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Operations on open book foliations

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We study b–arc foliation changes and exchange moves of open book foliations which generalize the corresponding operations in braid foliation theory. We also define a bypass move as an analogue of Honda's bypass attachment operation.

As applications, we study how open book foliations change under a stabilization of the open book. We also generalize Birman–Menasco's split/composite braid theorem: we show that closed braid representatives of a split (resp. composite) link in a certain open book can be converted to a split (resp. composite) closed braid by applying exchange moves finitely many times.

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1 Introduction

This is a sequel of the papers [18; 19; 20] on open book foliations, in which techniques to study the topology and contact structures of 3–manifolds are developed. The idea of an open book foliation originally came from the works of Bennequin [1] and Birman and Menasco [3; 4; 5; 6; 7; 8; 9; 10; 11].

In this paper we study three types of operations on open book foliations on surfaces that are realized by isotopies of the surfaces: the *b–arc foliation change* (Section 3), *bypass move* (Section 4) and *exchange move* (Section 5).

A b–arc foliation change and an exchange move are generalizations of Birman–Menasco's *foliation change* and exchange move in braid foliation theory. A bypass move can be seen as an analogue of Honda's *bypass attachment* in convex surface theory.

It is natural to expect our b–arc foliation change and exchange move on open book foliations to be more complex than Birman and Menasco's original moves on braid foliations. In fact, we need additional assumptions to make these operations actually work.

Roughly speaking, a b–arc foliation change and a bypass move are associated to isotopies interchanging the "heights" of a pair of adjacent saddle points of a surface. A b–arc foliation change treats the case that two saddles have the same sign whereas a bypass move treats the case with opposite signs.

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These isotopies are local in the sense that they take place in 3–balls. Hence both a b–arc foliation change and a bypass move are local operations on open book foliations. Under these operations, the total number of singularities of an open book foliation stays the same. Moreover, if there are braids passing through the 3–balls, the isotopies preserve the braid isotopy classes.

On the contrary, an isotopy realizing an exchange move may change the braid isotopy class. (The braid index and the transverse link type of the braid are preserved.) Also the number of singularities of an open book foliation decreases by an exchange move.

In the second half of the paper we discuss two applications:

We study the effect of (de)stabilizations of open books on open book foliations in Section 6. We show that the open book foliation of a surface changes in two ways after a stabilization of the open book. Next, we see that the resulting two open book foliations are related to each other by bypass moves and exchange moves.

As applications of b–arc foliation changes and exchange moves, in Section 7 we consider the split/composite closed braid theorems of Birman and Menasco [3] in the setting of general open books and prove them under certain conditions.

2 Preliminaries

We assume that the readers are familiar with the basic definitions and properties of open book foliations which can be found in [18; 19].

Let (S,ϕ) be an open book decomposition of a closed oriented 3–manifold M, where $S = S_{g,r}$ is a genus-g surface with r boundary components, and $\phi \in \mathrm{Diff}^+(S,\partial S)$ is an orientation-preserving diffeomorphism of S that fixes the boundary pointwise. The manifold M is often denoted by $M_{(S,\phi)}$. Let B denote the *binding* of the open book and $\pi\colon M \setminus B \to S^1$ the fibration whose fiber $S_t := \pi^{-1}(t)$ is a *page*.

An oriented link L in $M_{(S,\phi)}$ is called a *closed braid* with respect to the open book (S,ϕ) if L is disjoint from the binding B and positively transverse to each page S_t.

Let $F \subset M_{(S,\phi)}$ be an embedded, oriented surface possibly with boundary. If F has a boundary ∂F, we require that ∂F be a closed braid with respect to (S,ϕ). Up to perturbation of F the singular foliation

$$\mathcal{F}_{\mathrm{ob}}(F) = \{F \cap S_t \mid t \in [0,1]\}$$

satisfies the following conditions (see [19, Theorem 2.5]).

($\mathcal{F}$i) The binding B pierces the surface F transversely in finitely many points. Moreover, $p \in B \cap F$ if and only if there exists a disc neighborhood $N_p \subset \mathrm{Int}(F)$ of p on which the foliation $\mathcal{F}_{\mathrm{ob}}(N_p)$ is radial with the node p; see Figure 1(1), (2). We call p an *elliptic* point.

($\mathcal{F}$ii) The leaves of $\mathcal{F}_{\mathrm{ob}}(F)$ along ∂F are transverse to ∂F.

($\mathcal{F}$iii) All but finitely many fibers S_t intersect F transversely. Each exceptional fiber is tangent to F at a single point in $\mathrm{Int}(F)$. In particular, $\mathcal{F}_{\mathrm{ob}}(F)$ has no saddle–saddle connections.

($\mathcal{F}$iv) All the tangencies of F and fibers are of saddle type; see Figure 1(3), (4). We call them *hyperbolic* points.

We say that an elliptic point p is *positive* (respectively *negative*) if the binding B is positively (respectively negatively) transverse to F at p. The sign of the hyperbolic point q is *positive* (respectively *negative*) if the positive normal direction of F at q agrees (respectively disagrees) with the direction of t. We denote the sign of a singular point v by $\mathtt{sgn}(v)$. See Figure 1. We will describe an elliptic point by a hollowed circle with its sign inside, a hyperbolic point by a dot with the sign nearby. We often write a positive normal vector $\vec{n}_F$ to F, by dashed arrows.

Definition 2.1 We call each connected component of $F \cap S_t$ a *leaf*. We say a leaf l of $\mathcal{F}_{\mathrm{ob}}(F)$ is *regular* if l does not contain a tangency point and is *singular* otherwise. The regular leaves are classified into the following three types.

- a–arc: An arc where one endpoint lies on B and the other lies on ∂F.
- b–arc: An arc whose endpoints both lie on B.
- c–circle: A simple closed curve.

In order to study the topology and geometry of 3–manifolds $M_{(S,\phi)}$ it is often important to take the following homotopical properties of leaves into account.

Definition 2.2 [18] We say that a b–arc $b \subset S_t$ is *essential* (respectively *strongly essential*) if b is not boundary-parallel in $S_t \setminus (S_t \cap \partial F)$ (respectively S_t). An elliptic point v is called *strongly essential* if every b–arc that ends at v is strongly essential. An open book foliation $\mathcal{F}_{\mathrm{ob}}(F)$ is called (*strongly*) *essential* if all the b–arcs are (strongly) essential.

For a b–arc the conditions "*boundary-parallel in S_t*" and "*non*strongly essential" are equivalent. In this paper we prefer to use the former.

Essentiality is a natural condition in the sense that if F is incompressible then upon application of an isotopy (possibly the identity) that fixes ∂F (if it exists) we may assume F admits an essential open book foliation.

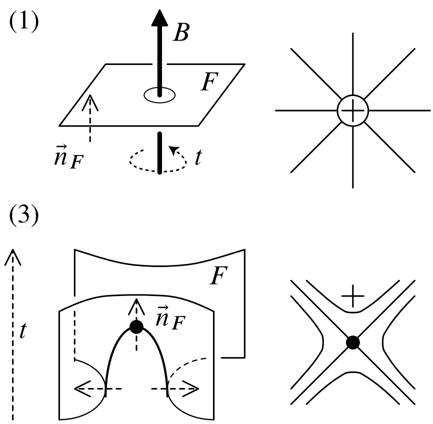

Figure 1: Signs of singularities and normal vectors $\vec{n}_F$ where (1) and (2) are elliptic points and (3) and (4) are hyperbolic points.

Definition 2.3 We say a b–arc b in the page S_t is *separating* if b separates the page S_t into two regions.

Clearly an inessential or boundary-parallel b–arc is separating. We will use this separating condition in Proposition 3.2, Lemmas 7.7 and 7.6 below.

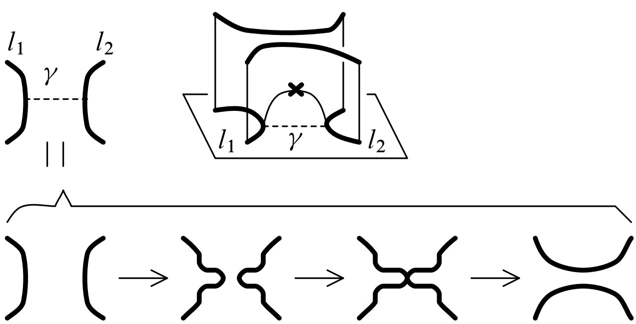

Figure 2: A describing arc (dashed) for a hyperbolic point

Definition 2.4 A hyperbolic point is regarded as a process of switching the configuration of leaves. As $t \in [0, 1]$ increases, two regular leaves l_1 and l_2 approach along an arc γ (the dashed arc in Figure 2) connecting l_1 and l_2. At a critical moment l_1 and l_2 form a hyperbolic point and the configuration changes. See the passage in Figure 2. The embedding near the hyperbolic point is determined by the isotopy class of the arc γ. We call γ a *describing arc* of the hyperbolic point and a dashed arc often depicts a describing arc.

Hyperbolic singularities in $\mathcal{F}_{\mathrm{ob}}(F)$ are classified into six types, according to the types of nearby regular leaves: Type aa, ab, bb, ac, bc and cc as depicted in Figure 3. Such a model neighborhood is called a *region*. We denote by $\mathtt{sgn}(R)$ the sign of the hyperbolic point contained in the region R.

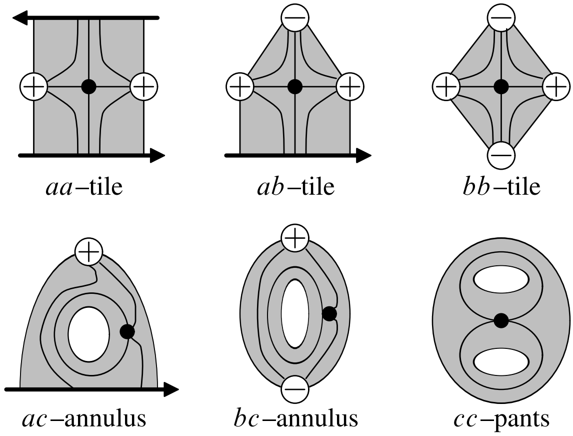

Figure 3: Six types of regions

3 b–arc foliation change

In this section we generalize Birman–Menasco's *foliation changes* of braid foliations [7, page 123] to *b–arc foliation changes* of open book foliations (Theorem 3.1).

Here is the set up for a b–arc foliation change: Let (S,ϕ) be an open book decomposition of a 3–manifold M and $\mathcal{F}_{\mathrm{ob}}(F)$ the open book foliation on F, where F is a closed surface in the complement of a closed braid L or a Seifert surface of a closed braid L.

We will use the underlined letter "$\underline{a}$" to indicate the image of an arc $a \subset S_t$ superimposed on S by the projection $S_t \ni (p,t) \mapsto p \in S$. This allows us to compare leaves in different pages. We assume that the region decomposition of F contains two tiles R_1, R_2 satisfying the following conditions (i)–(iv). See Figure 4(a):

(i) R_i $(i=1,2)$ is either an ab–tile or a bb–tile.

(ii) $\mathtt{sgn}(R_1)=\mathtt{sgn}(R_2)=\varepsilon \in \{+1,-1\}$.

(iii) R_1 and R_2 are adjacent exactly at one b–arc b.

Let v (respectively A) be the negative (respectively positive) elliptic point at the end of b, and $l_1, \dots, l_6$ be boundary arcs of $R_1 \cup R_2$ as depicted in Figure 4. Let $B \in \partial R_1$ and $C \in \partial R_2$ be positive elliptic points.

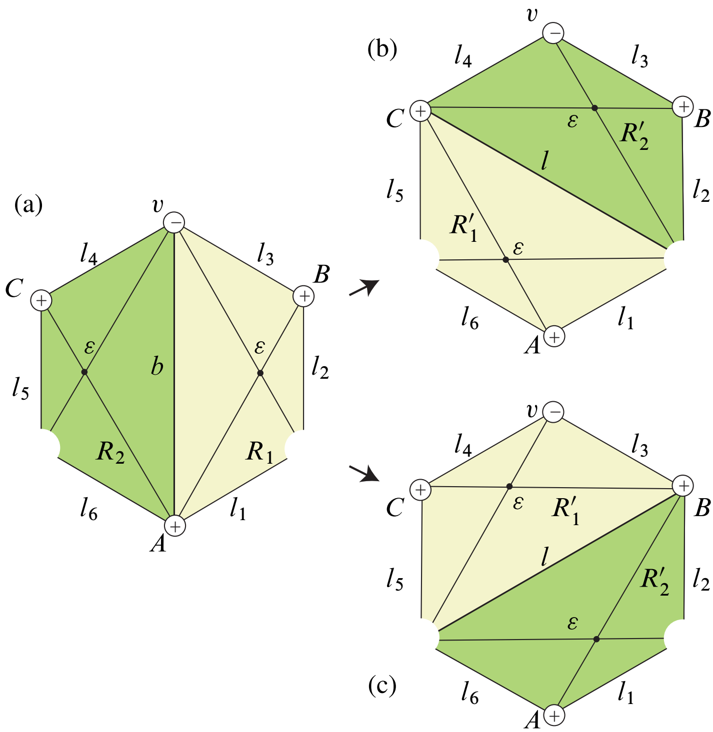

Figure 4: b–arc foliation change

Suppose that $l_k \subset S_{t_k}$, where $k = 1, \dots, 6$ and $t_k \in [0, 1)$, and the hyperbolic point of R_i is sitting on the page S_{τ_i}. The open book foliation $\mathcal{F}_{\mathrm{ob}}(R_1 \cup R_2)$ imposes the relations

$$\tau_1 < t_2,$$
$$\max\{t_1, t_3\} < \tau_1 < \tau_2 < \min\{t_4, t_6\},$$
$$t_5 < \tau_2.$$

In addition to the above conditions (i, ii, iii) we further require that

$$\max\{t_1, t_3, t_5\} < \tau_1 < \tau_2 < \min\{t_2, t_4, t_6\},$$

or

(iv) $t_1 = t_3 = t_5 < \tau_1 < \tau_2 < t_2 = t_4 = t_6$.

