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# $E_1$–formality of complex algebraic varieties

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Let $X$ be a smooth complex algebraic variety. Morgan showed that the rational homotopy type of $X$ is a formal consequence of the differential graded algebra defined by the first term $E_1(X, W)$ of its weight spectral sequence. In the present work, we generalize this result to arbitrary nilpotent complex algebraic varieties (possibly singular and/or non-compact) and to algebraic morphisms between them. In particular, our results generalize the formality theorem of Deligne, Griffiths, Morgan and Sullivan for morphisms of compact Kähler varieties, filling a gap in Morgan's theory concerning functoriality over the rationals. As an application, we study the Hopf invariant of certain algebraic morphisms using intersection theory.

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## 1 Introduction

Morgan [21] introduced mixed Hodge diagrams of differential graded algebras (DGAs for short) and proved, using Sullivan's theory of minimal models, the existence of functorial mixed Hodge structures on the rational homotopy groups of smooth complex algebraic varieties. This result was independently extended to the singular case by Hain [18] and Navarro [22]. Such a mixed Hodge diagram is given by a filtered DGA $(A_{\mathbb{Q}}, W)$ defined over the field $\mathbb{Q}$ of rational numbers, a bifiltered DGA $(A_{\mathbb{C}}, W, F)$ defined over the field $\mathbb{C}$ of complex numbers, together with a finite string of filtered quasi-isomorphisms $(A_{\mathbb{Q}}, W) \otimes \mathbb{C} \leftrightarrow (A_{\mathbb{C}}, W)$ over $\mathbb{C}$, in such a way that the cohomology $H(A_{\mathbb{Q}})$ is a graded mixed Hodge structure. We denote by **MHD** the category whose objects are mixed Hodge diagrams and whose morphisms are given by level-wise filtered morphisms that make the corresponding diagrams commute. This differs from Morgan's original definition, in which level-wise morphisms commute only up to a filtered homotopy.

In the context of sheaf cohomology of DGAs, Navarro [22] introduced the Thom–Whitney simple functor and used this construction to establish the functoriality of mixed Hodge diagrams associated with complex algebraic varieties. He defined a functor $\mathbb{H}dg$: $\mathbf{Sch}(\mathbb{C}) \to \mathrm{Ho}(\mathbf{MHD})$ from the category of complex reduced schemes

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that are separated and of finite type to the homotopy category of mixed Hodge diagrams (defined by inverting level-wise quasi-isomorphisms), in such a way that the rational component of $\mathbb{H}dg(X)$ is the Sullivan–de Rham functor of $X$.

To study the homotopy category Ho(**MHD**), we introduce a notion of minimal object in the category of mixed Hodge diagrams and prove the existence of enough models of such type, adapting the classical theory of Sullivan's minimal models of DGAs. In conjunction with Navarro's functorial construction of mixed Hodge diagrams, this provides an alternative proof of Morgan's result on the existence of functorial mixed Hodge structures in rational homotopy. A main difference with respect to Morgan's approach is that our models are objects of a well defined category. The complex component of our minimal model coincides with Morgan's bigraded model (see [21, Section 6]). However, we preserve the rational information, allowing functorial results over the rational numbers. Using Deligne's splitting of mixed Hodge structures on the minimal models, we prove that morphisms of nilpotent complex algebraic varieties are $E_1$–formal at the rational level: the rational homotopy type is entirely determined by the first term of the spectral sequence associated with the multiplicative weight filtration. This generalizes the formality theorem of Deligne, Griffiths, Morgan and Sullivan [9] for compact Kähler manifolds and a result due to Morgan (see [21, Theorem 10.1]) for smooth open varieties. The results agree with Grothendieck's yoga of weights and can be viewed as a materialization of his principle in rational homotopy. Indeed, the weight filtration expresses the way in which the cohomology of the variety is related to cohomologies of smooth projective varieties. In particular, $E_1$–formality implies that, at the rational level, nilpotent complex algebraic varieties have finite-dimensional models determined by cohomologies of smooth projective varieties.

A note of caution about base-point independence and homotopy groups. In this paper we study the rational $E_1$–homotopy type of complex algebraic varieties: the class of a filtered differential graded algebra over $\mathbb{Q}$ in a certain localized category, whose first term of its associated spectral sequence is the weight spectral sequence of the variety. Therefore a treatment of augmented mixed Hodge diagrams and the subject of base-point independence is not developed here. A more detailed study of the homotopy category of (augmented) mixed Hodge diagrams appears in [6], where the first author interprets the existence of minimal models as a multiplicative version of Beĭlinson's theorem on mixed Hodge complexes (see [1, Theorem 2.3]).

This paper is organized as follows. Section 2 is devoted to the homotopy theory of filtered differential graded commutative algebras. We introduce the notions of $E_r$–quasi-isomorphism and $E_r$–formality and study descent properties with respect to field extensions. In Section 3, we study the homotopy theory of mixed Hodge diagrams. The existence of minimal models is proven in Theorem 3.17 for objects, and in Theorem 3.19 for morphisms. In Section 4, we recall Navarro's construction of mixed Hodge diagrams associated with complex algebraic varieties. This leads to the main result of this paper (Theorem 4.5) on the $E_1$–formality of complex algebraic varieties. Lastly, Section 5 is devoted to an application: we study the Hopf invariant of certain algebraic morphisms in the context of algebraic geometry, using the weight spectral sequence and intersection theory.

# 2 Homotopy theory of filtered algebras

The homotopy theory of filtered DGAs over a field of characteristic 0 was first studied by Halperin and Tanré [20], who verified some of the axioms for Quillen model categories. Following their ideas, in this section we introduce $E_r$–cofibrant filtered DGAs and show that these satisfy a homotopy lifting property with respect to $E_r$–quasi-isomorphisms. This result allows us to understand the homotopy theory of filtered DGAs within the axiomatic framework of Cartan–Eilenberg categories of Guillén, Navarro, Pascual and Roig [16]. We introduce the notions of $E_r$–formality and $r$–splitting of filtered differential graded algebras and study their descent properties with respect to field extensions.

## 2.1 Filtered differential graded commutative algebras

The notion of a filtered DGA arises from the compatible combination of a filtered complex with the multiplicative structure of a DGA. For the basic definitions and results on the homotopy theory of DGAs, we refer to Bousfield and Gugenheim [4], and Félix, Halperin and Thomas [10]. All DGAs considered will be non-negatively graded and defined over a field $\boldsymbol{k}$ of characteristic 0.

Denote by $\mathbf{F}\mathsf{DGA}(\boldsymbol{k})$ the category of filtered DGAs over $\boldsymbol{k}$. The base field $\boldsymbol{k}$ is considered as a filtered DGA with the trivial filtration and the unit map $\eta\colon \boldsymbol{k} \to A$ is filtered. We will restrict to filtered DGAs $(A, W)$ whose filtration is regular and exhaustive: for each $n \geq 0$ there exists $q \in \mathbb{Z}$ such that $W_q A^n = 0$, and $A = \bigcup_p W_p A$.

