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The growth function of Coxeter dominoes and 2–Salem numbers

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By the results of Cannon, Wagreich and Parry, it is known that the growth rate of a cocompact Coxeter group in $\mathbb{H}^2$ and $\mathbb{H}^3$ is a Salem number. Kerada defined a j–Salem number, which is a generalization of Salem numbers. In this paper, we realize infinitely many 2–Salem numbers as the growth rates of cocompact Coxeter groups in $\mathbb{H}^4$. Our Coxeter polytopes are constructed by successive gluing of Coxeter polytopes, which we call Coxeter dominoes.

20F55; 20F65, 11K16

1 Introduction

Let $\mathbb{H}^n$ denote hyperbolic n–space. A Coxeter polytope $P \subset \mathbb{H}^n$ is a convex polytope of dimension n whose dihedral angles are all of the form π/m, where $m \geq 2$ is an integer. By a well-known result, the group W generated by reflections with respect to the hyperplanes bounding P is a discrete subgroup of the isometry group of $\mathbb{H}^n$ whose fundamental domain is P, and W itself is called a hyperbolic Coxeter group. If P is compact, W is called cocompact.

As typical quantities related to Coxeter groups, the growth series and their growth rates have been studied. The growth series is a formal power series (see (3-1)), and Steinberg [22] proved that the growth series of an infinite Coxeter group is an expansion of a rational function (see (3-4)). As a result, the growth rate (see (3-2)), which is defined as the reciprocal of the radius of convergence of the growth series, is an algebraic integer.

The focus of this work is on arithmetic properties of the growth rates of cocompact hyperbolic Coxeter groups. Cannon and Wagreich [2] and Parry [15] proved that the growth rates of cocompact hyperbolic Coxeter groups in $\mathbb{H}^2$ and $\mathbb{H}^3$ are Salem numbers. Here a Salem number is defined as a real algebraic integer $\alpha > 1$ such that α^{-1} is an algebraic conjugate of α and all algebraic conjugates of α other than α and α^{-1} lie on the unit circle. This definition includes quadratic units as Salem numbers, while the ordinary definition does not (see Section 3.2). We remark that the growth

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rates of cocompact hyperbolic Coxeter groups in $\mathbb{H}^4$ are not Salem numbers in general (see [7]).

As a kind of generalization of a Salem number, Kerada [9] defined a j–Salem number. In [25], T Zehrt and C Zehrt constructed infinitely many growth functions of cocompact hyperbolic Coxeter groups in $\mathbb{H}^4$, whose denominator polynomials have the same distribution of roots as 2–Salem polynomials. More precisely, their Coxeter polytopes are the Coxeter garlands in $\mathbb{H}^4$ built by the compact truncated Coxeter simplex described by the Coxeter graph on the left side in Figure 1 (cf [25, Figure 1]). This polytope is constructed from the extended Coxeter simplex (see Section 4.1) with two ultraideal vertices described by the Coxeter graph on the right side in Figure 1.

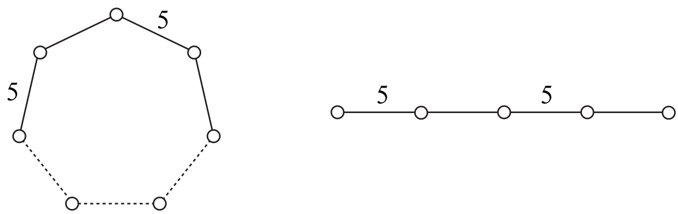

Figure 1: The Coxeter graphs of the compact truncated Coxeter 4–simplex and its underlying extended 4–simplex

Inspired by the work mentioned above, in the main result of this paper, we realize infinitely many 2–Salem numbers as the growth rates of hyperbolic Coxeter groups in $\mathbb{H}^4$ as follows.

We focus on the compact Coxeter polytope $T \subset \mathbb{H}^4$, whose Coxeter graph is on the left side in Figure 2, and which was first described by Schlettwein [19]. The nodes of the graph describe the facets of the Coxeter polytope, and the non-dotted edges describe the dihedral angles formed by two intersecting facets (see Section 2).

This polytope is constructed from the extended Coxeter simplex of dimension 4 with all vertices outside of $\mathbb{H}^4$, whose Coxeter graph is on the right side in Figure 2, as follows (see Section 4.1 and Section 4.2). We truncate all vertices of this simplex and replace them by facets orthogonal to all facets intersecting them, which we call orthogonal facets (see Section 4). This construction yields the compact Coxeter polytope T. This kind of construction is explained by Vinberg [23, Proposition 4.4], for example. Since we can glue many copies of this along isometric orthogonal facets, T is the building block for infinitely many compact Coxeter polytopes, which we call *Coxeter dominoes* (see Figure 3). Note that each orthogonal facet of T is one of the three types A, B and C (see Section 5.1), and type C arises only once.

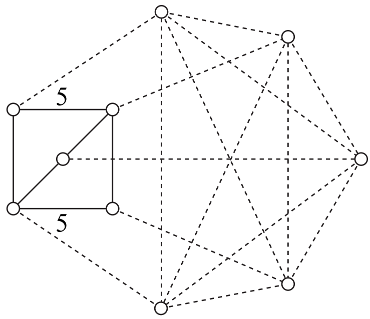

Figure 2: The Coxeter graphs of the compact totally truncated Coxeter 4–simplex T and its underlying extended 4–simplex

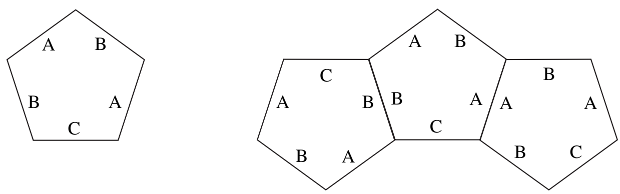

Figure 3: A Coxeter domino T and Coxeter dominoes

Then we prove that all cocompact hyperbolic Coxeter groups with respect to Coxeter dominoes built by $T \subset \mathbb{H}^4$ have growth functions whose denominator polynomials $Q_{\ell,m,n}(t)$ satisfy the following property: all the roots of $Q_{\ell,m,n}(t)$ are on the unit circle except two pairs of real roots (see Theorem 2). Finally, we prove our main result, providing infinite families whose growth rates are 2–Salem numbers under certain restrictions for (ℓ, m, n).

For the calculation of the growth functions of Coxeter dominoes, we adapt the formula developed by T Zehrt and C Zehrt [25] (see Proposition 3), which allows us to compute all the growth functions of such infinite families at once (see Corollary 1). For the study of the denominator polynomials, we use the adaption of Kempner's result [8] to palindromic polynomials, due to T Zehrt and C Zehrt, about the distribution of the roots of the polynomials. To show the irreducibility of denominator polynomials, we analyze various cases by elementary arithmetic (cf Theorem 3).

The paper is organized as follows. In Section 2, we review hyperbolic spaces and hyperbolic Coxeter groups. In Section 3.1, we collect some basic results of the growth functions and growth rates of Coxeter groups, and in Section 3.2, we introduce 2–Salem numbers and their properties. In Section 4, we describe the compact totally truncated Coxeter simplex T in $\mathbb{H}^4$ giving rise to Coxeter dominoes. Finally, in Section 5, we state and prove the main theorem. For the sake of completeness, consider the Coxeter garland associated to the Coxeter domino on the left side in Figure 1 (see [25]). By using the same ideas, it can be shown that the growth rates of the resulting Coxeter groups provide infinitely many distinct 2–Salem numbers as well. We omit the details.