Let γ_i denote the describing arc for the hyperbolic point in R_i $(i = 1, 2)$. We may assume that $\underline{\gamma_1}$ joins $\underline{l_1}$ and $\underline{l_3}$. See Figure 5.

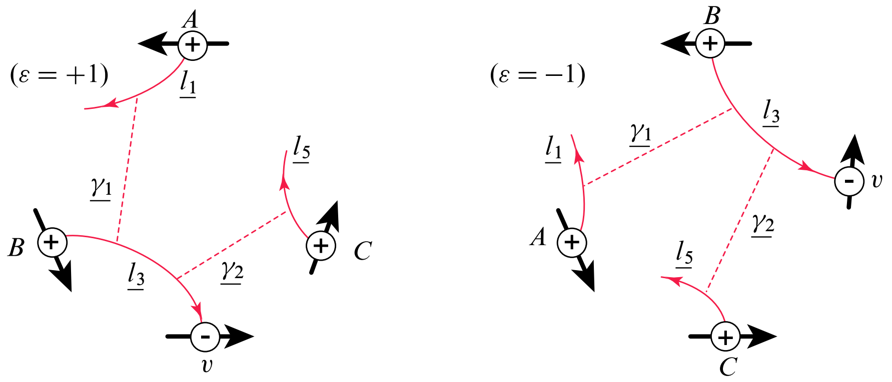

Figure 5: The superimposed graph in S: The bold (respectively thin) arrows represent parts of the oriented binding (respectively leaves).

By sliding $\underline{\gamma_2}$ along $\underline{b}$, we can further assume that $\underline{\gamma_2}$ joins $\underline{l_3}$ and $\underline{l_5}$. Because $\mathtt{sgn}(R_1) = \mathtt{sgn}(R_2) = \varepsilon$, if we walk along $\underline{l_3}$ from B to v, regardless of the sign ε, we meet $\underline{\gamma_1}$ first then $\underline{\gamma_2}$. Both $\underline{\gamma_1}, \underline{\gamma_2}$ lie on the same side of $\underline{l_3}$. In general the arcs $\underline{l_1}, \underline{l_3}, \underline{l_5}, \underline{\gamma_1}, \underline{\gamma_2}$ may intersect each other.

Theorem 3.1 (b–arc foliation change) *Assume that R_1, R_2 satisfy the above conditions (i)–(iv). Suppose that the graph $\underline{l_1} \cup \underline{l_3} \cup \underline{l_5} \cup \underline{\gamma_1} \cup \underline{\gamma_2}$ (see Figure 5) is a tree in S. Then there is an ambient isotopy $\Phi_\tau \colon M \to M$ supported on $M \setminus B$ such that:*

(1) *$F' = \Phi_1(F)$ admits an open book foliation $\mathcal{F}_{\mathrm{ob}}(F')$; if $\mathcal{F}_{\mathrm{ob}}(F)$ is essential, then so is $\mathcal{F}_{\mathrm{ob}}(F')$.*

(2) *The region decomposition of $\mathcal{F}_{\mathrm{ob}}(F')$ contains regions R'_1, R'_2 (see (b) and (c) in Figure 4) such that:*

   (a) *Their type is either aa, ab, or bb–tile.*

   (b) *$\mathtt{sgn}(R'_1) = \mathtt{sgn}(R'_2) = \varepsilon$ as in (ii).*

   (c) *$\Phi_1(R_1 \cup R_2) = R'_1 \cup R'_2$.*

   (d) *$R'_1 \cap R'_2$ is exactly one leaf l of type a or b.*

   (e) *The numbers of the hyperbolic points connected to v and A by a singular leaf decrease both by one, though the total number of hyperbolic points remains the same.*

(3) *Φ_t preserves the region decomposition of $F \setminus (R_1 \cup R_2)$.*

(4) *If ∂F is nonempty $\Phi_t(\partial F)$ is a closed braid with respect to (S, ϕ) for all $t \in [0, 1]$, ie $L = \partial F$ and $L' = \partial F'$ are braid isotopic.*

Proof Let $N = N(\underline{l_1} \cup \underline{l_3} \cup \underline{l_5} \cup \underline{\gamma_1} \cup \underline{\gamma_2}) \subset S$ be a regular neighborhood of the graph $G = \underline{l_1} \cup \underline{l_3} \cup \underline{l_5} \cup \underline{\gamma_1} \cup \underline{\gamma_2}$. Since G is a tree, N is planar and there is an embedding ι of N in D^2 such that $\iota(\partial S \cap N) \subset \partial D^2$. See Figure 6.

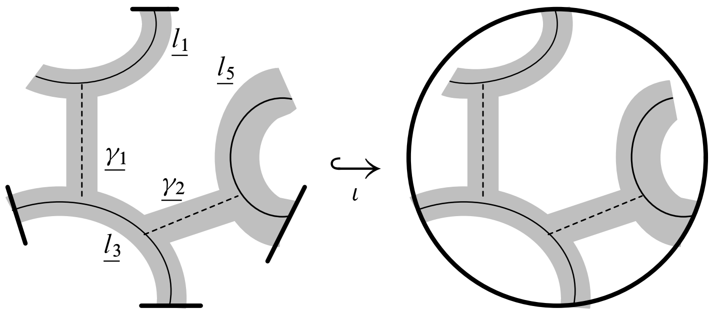

Figure 6: An embedding $\iota\colon N \hookrightarrow D^2$ when $\varepsilon = +1$.

We may assume that the region $R_1 \cup R_2$ is embedded in $N \times [t_1, t_2]$, hence also in $D^2 \times [t_1, t_2]$. The foliation on the surface $(\iota \times \mathrm{id})(R_1 \cup R_2) \subset D^2 \times [t_1, t_2]$ induced by the family of discs $\{D^2 \times \{t\} \mid t \in [t_1, t_2]\}$ is the same as that on $\mathcal{F}_{\mathrm{ob}}(R_1 \cup R_2)$. Theorem 2.1 of Birman and Finkelstein [2] guarantees the existence of a desired isotopy Φ_t.

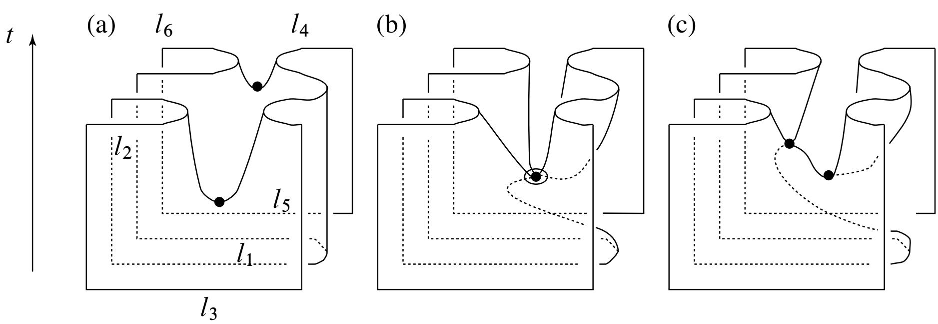

Figure 7: An isotopy Φ_t of $R_1 \cup R_2$ that realizes a b–arc foliation change.

Here we sketch the transition of $\Phi_t(R_1 \cup R_2)$ from $t = t_1$ to $t = t_2$ when $\varepsilon = +1$. Figure 7(a) depicts the interior of $R_1 \cup R_2$, where the two saddles lie on the different pages S_{τ_1} and S_{τ_2} of the open book. We perturb the surface so that the saddles get closer until amalgamated to a monkey saddle, or a valence 6 saddle; see Figure 7(b). By further perturbation the singular point splits into two hyperbolic points as shown in Figure 7(c). The isotopy replaces γ_1, γ_2 (the top row of Figure 8) with γ_1', γ_2' (cf the bottom row). This results in a change in the open book foliation of $R_1 \cup R_2$ as

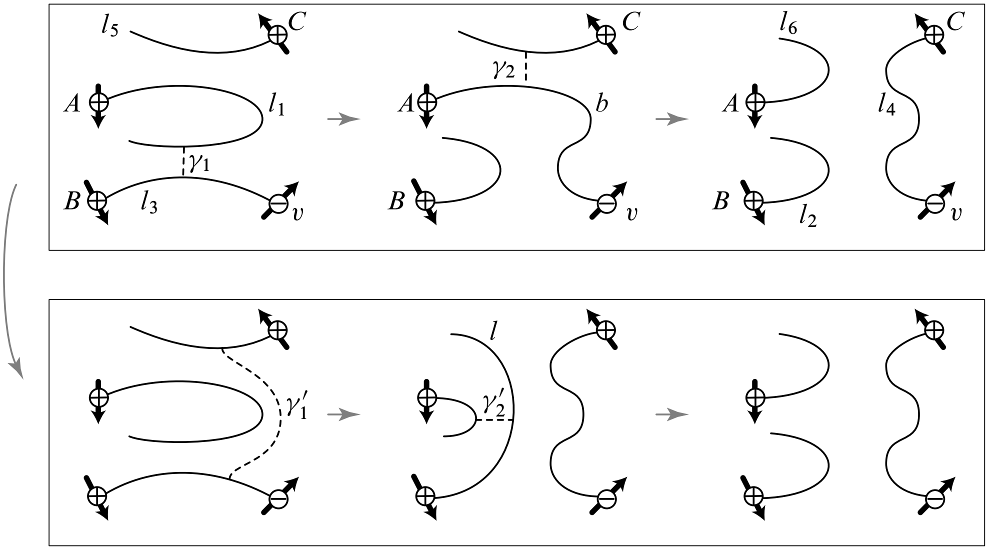

Figure 8: Replacing describing arcs where $\varepsilon = +1$

depicted in Figure 4. For example, Figure 8 corresponds to the passage (a)→(c) in Figure 4.

Finally it is easy to see the assertion (1). If $\mathcal{F}_{\mathrm{ob}}(F)$ is essential and $\mathcal{F}_{\mathrm{ob}}(F')$ is inessential then the leaf $l = R_1' \cap R_2'$ must be inessential. This implies that at least one of the leaves l_i must be inessential, which is a contradiction. (In the case of Figure 8, at least one of l_1 or l_4 is inessential.) □

In general, checking the assumption of Theorem 3.1 is not so simple, but there is one sufficient condition which is easier to check:

Proposition 3.2 *In addition to the conditions (i)–(iv), assume further that the common b–arc b of the tiles R_1 and R_2 is separating in the sense of Definition 2.3. Then the graph $\underline{l_1} \cup \underline{l_3} \cup \underline{l_5} \cup \underline{\gamma_1} \cup \underline{\gamma_2}$ is a tree in S.*

Proof Suppose that $\varepsilon = +1$ (for the case $\varepsilon = -1$ a parallel argument holds). Let $S \setminus \underline{b} = D \sqcup D'$, where D (respectively D') is the connected region on the left (right) side of $\underline{b}$ as we walk along $\underline{b}$ from the positive elliptic point A to the negative elliptic point v. See Figure 9.

Note that v and A lie on the same boundary component of S because b is separating. The vertex C (respectively B) lies on ∂D (respectively $\partial D'$) but not necessarily on the same boundary component on which v and A lie Since l_1, γ_1 and l_3 are

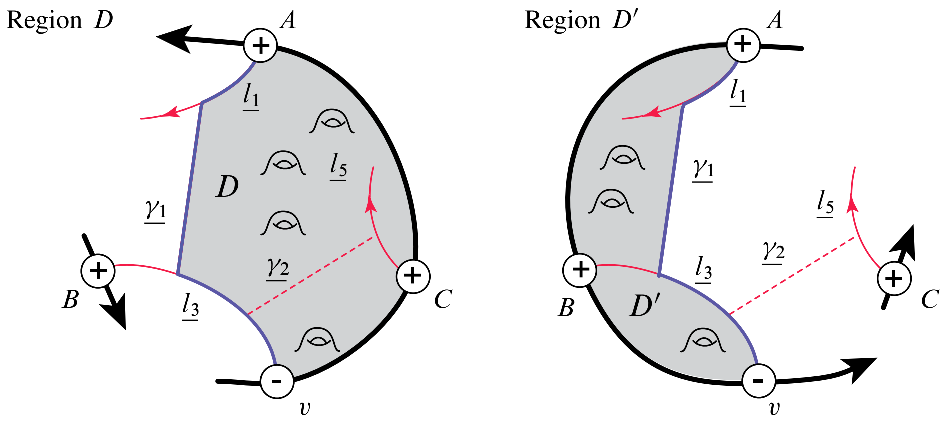

Figure 9: The regions D and D' when $\varepsilon = +1$. Vertices B and C may not be on the same binding component (bold arrows) where A and v lie.

contained in the same page S_{t_1}, their images $\underline{l_1}, \underline{l_3}, \underline{\gamma_1}$ form a tree in S, and the tree $\underline{l_1} \cup \underline{l_3} \cup \underline{\gamma_1}$ is disjoint from $\mathrm{Int}(D)$. The tree $\underline{\gamma_2} \cup \underline{l_5}$ is contained in D. Hence the graph $(\underline{l_1} \cup \underline{l_3} \cup \underline{\gamma_1}) \cup (\underline{\gamma_2} \cup \underline{l_5})$ is a tree in S. □

The above argument implies the following:

Corollary 3.3 *If b is boundary-parallel (so D or D' is a disc region) then l_3 or l_4 is boundary-parallel.*

Remark The essential point in the above proof is that $\mathrm{Int}(\underline{\gamma_2})$ and $\underline{l_1} \cup \underline{l_3}$ are disjoint, so our problem is reduced to a problem in braid foliation theory, a theory for the *trivial* open book (D^2, id). Suppose that $\underline{\gamma_2}$ is parallel to $\underline{\gamma_1}$ as in Figure 10.