The spectral sequence associated with a filtered DGA $A$ is compatible with the multiplicative structure. Hence for all $r \geq 0$, the term $E_r^{*,*}(A)$ is a bigraded DGA with $d_r$ of bidegree $(r, 1-r)$.

For the rest of this section, fix an integer $r \geq 0$. We adopt the following definition of [20].

**Definition 2.1** A morphism of filtered DGAs $f\colon A \to B$ is called a $E_r$–*quasi-isomorphism* if $E_r(f)\colon E_r(A) \to E_r(B)$ is a quasi-isomorphism (the map $E_{r+1}(f)$ is an isomorphism).

Since filtrations are assumed to be regular and exhaustive, every $E_r$–quasi-isomorphism is a quasi-isomorphism. Denote by $\mathcal{E}_r$ the class of $E_r$–quasi-isomorphisms, and by

$$\mathrm{Ho}_r(\mathbf{F}\mathsf{DGA}(\boldsymbol{k})) := \mathbf{F}\mathsf{DGA}(\boldsymbol{k})[\mathcal{E}_r^{-1}]$$

the corresponding localized category. This is the main object of study in the homotopy theory of filtered DGAs. Objects in this category are called *$E_r$–homotopy types*. We have functors

$$\mathrm{Ho}_r(\mathbf{F}\mathsf{DGA}(\boldsymbol{k})) \xrightarrow{E_r} \mathrm{Ho}_0(\mathbf{F}\mathsf{DGA}(\boldsymbol{k})) \xrightarrow{H} \mathsf{DGA}(\boldsymbol{k}).$$

Deligne's décalage functor of filtered complexes [7, Definition 1.3.3] is compatible with multiplicative structures. It defines a functor Dec: $\mathbf{F}\mathsf{DGA}(\boldsymbol{k}) \to \mathbf{F}\mathsf{DGA}(\boldsymbol{k})$, which is the identity on morphisms. It follows from [7, Proposition 1.3.4] that $\mathcal{E}_{r+1} = \mathrm{Dec}^{-1}(\mathcal{E}_r)$. Hence there is an induced functor

$$\mathrm{Dec}\colon \mathrm{Ho}_{r+1}(\mathbf{F}\mathsf{DGA}(\boldsymbol{k})) \to \mathrm{Ho}_r(\mathbf{F}\mathsf{DGA}(\boldsymbol{k})).$$

In a subsequent paper we will show that this is in fact an equivalence of categories. In particular, the study of $E_r$–homotopy types reduces to the case $r = 0$.

**Definition 2.2** Let $(V, W)$ be a non-negatively graded $\boldsymbol{k}$–vector space with a regular and exhaustive filtration. The *free filtered graded algebra* $\Lambda(V, W)$ defined by $(V, W)$ is the free graded algebra $\Lambda(V)$ endowed with the multiplicative filtration induced by the filtration of $V$. If $A$ has a differential compatible with its multiplicative filtration, then it is called a *free filtered DGA*.

We next introduce a notion of homotopy between morphisms suitable to the study of $E_r$–homotopy types of filtered DGAs.

**2.3** Let $\Lambda(t, dt)$ be the free DGA with generators $t$ and $dt$ of degrees 0 and 1, respectively. For $r \geq 0$, define an increasing filtration $\sigma[r]$ on $\Lambda(t, dt)$ by letting $t$ be of pure weight 0 and $dt$ of pure weight $-r$ and extending multiplicatively. Note that $\sigma[0]$ is the trivial filtration, and $\sigma[1]$ is the bête filtration.

**Definition 2.4** The *$r$–path* $P_r(A)$ of a filtered DGA $(A, W)$ is the DGA $A \otimes \Lambda(t, dt)$ with the filtration defined by the convolution of $W$ and $\sigma[r]$. We have

$$W_p P_r(A) = \sum_q W_{p-q} A \otimes \sigma[r]_q \Lambda(t, dt) = (W_p A \otimes \Lambda(t)) \oplus (W_{p+r} A \otimes \Lambda(t)\, dt).$$

For each $\lambda \in \boldsymbol{k}$, there is a map $\delta^\lambda\colon P_r(A) \to A$ defined by $t \mapsto \lambda$ and $dt \mapsto 0$.

The following lemma is easy to verify.

**Lemma 2.5** *Let $A$ be a filtered DGA. There are canonical isomorphisms*

$$E_r(P_r(A)) \cong E_r(A) \otimes \Lambda(t, dt), \quad \mathrm{Dec}(P_{r+1}(A)) \cong P_r(\mathrm{Dec}A).$$

**Definition 2.6** Let $f, g\colon A \to B$ be morphisms of filtered DGAs. An *$r$–homotopy* from $f$ to $g$ is a morphism of filtered DGAs $h\colon A \to P_r(B)$ satisfying $\delta^0 h = f$ and $\delta^1 h = g$. Denote such an $r$–homotopy by $h\colon f \underset{r}{\simeq} g$.

**Lemma 2.7** *If $f \underset{r}{\simeq} g$, then $f = g$ in $\mathrm{Ho}_r(\mathbf{F}\mathsf{DGA}(\boldsymbol{k}))$ and $E_{r+1}(f) = E_{r+1}(g)$.*

**Proof** Since the inclusion $\iota\colon A \to A \otimes \Lambda(t, dt)$ is a quasi-isomorphism for any given DGA $A$, by Lemma 2.5 the map $\iota\colon B \to P_r(B)$ is an $E_r$–quasi-isomorphism. Hence $E_{r+1}(\delta^0) = E_{r+1}(\delta^1)$. □

## 2.2 Cofibrant filtered algebras

We introduce $E_r$–cofibrant DGAs as an adaptation to the filtered setting of the classical notion of Sullivan DGA. The following is a simplified variant of the notion of $(R, r)$–extension of [20].

**Definition 2.8** Let $(A, W)$ be a filtered DGA. An *$E_r$–cofibrant extension of $(A, W)$ of degree $n \geq 0$ and weight $p \in \mathbb{Z}$* is a filtered DGA $A \otimes_\xi \Lambda V$, where $V$ is a filtered graded module concentrated in pure degree $n$ and pure weight $p$ and $\xi\colon V \to W_{p-r}A$ is a linear map of degree 1 such that $d \circ \xi = 0$. The differential and the filtration on $A \otimes_\xi \Lambda V$ are defined by multiplicative extension.

**Definition 2.9** An *$E_r$–cofibrant DGA over $\boldsymbol{k}$* is a filtered DGA defined by the colimit of a sequence of $E_r$–cofibrant extensions starting from the base field $\boldsymbol{k}$.