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2 Cocompact hyperbolic Coxeter groups

2.1 Hyperbolic convex polytopes

Let $\mathbb{R}^{n,1}$ be the real vector space $\mathbb{R}^{n+1}$ equipped with the *Lorentzian inner product*

$$x \circ y := x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1},$$

where $x = (x_1, \ldots, x_{n+1})$, $y = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$. A vector $x \in \mathbb{R}^{n,1}$ is *space-like* (resp. *light-like, time-like*) if $x \circ x > 0$ (resp. $x \circ x = 0$, $x \circ x < 0$). The set

$$C := \{x \in \mathbb{R}^{n,1} \mid x \circ x = 0\}$$

is a cone in $\mathbb{R}^{n+1}$ formed by all light-like vectors. Space-like vectors lie outside C and time-like vectors lie inside C. The set

$$\mathbb{H}^n := \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n,1} \mid x \circ x = -1, x_{n+1} > 0\}$$

is a hyperboloid lying inside the upper half of C and is called *hyperbolic n–space* when equipped with the metric

$$\cosh d_{\mathbb{H}}(x, y) := -x \circ y.$$

A nonempty subset $H \subset \mathbb{H}^n$ is a *hyperplane of* $\mathbb{H}^n$ if there exists an n–dimensional vector subspace $V \subset \mathbb{R}^{n+1}$ such that $H = V \cap \mathbb{H}^n$. It is equivalent to say that V is a *time-like subspace* of $\mathbb{R}^{n,1}$, that is, V contains a time-like vector. So we can represent V by the *Lorentzian orthogonal complement* $e^L := \{x \in \mathbb{R}^{n,1} \mid x \circ e = 0\}$ for some unit space-like vector $e \in \mathbb{R}^{n,1}$ (see Ratcliffe [16, Exercise 3.1, Number 10]). Hence

$H = e^L \cap \mathbb{H}^n$ and we often use the notation $\widehat{H} := e^L$. In a similar way, we denote by $(e^L)^- := \{x \in \mathbb{R}^{n,1} \mid x \circ e \le 0\}$ the half space of $\mathbb{R}^{n,1}$ bounded by $\widehat{H}$, and put $H^- := (e^L)^- \cap \mathbb{H}^n$ and $\widehat{H}^- := (e^L)^-$.

A *convex polytope* $P \subset \mathbb{H}^n$ of dimension n is defined by the intersection of finitely many closed half spaces of $\mathbb{H}^n$,

$$P = \bigcap_{i \in I} H_i^-, \tag{2-1}$$

containing a nonempty open subset of $\mathbb{H}^n$. We remark that $H_i = e_i^L \cap \mathbb{H}^n$ for a unit space-like vector $e_i \in \mathbb{R}^{n,1}$ such that e_i is directed outwards with respect to P. We always assume that for any proper subset $J \subsetneq I$, $P \subsetneq \bigcap_{j \in J} H_j^-$ is satisfied. For a subset $J \subset I$, the nonempty intersection $F := P \cap (\bigcap_{j \in J} H_j)$ is called a *face* of P. Observe that F itself can be considered as a convex hyperbolic polytope of some dimension. A face $F \subset P$ of dimension $n-1$, 1 or 0 is called a *facet, edge* or *vertex* of P respectively.

The mutual disposition of hyperplanes bounding P is as follows (cf [16, pages 67–71; 23, page 41]). Two hyperplanes H_i and H_j in $\mathbb{H}^n$ intersect each other if and only if $|e_i \circ e_j| < 1$, and the angle between H_i and H_j of P is defined as the real number $\theta_{ij} \in [0, \pi[$ satisfying

$$\cos \theta_{ij} = -e_i \circ e_j. \tag{2-2}$$

We call θ_{ij} the *dihedral angle* between H_i and H_j of P, and denote

$$\angle H_i^- H_j^- := \theta_{ij}.$$

The hyperplanes H_i and H_j do not intersect if and only if

$$e_i \circ e_j \le -1.$$

As a remark, $e_i \circ e_j \not\ge 1$ since $H_i^- \not\subset H_j^-$ for any $i \ne j$ under the above assumption. More precisely, if $\widehat{H}_i$ and $\widehat{H}_j$ do not intersect in $\mathbb{H}^n \cup C \setminus \{0\}$, then $\cosh d_{\mathbb{H}}(H_i, H_j) = -e_i \circ e_j$, where $d_{\mathbb{H}}(H_i, H_j)$ is the hyperbolic length of the unique geodesic orthogonal to H_i and H_j.

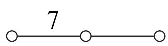
The *Gram matrix* $G(P)$ *of* P is defined as the symmetric matrix $(g_{ij}) := (e_i \circ e_j)$, $i, j \in I$, with $g_{ii} = 1$.

A convex polytope $P \subset \mathbb{H}^n$ of dimension n is called *acute-angled* if all of its dihedral angles are less than or equal to $\pi/2$. For acute-angled polytopes in $\mathbb{H}^n$, Vinberg [23] developed a complete combinatorial description in terms of its Gram matrix $G(P)$. We will discuss it in Section 4.1.

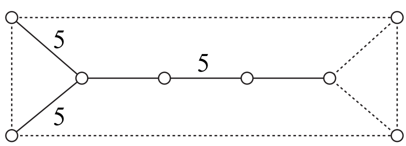
2.2 Hyperbolic Coxeter polytopes and their associated hyperbolic Coxeter groups

A *Coxeter polytope* $P \subset \mathbb{H}^n$ of dimension n is a convex polytope of dimension n all of whose dihedral angles are of the form π/m, where $m \geq 2$ is an integer. For the Coxeter polytope described by (2-1), the group W generated by the set $S = \{s_i \mid i \in I\}$ of reflections with respect to the bounding hyperplanes H_i in $\mathbb{H}^n$ is a discrete subgroup of the isometry group $Isom(\mathbb{H}^n)$ of $\mathbb{H}^n$, and P is a fundamental polytope of W. Furthermore, (W, S) is a Coxeter system (see Section 3.1) with relations $s_i^2 = \mathrm{id}$, and $(s_i s_j)^{m_{ij}} = \mathrm{id}$ for $i \neq j$, if $\angle H_i^- H_j^- = \pi/m_{ij}$. If P is compact, we call W a *cocompact hyperbolic Coxeter group*. It is convenient to associate a Coxeter graph Γ to a Coxeter polytope P. Represent each bounding hyperplane H_i (or reflection $s_i \in S$) by a node v_i, and join two nodes v_i and v_j by a single edge labeled m_{ij} if $\angle H_i^- H_j^- = \pi/m_{ij}$, $m_{ij} \geq 3$, or by a dotted edge if H_i and H_j do not intersect. We do not join v_i and v_j if H_i and H_j are orthogonal, and we omit the label for $m_{ij} = 3$.

Example 1 The following Coxeter graph describes a compact hyperbolic Coxeter triangle in $\mathbb{H}^2$ with angles $\pi/2, \pi/3$ and $\pi/7$:



Example 2 The following Coxeter graph represents a compact hyperbolic Coxeter polytope in $\mathbb{H}^4$ (cf [25, Example 2]):



3 Growth rates of cocompact hyperbolic Coxeter groups

3.1 Growth functions of Coxeter groups

In this section, we shall introduce the quantity τ for the hyperbolic Coxeter groups. At first, we shall consider the general situation. Let (W, S) be a pair consisting of a group W and its finite generating set S. The *word length* of an element $w \in W$ with respect to S is defined by

$$l_S(w) := \min\{k \in \mathbb{N} \cup \{0\} \mid w = s_1 s_2 \cdots s_k, s_i \in S\}.$$

By convention, $l_S(\mathrm{id}) = 0$. The *growth series* of (W, S) is defined by the power series

$$f_S(t) := \sum_{w \in W} t^{l_S(w)} = \sum_{k \geq 0} a_k t^k = 1 + |S| t + \cdots, \quad t \in \mathbb{C}, \tag{3-1}$$

where a_k is the number of elements $w \in W$ with $l_S(w) = k$ and $|S|$ denotes the number of S. The *growth rate* of (W, S) is defined by

$$\tau := \limsup_{k \to \infty} \sqrt[k]{a_k}, \tag{3-2}$$

which is the inverse of the radius of convergence R of $f_S(t)$, that is, $\tau = R^{-1}$.

From now on, we focus on the growth series and the growth rate of a Coxeter group, wherefore we review the notions of Coxeter groups. A pair (W, S) is called a *Coxeter system* if generators $s, t \in S$ satisfy relations of the type $(st)^{m_{s,t}} = \mathrm{id}$, $m_{s,t} = m_{t,s} \geq 2$ for $s \neq t$, and $m_{s,s} = 1$. We call the group W itself a *Coxeter group*. In the sequel, we often do not distinguish between Coxeter group and its underlying Coxeter system. It is convenient to use the *Coxeter graph* Γ associated to (W, S) whose nodes ν_s correspond to the generators $s \in S$, and two nodes ν_s, ν_t are joined as follows. They are joined by a single edge labeled $m_{s,t}$ if $m_{s,t} \geq 3$ and are not joined if $m_{s,t} = 2$. Usually, the label is omitted if $m_{s,t} = 3$. Two nodes ν_s and ν_t are joined by a dotted edge if the group generated by st is infinite. All connected Coxeter graphs with finite associated Coxeter groups have been classified (Coxeter [4]).

First, we consider a finite Coxeter group (W, S). It is obvious that its growth series (3-1) is a polynomial. Solomon [21, Corollary 2.3] gave an explicit formula for the growth polynomial of (W, S), which allows us to compute it by using the *exponents* $m_1, m_2, \ldots, m_k \in \mathbb{Z}$, $1 = m_1 \leq m_2 \leq \cdots \leq m_k$, of (W, S), that is,

$$f_S(t) = \prod_{i=1}^{k} [m_i + 1], \tag{3-3}$$

where $[m] = 1 + t + \cdots + t^{m-1}$.