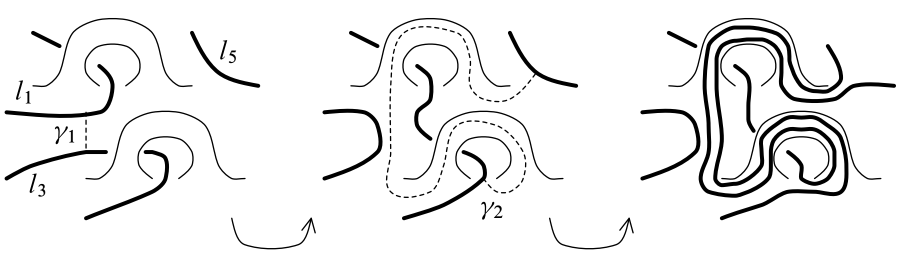

Figure 10: Nested saddles

The right sketch shows the saddles for γ_1 and γ_2 are *nested*. The saddle of γ_2 can exist only after the saddle of γ_1, so the trick of replacing the order of describing arcs (cf Figure 8) does not work.

The existence of nested saddles is a unique feature of open book foliations. In braid foliation theory no b–arcs are strongly essential because the page S is a disc, so by Proposition 3.2 if $\mathtt{sgn}(R_1) = \mathtt{sgn}(R_2)$ the graph $\underline{l_1} \cup \underline{l_3} \cup \underline{l_5} \cup \underline{\gamma_1} \cup \underline{\gamma_2}$ is always a tree in S and nested saddles do not exist.

Remark One might consider an *a–arc* foliation change under a similar setting where two tiles of the same sign are adjacent along an a–arc, instead of a b–arc. However, "a–arc foliation change" does *not* work in general. This is why we call our operation *b–arc* foliation change, rather than simply calling it foliation change. We thank Bill Menasco for pointing this out and informing us of the importance of the separating condition on the b–arc b in Proposition 3.2.

4 Bypass move

In the setting of a b–arc foliation change the two adjacent tiles R_1, R_2 must have the same sign. This raises a natural question: how about the case where two adjacent regions have *opposite* signs?

Birman and Menasco observed in braid foliation theory that the opposite sign case is more complicated than the same sign case: They found that the complement of the hexagon region $F \setminus (R_1 \cup R_2)$ or a closed braid may prevent the desired height exchange of the saddles (see [3, Figure 11b]). Thus validity of similar moves in open book foliation theory should reflect global features of the surface F.

In this section we study when the "heights" of hyperbolic points of opposite signs are exchangeable. A short answer to this question would be "when there exists a bypass-rectangle", which we define shortly. We start by defining *dividing sets* whose idea comes from Giroux's *dividing sets* for convex surfaces [15, Section 2]; see also Honda [16, Section 3.1.3].

Definition 4.1 (Dividing set) Let $F \subset M_{(S,\phi)}$ be a surface admitting an open book foliation $\mathcal{F}_{\mathrm{ob}}(F)$ with no c–circles. (In [19] we prove that by finger moves we can always get rid of c–circles.) Let $\Gamma \subset F$ be a set of properly embedded arcs and circles that decompose F into regions F_+ and F_- such that:

- $F \setminus \Gamma = F_+ \sqcup F_-$.
- As sets (forgetting orientations), $\Gamma = \partial F_+ \setminus \partial F = \partial F_- \setminus \partial F$.
- The leaves of $\mathcal{F}_{\mathrm{ob}}(F)$ along Γ are oriented out of the region F_+ and into F_-.
- F_+ contains all the positive singularities of $\mathcal{F}_{\mathrm{ob}}(F)$.
- F_- contains all the negative singularities of $\mathcal{F}_{\mathrm{ob}}(F)$.

We call Γ the *dividing set* of $\mathcal{F}_{\mathrm{ob}}(F)$.

Given an open book foliation $\mathcal{F}_{\rm ob}(F)$ with no c–circles, the region F_- can be identified, up to isotopy, with a collar neighborhood of the graph G_{--} of $\mathcal{F}_{\rm ob}(F)$ (see [19] for definition), hence Γ is uniquely determined up to isotopy.

Next we will define a bypass rectangle which is inspired by Honda's *bypass half–disc* [16, Section 3.4].

Definition 4.2 (Bypass rectangle) Let $\mathcal{D} \subset M_{(S,\phi)}$ be a rectangle such that:

(1) $\mathcal{F}_{\rm ob}(\mathcal{D})$ contains a hyperbolic point of sign ε (see Figure 11).

(2) The boundary $\partial\mathcal{D}$ consists of four piecewise smooth curves $\delta_1, \dots, \delta_4$ such that the oriented leaves are pointing out of (respectively into) $\mathcal{D}$ along δ_1, δ_3 (respectively δ_2, δ_4).

Denote the four corner points by p, p', q, q'. We call $\mathcal{D}$ a *bypass rectangle* of $\mathtt{sgn}(\mathcal{D}) = \varepsilon$.

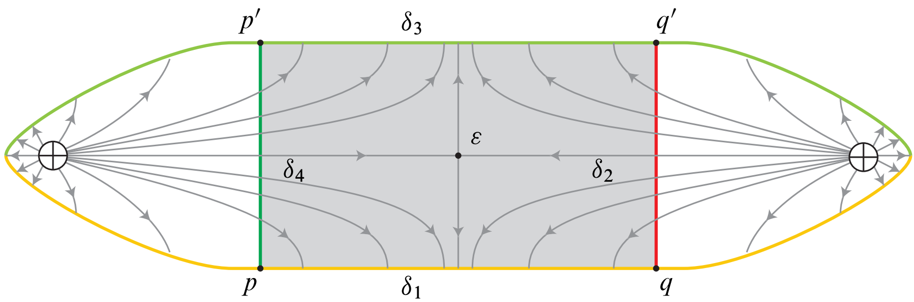

Figure 11: Bypass rectangle $\mathcal{D}$ (shaded) embedded in a degenerate aa–tile

Definition 4.3 ($\mathtt{Type1}$, $\mathtt{Type2}$ hexagon R) Let $F \subset M_{(S,\phi)}$ be a surface admitting an open book foliation. Suppose that F contains a hexagon region R consisting of two bb–tiles of opposite signs meeting along a b–arc as in Figure 12(1). We name the vertices (elliptic points) A, B, C, D, E, F counterclockwise. We may assume that $\mathtt{sgn}(A) = \mathtt{sgn}(C) = \mathtt{sgn}(E) = +1$ and $\mathtt{sgn}(B) = \mathtt{sgn}(D) = \mathtt{sgn}(F) = -1$. We require that the boundary b–arcs $\overline{AB}, \overline{CD}, \overline{EF}$ lie on the same page of the open book and $\overline{BC}, \overline{DE}, \overline{FA}$ lie on another same page.

Let $\boldsymbol{p}, \boldsymbol{q}$ denote the two hyperbolic points of R. From now on we assume that

$$\mathtt{sgn}(\boldsymbol{p}) = +1, \quad \mathtt{sgn}(\boldsymbol{q}) = -1.$$

(If $\mathtt{sgn}(\boldsymbol{p}) = -1$, $\mathtt{sgn}(\boldsymbol{q}) = +1$, similar statements hold.) With this sign assumption there are two possible movie presentations realizing the open book foliation $\mathcal{F}_{\rm ob}(R)$. See Figure 13. We call them $\mathtt{Type1}$ and $\mathtt{Type2}$.

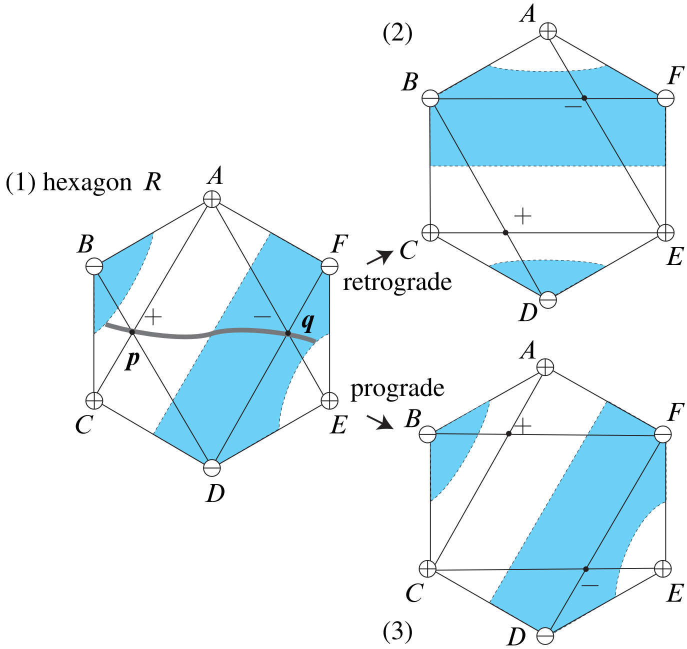

Figure 12: (1) Original hexagon R. (2) Hexagon after retrograde bypass move. (3) Hexagon after prograde bypass move; dashed arcs are dividing sets; shaded regions are negative and unshaded regions are positive.

Theorem 4.4 *Suppose there exists a bypass rectangle $\mathcal{D}$ in $M \setminus F$ such that:*

(1) *The union of arcs $\delta_1 \cup \delta_2 \cup \delta_4 \subset \partial\mathcal{D}$ is glued to the thick gray arc in Figure 12(1) that joins the dividing curves and contains $\boldsymbol{p}$ and $\boldsymbol{q}$.*

(2) *$p \in \partial\mathcal{D}$ is identified with $\boldsymbol{p} \in R$.*

(3) *$q \in \partial\mathcal{D}$ is identified with $\boldsymbol{q} \in R$.*

(4) *p and q' live on the same page of the open book (Figure 14(1)).*

(5) *p' and q live on the same page of the open book (Figure 14(6)).*

(6) $\mathtt{sgn}(\mathcal{D}) = \begin{cases} +1 & \textit{if } R \textit{ is of } \mathtt{Type1}, \\ -1 & \textit{if } R \textit{ is of } \mathtt{Type2}. \end{cases}$

Then by a local perturbation of F supported on a neighborhood of $R \cup \mathcal{D}$, the open book foliation changes in the following ways.

(i) *If R is of $\mathtt{Type1}$, $\mathcal{F}_{\mathrm{ob}}(R)$ changes as in the passage $(1) \to (2)$ of Figure 12 and the dividing set also changes.*

(ii) *If R is of $\mathtt{Type2}$, $\mathcal{F}_{\mathrm{ob}}(R)$ changes as in the passage $(1) \to (3)$ of Figure 12 but the dividing set stays the same.*

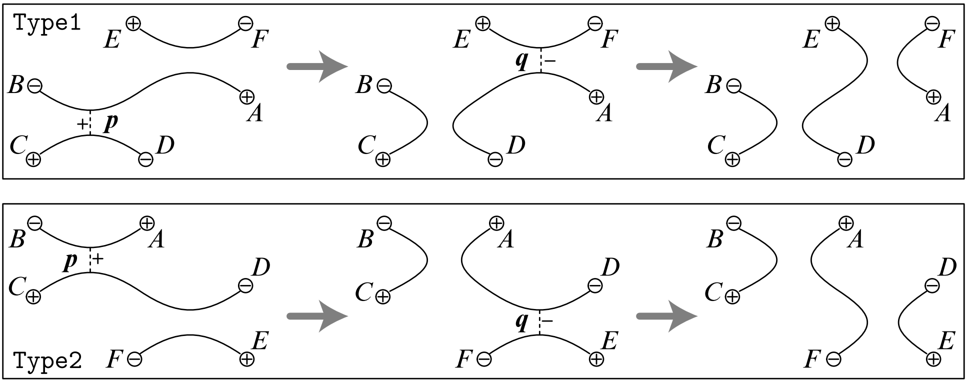

Figure 13: Movie presentations of `Type1` and `Type2` hexagon R

Proof We study the `Type1` case carefully. Similar arguments work for the `Type2` case. Figure 14 shows a movie presentation of a bypass rectangle D attached to a `Type1` hexagon R along $\delta_1, \delta_2, \delta_3$.

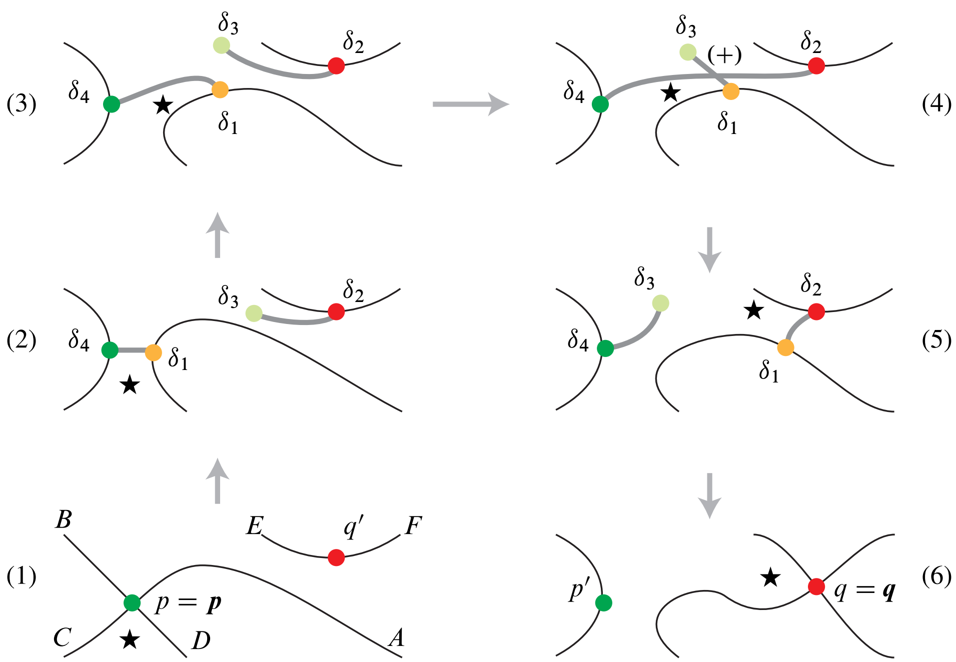

Figure 14: A movie presentation of a bypass rectangle $\mathcal{D}$ attached to a `Type1` hexagon R

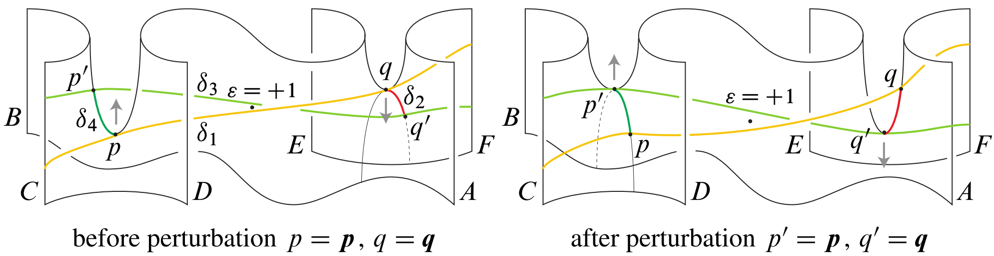

Figure 15: A `Type1` hexagon R slid along the bypass rectangle $\mathcal{D}$.