**Lemma 2.10** *Let $(A, W)$ be an $E_r$–cofibrant DGA. Then:*

(1) *$A = \Lambda(V, W)$ is a free filtered DGA and $d(W_p A) \subset W_{p-r}A$ for all $p \in \mathbb{Z}$.*

(2) *As bigraded vector spaces, $E_0(A) = \cdots = E_{r-1}(A) = E_r(A)$.*

**Proof** Assertion (1) follows directly from the definition. From (1), the induced differentials of the associated spectral sequence satisfy $d_0 = d_1 = \cdots = d_{r-1} = 0$. Hence (2) follows. □

**Lemma 2.11** *Let $(A, W)$ be an $E_{r+1}$–cofibrant DGA. Then:*

(1) *For all $n \geq 0$ and all $p \in \mathbb{Z}$, $\mathrm{Dec}W_p A^n = W_{p-n}A^n$.*

(2) *The filtered DGA $\mathrm{Dec}A$ is $E_r$–cofibrant.*

**Proof** Recall that $\mathrm{Dec}W_pA^n = W_{p-n}A^n \cap d^{-1}(W_{p-n-1}A^{n+1})$. By Lemma 2.10(1), we have $d(W_pA) \subset W_{p-1}A$. Hence (1) follows. To prove (2) it suffices to note that if $A \otimes_\xi \Lambda(V)$ is an $E_{r+1}$–cofibrant extension of weight $p$ of $A$, then $\mathrm{Dec}A \otimes_\xi \Lambda(\mathrm{Dec}V)$ is an $E_r$–cofibrant extension of weight $p-n$ of $\mathrm{Dec}A$. □

We next show that $E_r$–cofibrant DGAs are cofibrant in the sense of [16].

**Theorem 2.12** *Let $M$ be an $E_r$–cofibrant DGA. For any solid diagram*

$$\begin{tikzcd} & A \arrow[d, "w", "\wr"'] \\ M \arrow[ur, dotted, "g"] \arrow[r, "f"'] & B \end{tikzcd}$$

*in which $w$ is an $E_r$–quasi-isomorphism, there exists a lifting $g$ together with an $r$–homotopy $h$: $wg \underset{r}{\simeq} f$. The morphism $g$ is uniquely defined up to $r$–homotopy.*

**Proof** To prove the existence of $g$ and $h$, we use induction over $r \geq 0$. The case $r = 0$ is an adaptation to the filtered setting of the proof of Proposition 11.1 of Griffiths and Morgan [13]. We shall only indicate the main changes. Assume that $w$ is an $E_0$–quasi-isomorphism. Let $M = M' \otimes_d \Lambda(V)$ be an $E_0$–cofibrant extension of degree $n$ and weight $p$, and assume that we have defined $g'$: $M' \to A$ together with a 0–homotopy $h'$: $wg' \underset{0}{\simeq} fi$, where $i$: $M' \to M$ denotes the inclusion.

Denote by $C(w)$ the mapping cone of $w$, with filtration $W_pC(w) = W_pA[1] \oplus W_pB$ and differential $d(a,b) = (-da, w(a) + db)$. For each $v \in V$, define a cocycle

$$\widetilde{\theta}(v) := \left( g'(dv), f(v) + \int_0^1 h'(dv) \right) \in W_pC(w)^n.$$

The assignation $v \mapsto [\widetilde{\theta}(v)]$ defines a map $\theta$: $V \to H^n(W_pC(w))$. Since $f$ is an $E_0$–quasi-isomorphism and filtrations are regular, we have $H^n(W_pC(w)) = 0$ for all $p \in \mathbb{Z}$. Therefore $\widetilde{\theta}(v)$ must be exact. Hence there exists a linear map $(a,b)$: $V \to W_pC(w)^{n-1}$ such that $d(a,b) = \widetilde{\theta}$. Define a filtered morphism $g$: $M \to A$ extending $g'$ and a 0–homotopy $h$: $M \to P_0(B)$ extending $h'$ by letting

$$g(v) := a(v) \quad \text{and} \quad h(v) := \left( f(v) + \int_0^t h'(dv) + d(b(v) \otimes t) \right).$$

This ends the case $r = 0$. Let $w$ be an $E_{r+1}$–quasi-isomorphism. By [7, Proposition 1.3.4] the map $\mathrm{Dec}(w)$ is an $E_r$–quasi-isomorphism. By Lemma 2.11 the algebra $\mathrm{Dec}M$ is $E_r$–cofibrant. By induction there exists a map $g$: $\mathrm{Dec}M \to \mathrm{Dec}A$, compatible with $\mathrm{Dec}W$, together with an $r$–homotopy $h$: $wg \underset{r}{\simeq} f$ with respect to $\mathrm{Dec}W$.

Since $M$ is $E_{r+1}$–cofibrant, by Lemma 2.11(1) we have $\mathrm{Dec} W_p M^n = W_{p-n} M^n$. It follows that $g$ is compatible with $W$, and that $h$ is an $(r+1)$–homotopy with respect to $W$. This ends the inductive step.

To prove that $g$ is uniquely defined up to an $r$–homotopy, it suffices to show that if $f_0, f_1\colon M \to A$ are such that $h\colon wf_0 \underset{r}{\simeq} wf_1$, then $f_0 \underset{r}{\simeq} f_1$. Define the $r$–double mapping path $\mathcal{M}_r^2(w)$ of $w$ via the pull-back diagram:

$$\begin{tikzcd} \mathcal{M}_r^2(w) \arrow[r] \arrow[d] & P_r(B) \arrow[d, "(\delta^0,\delta^1)"] \\ A \times A \arrow[r, "w\times w"] & B \times B \end{tikzcd}$$

The map $\overline{w}\colon P_r(A) \to \mathcal{M}_r^2(w)$ induced by $(\delta^0, \delta^1, P_r(w))$ is an $E_r$–quasi-isomorphism. We have a solid diagram

$$\begin{tikzcd} & P_r(A) \arrow[d, "\overline{w}", "\wr"'] \\ M \arrow[r, "H"'] \arrow[ur, dotted, "G"] & \mathcal{M}_r^2(w) \end{tikzcd}$$

where $H = (f_0, f_1, h)$. By the existence of lifts proven above there is a morphism $G$ such that $\overline{w}G \underset{r}{\simeq} H$. Then $G\colon f_0 \underset{r}{\simeq} f_1$ is an $r$–homotopy from $f_0$ to $f_1$. □

## 2.3 Splittings and formality

We next introduce the notions of $r$–splitting and $E_r$–formality and study their descent properties. The notion of $E_r$–formality is a homotopic version of the existence of $r$–splittings, and generalizes the classical notion of Sullivan [25], and Halperin and Stasheff [19], of formality of DGAs to the filtered setting.

**Definition 2.13** An *$r$–splitting* of a filtered DGA $(A, W)$ is a direct sum decomposition $A = \bigoplus A^{p,q}$ into subspaces $A^{p,q}$ such that for all $p, q \in \mathbb{Z}$,

$$d(A^{p,q}) \subset A^{p+r,q-r+1}, \quad A^{p,q} \cdot A^{p',q'} \subset A^{p+p',q+q'} \quad \text{and} \quad W_m A^n = \bigoplus_{p\le m} A^{-p,n+p}.$$

The $r^{\text{th}}$ term of the spectral sequence associated with a filtered DGA admits a natural filtration

$$W_p E_r(A) := \bigoplus_{i\le p} E_r^{-i,*}(A).$$

Hence $(E_r(A), d_r, W)$ is a filtered DGA with an $r$–splitting. The following result is straightforward.