Next we consider an infinite Coxeter group (W, S). Steinberg [22, Corollary 1.29] derived a formula for the growth series $f_S(t)$ of (W, S), which allows us to compute it by using the growth polynomial $f_T(t)$ of its finite Coxeter subgroups (W_T, T) generated by a subset $T \subset S$, that is,

$$\frac{1}{f_S(t^{-1})} = \sum_{\substack{T \subset S \\ |W_T| < \infty}} \frac{(-1)^{|T|}}{f_T(t)} = 1 - \frac{|S|}{[2]} + \cdots. \tag{3-4}$$

Here each $f_T(t)$ is of the form (3-3). This means that $1/f_S(t^{-1})$ is expressible as a rational function, say

$$\frac{1}{f_S(t^{-1})} = \frac{\widetilde{q}(t)}{\widetilde{p}(t)},$$

where $\widetilde{q}(t), \widetilde{p}(t) \in \mathbb{Z}[t]$ are relatively prime and monic of the same degree, say n. Furthermore, the constant term of $\widetilde{p}(t)$ is ± 1 by (3-3) and (3-4). Hence we have

$$f_S(t) = \frac{p(t)}{q(t)}, \tag{3-5}$$

where $p(t) = t^n \widetilde{p}(t^{-1}), q(t) = t^n \widetilde{q}(t^{-1}) \in \mathbb{Z}[t]$. Both polynomials $p(t)$ and $q(t)$ are relatively prime over $\mathbb{Z}$, and we call the growth series $f_S(t)$ described by the form (3-5) the *growth function* of (W, S). Observe that the smallest positive root of $q(t)$ equals the radius of convergence R. This means $q(R) = R^n \widetilde{q}(R^{-1}) = 0$, and since $\widetilde{q}(t) \in \mathbb{Z}[t]$ is monic, the growth rate $\tau = R^{-1}$ is an algebraic integer.

Example 3 The growth function of the cocompact hyperbolic Coxeter group described by the Coxeter graph in Example 1 is

$$f_S(t) = \frac{(t+1)^2(t^2+t+1)(t^6+t^5+t^4+t^3+t^2+t+1)}{t^{10}+t^9-t^7-t^6-t^5-t^4-t^3+t+1}. \tag{3-6}$$

Example 4 The growth function of the cocompact hyperbolic Coxeter group described by the Coxeter graph in Example 2 is (cf [25, Theorem 1, $n = 1$])

$$f_S(t) = \frac{(t+1)^4(t^2-t+1)(t^2+t+1)(t^4-t^3+t^2-t+1)(t^4+t^3+t^2+t+1)}{t^{16}-4t^{15}+t^{14}+t^{12}+t^{11}+2t^9+2t^7+t^5+t^4+t^2-4t+1}. \tag{3-7}$$

3.2 Growth rates and 2–Salem numbers

In Section 3.1, we see that the growth rate of a Coxeter group is an algebraic integer. Cannon, Wagreich and Parry (see [15], for example) showed that the growth rate of a cocompact hyperbolic Coxeter group in $\mathbb{H}^2$ or $\mathbb{H}^3$ is a Salem number or a quadratic unit. We recall the notion of Salem numbers.

A *Salem number* is a real algebraic integer $\alpha > 1$, all of whose other conjugate roots ω satisfy $|\omega| \le 1$, and at least one of whose conjugate roots is on the unit circle (cf Bertin, Decomps-Guilloux, Grandet-Hugot, Pathiaux-Delefosse and Schreiber [1, Definition 5.2.2], and Ghate and Hironaka [5, page 293]). Call the minimal polynomial $p_\alpha(t)$ of α a *Salem polynomial*. The following proposition is a well-known fact about Salem polynomials (cf [5, page 294]).

Proposition 1 *A Salem polynomial $p_\alpha(t)$ is palindromic of even degree, that is, $p_\alpha(t) = t^n p_\alpha(t^{-1})$, where n is the degree of $p_\alpha(t)$.*

Proof Since $p_\alpha(t) \in \mathbb{Z}[t]$ has a root ω_0 on the unit circle, then $\overline{\omega_0} = \omega_0^{-1}$ is also a root of $p_\alpha(t)$. Hence, $t^n p_\alpha(t^{-1})$ is also the minimal polynomial of ω_0 and then there exists a constant $c \in \mathbb{Z}$ such that

$$p_\alpha(t) = c t^n p_\alpha(t^{-1}).$$

As a consequence, α^{-1} is also a root of $p_\alpha(t)$, so that any root ω of $p_\alpha(t)$ except α and α^{-1} is on the unit circle, where ω^{-1} is also a root of $p_\alpha(t)$. Therefore the constant term of $p_\alpha(t)$ should be 1, which implies that $c = 1$, that is, $p_\alpha(t) = t^n p_\alpha(t^{-1})$. We conclude that $p_\alpha(t)$ is a palindromic polynomial of even degree. □

Example 5 The growth rate of the cocompact hyperbolic Coxeter group described by the Coxeter graph in Example 1 is a Salem number. More precisely, the denominator polynomial

$$L(t) = t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1$$

of its growth function (3-6) is a Salem polynomial, and its positive root $\alpha_L \approx 1.17628$, which is a Salem number, is the growth rate. Note that $L(t)$ and α_L are known as the Lehmer polynomial and its Lehmer number, respectively (cf Lehmer [11] and Figure 4).

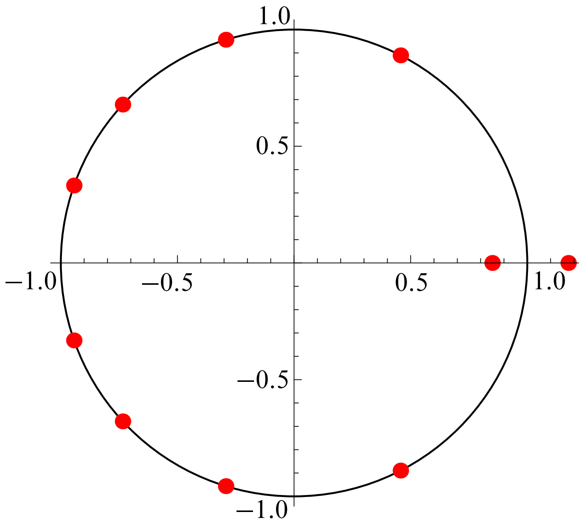


Figure 4: The distribution of the roots of $L(t)$

As we see, the minimal polynomial of a Salem number has roots on the unit circle except for a reciprocal pair of positive real roots. Focussing on the numbers of roots outside the unit circle and the existence of a root on the unit circle, Kerada defined a j–Salem number in [9, Definition 2.1], which is a generalization of a Salem number. In this paper, we focus on a 2–Salem number. Note that Samet [17] also defined a set of algebraic integers corresponding to 2–Salem numbers.

Definition 1 A 2–*Salem number* is an algebraic integer α such that $|\alpha| > 1$ and α has one conjugate root β different from α satisfying $|\beta| > 1$, while other conjugate roots ω satisfy $|\omega| \leq 1$ and at least one of them is on the unit circle. Call the minimal polynomial $p_\alpha(t)$ of α a 2–*Salem polynomial*.

As in the case of a Salem polynomial, a 2–Salem polynomial $p_\alpha(t)$ is a palindromic polynomial of even degree. As a consequence, α^{-1} and β^{-1} are roots of $p_\alpha(t)$, and all roots different from $\alpha, \alpha^{-1}, \beta, \beta^{-1}$ lie on the unit circle and are complex (cf Figure 5). So a 2–Salem polynomial has even degree $n \geq 6$, and note that β is also a 2–Salem number.

In [25], T Zehrt and C Zehrt found infinitely many cocompact Coxeter groups in $\mathbb{H}^4$ whose denominators $q(t)$ of the growth functions $f_S(t)$ have the following property: all the roots of the polynomial $q(t)$ are on the unit circle except exactly two pairs of real roots. We notice that if $q(t)$ is irreducible, it is a 2–Salem polynomial. This motivated us to investigate whether 2–Salem numbers appear as growth rates of such groups. As a first observation, we get the following proposition.