Locally $\mathcal{D}$ and R are embedded as in the left sketch of Figure 15.

The bypass plays a role of "stopper" that blocks other surfaces or braids (indicated by ★ in Figure 14) from coming from the region between C and D, moving through p and q, and then escaping into the region between A and F. Therefore we can slide the hexagon R along the rectangle $\mathcal{D}$. This perturbation slides the hyperbolic point $\boldsymbol{p}$ along the arc δ_4 from p to p'. Similarly $\boldsymbol{q}$ is slid along δ_2 from q to q'. After the perturbation the arc δ_3 sits on the new R but δ_1 no longer sits on the new R. See Figure 16. □

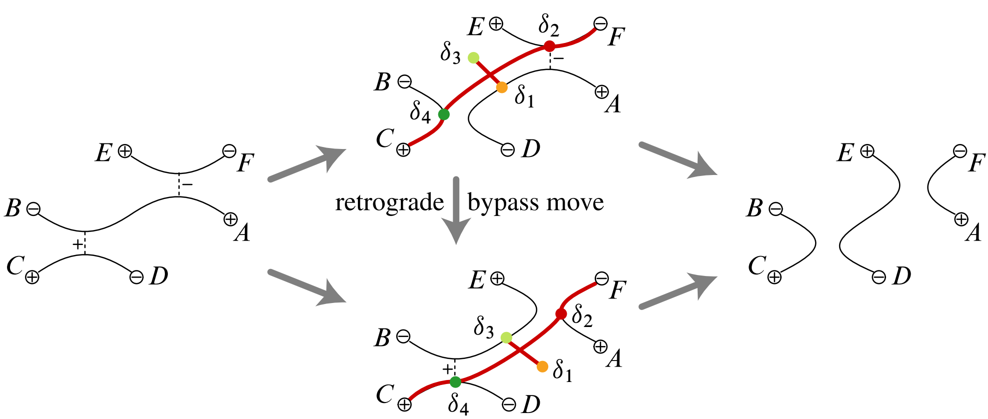

Figure 16: A bypass rectangle $\mathcal{D}$ (indicated by thick arcs) before/after a retrograde bypass move applied on a `Type1` hexagon: $\mathtt{sgn}(\mathcal{D}) = +1$.

Definition 4.5 (Retrograde/prograde bypass moves) The above perturbation of a `Type1` hexagon region interchanges the "heights" of the hyperbolic points $\boldsymbol{p}$, $\boldsymbol{q}$. At one moment $\boldsymbol{p}$, $\boldsymbol{q}$ have the same height. That is, $\boldsymbol{p}$ and $\boldsymbol{q}$ lie on the same page of the open book and are joined by a singular leaf of the open book foliation. The singular leaf is oriented from $\boldsymbol{q}$ to $\boldsymbol{p}$. Recall that $\mathtt{sgn}(\boldsymbol{p}) = +1$, $\mathtt{sgn}(\boldsymbol{q}) = -1$. Namely the

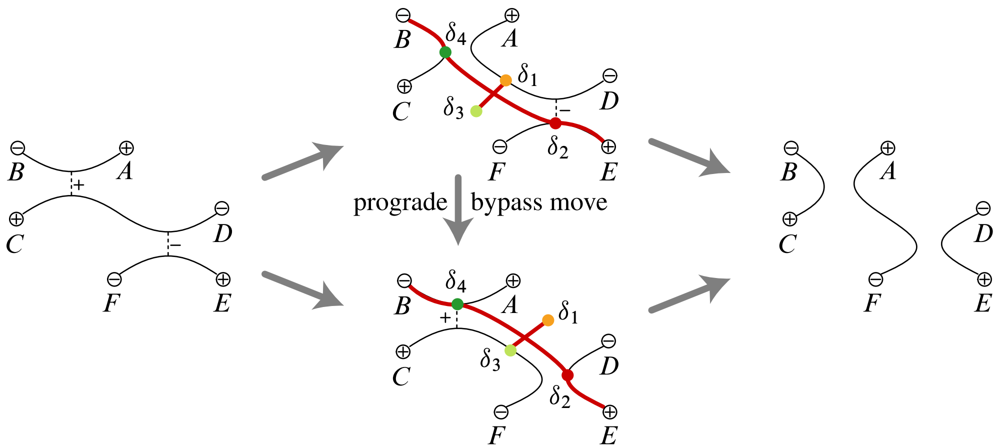

Figure 17: A bypass rectangle $\mathcal{D}$ before/after a prograde bypass move applied on a Type2 hexagon: $\mathtt{sgn}(\mathcal{D}) = -1$.

singular leaf is oriented from a negative hyperbolic point to a positive hyperbolic point. Such a singular leaf is called a *retrograde saddle–saddle connection*. Thus we call the foliation change depicted in (1)→(2) of Figure 12 a *retrograde bypass move*.

On the other hand, for a corresponding perturbation of a Type2 hexagon, the saddle–saddle connection is prograde, that is, the singular leaf is oriented from a positive hyperbolic point to a negative hyperbolic point. Thus we call the change in foliation depicted in (1)→(3) of Figure 12 a *prograde bypass move*.

We name the rectangle $\mathcal{D}$ a *bypass* because our retrograde bypass move and Honda's bypass attachment in convex surface theory yield exactly the same configuration change in dividing sets (compare Honda's [16, Figure 6] with our Figure 12).

Remark In his thesis [21, pages 123-124], LaFountain observes that the "nonstandard" change of braid foliation of Birman–Menasco which does change the graph G_{++} is accomplished through a bypass.

In [12], Dynnikov and Prasolov introduce a bypass for a rectangular diagram, a certain diagrammatic expression of a (Legendrian) link in the standard contact S^3. Their bypass can be turned into a Honda–bypass for the corresponding Legendrian link.

Although in [21; 12] techniques of braid foliations are extensively used we remark that our bypass and their bypasses have differences. For example, we have two types of bypass moves, prograde and retrograde.

5 Exchange moves

In this section we study *exchange moves* of open book foliations and closed braids.

First we recall the exchange move in braid foliation theory, which is one of the most fundamental operations on braid foliations and has numerous applications to the study of knots and links in S^3 and transverse links in the standard contact S^3 [11].

An *exchange move* is a move of a closed braid in $S^3 = M_{(D^2,\mathrm{id})}$ as depicted in Figure 18. It is a composition of a positive stabilization, braid isotopy and a positive destabilization. Suppose that braids L, L' are related to each other by an exchange move. The conjugacy classes of L and L' are different in general but L and L' have clearly the same braid index and the same transverse link type [11].

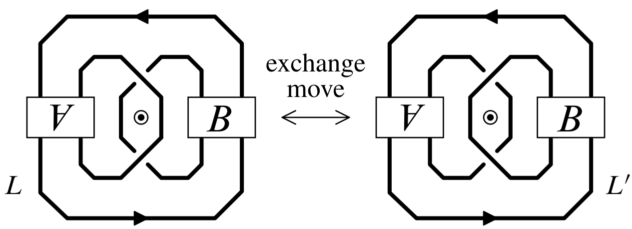

Figure 18: Exchange move of a closed braid in S^3

Let F be a Seifert surface of L or an incompressible closed surface in $S^3 \setminus L$. An exchange move of L is related to an isotopy of the surface F. Consider a situation as depicted in Figure 19(1).

We isotope L as in the passage (1)$\to$(2). As a consequence inessential b–arcs appear in the braid foliation. Next we push down the surface to remove the inessential b–arcs (Figure 19(3)).

The exchange move simplifies the braid foliation of F. It removes two elliptic points of opposite signs and two hyperbolic points of opposite signs in the (shaded) disc region of F, as described in Sketch (4)$\to$(5) of Figure 19 but it preserves the braid foliation on the rest of the surface.

The next theorem generalizes Birman–Menasco's exchange move.

Theorem 5.1 (Exchange moves in general open books) *Let L be a closed braid in $M_{(S,\phi)}$ and F be a Seifert surface of L. Assume that there exists a nonstrongly essential elliptic point $v \in \mathcal{F}_{\mathrm{ob}}(F)$ where exactly two regions R_1 and R_2 meet and satisfy the following:*

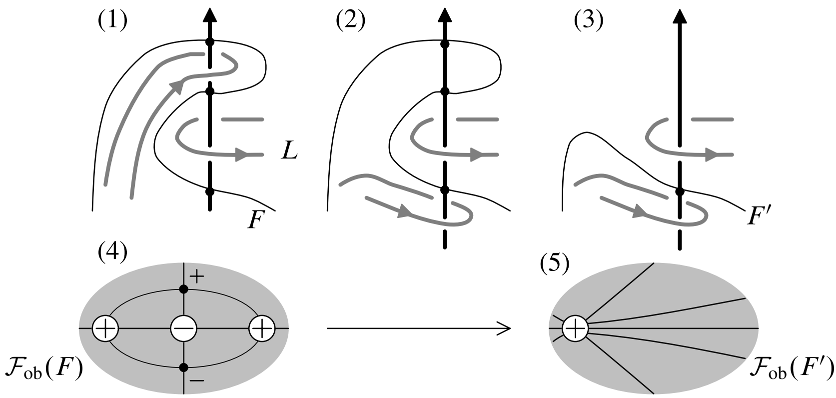

Figure 19: Braid foliation before and after an exchange move of L

- $\mathtt{sgn}(R_1) = -\mathtt{sgn}(R_2)$.
- $\text{type}(R_1) = \text{type}(R_2) = bb$ *when* $\mathtt{sgn}(v) = +1$.
- $\text{type}(R_1), \text{type}(R_2) \in \{ab, bb\}$ *when* $\mathtt{sgn}(v) = -1$.

Then there exists an isotopy $\Phi_t\colon M \to M$ *that takes* $F = \Phi_0(F)$ *to* $F' = \Phi_1(F)$ *and* $L = \Phi_0(L)$ *to* $L' = \Phi_1(L)$ *with the following properties:*

(1) *There exist discs* $D \subset F$ *and* $D' \subset F'$ *such that:*

 (a) $\mathcal{F}_{\mathrm{ob}}(F \setminus D)$ *is topologically conjugate to* $\mathcal{F}_{\mathrm{ob}}(F' \setminus D')$.

 (b) $\mathcal{F}_{\mathrm{ob}}(D)$ *has* $\pm$ *elliptic points and* $\pm$ *hyperbolic points as in Figure 20(1), but* $\mathcal{F}_{\mathrm{ob}}(D')$ *has no singularities as in Figure 20(3).*

(2) L *and* L' *have the same braid index with respect to the open book* (S, ϕ) *(but they may not be isotopic in the complement of the binding).*

(3) L *and* L' *are transversely isotopic links in the contact structure* $\xi_{(S,\phi)}$.

Definition 5.2 (Exchange moves) (i) We call the change $\mathcal{F}_{\mathrm{ob}}(F) \to \mathcal{F}_{\mathrm{ob}}(F')$ (in Theorem 5.1) an *exchange move* of the open book foliation.

(ii) We call the braid move $L \to L'$ (in Theorem 5.1) the *exchange move of* L *subordinate to* the exchange move $\mathcal{F}_{\mathrm{ob}}(F) \to \mathcal{F}_{\mathrm{ob}}(F')$.

Remark With a slight modification a similar statement as in Theorem 5.1 holds when F is a closed surface in $M \setminus L$. In fact in Section 7 we study a case where $F \simeq S^2$ and $\mathcal{F}_{\mathrm{ob}}(F)$ admits exchange moves.

Remark Although in braid foliation theory $\mathcal{F}_{\mathrm{ob}}(F)$ is necessarily essential, here we do not require essentiality of $\mathcal{F}_{\mathrm{ob}}(F)$.

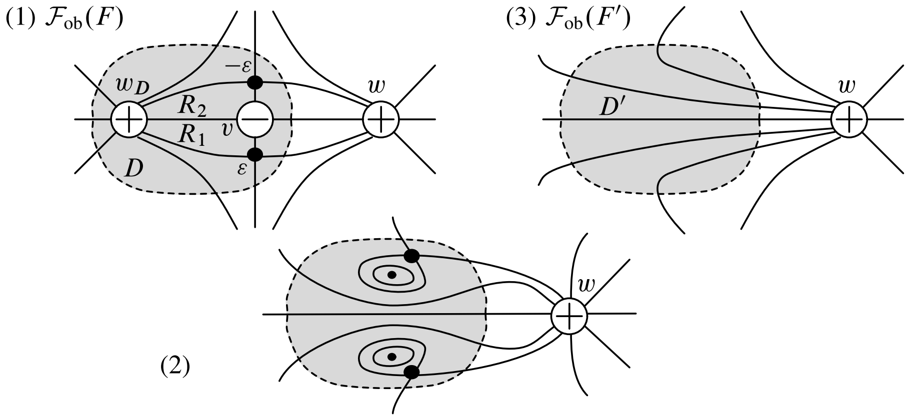

Figure 20: An exchange move of an open book foliation

Proof We may assume that $\mathtt{sgn}(v) = -1$ and $\mathtt{sgn}(R_1) = -\mathtt{sgn}(R_2) = +1$. Similar arguments hold for other cases. Here is an outline of the proof:

In Step 1, we define a surface F'' embedded in $M_{(S,\phi)}$ such that $\mathcal{F}_{\text{ob}}(F'') = \mathcal{F}_{\text{ob}}(F)$ topologically conjugate. In Step 2, we find a continuous family of surfaces $\{F_t\}$ embedded in $M_{(S,\phi)}$ such that $F_0 = F$ and $F_1 = F''$. In Step 3, we construct F' from F'' and verify (1) and (2). In Step 4, we verify (3).