**Proposition 2.14** *If a filtered DGA $(A,W)$ admits an $r$–splitting $A=\bigoplus A^{p,q}$, then the differentials of its spectral sequence satisfy $d_0=\cdots=d_{r-1}=0$, and there is an isomorphism of filtered DGAs $\pi\colon (A,d,W)\xrightarrow{\cong}(E_r(A),d_r,W)$, such that $\pi(A^{-p,n+p})=\mathrm{Gr}^W_p A^n=E_r^{-p,n+p}(A)$.*

The following example exhibits the relation between 1–splittings and formality.

**Example 2.15** Let $A$ be a filtered DGA, where $W$ is the trivial filtration $0=W_{-1}A\subset W_0A=A$. The bigraded model $M\to A$ of Halperin–Stasheff [19, 3.4] is $E_1$–cofibrant, and $A$ is formal (the DGAs $(A,d)$ and $(H(A),0)$ have the same Sullivan minimal model) if and only if $M$ admits a 1–splitting.

**Definition 2.16** A filtered DGA $(A,W)$ is said to be *$E_r$–formal* if there exists an isomorphism $(A,d,W)\to(E_r(A),d_r,W)$ in the homotopy category $\mathrm{Ho}_r(\mathbf{F}\mathsf{DGA}(\boldsymbol{k}))$.

In particular, if a filtered DGA is connected by a string of $E_r$–quasi-isomorphisms to a DGA admitting an $r$–splitting, then it is $E_r$–formal.

The previous definitions are naturally extended to morphisms.

**Definition 2.17** Let $f\colon A\to B$ be a morphism of filtered DGAs. We say that $f$ *admits an $r$–splitting* if $A$ and $B$ admit $r$–splittings and $f$ is compatible with them.

**Definition 2.18** A morphism of filtered DGAs $f\colon A\to B$ is said to be *$E_r$–formal* if there exists a commutative diagram

$$\begin{tikzcd}
(A,d,W) \arrow[r, "\cong"] \arrow[d, "f"] & (E_1(A),d_1,W) \arrow[d, "E_1(f)"] \\
(B,d,W) \arrow[r, "\cong"] & (E_1(B),d_1,W)
\end{tikzcd}$$

in the homotopy category $\mathrm{Ho}_r(\mathbf{F}\mathsf{DGA}(\boldsymbol{k}))$, where the horizontal arrows are isomorphisms.

## 2.4 Descent of splittings

The descent of formality of nilpotent DGAs from $\mathbb{C}$ to $\mathbb{Q}$ is proved in [25, Theorem 12.1]. The proof is based on the fact that the existence of certain grading automorphisms does not depend on the base field. Following this scheme, we characterize the existence of $r$–splittings of finitely generated $E_r$–cofibrant DGAs in terms of the existence of lifts of certain $r$–bigrading automorphisms.

**2.19** Let us fix some notation about group schemes. Given a filtered DGA $(A, W)$ denote by $\mathrm{Aut}_W(A)$ the set of its filtered automorphisms. Likewise, denote by $\mathrm{Aut}(E_r(A))$ the set of morphisms of bigraded DGAs from $E_r(A)$ to itself. We have a morphism $E_r\colon \mathrm{Aut}_W(A) \to \mathrm{Aut}(E_r(A))$.

Let $\boldsymbol{k} \to R$ be a commutative $\boldsymbol{k}$–algebra. The extension of scalars $A \otimes_{\boldsymbol{k}} R$ is a filtered DGA over $R$, and the correspondence

$$R \mapsto \mathbf{Aut}_W(A)(R) := \mathrm{Aut}_W(A \otimes_{\boldsymbol{k}} R)$$

defines a functor $\mathbf{Aut}_W(A)\colon \mathbf{alg}_{\boldsymbol{k}} \to \mathbf{Gr}$ from the category $\mathbf{alg}_{\boldsymbol{k}}$ of commutative $\boldsymbol{k}$–algebras to the category $\mathbf{Gr}$ of groups. It is clear that $\mathbf{Aut}_W(A)(\boldsymbol{k}) = \mathrm{Aut}_W(A)$.

**Proposition 2.20** *Let $(A, W)$ be a finitely generated $E_r$–cofibrant DGA over $\boldsymbol{k}$. Then:*

(1) $\mathrm{Aut}_W(A)$ *is an algebraic matrix group over $\boldsymbol{k}$.*

(2) $\mathbf{Aut}_W(A)$ *is an algebraic affine group scheme over $\boldsymbol{k}$ represented by* $\mathrm{Aut}_W(A)$.

(3) $E_r$ *defines a morphism $\boldsymbol{E_r}\colon \mathbf{Aut}_W(A) \to \mathbf{Aut}(E_r(A))$ of algebraic affine group schemes.*

(4) *The kernel $\boldsymbol{N} := \ker(\boldsymbol{E_r}\colon \mathbf{Aut}_W(A) \to \mathbf{Aut}(E_r(A)))$ is a unipotent algebraic affine group scheme over $\boldsymbol{k}$.*

**Proof** Since $A$ is finitely generated, for a sufficiently large $N \geq 0$, $\mathrm{Aut}_W(A)$ is the closed subgroup of $\mathbf{GL}_N(\boldsymbol{k})$ defined by the polynomial equations that express compatibility with differentials, products and filtrations. Thus $\mathrm{Aut}_W(A)$ is an algebraic matrix group. Moreover, $\mathbf{Aut}_W(A)$ is obviously the algebraic affine group scheme represented by $\mathrm{Aut}_W(A)$. Hence (1) and (2) are satisfied. For every commutative $\boldsymbol{k}$–algebra $R$, the map

$$\mathbf{Aut}_W(A)(R) = \mathrm{Aut}_W(A \otimes_{\boldsymbol{k}} R) \to \mathrm{Aut}(E_r(A) \otimes_{\boldsymbol{k}} R) = \mathbf{Aut}(E_r(A))(R)$$

is a morphism of groups which is natural in $R$. Thus (3) follows. Since, by (2), both of the groups $\mathbf{Aut}_W(A)$ and $\mathbf{Aut}(E_r(A))$ are algebraic, and $\boldsymbol{k}$ has zero characteristic, the kernel $\boldsymbol{N}$ is represented by an algebraic matrix group defined over $\boldsymbol{k}$ (see Borel [2, Corollary 15.4]). Therefore to prove (4) it suffices to verify that all elements in $\boldsymbol{N}(\boldsymbol{k})$ are unipotent. Given $f \in \boldsymbol{N}(\boldsymbol{k})$, consider the multiplicative Jordan decomposition $f = f_s \cdot f_u$ into semi-simple and unipotent parts. By [2, Theorem 4.4] we have $f_s, f_u \in \mathbf{Aut}_W(A)(\boldsymbol{k})$. Since $E_r(f) = 1$ and an algebraic group morphism preserves semi-simple and unipotent parts, it follows that $E_r(f_s) = E_r(f_u) = 1$. Let $A_1 = \mathrm{Ker}(f_s - I)$ and decompose $A$ into $f$–invariant subspaces $A = A_1 \oplus B$. Since $df_s = f_s d$, this decomposition satisfies $d(A_1) \subset A_1$ and $dB \subset B$. Hence both $A_1$ and $B$ are filtered subcomplexes of $A$ satisfying $d(W_pA_1) \subset W_{p-r}A_1$ and $d(W_pB) \subset W_{p-r}B$. Therefore we have

$$E_r(A) = E_0(A) = E_0(A_1) \oplus E_0(B) = E_r(A_1) \oplus E_r(B).$$

Since $E_r(A)$ contains nothing but the eigenspaces of eigenvalue $1$, we have $E_r(B) = E_0(B) = 0$, and so $B = 0$. Therefore $f_s = 1$ and $f$ is unipotent. □