Proposition 2 *The growth rate of the cocompact hyperbolic Coxeter group described by the Coxeter graph in Example 2 is a 2–Salem number.*

Proof We prove that the denominator polynomial

$$D(t) = t^{16} - 4t^{15} + t^{14} + t^{12} + t^{11} + 2t^9 + 2t^7 + t^5 + t^4 + t^2 - 4t + 1$$

of the growth function (3-7) is a 2–Salem polynomial. First, $D(t)$ has six pairs of complex roots on the unit circle and two pairs of positive roots α, α^{-1} and β, β^{-1}, where $\alpha, \beta > 1$ (see [25, Theorem 2, $n = 1$] and Figure 5).

The irreducibility of $D(t)$ in $\mathbb{Z}[t]$ is proved by Cohn's Theorem (see Murty [14, Theorem 1]) as follows. For a polynomial $f(t) = t^m + a_{m-1}t^{m-1} + \cdots + a_1 t + a_0$ of degree m in $\mathbb{Z}[t]$, set

$$H = \max_{0 \leq i \leq m-1} |a_i|.$$

If $f(n)$ is prime for some integer $n \geq H + 2$, then $f(t)$ is irreducible in $\mathbb{Z}[t]$ by Cohn's criterion. In fact, $H = 4$ for $D(t)$, and $D(186) = 2008067839285267472384758820173242349$ is a prime number. So $D(t)$ is irreducible. Furthermore, two positive roots $\alpha \approx 3.70422$ and $\beta \approx 1.24202$ are 2–Salem numbers. □

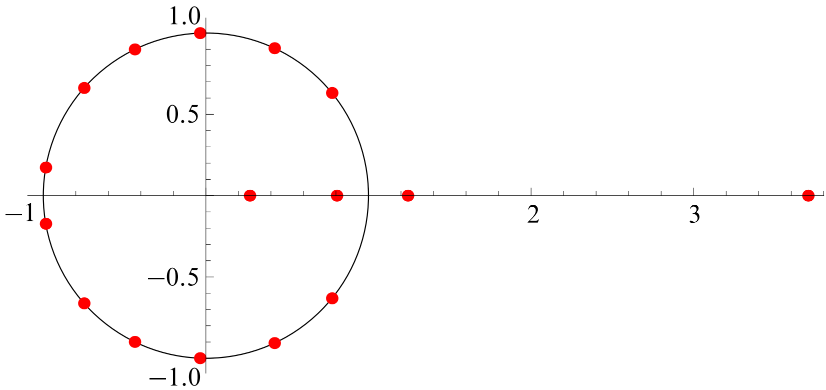


Figure 5: The distribution of the roots of $D(t)$

In the next chapter, we construct families of infinitely many cocompact hyperbolic Coxeter groups in $\mathbb{H}^4$ that are different from those constructed by T Zehrt and C Zehrt, and prove that their growth rates are 2–Salem numbers.

4 Construction of a Coxeter domino T

T Zehrt and C Zehrt [25, Theorem 2] described the characteristic distribution of roots of the denominator polynomials of the growth functions of Coxeter garlands in $\mathbb{H}^4$ built by a particular compact truncated Coxeter 4–simplex with two orthogonal facets. In this paper, we have similar, but more detailed results for the denominator polynomials of the growth functions of Coxeter dominoes constructed by the Coxeter domino $T \subset \mathbb{H}^4$. This leads us to a connection with 2–Salem numbers.

To begin with, we are interested in the Coxeter system (W, S) having the following Coxeter graph Γ (see Figure 6). It was first described by Schlettwein in [19] (unpublished).

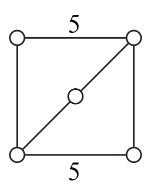


Figure 6

At first, notice that this graph describes an infinite Coxeter group since it does not belong to the well-known list of connected Coxeter graphs of finite Coxeter groups. In Section 4.1, we will explain in detail how to realize (W, S) as a geometric Coxeter group in $\mathbb{H}^4$.

4.1 The Coxeter simplex $P \subset \mathbb{H}^4$ with ultraideal vertices

We consider the matrix G related to the Coxeter graph Γ in Figure 6, which is defined by $(g_{ij}) = (-\cos(\pi/m_{ij}))$:

$$G = \begin{pmatrix} 1 & -\cos\frac{\pi}{5} & 0 & -\frac{1}{2} & 0 \\ -\cos\frac{\pi}{5} & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & -\cos\frac{\pi}{5} & 0 \\ -\frac{1}{2} & 0 & -\cos\frac{\pi}{5} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{pmatrix}. \tag{4-1}$$

The signature of G equals $(4,1)$. In fact, $\det G < 0$ and the principal submatrix formed by the first three rows and columns is positive definite (and related to the Cartan matrix of H_3). Therefore, G has precisely one negative and four positive eigenvalues (see Satake [18], for example). By a result of Vinberg [23, Theorem 2.1], an indecomposable symmetric real matrix $G = (g_{ij})$, with $g_{ii} = 1$ and $g_{ij} \le 0$ for $i \ne j$, is the Gram matrix of an acute-angled polytope of dimension n in $\mathbb{H}^n$ (up to isometry) if the signature of G equals $(n,1)$. Therefore, G is the Gram matrix of a Coxeter polytope $P \subset \mathbb{H}^4$ given by the Coxeter graph Γ (see Figure 6). However, P is not compact since it is not in the list of Coxeter graphs of compact Coxeter simplices in $\mathbb{H}^4$ classified by Lannér [10] (cf Humphreys [6, page 141]). In the sequel, we associate to P a new compact polytope $T \subset \mathbb{H}^4$ by truncation.

Let $P = \bigcap_{i\in I} H_i^- \subset \mathbb{H}^4$ be the Coxeter polytope having a Coxeter graph Γ as in Figure 6, where $I = \{1,\dots,5\}$. Represent $H_i = e_i^L \cap \mathbb{H}^4$ as usual, such that $G = (e_i \circ e_j)_{i,j\in I}$. It follows that P is an *extended* 4*–simplex* (or a 4*–simplex with* 5 *ultraideal vertices*) in $\mathbb{R}^{4,1}$, bounded by five hyperplanes, and of infinite volume in $\mathbb{H}^4$ (see [23, Chapter I, Section 4, section 8, line 5]). In fact, all vertices of P lie outside of $\mathbb{H}^4$. To see this, let us use Vinberg's description of faces in terms of submatrices of G.

Observe that all principal submatrices of order four in G are of signature $(3,1)$ and described by the Coxeter graphs in Figure 9 (cf [10, page 53]). Furthermore, all other principal submatrices of G, that is, those of order less than four, are positive definite and described by positive definite graphs, that is, by certain finite Coxeter groups. Then, by a result of Vinberg [23, Theorem 3.1], all the faces of P of positive dimension are hyperbolic polytopes, while the vertices of P do not belong to $\mathbb{H}^4$. In fact, $\bigcap_{j\in I_i} H_j = \varnothing$, for $I_i := I \setminus \{i\}$, $i = 1,\dots,5$. However, the set $\bigcap_{j\in I_i} \widehat{H}_j$ is a one–dimensional subspace in $\mathbb{R}^{4,1}$ since $e_i, i \in I$, are linearly independent. This line can be represented by a positive space-like unit vector $v_i \in \mathbb{R}^{4,1}$ such that v_i is a vertex of P outside $\mathbb{H}^4$ (see also (4-3)). We call v_i an *ultraideal vertex* of P.

4.2 Construct the compact totally truncated Coxeter simplex $T \subset \mathbb{H}^4$ from P

Now we will truncate all ultraideal vertices off from P in order to obtain a compact Coxeter polytope T as follows. The set $\bigcap_{j \in I_i} \widehat{H}_j^-$ is a 4–dimensional simplicial cone in $\mathbb{R}^{4,1}$, whose apex v_i is outside of $\mathbb{H}^4$. Moreover, the hyperplane $H_{v_i} := v_i^L \cap \mathbb{H}^4$ intersects all of the hyperplanes H_j, $j \in I_i$, orthogonally, that is,

$$\angle H_{v_i}^- H_j^- = \frac{\pi}{2}, \quad j \in I_i, \tag{4-2}$$

since $v_i \circ e_j = 0$ for all $j \in I_i$ by definition of v_i, and by (2-2). In this situation, by a result of Vinberg [23, Proposition 4.4], $P \cap H_{v_i}^-$ is also a convex polytope in $\mathbb{H}^4$. By performing this operation for each ultraideal vertex v_i, $i = 1, \dots, 5$, we get a new polytope

$$T := P \cap \left(\bigcap_{i \in I} H_{v_i}^- \right) \subset \mathbb{H}^4$$

that we shall call *totally (orthogonally) truncated simplex* (see Figure 7). A facet $P \cap H_{v_i}$ is called an *orthogonal facet* of T.