Step 1 For $i = 1, 2$, let h_i denote the hyperbolic point in R_i and let S_{t_i} be the singular fiber that contains h_i. For $t \neq t_1, t_2$, let $b_t \subset S_t$ be the b–arc of $\mathcal{F}_{\text{ob}}(F)$ that ends at v. Since v is nonstrongly essential, we may assume that $0 < t_1 < 0.5 < t_2 < 1$ and b_t is nonstrongly essential for $t \in (t_1, t_2)$, thus b_t and a binding component cobound a disc Δ_t in S_t.

Figure 21 shows a movie presentation of F in a neighborhood of $R_1 \cup R_2$, where $\epsilon > 0$ is a very small number, w, w_D denote the positive elliptic points from which $b_0, b_{0.5}$ start, and each box may be empty or contain part of a–arcs, b–arcs, c–circles, and a singular leaf. Triple parallel arcs represent some number (possibly zero) of arcs, and the shaded regions indicate a neighborhood of $X \cup b_0$, where

$$X := \bigcup_{t_1 < t < t_2} \Delta_t \cong D^2 \times (t_1, t_2).$$

We define the surface F'' by replacing the part of F depicted in Figure 21 by the description in Figure 22, which is obtained by moving the boxes $B_1, \ldots, B_6$ and their foot between v and w_D to the negative side of w.

By construction, F and F'' are homeomorphic and their open book foliations $\mathcal{F}_{\text{ob}}(F)$ and $\mathcal{F}_{\text{ob}}(F'')$ are topologically conjugate. If $\mathcal{F}_{\text{ob}}(F)$ is essential then B_i are nonempty,

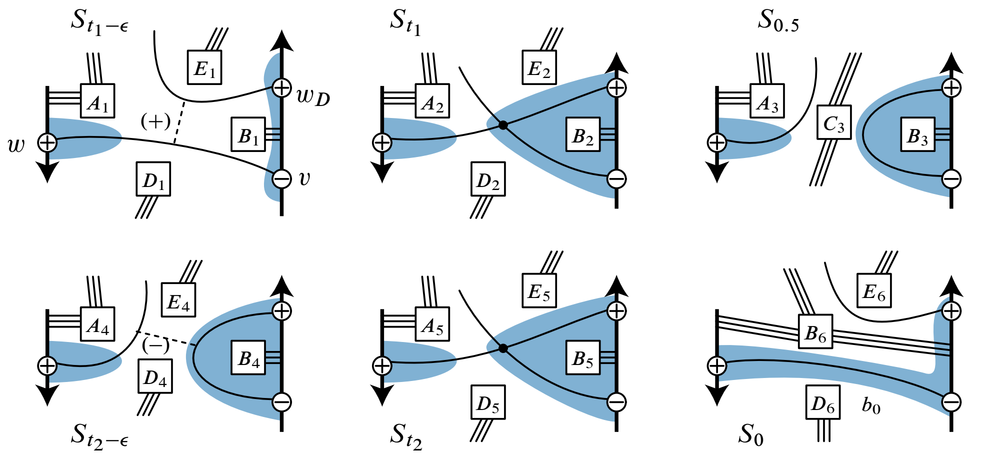

Figure 21: A movie presentation of F near $X \cup b_0$: Shaded regions represent a neighborhood of $X \cup b_0$.

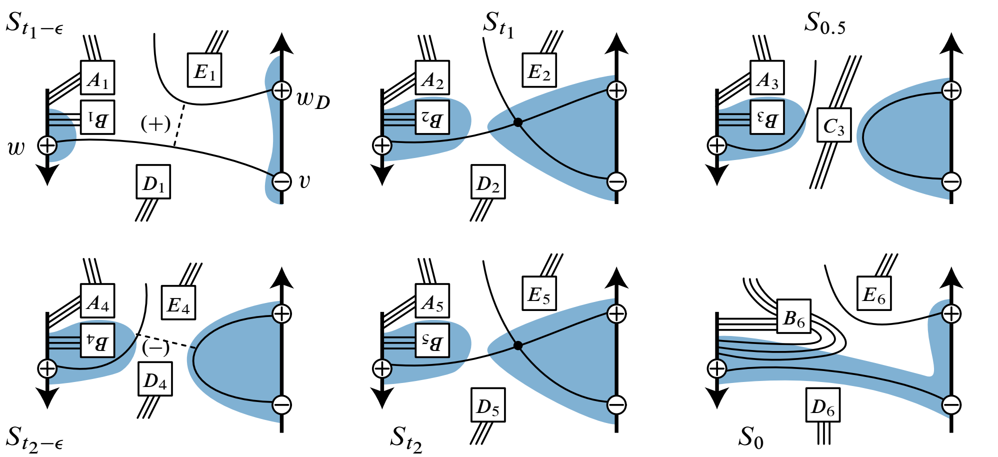

Figure 22: (Step 1) A movie presentation of F'' near $X \cup b_0$

but in general all of B_i can be empty and in that case $F''=F$. In any case the braids $L=\partial F$ and $\partial F''$ have the same braid index.

Step 2 By [3, Lemma 4; 6, Lemma 5] (see also [2, Theorem 2.2, Figure 2.19]), there is an isotopy $\Phi'_t\colon M \to M$ that takes $F \cap X$ out of X and moves along b_0 down to the negative side of w; see, for example, the items (1)–(6) in Figure 24. We have $\Phi'_1(F)=F''$.

Step 3 By the construction of F'' the b–arcs b_t of $\mathcal{F}_{\mathrm{ob}}(F'')$ for $t \in (t_1,t_2)$ are inessential (see Figure 22). Push F'' along a disc Δ_t for some $t \in (t_1,t_2)$ as shown in

Figure 23 to remove the inessential b–arcs and the elliptic points w and v. Call the surface F'''.

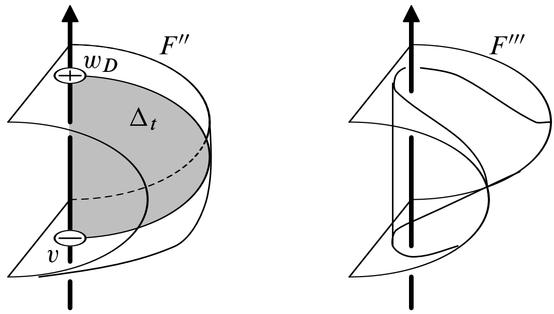

Figure 23: (Step 3) Push F'' along Δ_t

The surface F''' does not admit an open book foliation as it has two local extrema; see Figure 20(2). Flatten the two pairs of local extremum and saddle tangency and call the resulting surface F', whose open book foliation is depicted in Figure 20(3). This concludes the statement (1).

If the boxes $B_1, \dots, B_6$ are empty, the surface change $F \to F'$ is (the inverse of) what is called a *finger move* in [19]. During the process

$$F'' \xrightarrow{\text{pushing}} F''' \xrightarrow{\text{flatten}} F'$$

the boundary is fixed, so $L' = \partial F'$ and $\partial F''$ have the same braid index. With the observation at the end of Step 1, we verify the statement (2).

Step 4 It remains to show the statement (3), that is, $L = \partial F$ and $L' = \partial F'$ are indeed transversely isotopic. So far we have three isotopies: Φ'_t, pushing along Δ_t, and the flattening. Denote the concatenation of the three by $\Phi_t \colon M \to M$, hence $\Phi_0(F) = F$ and $\Phi_1(F) = F'$. Note that $L_t = \Phi_t(L)$ may not be in a braid position relative to the open book for some $t \in (0, 1)$.

To prove L and L' are transversely isotopic, we relate them by a sequence of positive (de)stabilizations and braid isotopy, all of which preserve transverse link types. We use an idea of Birman and Menasco in [10, page 421]: First we positively stabilize the part of L that goes through X along the b–arc $b_{t_1-\varepsilon}$.

See Figure 24, where all the braid strands may be weighted and boxes contain braidings. After braid isotopy, we positively destabilize it so that the resulting braid L' does not go through X. If L does not go through X then clearly $L = L'$. □

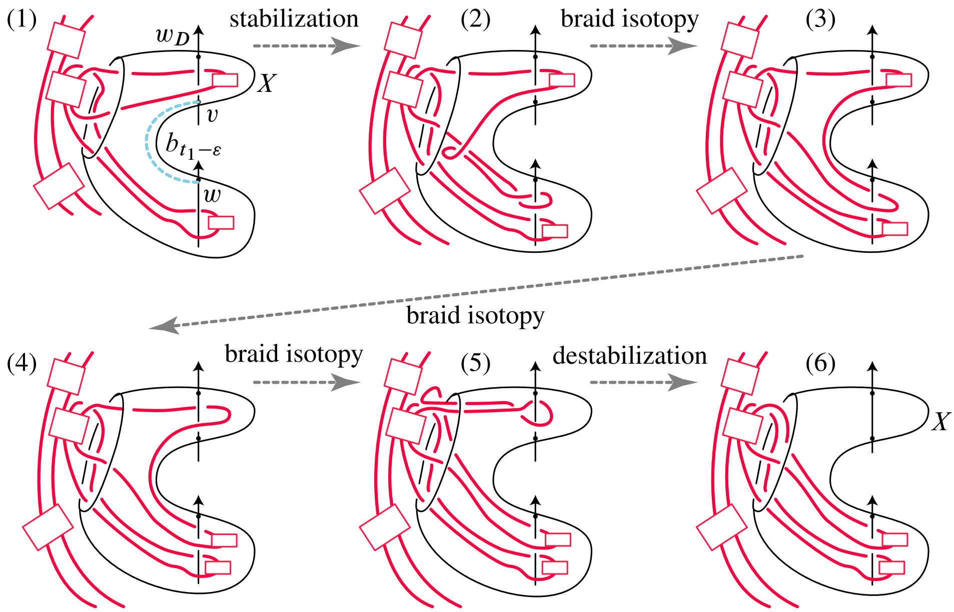

Figure 24: (Step 4) Realizing an exchange move as transverse isotopy

Our exchange move is related to Giroux's elimination of a pair of elliptic and hyperbolic points of the same sign and connected by a singular leaf in a characteristic foliation. In a neighborhood of $R_1 \cup R_2$ we may identify the open book foliation and the characteristic foliation by the structural stability theorem in [19]. We see two elimination pairs in the shaded region of Figure 20(1). Applying Giroux elimination twice, we get a characteristic foliation topologically conjugate to Figure 20(3). Despite this fact, an exchange move and a Giroux elimination are different in the following sense:

- A Giroux elimination can be achieved by a C^0–small perturbation that is supported on a small neighborhood of the singular leaf joining the elimination pair, whereas the exchange move requires global isotopy (ie not C^0–small and not supported on a small neighborhood of D). Moreover the latter might change the braid isotopy class of $L = \partial F$ though it preserves the transverse link type.
- One can apply a Giroux elimination without the nonstrongly essential condition on the elliptic point v, but for an exchange move this assumption is necessary.
- An exchange move on $\mathcal{F}_{\mathrm{ob}}(F)$ eliminates two pairs of elliptic and hyperbolic points at the same time. It is, in general, impossible to eliminate only one of the two pairs. But a Giroux elimination can apply to each pair separately. (In braid/open book foliation theory an operation called destabilization of a closed braid eliminates one pair.)

6 Stabilization and open book foliations

In this section we study how the open book foliation $\mathcal{F}_{\mathrm{ob}}(F)$ of a surface $F \subset M_{(S,\phi)}$ changes under a stabilization of the open book.

Let (S,ϕ) be an open book. Let $\alpha \subset S$ be a properly embedded arc in S. Let S' denote the surface S with an annulus A plumbed along α. Let

$$\phi'_{\pm} := D_{\alpha}^{\pm} \circ \widetilde{\phi} \in \mathrm{Diff}^{+}(S', \partial S')$$

where D_{α} is the positive Dehn twist along a core circle of the attached annulus A, and $\widetilde{\phi}\colon S' \to S'$ is an extension of $\phi\colon S \to S$ such that $\widetilde{\phi} = \phi$ on S and $\widetilde{\phi} = \mathrm{id}$ on $S' \setminus S$. We call the new open book $(S', \phi'_{\pm})$ a *positive/negative stabilization* of (S,ϕ) and the arc α a *stabilization arc*. It is known that (see Etnyre's survey [13] for example) $M_{(S',\phi')}$ and $M_{(S,\phi)}$ are homeomorphic.

To compare and relate open book foliations with respect to the different open books (S,ϕ) and (S',ϕ), we view the page S_t as a subsurface of S'_t as follows. Since S' is the surfaces S and A plumbed along α there is a natural inclusion map

$$\iota\colon S_t \to S'_t$$

for each page. Cutting open the manifold $M_{(S,\phi)}$ (respectively $M_{(S',\phi')}$) along (the closure of) the page S_0 (respectively S'_0) we get a product region $S\times[0,1]$ (respectively $S'\times[0,1]$). With the inclusion map ι we may regard $\iota(S\times[0,1]) \subsetneq S'\times[0,1]$.

In the following we construct a surface F' in $M_{(S',\phi')}$ homeomorphic to F by modifying the slices $\iota(S_t \cap F) \subset S'_t$. We start with a trivial case. Let $\alpha_t := \alpha \times \{t\} \subset S_t$.