**Definition 2.21** Let $\alpha \in \boldsymbol{k}^*$ (not a root of unity). The *$r$–bigrading automorphism of $E_r(A)$ associated with* $\alpha$ is the automorphism $\psi_\alpha\colon E_r(A) \to E_r(A)$ defined by

$$\psi_\alpha(a) = \alpha^{nr+p} a \quad \text{for } a \in E_r^{-p,n+p}(A).$$

**Lemma 2.22** *Let $(A, W)$ be a finitely generated $E_r$–cofibrant DGA over $\boldsymbol{k}$. The following are equivalent:*

1. *The filtered DGA $(A, W)$ admits an $r$–splitting.*
2. *The morphism $E_r\colon \mathrm{Aut}_W(A) \to \mathrm{Aut}(E_r(A))$ is surjective.*
3. *There exists $\alpha \in \boldsymbol{k}^*$ (not root of unity) together with an automorphism $\Phi \in \mathrm{Aut}_W(A)$ such that $E_r(\Phi) = \psi_\alpha$ is the $r$–bigrading automorphism of $E_r(A)$ associated with $\alpha$.*

**Proof** By Proposition 2.14, it follows that (1) implies (2). It is trivial that (2) implies (3). We show that (3) implies (1). Let $\Phi \in \mathrm{Aut}_W(A)$ be such that $E_r(\Phi) = \psi_\alpha$. Consider the multiplicative Jordan decomposition $\Phi = \Phi_s \cdot \Phi_u$. By [2, Theorem 4.4] we have that $\Phi_s, \Phi_u \in \mathrm{Aut}_W(A)$. Since $A$ is finitely generated there is a vector space decomposition of the form $A = A' \oplus B$, where

$$A' = \bigoplus A^{p,q} \quad \text{with } A^{-p,n+p} := \mathrm{Ker}(\Phi_s - \alpha^{nr+p} I) \cap A^n$$

and $B$ is the complementary subspace corresponding to the remaining factors of the characteristic polynomial of $\Phi_s$. Since $dA^n \subset A^{n+1}$ and $d\Phi_s = \Phi_s d$, this decomposition satisfies

$$d(A^{p,q}) \subset A^{p+r,q-r+1} \quad \text{and} \quad dB \subset B.$$

As in the proof of Proposition 2.20(4), one concludes that $B = 0$.

To show that $W_pA = \bigoplus_{i \le p} A^{-i,*}$ it suffices to see that $A^{-p,*} \subset W_pA$. For $x \in A^{-p,n+p}$, let $q$ be the smallest integer such that $x \in W_qA$. Then $x$ defines a class $x + W_{q-1}A \in E_r^{-q,n+q}(A)$, and

$$\psi_\alpha(x + W_{q-1}A) = \alpha^{nr+q} x + W_{q-1}A = \Phi(x) + W_{q-1}A = \alpha^{nr+p} x + W_{q-1}A.$$

It follows that $(\alpha^q-\alpha^p)\alpha^{nr}x\in W_{q-1}A$. Since $x\notin W_{q-1}A$, we have $q=p$, hence $x\in W_pA$. Since $\Phi$ is multiplicative we have $A^{p,q}\cdot A^{p',q'}\subset A^{p+p',q+q'}$, and hence the above decomposition is an $r$–splitting of $A$. $\square$

Based on Sullivan's formality criterion of [11, Theorem 1], the descent of formality for morphisms of DGAs is proved in [24, Theorem 3.2]. We follow the same scheme to characterize the existence of $r$–splittings of morphisms of filtered DGAs.

**2.23** Let $f\colon A\to B$ be a morphism of filtered DGAs. Denote by $\mathrm{Aut}_W(f)$ the set of pairs $(F^A,F^B)$, where $F^A\in\mathrm{Aut}_W(A)$ and $F^B\in\mathrm{Aut}_W(B)$ are such that $fF^A=F^Bf$. The set $\mathrm{Aut}(E_r(f))$ is defined analogously. We have a map $E_r\colon\mathrm{Aut}_W(f)\to\mathrm{Aut}(E_r(f))$. Let $\boldsymbol{k}\to R$ be a commutative $\boldsymbol{k}$–algebra. As in 2.19, the correspondence

$$R\mapsto\mathbf{Aut}_W(f)(R):=\mathrm{Aut}_W(f\otimes_{\boldsymbol{k}}R)$$

defines a functor $\mathbf{Aut}_W(f)\colon\mathbf{alg}_{\boldsymbol{k}}\to\mathbf{Gr}$ satisfying $\mathbf{Aut}_W(f)(\boldsymbol{k})=\mathrm{Aut}_W(f)$.

**Proposition 2.24** *Let $f$ be a map of finitely generated $E_r$–cofibrant DGAs over $\boldsymbol{k}$.*

1. $\mathrm{Aut}_W(f)$ *is an algebraic matrix group over* $\boldsymbol{k}$.
2. $\mathbf{Aut}_W(f)$ *is an algebraic affine group scheme over $\boldsymbol{k}$ represented by* $\mathrm{Aut}_W(f)$.
3. $E_r$ *defines a map $\boldsymbol{E_r}\colon\mathbf{Aut}_W(f)\to\mathbf{Aut}(E_r(f))$ of algebraic affine group schemes.*
4. *The kernel $N:=\ker(\boldsymbol{E_r}\colon\mathbf{Aut}_W(f)\to\mathbf{Aut}(E_r(f)))$ is a unipotent algebraic affine group scheme over $\boldsymbol{k}$.*

**Proof** The proof follows analogously to that of Proposition 2.20. $\square$

**Lemma 2.25** *Let $f\colon A\to B$ be a morphism of finitely generated $E_r$–cofibrant DGAs over $\boldsymbol{k}$. The following are equivalent:*

1. *The morphism $f\colon A\to B$ admits an $r$–splitting.*
2. *The morphism $E_r\colon\mathrm{Aut}_W(f)\to\mathrm{Aut}(E_r(f))$ is surjective.*
3. *There exists $\alpha\in\boldsymbol{k}^*$ (not root of unity) together $\Phi\in\mathrm{Aut}_W(f)$ such that $E_r(\Phi)=\psi_\alpha$ is induced by the level-wise $r$–bigrading automorphism associated with $\alpha$.*

**Proof** The proof follows analogously to that of Lemma 2.22. $\square$

**Theorem 2.26** *Let $f\colon A\to B$ be a morphism of finitely generated $E_r$–cofibrant DGAs over $\boldsymbol{k}$, and let $\boldsymbol{k}\subset\boldsymbol{K}$ be a field extension. Then $f$ admits an $r$–splitting if and only if $f_{\boldsymbol{K}}:=f\otimes_{\boldsymbol{k}}\boldsymbol{K}$ admits an $r$–splitting.*

