \xrightarrow{q^n} T^n$ implies that $i^{n+1}\colon D^{n+1} \to D^n$ is zero. Finally, the above identifies the image of D^n in E^n with the image of $\mathfrak{B}^{n-2}$. This completes the inductive step. □

Remark A.3 The proof only requires that $\mathrm{Ker}(\mathfrak{B}^{n-2}) = \mathrm{Ker}(\partial^{n-2})$; for the application, the equivalent homological formulation is convenient.

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Laboratoire Angevin de Recherche en Mathématiques, UMR 6093, Faculté des Sciences
Université d'Angers, 2 Boulevard Lavoisier, 49045 Angers, France

Geoffrey.Powell@math.cnrs.fr

http://math.univ-angers.fr/~powell/

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