Proposition 6.1 *If $F \subset M_{(S,\phi)}$ does* not *intersect α_0 then there exists a surface $F' \subset M_{(S',\phi')}$ such that $F \simeq F'$ are homeomorphic and $\mathcal{F}_{\mathrm{ob}}(F) \simeq \mathcal{F}_{\mathrm{ob}}(F')$ are topologically conjugate.*

Proof We construct F' so that $F' \cap S'_t = \iota(F \cap S_t)$ for every $t \in [0,1)$. Then F' does not intersect the arc $\iota(\alpha_0) \subset S'_0$. Therefore

$$F' \cap S'_0 = D_{\alpha}^{\pm 1}(F' \cap S'_0) = D_{\alpha}^{\pm 1} \circ \iota \circ \phi(F \cap S_1) = D_{\alpha}^{\pm 1} \circ \widetilde{\phi} \circ \iota(F \cap S_1) = \phi'(F' \cap S'_1),$$

so we can identify the multicurves $F' \cap S'_0$ and $F' \cap S'_1$ by the monodromy ϕ' and obtain a surface $F' \subset M_{(S',\phi')}$. Then from the construction it is clear that $F \simeq F'$ and $\mathcal{F}_{\mathrm{ob}}(F) = \mathcal{F}_{\mathrm{ob}}(F')$. ∎

Next we consider the case where F intersects the stabilization arc $\alpha_0 \subset S_0$. Let $\overline{\alpha}_t \subset S_t$ be a collar neighborhood of α_t. Assume that F intersects $\overline{\alpha}_0$ in m disjoint arcs $\beta_i \times \{0\}$,

$$F \cap \overline{\alpha}_0 = (\beta_1 \cup \cdots \cup \beta_m) \times \{0\},$$

where:

- $\beta_i \subset S$ is an arc traversing the plumbed annulus A (see Figure 26).
- The geometric intersection number $i(\beta_i, \alpha) = 1$.
- $\beta_i \times \{0\}$ is a subarc of some b–arc b_i of the open book foliation $\mathcal{F}_{\mathrm{ob}}(F)$, possibly $b_i = b_j$ for some $i \neq j$.

We construct surfaces F' and F'' in the stabilized open book (S', ϕ') that are homeomorphic to F.

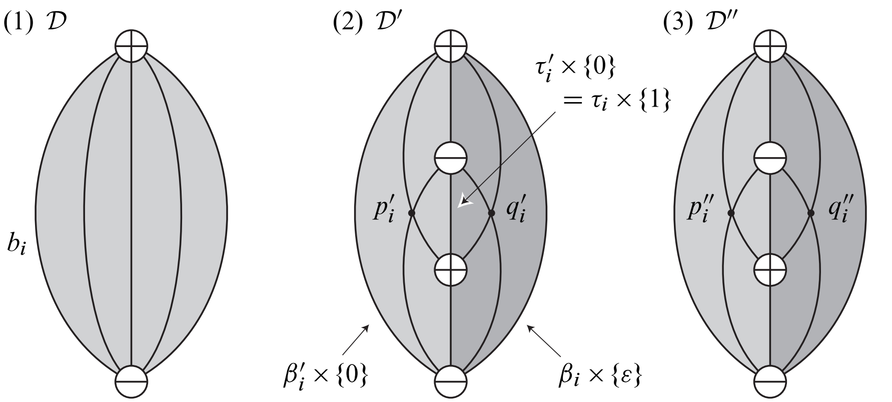

Figure 25: (1) A bigon in $\mathcal{D} \subset F$, (2) two adjacent bb–tiles forming a bigon in $\mathcal{D}' \subset F'$, (3) two adjacent bb–tiles forming a bigon in $\mathcal{D}'' \subset F''$; the hyperbolic points satisfy $\mathtt{sgn}(p_i') = -\mathtt{sgn}(p_i'') = -\mathtt{sgn}(q_i') = \mathtt{sgn}(q_i'')$ for $i = 1, \ldots, m$.

Proposition 6.2 *Suppose that $F \subset M_{(S,\phi)}$ intersects nontrivially the stabilization arc α_0 in m points. We further assume that $b_i \neq b_j$ for $i \neq j$, that is, every b–arc in S_0 intersects α_0 in at most one point. Then there exist surfaces F' and $F'' \subset M_{(S',\phi')}$ such that*

(6-1) $$F \simeq F' \simeq F''$$

are homeomorphic, and

(6-2) $$\mathcal{F}_{\mathrm{ob}}(F \setminus \mathcal{D}) \simeq \mathcal{F}_{\mathrm{ob}}(F' \setminus \mathcal{D}') \simeq \mathcal{F}_{\mathrm{ob}}(F'' \setminus \mathcal{D}'')$$

are topologically conjugate, where:

- $\mathcal{D} \subset F$ *is a disjoint union of* m *bigons foliated only by* b*–arcs; see Figure 25.*
- $\mathcal{D}' \subset F'$ *is a disjoint union of* m *bigons each of which consists of two adjacent* bb*–tiles of opposite signs.*
- $\mathcal{D}'' \subset F''$ *is exactly the same as* D' *after exchanging the signs of the* bb*–tiles for each bigon.*

Remark If $b_{i_1} = b_{i_2} = \cdots = b_{i_k}$ for some $1 \leq i_1 < \cdots < i_k \leq m$ that is, if some b–arc in S_0 intersects the stabilization arc α_0 in more than one point, after some modification of the descriptions of $\mathcal{D}, \mathcal{D}', \mathcal{D}''$, the same results (6-1) and (6-2) still hold: For example, $|\mathcal{D}| = |\mathcal{D}'| = |\mathcal{D}''|$ is no longer m but it becomes less than m. Also Sketches (2) and (3) of Figure 25 become more complicated and each should contain $2(k+1)$ bb–tiles.

Proof We prove Proposition 6.2 only for the case $\phi' = \phi'_+$ (positive stabilization) since a parallel argument holds for the case $\phi' = \phi'_-$.

We may assume that there exists $\varepsilon > 0$ such that $\mathcal{F}_{\mathrm{ob}}(F)$ has no hyperbolic points in the family of pages $\{S_t\}_{0 \leq t \leq \varepsilon}$ and

$$F \cap \overline{\alpha}_t = (\beta_1 \cup \cdots \cup \beta_m) \times \{t\} \quad \text{for } 0 \leq t \leq \varepsilon.$$

We assume $\beta_1, \ldots, \beta_m$ are lined up from the left to the right as in Figure 26. Recall that $\beta_i \times \{0\}$ is a subarc of a b–arc $b_i \subset \mathcal{F}_{\mathrm{ob}}(F)$. The orientation of b_i induces an orientation of β_i. Let $\tau_1, \ldots, \tau_m$ be essential arcs of A lined up from the right to the left as in the left sketch of Figure 26. We orient τ_i in the opposite direction to the orientation of β_i (ie if β_i is oriented "upward" then τ_i is oriented "downward" and vice versa.)

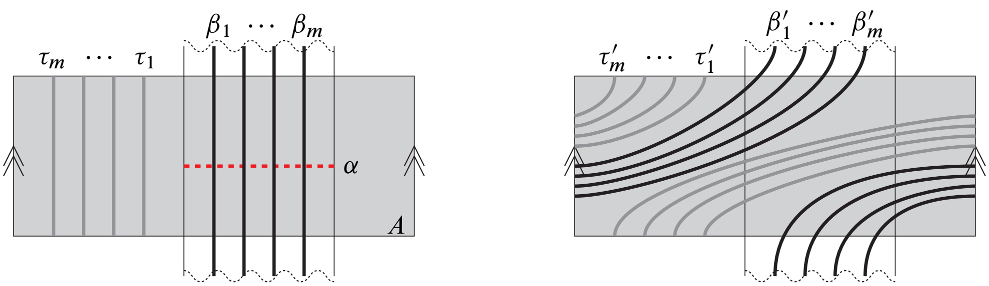

Figure 26: The arcs $\beta_i, \tau_i, \beta'_i = D_\alpha(\beta_i)$ and $\tau'_i = D_\alpha(\tau_i)$; the dashed line represents the stabilization arc α; the shaded rectangle where the left and the right edges are identified represents the plumbed annulus A.

We construct F' and F'' by defining intersections with the pages S'_t.

For $\varepsilon \le t \le 1$ let

$$F' \cap S'_t = F'' \cap S'_t = \iota(F \cap S_t) \cup \bigcup_{i=1}^{m} (\tau_i \times \{t\}). \tag{6-3}$$

Viewing the arc $\tau_i \times \{t\}$ as a b–arc of the open book foliation, the orientation of τ_i determines signs of the elliptic points at $\partial \tau_i \subset \partial A$.

For $t = 0$ let

$$F' \cap S'_0 = F'' \cap S'_0 = D_\alpha(\iota(F \cap S_0)) \cup \bigcup_{i=1}^{m} (\tau'_i \times \{0\}) = \phi'(F' \cap S'_1),$$

where $\tau'_i := D_\alpha(\tau_i)$ as in Figure 26.

For $0 \le t \le \varepsilon$ we define F' and F'' by movie presentations. Let $\beta'_i := D_\alpha(\beta_i)$. We make β'_i and τ'_i come closer

- for F' starting from $i = 1$ to m along the describing arcs in Figure 27(1),
- for F'' starting from $i = m$ to 1 along the describing arcs in Figure 27(2).

Call the resulting saddle points $p'_i \in F'$ and $p''_i \in F''$ respectively. Notice that we set the orientation of τ_i so that the hyperbolic points p'_i and p''_i have opposite signs. We further form hyperbolic points $q'_m, \dots, q'_1$ for $\mathcal{F}_{\mathrm{ob}}(F')$ and $q''_1, \dots, q''_m$ for $\mathcal{F}_{\mathrm{ob}}(F'')$ by using the describing arcs as depicted in Figures 27(3) and (4) respectively. On the level $t = \varepsilon$ the condition (6-3) is satisfied. We have

$$\mathtt{sgn}(p'_i) = -\mathtt{sgn}(p''_i) = -\mathtt{sgn}(q'_i) = \mathtt{sgn}(q''_i)$$

and the bb–tiles of $\mathcal{F}_{\mathrm{ob}}(F')$ (respectively $\mathcal{F}_{\mathrm{ob}}(F'')$) containing p'_i and q'_i (respectively p''_i and q''_i) are adjacent and form a bigon as depicted in Figure 25(2) (respectively (3)). □

We find pictorial similarity between the passage (2)→(1) in Figure 25 and the passage (1)→(3) in Figure 20. The former is the consequence of the destabilization $(S', \phi') \to (S, \phi)$ and the latter is caused by an exchange move. Important differences are:

- For an exchange move the b–arc corresponding to $\tau'_i \times \{0\}$ must be boundary-parallel, whereas for a destabilization $\tau'_i \times \{0\}$ is an essential arc.
- Under an exchange move the open book (S, ϕ) stays the same, but not under a destabilization.

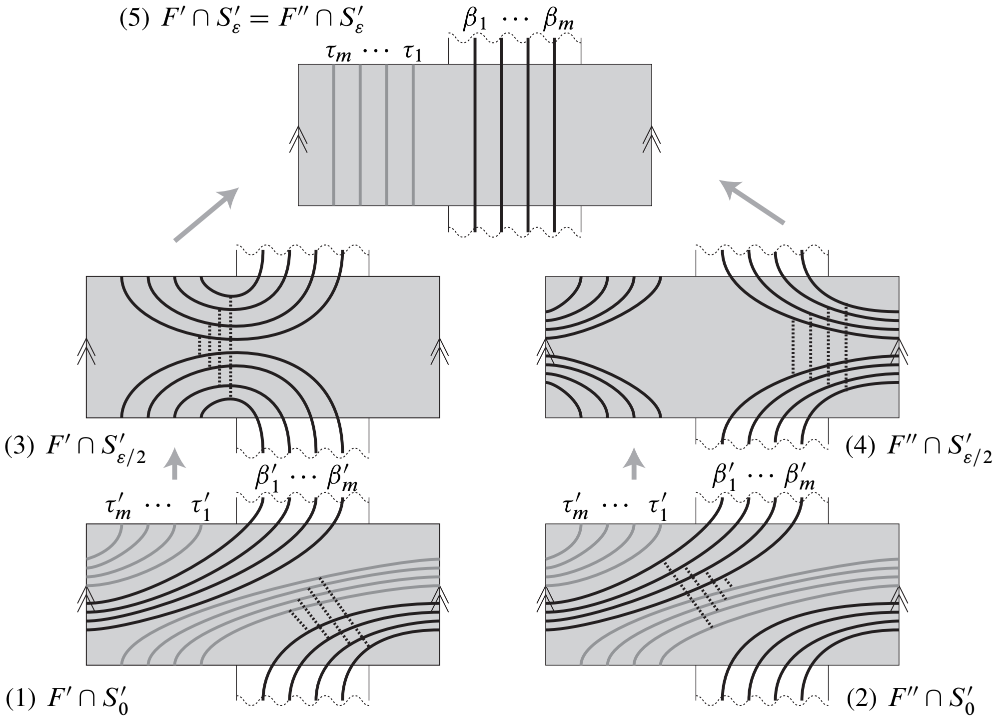

Figure 27: (1) $\to$ (3) $\to$ (5): Movie presentation of F' for $0 \le t \le \varepsilon$; (2) $\to$ (4) $\to$ (5): Movie presentation of F'' for $0 \le t \le \varepsilon$

We have constructed two different surfaces F' and F'' homeomorphic to F in a stabilized open book. They are related to each other in the following way:

Proposition 6.3 *The two surfaces $F', F'' \subset M_{(S',\phi')}$ constructed in the proof of Proposition 6.2 are isotopic to each other. For example, they can be related to each other by exchange moves and bypass moves (see Figure 28):*

$$F' \text{ sketch(1)} \xrightarrow{\text{exchange}^{-1}} \text{sketch(2)} \xrightarrow{\text{bypass}} \xrightarrow{\text{bypass}} \text{sketch(3)} \xrightarrow{\text{exchange}} F'' \text{ sketch(4)}.$$

Proof For simplicity we assume $m=1$, ie the number of bigon regions $|\mathcal{D}'| = |\mathcal{D}'| = 1$ and we call the bigons $\mathcal{D}'$ and $\mathcal{D}''$, respectively, by abusing the notation. (If $m>1$ each arc in Figure 28 is replaced by parallel m arcs and we apply similar constructions.) There are many ways to relate $\mathcal{D}'$ and $\mathcal{D}''$. In the following we present one of the ways.