**Proof** We may assume that $\boldsymbol{K}$ is algebraically closed. If $f_{\boldsymbol{K}}$ admits an $r$–splitting, the map $\mathbf{Aut}_W(f)(\boldsymbol{K}) \to \mathbf{Aut}(E_1(f))(\boldsymbol{K})$ is surjective by Lemma 2.22. From Section 18.1 of [26] there is an exact sequence of groups

$$1 \to \boldsymbol{N}(\boldsymbol{k}) \to \mathbf{Aut}_W(f)(\boldsymbol{k}) \to \mathbf{Aut}(E_1(f))(\boldsymbol{k}) \to H^1(\boldsymbol{K}/\boldsymbol{k}, \boldsymbol{N}) \to \cdots$$

where $\boldsymbol{N}$ is unipotent by Proposition 2.24. Since $\boldsymbol{k}$ has characteristic zero, the group $H^1(\boldsymbol{K}/\boldsymbol{k}, \boldsymbol{N})$ is trivial (see Example 18.2.e of [26]). This gives the exact sequence

$$1 \to \boldsymbol{N}(\boldsymbol{k}) \to \mathrm{Aut}_W(f) \to \mathrm{Aut}(E_1(f)) \to 1.$$

Hence the middle arrow is surjective, and $f$ admits a 1–splitting by Lemma 2.25. □

# 3 Homotopy theory of mixed Hodge diagrams

In this section we prove the existence of minimal models of mixed Hodge diagrams, as an adaptation of the classical construction of Sullivan's minimal models. We then use Deligne's splitting of mixed Hodge structures to prove $E_1$–formality for the rational component of mixed Hodge diagrams.

## 3.1 Mixed Hodge diagrams

Throughout this section we let $I = \{0 \to 1 \leftarrow 2 \to \cdots \leftarrow s\}$ be a finite category of zig-zag type and fixed length $s$. The following is a multiplicative version of the original notion of mixed Hodge complex [8, 8.1].

**Definition 3.1** A *mixed Hodge diagram (of DGAs over $\mathbb{Q}$ of type $I$)* consists of:

(i) A filtered DGA $(A_{\mathbb{Q}}, W)$ over $\mathbb{Q}$.

(ii) A bifiltered DGA $(A_{\mathbb{C}}, W, F)$ over $\mathbb{C}$.

(iii) An $E_1$–quasi-isomorphism $\varphi_u\colon (A_i, W) \to (A_j, W)$ over $\mathbb{C}$, for each $u\colon i \to j$ of $I$, with $A_0 = A_{\mathbb{Q}} \otimes \mathbb{C}$ and $A_s = A_{\mathbb{C}}$.

In addition, the following axioms are satisfied:

($\mathrm{MH}_0$) The weight filtrations $W$ are regular and exhaustive. The Hodge filtration $F$ is biregular. The cohomology $H(A_{\mathbb{Q}})$ has finite type.

($\mathrm{MH}_1$) For all $p \in \mathbb{Z}$, the differential of $\mathrm{Gr}_p^W A_{\mathbb{C}}$ is strictly compatible with $F$.

($\mathrm{MH}_2$) For all $n \geq 0$ and all $p \in \mathbb{Z}$, the filtration $F$ induced on $H^n(\mathrm{Gr}_p^W A_{\mathbb{C}})$ defines a pure Hodge structure of weight $p + n$ on $H^n(\mathrm{Gr}_p^W A_{\mathbb{Q}})$.

Such a diagram is denoted as

$$A = \big((A_{\mathbb{Q}}, W) \overset{\varphi}{\dashleftarrow\dashrightarrow} (A_{\mathbb{C}}, W, F)\big).$$

Note that axiom ($\mathrm{MH}_2$) implies that for all $n \geq 0$ the triple $(H^n(A_{\mathbb{Q}}), \mathrm{Dec}W, F)$ is a mixed Hodge structure over $\mathbb{Q}$.

**Definition 3.2** A *pre-morphism $f$ of mixed Hodge diagrams from $A$ to $B$* consists of:

(i) A morphism of filtered DGAs $f_{\mathbb{Q}}\colon (A_{\mathbb{Q}}, W) \to (B_{\mathbb{Q}}, W)$ over $\mathbb{Q}$.

(ii) A morphism of bifiltered DGAs $f_{\mathbb{C}}\colon (A_{\mathbb{C}}, W, F) \to (B_{\mathbb{C}}, W, F)$ over $\mathbb{C}$.

(iii) A family of morphisms of filtered DGAs $f_i\colon (A_i, W) \to (B_i, W)$ over $\mathbb{C}$, for each $i \in I$, with $f_0 = f_{\mathbb{Q}} \otimes \mathbb{C}$ and $f_s = f_{\mathbb{C}}$.

**Definition 3.3** A pre-morphism $f$ is said to be a *quasi-isomorphism* if $f_{\mathbb{Q}}$, $f_{\mathbb{C}}$ and $f_i$ are quasi-isomorphisms (the induced morphisms $H^*(f_{\mathbb{Q}})$, $H^*(f_i)$ and $H^*(f_{\mathbb{C}})$ are isomorphisms).

The following result is an easy consequence of [8, Scholie 8.1.9], stating that the spectral sequences associated with the Hodge and the weight filtrations degenerate at the stages $E_1$ and $E_2$, respectively.

**Lemma 3.4** *Let $f$ be a quasi-isomorphism of mixed Hodge diagrams. Then $f_{\mathbb{Q}}$ and $f_i$ are $E_1$–quasi-isomorphisms and $f_{\mathbb{C}}$ is an $E_{1,0}$–quasi-isomorphism (the induced morphisms $E_2(f_{\mathbb{Q}})$, $E_2(f_i)$ for all $i \in I$ and $E_2(\mathrm{Gr}_F^p f_{\mathbb{C}})$ for all $p \in \mathbb{Z}$ are isomorphisms).*

**Definition 3.5** A *morphism of mixed Hodge diagrams* $f\colon A \to B$ is a pre-morphism such that for all $u\colon i \to j$ of $I$ the diagram

$$\begin{tikzcd} (A_i, W) \arrow[r, "\varphi_u^A"] \arrow[d, "f_i"'] & (A_j, W) \arrow[d, "f_j"] \\ (B_i, W) \arrow[r, "\varphi_u^B"] & (B_j, W) \end{tikzcd}$$

commutes. We denote such a morphism by $f\colon A \to B$.

Denote by **MHD** the category of mixed Hodge diagrams over $\mathbb{Q}$ of a fixed type $I$ and by Ho(**MHD**) the localized category of mixed Hodge diagrams with respect to the class of quasi-isomorphisms. By Lemma 3.4, the forgetful functor $U_{\mathbb{Q}}\colon \mathbf{MHD} \to \mathbf{F}\mathsf{DGA}(\mathbb{Q})$

defined by sending every mixed Hodge diagram $A$ to the filtered DGA $(A_{\mathbb{Q}}, W)$ sends quasi-isomorphisms of mixed Hodge diagrams to $E_1$–quasi-isomorphisms of filtered DGAs. Hence there is an induced functor

$$U_{\mathbb{Q}}\colon \mathrm{Ho}(\mathbf{MHD}) \to \mathrm{Ho}_1(\mathbf{F}\mathsf{DGA}(\mathbb{Q})).$$

For the construction of minimal models, we shall need a broader class of maps between mixed Hodge diagrams, defined by level-wise morphisms commuting only up to 1–homotopy.