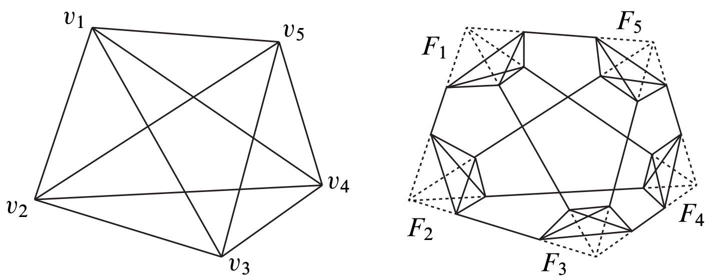


Figure 7: The passage to the compact totally truncated Coxeter 4–simplex T where each ultraideal vertex v_i is replaced by the orthogonal facet F_i

Let us describe the ultraideal vertices $v_1, \dots, v_5$ of P in the following explicit way (cf [12]). Let $\mathrm{cof}_{ij}(G) := (-1)^{i+j} \det G_{ij}$ be the cofactor of G, where G_{ij} is the submatrix obtained by removing the i^{th} row and j^{th} column of G. Since the ij^{th} coefficient of G^{-1} equals $(1/\det G)\,\mathrm{cof}_{ji}(G)$, then by comparing the ij^{th} coefficient of $G^{-1}G = I$,

$$\frac{1}{\det G} \sum_{k=1}^{5} \mathrm{cof}_{ik}(G) g_{kj} = \delta_{ij}, \quad \text{and then} \quad \left(\sum_{k=1}^{5} \mathrm{cof}_{ik}(G) e_k \right) \circ e_j = \delta_{ij} \det G.$$

If we set $w_i := \sum_{k=1}^{5} \mathrm{cof}_{ik}(G) e_k$, $1 \le i \le 5$, then

$$w_i \circ w_j = \mathrm{cof}_{ij}(G) \det G \quad \text{is} \quad \begin{cases} < 0 & \text{if } i \neq j, \\ > 0 & \text{if } i = j. \end{cases}$$

Hence, $\{w_1, \dots, w_5\}$ is a linearly independent set. Now, consider

$$v_i = \frac{w_i}{\sqrt{w_i \circ w_i}} = \frac{w_i}{\sqrt{\mathrm{cof}_{ii}(G) \det G}}. \tag{4-3}$$

Then, for each i,

$$v_i \circ v_i = 1, \qquad v_i \circ e_j = 0 \text{ for } j \neq i, \qquad v_i \circ e_i = -\sqrt{\frac{\det G}{\mathrm{cof}_{ii}(G)}} < 0. \tag{4-4}$$

So, for each i, $\{v_j \mid j \in I_i\}$ spans $\widehat{H}_i$ since it is a linearly independent set, and v_i spans $\bigcap_{j \in I_i} \widehat{H}_j$. Hence v_i is indeed an ultraideal vertex of P.

By direct calculation, based on (4-1) and (4-4), we get $v_i \circ e_i < -1$ for each $i \in I$, which means that the orthogonal facet H_{v_i} and the facet H_i opposite to it are disjoint in T [16, Section 3]:

$$H_{v_i} \cap H_i = \varnothing \tag{4-5}$$

Similarly,

$$v_i \circ v_j = \frac{-\mathrm{cof}_{ij}(G)}{\sqrt{\mathrm{cof}_{ii}(G)\, \mathrm{cof}_{jj}(G)}} < -1$$

for each $j \in I_i$, which means that the orthogonal facets are mutually disjoint:

$$H_{v_i} \cap H_{v_j} = \varnothing \quad \text{if } i \neq j \tag{4-6}$$

By combining (4-2), (4-5) and (4-6), it follows that T is described by the Coxeter graph Γ^* in Figure 8. In particular, T is a Coxeter polytope.

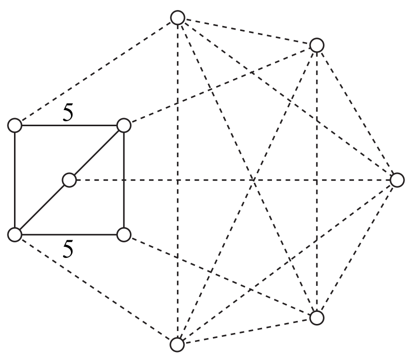


Figure 8: The Coxeter graph Γ^* of the compact totally truncated Coxeter 4–simplex T

Let $I^* = \{1, \dots, 10\}$ be the indexed set of ten facets of T. For the compactness of T, observe first that all order four subgraphs $\Gamma(4)$ of Γ are of signature $(3,1)$, and each subgraph $\Gamma(2)$ of type ○┈┈┈○ in Γ^* is of signature $(1,1)$. Next, let $J \subset I^*$ be a subset corresponding to one of the subgraphs $\Gamma(j)$, $j = 2, 4$, above. Consider $J' \subset I^*$ indexing all nodes in Γ^* that are *not* connected to nodes in J. Form $N(J) := J \cup J'$. By applying Vinberg's criterion [23, Theorem 4.1], T is compact because it is easily checked that $\bigcap_{i \in N(J)} \widehat{H}_i = \{0\}$.

Since T will be the building block for new Coxeter polytopes (see Section 5.1), we call T a *Coxeter domino*.

5 Main theorems

5.1 Growth functions of Coxeter dominoes

As we see, T has five orthogonal facets and each of them is one of the three kinds of Coxeter 3–simplices in $\mathbb{H}^3$ described by the Coxeter graphs in Figure 9. More precisely, T has two orthogonal facets of type A and B, and one orthogonal facet of type C.

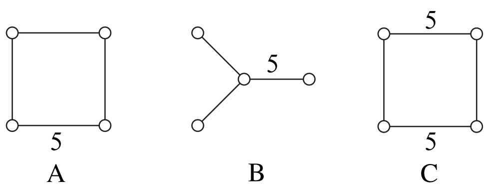


Figure 9: The three Coxeter graphs of the hyperbolic Coxeter simplex of dimension 3 that describe the figures of three types of orthogonal facets of T

By gluing copies of T along the orthogonal facets of the same type, we obtain a new polytope, which is again a Coxeter polytope. By gluing over and over again, we obtain infinitely many Coxeter polytopes in $\mathbb{H}^4$. This construction is established by Makarov [13] and is also explained in [23, Chapter II, Section 5]. Vinberg uses the term "garlands" (cf [23, page 62]) for the resulting Coxeter polytopes obtained by gluing together the truncated Coxeter simplices having two orthogonal facets (see also [25]).

Let us consider the growth functions of the Coxeter groups with respect to the Coxeter dominoes built by T. First, the growth function $W(t)$ of the Coxeter domino T itself is calculated by using the formula (3-3) and (3-4):

$$\begin{aligned}(5\text{-}1)\quad \frac{1}{W(t^{-1})} = \frac{1}{W(t)} = 1 - \frac{10}{[2]} &+ \left\{\frac{2}{[2,5]} + \frac{4}{[2,3]} + \frac{24}{[2,2]}\right\} \\ &- \left\{\frac{6}{[2,6,10]} + \frac{3}{[2,3,4]} + \frac{6}{[2,2,5]} + \frac{12}{[2,2,3]} + \frac{13}{[2,2,2]}\right\} \\ &+ \left\{\frac{12}{[2,2,6,10]} + \frac{6}{[2,2,3,4]} + \frac{2}{[2,2,2,2]}\right\},\end{aligned}$$

where the first equality of (5-1) comes from [20] (and [3, Corollary]) since T is compact. Then $W(t) = P(t)/Q(t)$, where

(5-2) $P(t) = [2,4,6,10] = \Phi_2(t)^4\Phi_3(t)\Phi_4(t)\Phi_5(t)\Phi_6(t)\Phi_{10}(t)$,

(5-3) $Q(t) = t^{18} - 6t^{17} + 3t^{16} - 5t^{15} + 5t^{14} - t^{13} + 9t^{12} + 11t^{10} - 2t^9 + 11t^8 + 9t^6 - t^5 + 5t^4 - 5t^3 + 3t^2 - 6t + 1$,

which is a palindromic polynomial of degree 18. Here $\Phi_i(t)$ is the i^{th} cyclotomic polynomial. To compute the growth functions for Coxeter dominoes, we use the following key formula.

Proposition 3 [25, Corollary 2] *Consider two Coxeter n–polytopes P_1 and P_2 having the same orthogonal facet of type F that is a Coxeter $(n-1)$–polytope, and their growth functions are $W_1(t)$, $W_2(t)$ and $F(t)$, respectively. Then the growth function $(W_1 *_F W_2)(t)$ of the Coxeter polytope obtained by gluing P_1 and P_2 along F is given by*

$$\frac{1}{(W_1 *_F W_2)(t)} = \frac{1}{W_1(t)} + \frac{1}{W_2(t)} + \frac{t-1}{t+1}\frac{1}{F(t)}.$$

Now we have the formula for the growth functions of the Coxeter dominoes built by T as follows.