Denote the elliptic points of $\mathcal{D}'$ by A, B, C, D as in Sketch (1) of Figure 28 such that $\mathtt{sgn}(A) = \mathtt{sgn}(C) = -\mathtt{sgn}(B) = -\mathtt{sgn}(D) = +1$. We apply the inverse of an

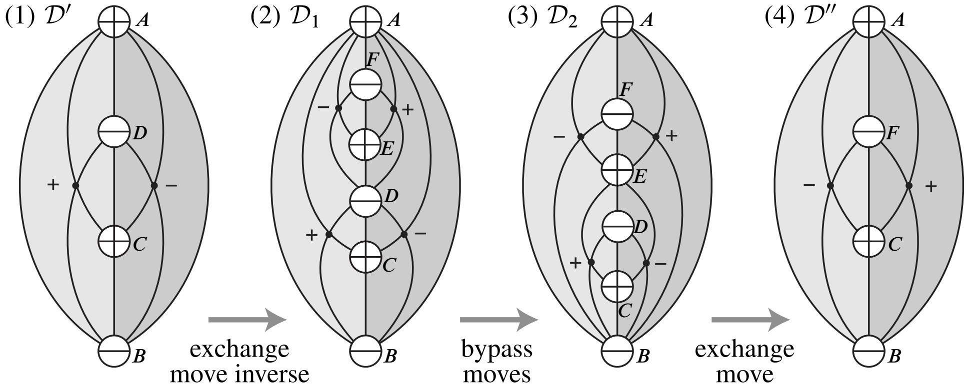

Figure 28: (1) bigon $\mathcal{D}' \subset F'$, (2) bigon $\mathcal{D}_1$, (3) bigon $\mathcal{D}_2$, (4) bigon $\mathcal{D}'' \subset F''$

exchange move to $\mathcal{D}'$ to insert two adjacent bb–tiles between A and D as in Sketch (2), where E and F denote new positive and negative elliptic points, respectively. We call the resulting bigon of four bb–tiles $\mathcal{D}_1$.

Next we apply a retrograde bypass move to the left half of $\mathcal{D}_1$ and then apply a prograde bypass move to the right half of $\mathcal{D}_1$. Detailed movie presentation and bypass rectangles of the transition from $\mathcal{D}_1$ to $\mathcal{D}_2$ are depicted in Figure 29.

Finally we get rid of two bb–tiles of $\mathcal{D}_2$ that share the elliptic points D and E by an exchange move and we obtain the bigon $\mathcal{D}''$. □

7 Split closed braid theorem and composite closed braid theorem

In this section we prove the split/composite braid theorem by using the b–arc foliation change and exchange move.

Definition 7.1 Let L be a link in a closed oriented 3–manifold M. We say that L is a *split link* if there exists a 2–sphere that separates components of L. We call such a sphere a *separating sphere* for L.

Similarly, we say that L is a *composite link* if there exists a 2–sphere that intersects L in exactly two points and decomposes L as a connected sum of two nontrivial links. We call such a sphere a *decomposing sphere* for L.

The above notions of split/composite link are extended to those for closed braids relative to open books. (For braid foliations they are defined in [3].)

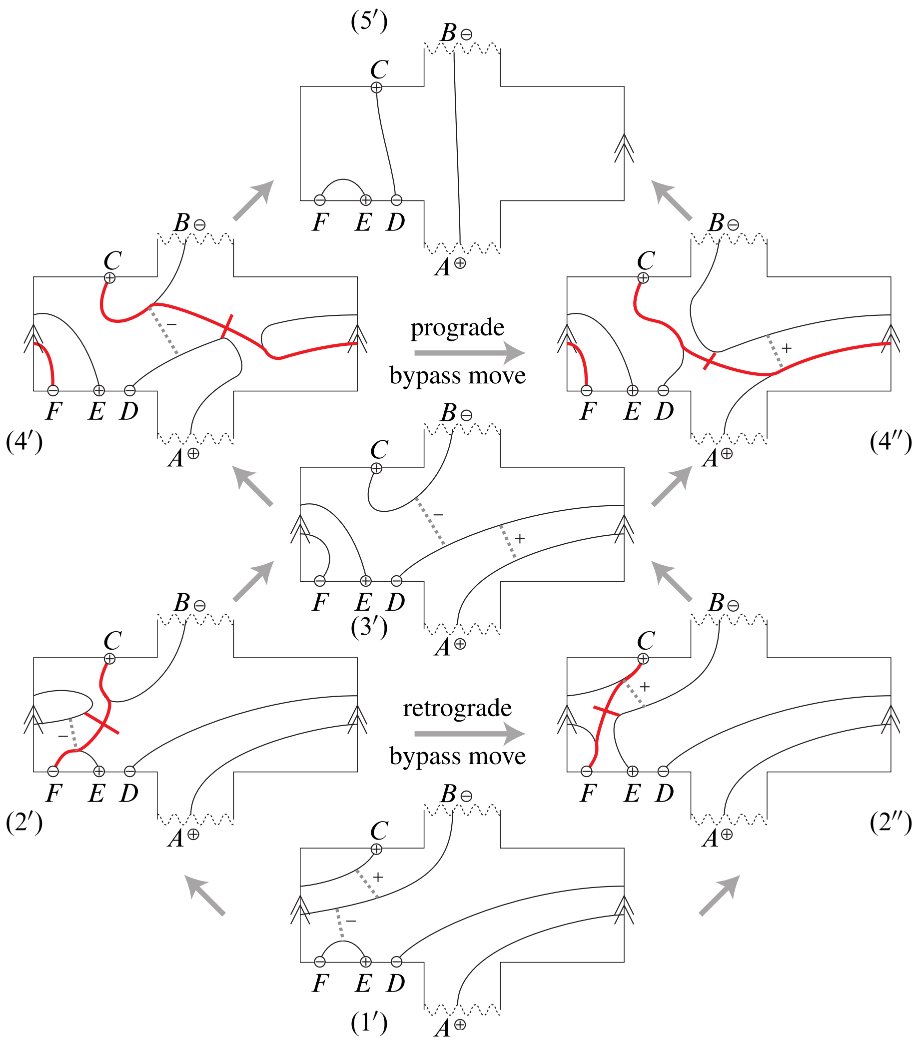

Figure 29: ($1' \to 2' \to 3' \to 4' \to 5'$) movie presentation of $\mathcal{D}_1$; ($1' \to 2'' \to 3' \to 4'' \to 5'$) movie presentation of $\mathcal{D}_2$; thick arcs (red) represent bypasses.

Definition 7.2 Let $L \subset M_{(S,\phi)}$ be a closed braid with respect to (S,ϕ). We say that L is a *split/composite closed braid* if there exists a separating/decomposing sphere F for L such that $\mathcal{F}_{ob}(F)$ has exactly one positive elliptic point, one negative elliptic point and no hyperbolic points, namely F intersects the binding in two points.

Clearly a split/composite closed braid with respect to (S,ϕ) is a split/composite link in $M_{(S,\phi)}$, but the converse is not true in general. This is because a separating/decomposing sphere might be embedded in a complicated way relative to (S,ϕ).

In fact, for the special case where $M_{(D^2,\mathrm{id})} \simeq S^3$ Birman and Menasco construct an example of split link and its 4–braid representative that cannot be isotopic to a split closed braid in the complement of the braid axis [3, page 116]. Also in [22] Morton finds a 5–braid representative of a composite link that is not conjugate to a composite 5–braid.

However, if we are allowed to use exchange moves the converse holds: In [3] Birman and Menasco prove that any closed braid representative of a split/composite link in S^3 with the open book (D^2, id) can be modified to a split/composite braid by applying a sequence of exchange moves. As a corollary, they prove the additivity of the minimum braid index of knots and links in $\mathbb{R}^3$.

We extend the above result of Birman and Menasco to closed braids in general open books with additional assumptions. Let $C \subset \partial S$ be a boundary component of S. We denote by $c(\phi, C)$ the fractional Dehn twist coefficient of ϕ with respect to C, which is defined by Honda, Kazez and Matić in [17] (cf Gabai and Oertel [14]).

Theorem 7.3 (Split/composite closed braid theorem) *Let L be a closed braid representative of a split/composite link in $M_{(S,\phi)}$. Let F be a separating/decomposing sphere for L. Assume the following:*

(1) *$\mathcal{F}_{\mathrm{ob}}(F)$ is essential and all of whose b–arcs are separating.*
(2) *If a binding component $C \subset \partial S$ intersects F then $|c(\phi, C)| > 1$.*

Then there exists a sequence of exchange moves of closed braids

$$L \to L_1 \to \cdots \to L_m$$

such that L_m is a split/composite closed braid.

Remark Before proceeding to a proof, we give remarks on the assumptions and the statement of Theorem 7.3.

(i) The braid L_m is split/composite and transversely isotopic to L. However, we do not assert that a separating/decomposing sphere F_m for L_m is isotopic to F.
(ii) If (S, ϕ) has connected binding then by [18, Theorem 7.2] conditions (1), (2) imply that the sphere F (hence F_m) bounds a 3–ball in M.
(iii) In braid foliation theory condition (1) always holds but $c(\mathrm{id}, \partial D^2) = 0$. To treat braid foliation case uniformly, it is often convenient to regard $c(\mathrm{id}, \partial D^2) = +\infty$. This is also true for other results like [18, Corollaries 7.3, 7.4 and Theorem 8.3].

Example 7.4 In general, without assuming conditions (1) or (2), there may exist a closed braid representative L of a split/composite link type whose separating/decomposing sphere does not admit a sequence of exchange moves that turns L into a split/composite closed braid.

For example let $\phi = \mathrm{id}_S$ (ie $c(\phi, C) = 0$) and F be a splitting sphere of L defined by the movie presentation in Figure 30. The open book foliation $\mathcal{F}_{\mathrm{ob}}(F)$ consists of two bb–tiles. Since all the b–arcs are strongly essential F does not admit exchange moves.

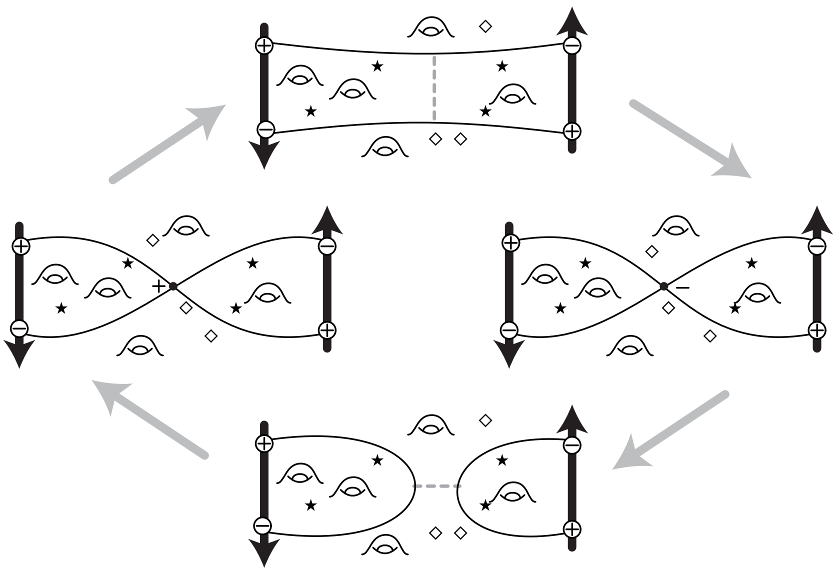

Figure 30: (Example 7.4) A movie presentation of the separating sphere F, where $\star$ and $\diamond$ represent distinct components of L separated by F.

We have three lemmas, where conditions (1) or (2) are not assumed. The first lemma is proven in [18].

Lemma 7.5 [18, Lemma 5.1] *Let (S, ϕ) be a general open book and F a closed, incompressible surface in $M_{(S,\phi)}$. Let v be a strongly essential elliptic point of $\mathcal{F}_{\mathrm{ob}}(F)$ that lies on a boundary component $C \subset \partial S$, and P (respectively N) be the number of the positive (respectively negative) hyperbolic points that are connected to v by a singular leaf. Then*

$$\begin{cases} -P \leq c(\phi, C) \leq N & \text{if } \mathrm{sgn}(v) = -1, \\ -N \leq c(\phi, C) \leq P & \text{if } \mathrm{sgn}(v) = +1. \end{cases}$$

Lemma 7.6 *Let v be an elliptic point in the open book foliation $\mathcal{F}_{\mathrm{ob}}(F)$. Assume that all the regions meeting at v are bb–tiles, and that all the b–arcs that end at v are separating. Then there exist both positive and negative hyperbolic points connected to v by a singular leaf.*

Proof Let $h_1, \dots, h_n$ be the hyperbolic points that are connected to v by a singular leaf. We assume that $\mathtt{sgn}(v) = -1$ and $\mathtt{sgn}(h_i) = +1$ for all $i = 1, \dots, n$ (parallel arguments hold for other cases) and deduce a contradiction.

Let $w_1, \dots, w_n$ be the positive elliptic points that are connected to v by a b–arc and ordered clockwise; see Figure 31. Let b_i be a b–arc in the page S_{t_i} connecting w_i and v, so $0 < t_1 < t_2 < \cdots < t_n < 1$.