**Definition 3.6** A *ho-morphism of mixed Hodge diagrams* is a pre-morphism $f$ together with a family of 1–homotopies (see Definition 2.6) $F_u\colon f_j\varphi_u^A \simeq \varphi_u^B f_i$ for all $u\colon i \to j$ of $I$, making the diagram

$$\begin{tikzcd}
(A_i, W) \arrow[r, "\varphi_u^A"] \arrow[d, "f_i"'] \arrow[dr, Rightarrow, "F_u"] & (A_j, W) \arrow[d, "f_j"] \\
(B_i, W) \arrow[r, "\varphi_u^B"] & (B_j, W)
\end{tikzcd}$$

1–homotopy commute. We denote such a ho-morphism by $f\colon A \rightsquigarrow B$.

In general, ho-morphisms cannot be composed. Therefore unlike (strictly commutative) morphisms, they do not define a category. However, homotopy classes of ho-morphisms between minimal cofibrant define a category (see [6]).

We next introduce the mapping cone of a ho-morphism of mixed Hodge diagrams and show that under the choice of certain filtrations, the mapping cone is a mixed Hodge complex.

**Definition 3.7** Let $f\colon A \to B$ be a morphism of filtered DGAs. The *$r$–cone of $f$* is the filtered complex $C_r(f)$ defined by

$$W_p C_r(f) := W_{p-r} A^{n+1} \oplus W_p B^n, \quad \text{with } d = (-da, f(a) + db).$$

For a bifiltered morphism $f\colon A \to B$, the *$(r,s)$–cone* $C_{r,s}(f)$ is defined analogously:

$$W_p F^q C_r(f) := W_{p-r} F^{q+s} A^{n+1} \oplus W_p F^q B^n.$$

**3.8** Let $f\colon A \rightsquigarrow B$ be a ho-morphism of mixed Hodge diagrams. For each $u\colon i \to j$ of $I$, the 1–homotopy of filtered DGAs $F_u\colon A_i \to P_1(B_j)$ from $f_j\varphi_u^A$ to $\varphi_u^B f_i$ of the ho-morphism $f$ gives rise to a homotopy

$$W_p \int_0^1 F_u\colon W_p A_i \to W_{p+1} B_j[-1]$$

at the level of underlying complexes of vector spaces (see [13, Section 11.1]). This allows us to define filtered morphisms $\varphi_u^f\colon C_1(f_i) \to C_1(f_j)$ by letting

$$(a,b) \mapsto \left( \varphi_u^A(a), \varphi_u^B(b) + \int_0^1 F_u(a) \right).$$

**Definition 3.9** The *mixed cone* of a ho-morphism $f\colon A \rightsquigarrow B$ of mixed Hodge diagrams is the diagram of filtered complexes given by

$$C(f) = \Big( (C_1(f_{\mathbb{Q}}), W) \overset{\varphi^f}{\longleftrightarrow} (C_{1,0}(f_{\mathbb{C}}), W, F) \Big).$$

**Lemma 3.10** (cf [23, Theorem 3.22]) *Let $f\colon A \rightsquigarrow B$ be a ho-morphism of mixed Hodge diagrams. The mixed cone of $f$ is a mixed Hodge complex.*

**Proof** Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & B_i \arrow[r] \arrow[d, "\varphi_u^B"] & C(f_i) \arrow[r] \arrow[d, "\varphi_u^f"] & A_i[1] \arrow[r] \arrow[d, "\varphi_u^A"] & 0 \\
0 \arrow[r] & B_j \arrow[r] & C(f_j) \arrow[r] & A_j[1] \arrow[r] & 0
\end{tikzcd}$$

By the five lemma, $\varphi_u^f$ is an $E_1$–quasi-isomorphism. Condition ($\mathrm{MH}_0$) is trivial. For all $p \in \mathbb{Z}$,

$$\mathrm{Gr}_p^W C(f_{\mathbb{C}}) = \mathrm{Gr}_{p-1}^W A_{\mathbb{C}}[1] \oplus \mathrm{Gr}_p^W B_{\mathbb{C}}.$$

Hence at the graded level, the contribution of $f_{\mathbb{C}}$ to the differential of $C(f_{\mathbb{C}})$ vanishes. Therefore we have a direct sum decomposition of complexes compatible with the Hodge filtration $F$, and ($\mathrm{MH}_1$) and ($\mathrm{MH}_2$) follow. □

## 3.2 Minimal models

The following technical results will be of use for the construction of minimal models of mixed Hodge diagrams. Let us first recall Deligne's splitting [7, 1.2.11]; see also Griffiths and Schmid [14, Lemma 1.12]. This is a global decomposition for any given mixed Hodge structure, which generalizes the decomposition of pure Hodge structures.

**Lemma 3.11** [7, 1.2.11] *Let $(V, W, F)$ be a mixed Hodge structure defined over $\boldsymbol{k}$. Then $V_{\mathbb{C}} = V \otimes_{\boldsymbol{k}} \mathbb{C}$ admits a direct sum decomposition $V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$ such that the filtrations $W$ and $F$ defined on $V_{\mathbb{C}}$ are given by*

$$W_m V_{\mathbb{C}} = \bigoplus_{p+q \le m} I^{p,q} \quad \text{and} \quad F^l V_{\mathbb{C}} = \bigoplus_{p \ge l} I^{p,q}.$$

*The above decomposition is functorial for morphisms of mixed Hodge structures.*

**Lemma 3.12** *Let $A$ be a mixed Hodge diagram.*

(1) *There are sections $\sigma^n_{\mathbb{Q}}\colon H^n(A_{\mathbb{Q}}) \to Z^n(A_{\mathbb{Q}})$ and $\sigma^n_i\colon H^n(A_i) \to Z^n(A_i)$ of the projection, which are compatible with the weight filtration $W$.*

(2) *There exists a section $\sigma^n_{\mathbb{C}}\colon H^n(A_{\mathbb{C}}) \to Z^n(A_{\mathbb{C}})$ of the projection, which is compatible with both filtrations $W$ and $F$.*

**Proof** Since the differential of $A_{\mathbb{Q}}$ is strictly compatible with the filtration $\mathrm{Dec}W$, there is a section $\sigma_{\mathbb{Q}}\colon H^n(A_{\mathbb{Q}}) \to Z^n(A_{\mathbb{Q}})$ compatible with $\mathrm{Dec}W$. Since

$$\mathrm{Dec}W_p H^n(A_{\mathbb{Q}}) = W_{p-n} H^n(A_{\mathbb{Q}}),$$

the map $\sigma_{\mathbb{Q}}$ is compatible with $W$. For $\sigma_i$ the proof is analogous. This proves (1).