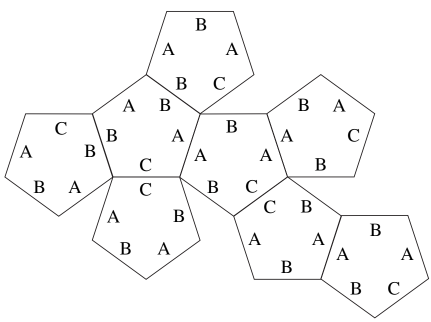


Figure 10: The Coxeter polytope obtained by gluing T 7 times, satisfying $(\ell,m,n) = (3,2,7)$

Corollary 1 *Consider $n+1$ copies of T, and obtain from them one Coxeter polytope by gluing n times. If this polytope is obtained by gluing ℓ times along the orthogonal facet of type A, m times along B and $(n-\ell-m)$ times along C, where $\ell+m \le n$ (cf Figure 10), then $n-\ell-m \le (n+1)/2$, and its growth function $W_{\ell,m,n}(t)$ is given by*

(5-4) $$\frac{1}{W_{\ell,m,n}(t)} = \frac{n+1}{W(t)} + \frac{t-1}{t+1}\left(\frac{\ell}{A(t)} + \frac{m}{B(t)} + \frac{n-\ell-m}{C(t)}\right).$$

Indeed, the property $n-\ell-m \le (n+1)/2$ in Corollary 1 follows from the fact that T has only one facet of type C. Therefore, consecutive gluing along C is not possible. Note that $W_{\ell,m,n}(t)$ only depends on (ℓ,m,n) and does not directly depend on the resulting polytope.

Theorem 1 *The growth function $W_{\ell,m,n}(t)$ is the rational function*

$$\frac{P_{\ell,m,n}(t)}{Q_{\ell,m,n}(t)},$$

where

$$\begin{aligned}
\text{(5-5)}\quad P_{\ell,m,n}(t) &= [2,4,6,10] \\
&= \Phi_2(t)^4\Phi_3(t)\Phi_4(t)\Phi_5(t)\Phi_6(t)\Phi_{10}(t)
\end{aligned}$$

$$\begin{aligned}
\text{(5-6)}\quad Q_{\ell,m,n}(t) = {} & t^{18}-(4n+6)t^{17}+(2n-m+3)t^{16}-(3n-m+\ell+5)t^{15} \\
&+(5n-3m+5)t^{14}-(n-4m+1)t^{13}+(8n-4m+\ell+9)t^{12} \\
&+(5m-\ell)t^{11}+(10n-5m+\ell+11)t^{10}-(2n-6m+2)t^{9} \\
&+(10n-5m+\ell+11)t^{8}+(5m-\ell)t^{7}+(8n-4m+\ell+9)t^{6} \\
&-(n-4m+1)t^{5}+(5n-3m+5)t^{4}-(3n-m+\ell+5)t^{3} \\
&+(2n-m+3)t^{2}-(4n+6)t+1.
\end{aligned}$$

Proof The growth function $W(t)=P(t)/Q(t)$ of T is already given by (5-2) and (5-3). The growth functions $A(t)$, $B(t)$ and $C(t)$ of the orthogonal facets of types A, B and C in Figure 9 are calculated by using the formula (3-3) and (3-4) as follows (cf Worthington [24]):

$$A(t) = -\frac{(t+1)^3(t^2+1)(t^2-t+1)(t^4-t^3+t^2-t+1)}{(t-1)(t^{10}-2t^9+t^8-2t^6+2t^5-2t^4+t^2-2t+1)}$$

$$B(t) = -\frac{(t+1)^3(t^2+1)(t^2-t+1)(t^4-t^3+t^2-t+1)}{(t-1)(t^{10}-2t^9+2t^8-2t^7+t^6-t^5+t^4-2t^3+2t^2-2t+1)}$$

$$C(t) = -\frac{(t+1)^3(t^2-t+1)(t^4-t^3+t^2-t+1)}{(t-1)(t^2+1)(t^6-2t^5-t^4+3t^3-t^2-2t+1)}$$

Then by (5-4) in Corollary 1, (5-5) and (5-6) are obtained. □

Remark 1 The polynomials $P_{\ell,m,n}$ and $Q_{\ell,m,n}$ in (5-5) and (5-6) are relatively coprime. In fact, Theorem 2 below rules out common linear and quadratic factors, while an adaption of the proof of Theorem 3 below (see (5-10) and (5-11)) allows us to rule out common quartic factors (of type Φ_5 and Φ_{10}) as follows. Consider $\Phi_5(t)=t^4+t^3+t^2+t+1$ and $\Phi_{10}(t)=t^4-t^3+t^2-t+1$ and suppose that Φ_5, Φ_{10} divide $Q_{\ell,m,n}$ according to (5-10). Then, for Φ_5, one gets $g_{\ell,m,n}(1,1)=-2-2n=0$ (see (2) following (5-11)) with unique solution $n=-1$. This contradicts $n\in\mathbb{N}\cup\{0\}$. For Φ_{10}, one gets $g_{\ell,m,n}(-1,1)=-6-2\ell-2m-2n=0$ leading to $\ell+m+n=-3$. This contradicts $\ell,m,n\in\mathbb{N}\cup\{0\}$.

5.2 Growth rates of Coxeter dominoes and 2–Salem numbers

Now, we will show that there are infinitely many 2–Salem numbers as growth rates of $W_{\ell,m,n}(t)$. Recall that $\ell,m,n\in\mathbb{N}\cup\{0\}$ satisfy the inequalities (see Corollary 1):

$$\begin{cases}\ell+m\le n\\ n-\ell-m\le (n+1)/2\end{cases}\tag{5-7}$$

The next proposition is an adaption of Kempner's result [8] to a palindromic polynomial in $\mathbb{Z}[t]$ to investigate its number of positive real roots and roots on the unit circle. It is due to T Zehrt and C Zehrt.

Proposition 4 [25, Proposition 1 and Corollary 1] *Let $f\in\mathbb{Z}[t]$ be a palindromic polynomial of even degree $n\ge 2$ with $f(\pm1)\ne 0$, and let*

$$g(t)=(t-i)^n f\left(\frac{t+i}{t-i}\right)=(t+i)^n f\left(\frac{t-i}{t+i}\right).$$

Then $g(t)$ is a polynomial in $\mathbb{Z}[t]$ of degree n and an even function. Furthermore, if we consider $g(t)$ as a function of $u=t^2$, then the roots of $f(t)$ and $g(u)$ are related as follows.

(1) *$f(t)$ has $2k$ roots on the unit circle if and only if $g(u)$ has k positive real roots.*
(2) *$f(t)$ has 2ℓ real roots if and only if $g(u)$ has ℓ negative real roots.*

Applying Proposition 4, we have the following result about the roots of $Q_{\ell,m,n}(t)$.

Theorem 2 (1) *$Q_{\ell,m,n}(t)$ has exactly seven pairs of complex roots on the unit circle and exactly two pairs of real roots.*

(2) *The two pairs of real roots $(\alpha_{\ell,m,n},1/\alpha_{\ell,m,n})$ and $(\beta_{\ell,m,n},1/\beta_{\ell,m,n})$ satisfy*

$$0<\frac{1}{\alpha_{\ell,m,n}}<\frac{1}{\beta_{\ell,m,n}}<1<\beta_{\ell,m,n}<\alpha_{\ell,m,n}=\tau_{\ell,m,n},$$

where $\tau_{\ell,m,n}$ is the growth rate. Furthermore, the sequence $\{\tau_{\ell,m,n}\}$ converges to ∞ as $n\to\infty$.