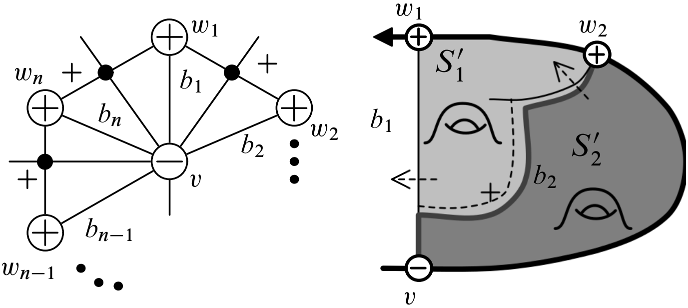

Figure 31: An illustration of Lemma 7.6

Since b_i is separating the elliptic points v and w_i lie on the same binding component, ie v and $w_1, \dots, w_n$ lie on the same binding component. Let $S'_i \subset S_{t_i}$ be the subsurface that lies on the left side of b_i as we walk from w_i to v. Since $\mathtt{sgn}(h_i) = +1$ by a standard argument (or the argument as in the proof of [18, Lemma 5.1]) the describing arc of h_i is contained in S'_i. Therefore $w_{i+1} \in S'_i$, hence $S'_i \supsetneq S'_{i+1}$ (see Figure 31). In particular $w_1 (= w_{n+1}) \in S'_n$. However, $S'_1 \supsetneq S'_2 \supsetneq \cdots \supsetneq S'_n$ and $w_1 \in S'_1 \setminus S'_2$. This is a contradiction. □

Lemma 7.7 *Let $F \subset M_{(S,\phi)}$ be a closed incompressible surface in the complement of a closed braid L. We may assume that $\mathcal{F}_{\mathrm{ob}}(F)$ is essential by [18, Theorem 3.2]. Let R be a degenerate bc–annulus in $\mathcal{F}_{\mathrm{ob}}(F)$; see Figure 32. Let $\mathcal{C} \subset S_{t_0}$ be the c–circle boundary of R and $C \subset \partial S$ be a binding component that intersects R. If all the b–arcs in $\mathcal{F}_{\mathrm{ob}}(R)$ are separating then $\mathcal{C}$ is essential in S_{t_0} and $|c(\phi, C)| \leq 1$.*

Proof Assume to the contrary that $\mathcal{C}$ bounds a disc $\Delta_{t_0} \subset S_{t_0}$, ie every c–circle of $\mathcal{F}_{\mathrm{ob}}(R)$ also bounds a disc $\Delta_t \subset S_t$. Since F is incompressible in $M - L$, the disc Δ_{t_0}

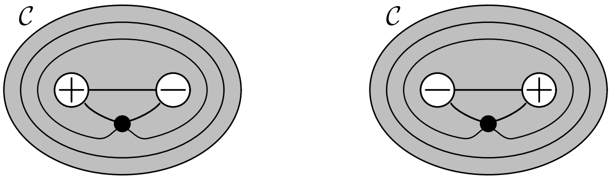

Figure 32: Degenerate bc–annuli R

must be pierced by L at least once. Since each b–arc $b_t \subset S_t \cap R$ is separating, b_t cobounds a subsurface $S_t' \subset S_t$ that is disjoint from Δ_t. Hence $R \cup \Delta_{t_0}$ bounds a compact region $M' \subset M$ which is the union of various S_t' and discs Δ_t. Thus the algebraic intersection number of L and $R \cup \Delta_{t_0}$ must be zero.

On the other hand, since L is a closed braid all the intersections of L with Δ_{t_0} are positive. But L and R never intersect, thus the algebraic intersection number of L and $R \cup \Delta_{t_0}$ must be positive, which is a contradiction. This concludes that $\mathcal{C}$ is essential in S_{t_0}.

Moreover, if $\mathcal{C}$ is essential, then all the b–arcs in R are strongly essential [18, Claim 6.8], hence by Lemma 7.5 we have $|c(\phi, C)| \leq 1$. □

Now we are ready to prove Theorem 7.3. Our proof is similar to Birman and Menasco's original one [3], but ours requires a more careful and different approach, especially when we show nonexistence of c–circles (in Case II below). More importantly, we need to be aware of the homotopical properties of b–arcs: essential, strongly essential or separating, since these properties are assumptions for b–arc foliation change and exchange move.

Proof of the split closed braid theorem Let F be a separating 2–sphere with the essential open book foliation $\mathcal{F}_{\text{ob}}(F)$. Let $e(F)$ be the number of elliptic points of $\mathcal{F}_{\text{ob}}(F)$. We prove the theorem by induction on $e(F)$. We show that if L is not a split closed braid (ie $e(F) > 2$) then after applying a b–arc foliation change and an exchange move $e(F)$ decreases. Eventually we obtain $e(F) = 2$, that is, L is a split closed braid. We study the following two cases:

Case I: $\mathcal{F}_{\text{ob}}(F)$ contains no c–circle leaves In this case, the region decomposition of F consists of bb–tiles only and it induces a cell decomposition of F. Let $V(i)$ ($i > 1$) be the number of 0–cells (elliptic points) of valence i, E the number of 1–cells, and R the number of 2–cells (bb–tiles). By the definition of bb–tiles, the valence of a 0–cell, v, is equal to the number of hyperbolic points that is connected to v by a singular leaf. Notice that $V(1) = 0$ because existence of a 0–cell of valence 1 implies existence of a degenerate bb–tile which never exists. Since each 1–cell is a common boundary of distinct two 2–cells and each 2–cell has distinct four 1–cells on its boundary we have

$$2E = 4R. \tag{7-1}$$

Since the end points of each 1–cell are distinct two 0–cells we have

$$\sum_{i>1} i\,V(i) = 2E. \tag{7-2}$$

The Euler characteristic of F is

$$\sum_{i>1} V(i) - E + R = \chi(F) = 2. \tag{7-3}$$

From (7-1), (7-2) and (7-3), we get

$$\sum_{i>1} (4-i)V(i) = 8. \tag{7-4}$$

The equality (7-4) implies

$$2V(2) + V(3) = 8 + \sum_{i\geq 4}(i-4)V(i). \tag{7-5}$$

This shows that there exist vertices of valence less than or equal to 3.

Assume that v has valence 3. Let h_1, h_2, h_3 be the hyperbolic points that are connected to v by a singular leaf. We may assume that $\mathtt{sgn}(h_1) = \mathtt{sgn}(h_2)$. Let R_i denote the bb–tile that contains h_i. By condition (1), the common b–arc of R_1 and R_2 is separating, so by Proposition 3.2 and Theorem 3.1 we can apply a b–arc foliation change to $R_1 \cup R_2$, which lowers the valence of v but preserves $e(F)$ and no c–circles are introduced.

Hence we may assume that there exists a vertex of valence equal to 2. Call it v. Let C be the boundary component of S on which v lies. By condition (1) and Lemma 7.6 the two hyperbolic points around v have opposite signs. If v is strongly essential, Lemma 7.5 implies $|c(\phi, C)| \leq 1$. This contradicts condition (2), so v is nonstrongly essential. Hence by an exchange move on $\mathcal{F}_{\mathrm{ob}}(F)$ that involves an exchange move on L we can remove v and get a new splitting sphere F' with $e(F') = e(F) - 2$. We can repeat this procedure until we get F with $e(F) = 2$.

Case II: $\mathcal{F}_{\mathrm{ob}}(F)$ contains c–circle leaves In this case the region decomposition of F contains bc–annuli (and possibly cc–pants). Let R be an *innermost* bc–annulus; here by "innermost" we mean that the c–circle boundary of R bounds a disc D such that $R \subset D \subset F$ and $D \setminus R$ contains no c–circles. Because F is a sphere such an R necessarily exists and also a cc–pants cannot be innermost.

If R is degenerate (ie $D = R$) then by Lemma 7.7 we get a contradiction.

Suppose that R is nondegenerate. Then the region decomposition of $D_\circ := D \setminus R$ consists only of bb–tiles. We can verify that the formula (7-5) also holds for $\mathcal{F}_{\text{ob}}(D_\circ)$. We apply a similar argument as in Case I to $D_\circ$ repeatedly until all the 0–cells in $\text{Int}(D_\circ)$ disappear. Now the region R is a degenerate bc–annulus, which is a contradiction.

Therefore, under conditions (1), (2) of the theorem, $\mathcal{F}_{\text{ob}}(F)$ actually does not contain c–circles. □

Proof of the composite closed braid theorem We prove the composite closed braid theorem in the same way as the split closed braid theorem (SCBT). The main difference between the two theorems is that a decomposing sphere F has intersections with L but a splitting sphere does not.

By the same argument as in the embedded surface case [18, Theorem 3.2], using Novikov–Roussarie–Thurston's general position argument [23] we can put F so that it admits an essential open book foliation.

If the region decomposition of F consists only of bb–tiles then the above equality (7-5) holds. By the same argument as in Case I we may assume that $V(2) > 0$. Except for the case $V(2) = 4$ and $V(i) = 0$ for $i = 3, 4, \dots$, we can move the intersection points $L \cap F$ by following the guideline in Birman and Menasco [9, Lemma 1] outside the region we attempt to apply an exchange move (the shaded region in Figure 20(1)). Then we apply an exchange move. The number $e(F)$ decreases by 2 and no new c–circles are introduced. We repeat this procedure until F satisfies $V(2) = 4$ and $V(i) = 0$ for $i = 3, 4, \dots$ This case is depicted in [3, Figure 22] by Birman and Menasco. The only difference is the two b–arcs joining p_2, p_3 and p_1, p_3 in that figure may be strongly essential in our situation. By the argument in [3, page 136] our sphere F admits one more exchange move and we obtain $e(F) = 2$.

We need to treat the case where $\mathcal{F}_{\text{ob}}(F)$ contains c–circles. Let $R \subset F$ be an innermost bc–annuli. As in the proof of the SCBT, after exchange moves and b–arc foliation changes R becomes a degenerate bc–annulus. By the proof of Lemma 7.7, R must have one nonempty intersection with L. We note that $\mathcal{F}_{\text{ob}}(F)$ contains no cc–pants, because otherwise F is capped off by (at least) three degenerate bc–annuli and all but two are not pierced by L which contradicts Lemma 7.7.

Therefore up to isotopy we may consider that F consists of two degenerate bc–annuli R_1 and R_2, each of which is pierced by L (Figure 33(1)). We observe that all the b–arcs of $\mathcal{F}_{\mathrm{ob}}(F)$ are boundary-parallel, because otherwise by Lemma 7.5 condition (2) will be violated. All the c–circles of $\mathcal{F}_{\mathrm{ob}}(F)$ bound discs in their pages, because otherwise there must exist strongly essential b–arcs. Moreover each disc bounded by a c–circle is pierced by L in one point. We replace F with the degenerate bc–annulus R_1 capped off by the disc. We perturb the disc to be foliated by concentric circles and has a local extremal point (Figure 33(2)). Then flatten the extremal point paired with the hyperbolic point in R_1, this will turn F into a desired decomposition sphere (Figure 33(3)). During these operations the braid L is fixed. □

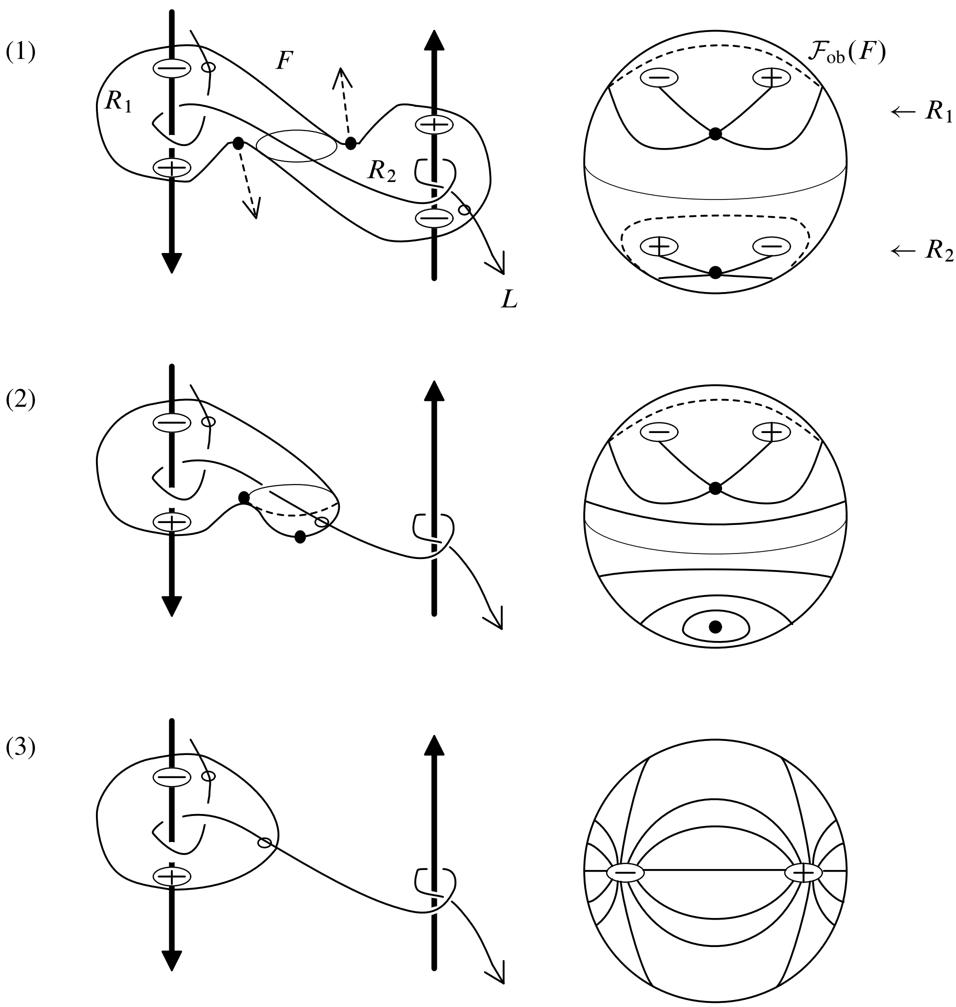

Figure 33: Special case: A decomposing sphere consisting of two degenerate bc–annuli

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