(3) *$Q_{\ell,m,n}(t)$ does not have a quadratic factor in $\mathbb{Z}[t]$.*

Proof (1) We adapt Proposition 4 (cf [25, Theorem 2]) to $Q_{\ell,m,n}(t)$. At first,

$$Q_{\ell,m,n}(\pm 1) \neq 0.$$

Consider the polynomial

$$K_{\ell,m,n}(t) := (t-i)^{18} Q_{\ell,m,n}\left(\frac{t+i}{t-i}\right),$$

and replace $u = t^2$. Then $K_{\ell,m,n}(u)$ can be written as follows.

$$\begin{aligned} K_{\ell,m,n}(u) = 4\{&(8n+8)u^9 + (147n+45m+30\ell+207)u^8 \\ &- (3068n+360m+160\ell+3148)u^7 + (11256n+364m-184\ell+7208)u^6 \\ &- (10124n-616m-480\ell-6724)u^5 - (7162n+722m-532\ell+32018)u^4 \\ &+ (12268n+40m-96\ell+27964)u^3 - (4608n-428m+120\ell+8528)u^2 \\ &+ (532n-168m+32\ell+836)u - (17n-13m+2\ell+21)\} \end{aligned}$$

By considering the signs of $K_{\ell,m,n}(u)$ on the real line, it has seven positive real roots and two negative real roots:

u	-41	-31	0	$\frac{1}{10}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	9
$\operatorname{sign}(K_{\ell,m,n}(u))$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$

For example, by (5-7),

$$\tfrac{1}{4}K_{\ell,m,n}(0) = -21-2\ell+13m-17n \leq -21-2\ell+13n-17n = -21-2\ell-4n < 0.$$

Then by Proposition 4, $Q_{\ell,m,n}(t)$ has exactly four real roots, and all other roots are complex and on the unit circle.

(2) Observe that $Q_{\ell,m,n}(0) = 1 > 0$, $Q_{\ell,m,n}(\frac{1}{2}) < 0$ and $Q_{\ell,m,n}(1) = 32+32n > 0$. Hence

$$0 < \frac{1}{\alpha_{\ell,m,n}} < \frac{1}{2} < \frac{1}{\beta_{\ell,m,n}} < 1 < \beta_{\ell,m,n} < \alpha_{\ell,m,n}.$$

Furthermore,

$$Q_{\ell,m,n}\left(\frac{1}{4n+5}\right) < 0 \quad \text{and} \quad Q_{\ell,m,n}\left(\frac{1}{4n+m+\ell+6}\right) > 0,$$

that is,

$$\frac{1}{4n+m+\ell+6} < \frac{1}{\alpha_{\ell,m,n}} = \frac{1}{\tau_{\ell,m,n}} < \frac{1}{4n+5},$$

for all $n \geq 0$. This implies $\tau_{\ell,m,n} \to \infty$ as $n \to \infty$.

(3) Let us fix (ℓ, m, n). Assume that $Q_{\ell,m,n}(t)$ is written as

$$Q_{\ell,m,n}(t) = (1+at+t^2)\left(1+\sum_{k=1}^{8} b_k t^k + \sum_{k=1}^{7} b_{8-k} t^{k+8} + t^{16}\right),$$

where a and b_k are integers. Then by comparing the coefficients of both sides, we get the following simultaneous equations for $a, b_1, b_2, \dots, b_8$:

$$(5\text{-}8)\qquad \begin{cases} b_1 = -a + (-6-4n) \\ b_2 = -ab_1 - 1 + (3-m+2n) \\ b_3 = -ab_2 - b_1 + (-5-\ell+m-3n) \\ b_4 = -ab_3 - b_2 + (5-3m+5n) \\ b_5 = -ab_4 - b_3 + (-1+4m-n) \\ b_6 = -ab_5 - b_4 + (9+\ell-4m+8n) \\ b_7 = -ab_6 - b_5 + (-\ell+5m) \\ b_8 = -ab_7 - b_6 + (11+\ell-5m+10n) \\ ab_8 + 2b_7 - (-2+6m-2n) = 0 \end{cases}$$

By using the method of successive substitution inductively, the last equation of (5-8) is

$$(5\text{-}9)\qquad \begin{aligned} f_{\ell,m,n}(a) := a^9 &+ (4n+6)a^8 + (2n-m-6)a^7 + (-29n-m+\ell-43)a^6 \\ &+ (-9n+4m+11)a^5 + (63n+2m-6k+91)a^4 \\ &+ (11n-3m+\ell-4)a^3 + (-41n+2m+10\ell-55)a^2 \\ &+ (-3n-m-2\ell-3)a + (6n-2m-4\ell+6) \\ = 0.& \end{aligned}$$

On the other hand, the signs of $f_{\ell,m,n}(t)$ on the real line are as follows:

t	$-(4n+6)$	$-(4n+5)$	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{8}{5}$	2
$\operatorname{sign}(f_{\ell,m,n}(t))$	$-$	$+$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$

For example:

- $f_{\ell,m,n}(-3) = 5784 + 308\ell + 748m + 7368n > 0$
- $f_{\ell,m,n}(-2) = -32 - 32n < 0$

Hence $f_{\ell,m,n}(t)$ has one root in the open interval $(-(4n+6), -(4n+5))$, and has eight roots in the open interval $(-3, 2)$, while $-2, -1, 0, 1$ are not roots of $f_{\ell,m,n}(t)$. Therefore $f_{\ell,m,n}(t) = 0$ cannot have an integer solution, which contradicts (5-9). □

Now we are interested in whether the growth rates $\tau_{\ell,m,n}$ are 2–Salem numbers or not. To show this, by means of Theorem 2(1), it is sufficient to prove $Q_{\ell,m,n}(t)$ is irreducible over $\mathbb{Z}$. It is already shown in Theorem 2(3) that $Q_{\ell,m,n}(t)$ is not described as a product of two palindromic polynomials of degree two and sixteen. For the irreducibility of $Q_{\ell,m,n}(t)$, the next proposition plays an important role in view of the main theorem.

Proposition 5 *If $Q_{\ell,m,n}(t)$ is not irreducible, then $Q_{\ell,m,n}(t)$ is described as a product of two distinct monic palindromic polynomials in $\mathbb{Z}[t]$ of even degree.*

Proof By Theorem 2(1), $Q_{\ell,m,n}(t)$ can be written as

$$Q_{\ell,m,n}(t) = (t-\alpha_{\ell,m,n})\Big(t-\frac{1}{\alpha_{\ell,m,n}}\Big)(t-\beta_{\ell,m,n})\Big(t-\frac{1}{\beta_{\ell,m,n}}\Big)(t-\omega_1)(t-\overline{\omega_1}) \times\cdots\times(t-\omega_7)(t-\overline{\omega_7}),$$

where $\alpha_{\ell,m,n}, 1/\alpha_{\ell,m,n}, \beta_{\ell,m,n}, 1/\beta_{\ell,m,n}$ are two pairs of real roots and $\omega_1, \overline{\omega_1}, \dots, \omega_7, \overline{\omega_7}$ are seven pairs of complex roots lying on the unit circle, and each pair of roots that are an inversive pair are algebraically conjugate to each other. Hence if $Q_{\ell,m,n}(t)$ is not irreducible, each of its factors is of even degree, and the claim follows. □

It is not easy to examine the irreducibility for all (ℓ,m,n), but we have the following result under certain restrictions.

Theorem 3 *For $n \equiv 1 \pmod 3$, $Q_{0,n,n}(t)$ and $Q_{n,0,n}(t)$ are irreducible over $\mathbb{Z}$. As a consequence, the growth rates $\tau_{0,n,n}$ and $\tau_{n,0,n}$ are 2–Salem numbers.*

Proof It is sufficient by Proposition 5 to prove that $Q_{0,n,n}(t)$ and $Q_{n,0,n}(t)$ do not have a monic palindromic factor of degree 4, 6 or 8 for $n \equiv 1 \pmod 3$. Since the method for each case is the same, we will explain the case of degree 4, only.

Let us fix (ℓ,m,n). Suppose that

$$Q_{\ell,m,n}(t) = (1+at+bt^2+at^3+t^4)\Big(1+\sum_{k=1}^{7} c_k t^k + \sum_{k=1}^{6} c_{7-k} t^{k+7} + t^{14}\Big) \tag{5-10}$$

where a, b and c_k are integers. Then by comparing the coefficients of both sides, we get the simultaneous equations for $a, b, c_1, c_2, \dots, c_7$:

$$
(5\text{-}11)\qquad
\begin{cases}
c_1 = -a + (-6 - 4n) \\
c_2 = -ac_1 - b + (3 - m + 2n) \\
c_3 = -ac_2 - bc_1 - a + (-5 - \ell + m - 3n) \\
c_4 = -ac_3 - bc_2 - ac_1 - 1 + (5 - 3m + 5n) \\
c_5 = -ac_4 - bc_3 - ac_2 - c_1 + (-1 + 4m - n) \\
c_6 = -ac_5 - bc_4 - ac_3 - c_2 + (9 + \ell - 4m + 8n) \\
c_7 = -ac_6 - bc_5 - ac_4 - c_3 + (-\ell + 5m) \\
c_6 = -ac_7 - bc_6 - ac_5 - c_4 + (11 + \ell - 5m + 10n) \\
c_5 = -ac_6 - bc_7 - ac_6 - c_5 + (-2 + 6m - 2n)
\end{cases}
$$

By using the method of successive substitution inductively, the last two equations in (5-11) are described as follows:

$$
\begin{aligned}
(1)\quad f_{\ell,m,n}(a,b) :=\; & -1-a^8-b^4-m+a^7(-6-4n)+b^2(1+2m-3n) \\
& + a^6(-8+7b+m-2n)+b(\ell+m-n)+n+b^3(2-m+2n) \\
& + a^4\big(-11-15b^2+6m-11n+b(30-5m+10n)\big)+a\big(15+2\ell \\
& + 2m+b(-46-8m-30n)+b^2(-15-3\ell+3m-9n)+9n \\
& + b^3(24+16n)\big)+a^2\big(2+10b^3-\ell+3m \\
& + b^2(-24+6m-12n)-5n+b(9-12m+21n)\big) \\
& + a^5\big(-29-\ell+m-19n+b(36+24n)\big)+a^3\big(13-2\ell+6m \\
& + b^2(-60-40n)+9n+b(68+4\ell-4m+44n)\big) \\
=\; & 0
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad g_{\ell,m,n}(a,b) :=\; & 12+a^7(2-b)+2m+b^2(-23-4m-15n) \\
& + b^3(-5-\ell+m-3n)+8n+b^4(6+4n)+a^5\big(12+6b^2-2m \\
& + b(-18+m-2n)+4n\big)+a^6\big(12+b(-6-4n)+8n\big) \\
& + b(15+2\ell+2m+9n)+a^3\big(6-10b^3-8m \\
& + b(-35+12m-23n)+14n+b^2(36-4m+8n)\big) \\
& + a\big(4b^4+2\ell+2m+b(4-\ell+7m-11n)+b^3(-14+3m-6n) \\
& - 2n+b^2(10-10m+18n)\big)+a^4\big(34+2\ell-2m \\
& + b(-77-\ell+m-51n)+22n+b^2(30+20n)\big) \\
& + a^2(-46-8m+b^3(-36-24n)+b(-7-6\ell+10m-3n) \\
& - 30n+b^2(87+3\ell-3m+57n)) \\
=\; & 0
\end{aligned}
$$

First, consider the case $(\ell, m, n) = (0, n, n)$, that is, $Q_{0,n,n}(t)$, which is the denominator of the growth function of the Coxeter polytope obtained by gluing together $n+1$ copies of T only along the facets of type B (cf Figure 11).

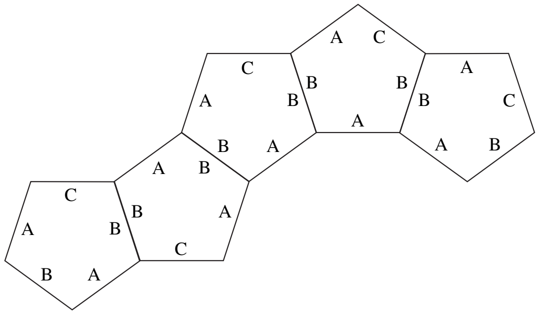


Figure 11: The Coxeter polytope obtained by gluing T 4 times, satisfying $(\ell, m, n) = (0, 4, 4)$

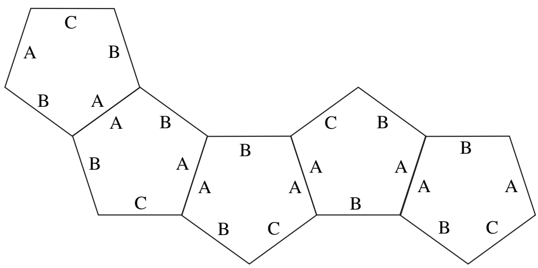


Figure 12: The Coxeter polytope obtained by gluing T 4 times, satisfying $(\ell, m, n) = (4, 0, 4)$

By the assumption, there exist $a, b \in \mathbb{Z}$ that depend on n and satisfy the two equations (1) and (2). If $(a, b) \equiv (0, 0) \pmod 3$, then $f_{0,n,n}(a, b) \equiv f_{0,n,n}(0, 0) = -1 \pmod 3$, which contradicts (1). Hence $(a, b) \equiv (0, 0) \pmod 3$ is impossible. If $(a, b) \equiv (1, 0) \pmod 3$, then $f_{0,n,n}(a, b) \equiv f_{0,n,n}(1, 0) = -(26 + 4n) \equiv 1 - n \pmod 3$. So it is possible to satisfy (1) only if $n \equiv 1 \pmod 3$. On the other hand, $g_{0,n,n}(a, b) \equiv g_{0,n,n}(1, 0) = 32 + 8n \equiv -(1 + n)$. Hence it is possible to satisfy (2) only if $n \equiv -1 \pmod 3$, which is contradiction. So $(a, b) \equiv (1, 0) \pmod 3$ is also impossible. Table 1 lists the values of $f_{0,n,n}(a, b)$ and $g_{0,n,n}(a, b)$, and whether it is possible for (a, b) to satisfy (1) and (2), for all the cases of (a, b) modulo 3. (We leave the box for the value of $g_{0,n,n}(a, b)$ empty if the value of $f_{0,n,n}(a, b)$ gives us sufficient information.)

Therefore there are no integers a and b satisfying

$$f_{0,n,n}(a, b) = g_{0,n,n}(a, b) = 0$$

if $n \equiv 1 \pmod 3$, which implies that $Q_{0,n,n}(t)$ has no monic palindromic factor of degree 4 if $n \equiv 1 \pmod 3$.

(a,b)	$f_{0,n,n}(a,b)$	$g_{0,n,n}(a,b)$	
$(0,0)$	-1		impossible
$(1,0)$	$-26-4n \equiv 1-n$	$32+8n \equiv -(1+n)$	impossible
$(-1,0)$	$-12-12n \equiv 0$	$-8-8n \equiv 1+n$	possible only if $n \equiv -1$
$(0,1)$	1		impossible
$(0,-1)$	$-3-2n \equiv n$	$-15-14n \equiv n$	possible only if $n \equiv 0$
$(1,1)$	0	$-2-2n \equiv 1+n$	possible only if $n \equiv -1$
$(-1,1)$	0	$-6-4n \equiv -n$	possible only if $n \equiv 0$
$(1,-1)$	$-280-114n \equiv -1$		impossible
$(-1,-1)$	$48+54n \equiv 0$	$78+82n \equiv n$	possible only if $n \equiv 0$

Table 1: The list of the values of $f_{0,n,n}(a,b)$ and $g_{0,n,n}(a,b)$, and whether it is possible for (a,b) to satisfy $f_{0,n,n}(a,b) = g_{0,n,n}(a,b) = 0$, for all the cases of (a,b) modulo 3

(a,b)	$f_{n,0,n}(a,b)$	$g_{n,0,n}(a,b)$	
$(0,0)$	$-1+n$	$12+8n \equiv -n$	impossible
$(1,0)$	$-26-24n \equiv 1$		impossible
$(-1,0)$	$-12-12n \equiv 0$	$-8-8n \equiv 1+n$	possible only if $n \equiv -1$
$(0,1)$	1		impossible
$(0,-1)$	$-3-4n \equiv -n$	$-15-10n \equiv -n$	possible only if $n \equiv 0$
$(1,1)$	0	$-2-2n \equiv 1+n$	possible only if $n \equiv -1$
$(-1,1)$	0	$-6-4n \equiv -n$	possible only if $n \equiv 0$
$(1,-1)$	$-280-182n \equiv -1+n$	$378+248n \equiv -n$	impossible
$(-1,-1)$	$48+50n \equiv -n$	$78+74n \equiv -n$	possible only if $n \equiv 0$

Table 2: The list of the values of $f_{n,0,n}(a,b)$ and $g_{n,0,n}(a,b)$, and whether it is possible for (a,b) to satisfy $f_{n,0,n}(a,b) = g_{n,0,n}(a,b) = 0$, for all the cases of (a,b) modulo 3

Next consider the case $(\ell,m,n) = (n,0,n)$, that is, $Q_{n,0,n}(t)$, which is the denominator of the growth function of the Coxeter polytope obtained by gluing together $n+1$ copies of T only along the facets of type A (cf Figure 12).

Table 2 is the list for all the cases of (a,b) modulo 3 as for the case $(\ell,m,n) = (n,n,0)$.

Therefore there are no integers a and b satisfying $f_{n,0,n}(a,b) = g_{n,0,n}(a,b) = 0$ if $n \equiv 1 \pmod 3$, which implies that $Q_{n,0,n}(t)$ has no monic palindromic factor of degree 4 if $n \equiv 1 \pmod 3$. □

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