Let us prove (2). Since $(H^n(A_{\mathbb{Q}}), \mathrm{Dec}W, F)$ is a mixed Hodge structure, by Lemma 3.11 there is a direct sum decomposition $H^n(A_{\mathbb{C}}) = \bigoplus I^{p,q}$ with

$$I^{p,q} \subset \mathrm{Dec}W_{p+q} F^p H^n(A_{\mathbb{C}})$$

such that

$$W_m H^n(A_{\mathbb{C}}) = \mathrm{Dec}W_{m+n} H^n(A_{\mathbb{C}}) = \bigoplus_{\substack{p+q\\ \leq m+n}} I^{p,q} \quad \text{and} \quad F^l H^n(A_{\mathbb{C}}) = \bigoplus_{p\geq l} I^{p,q}.$$

Therefore it suffices to define sections $\sigma^{p,q}\colon I^{p,q} \to Z^n(A_{\mathbb{C}})$. By [8, Scholie 8.1.9], the four spectral sequences

$$\begin{tikzcd}
E_1(\mathrm{Gr}^{\mathrm{Dec}W}_{\bullet} A_{\mathbb{C}}, F) \arrow[r, Rightarrow] \arrow[d, equal] & E_1(A_{\mathbb{C}}, \mathrm{Dec}W) \arrow[dr, Rightarrow] & \\
& & H(K) \\
E_1(\mathrm{Gr}_F^{\bullet} A_{\mathbb{C}}, \mathrm{Dec}W) \arrow[r, Rightarrow] & E_1(A_{\mathbb{C}}, F) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

degenerate at $E_1$. It follows that the induced filtrations in cohomology are given by

$$\mathrm{Dec}W_p F^q H^n(A_{\mathbb{C}}) = \mathrm{Im}\{H^n(\mathrm{Dec}W_p F^q A_{\mathbb{C}}) \to H^n(A_{\mathbb{C}})\}.$$

Since $I^{p,q} \subset \mathrm{Dec}W_{p+q} F^p H^n(A_{\mathbb{C}})$, we have $\sigma^{p,q}(I^{p,q}) \subset \mathrm{Dec}W_{p+q} F^p A_{\mathbb{C}}$. Define

$$\sigma^n_{\mathbb{C}} := \bigoplus \sigma^{p,q}\colon H^n(A_{\mathbb{C}}) \to A_{\mathbb{C}}.$$

For the weight filtration, we have

$$\sigma^n_{\mathbb{C}}(W_m H^n(A_{\mathbb{C}})) = \bigoplus_{p+q\leq m+n} \sigma^{p,q}(I^{p,q}) \subset \bigoplus_{p+q\leq m+n} \mathrm{Dec}W_{p+q} A^n_{\mathbb{C}} \subset W_m A^n_{\mathbb{C}}.$$

Therefore $\sigma_{\mathbb{C}}$ is compatible with $W$. For the Hodge filtration, we have

$$\sigma_{\mathbb{C}}^n(F^l H^n(A_{\mathbb{C}})) = \bigoplus_{p\geq l} \sigma^{p,q}(I^{p,q}) \subset \sum_{p\geq l} F^p A_{\mathbb{C}} \subset F^l A_{\mathbb{C}}.$$

Therefore $\sigma_{\mathbb{C}}$ is compatible with $F$. □

**Definition 3.13** A mixed Hodge diagram $A$ is called $0$*–connected* if the unit map $\eta\colon \mathbb{Q} \to A_{\mathbb{Q}}$ induces an isomorphism $\mathbb{Q} \cong H^0(A_{\mathbb{Q}})$.

**Definition 3.14** A *mixed Hodge DGA* is a filtered DGA $(A, W)$ over $\mathbb{Q}$, together with a filtration $F$ on $A_{\mathbb{C}} := A \otimes_{\mathbb{Q}} \mathbb{C}$, such that for each $n \geq 0$, the triple $(A^n, \mathrm{Dec}W, F)$ is a mixed Hodge structure and the differentials $d\colon A^n \to A^{n+1}$ and products $A^n \otimes A^m \to A^{n+m}$ are morphisms of mixed Hodge structures.

The cohomology of every mixed Hodge diagram is a mixed Hodge DGA with trivial differential. Conversely, since the category of mixed Hodge structures is abelian [7, Theorem 2.3.5], every mixed Hodge DGA is a mixed Hodge diagram in which the comparison morphisms are identities. We will show that every $0$–connected mixed Hodge diagram is quasi-isomorphic to a mixed Hodge DGA satisfying the following minimality condition.

**Definition 3.15** Let $(A, W, F)$ be a mixed Hodge DGA. A *mixed Hodge extension* of $A$ of degree $n$ is a mixed Hodge DGA $A \otimes_\xi \Lambda(V)$, where $(V, W)$ is a filtered graded module concentrated in pure degree $n$ and $\xi\colon V \to A$ is a linear map of degree 1 such that $d \circ \xi = 0$ and $\xi(W_p V) \subset W_{p-1} A$. In addition, the vector space $V \otimes \mathbb{C}$ has a filtration $F$ compatible with $\xi$ making the triple $(V, \mathrm{Dec}W, F)$ into a mixed Hodge structure. The differentials and filtrations on $A \otimes_\xi \Lambda(V)$ are defined by multiplicative extension. Such an extension is said to be *minimal* if $A$ is augmented and $\xi(V) \subset A^+ \cdot A^+$.

**Definition 3.16** A mixed Hodge DGA is said to be *minimal* if it is the colimit of a sequence of minimal mixed Hodge extensions starting from the base field $\mathbb{Q}$ endowed with the trivial mixed Hodge structure.

Thus every minimal mixed Hodge DGA is a Sullivan minimal $E_1$–cofibrant DGA. To construct minimal models for $0$–connected mixed Hodge diagrams, we adapt the classical step by step construction of Sullivan minimal models for $0$–connected DGAs.

**Theorem 3.17** *For every $0$–connected mixed Hodge diagram $A$, there exists a minimal mixed Hodge DGA $M$ together with a ho-morphism $\rho\colon M \rightsquigarrow A$ that is a quasi-isomorphism.*

**Proof** We will define inductively over $n \geq 1$ and $q \geq 0$ a sequence of mixed Hodge DGAs $M(n,q)$ together with ho-morphisms $\rho(n,q)\colon M(n,q) \rightsquigarrow A$ satisfying the following conditions:

($a_{1,0}$) $M(1,0) = \mathbb{Q}$ has a mixed Hodge structure defined by the trivial filtrations.

($a_{n,q}$) If $q > 0$, then $M(n,q) = M(n,q-1) \otimes_\xi \Lambda(V)$ is a minimal extension of degree $n$. The morphism $\rho(n,q)^*_{\mathbb{Q}}\colon H^i(M(n,q)) \to H^i(A_{\mathbb{Q}})$ is an isomorphism for all $i \leq n$, and the morphism $i^*\colon H^n(C(\rho(n,q-1))) \to H^n(C(\rho(n,q)))$ is trivial.

($a_{n,0}$) If $n > 1$, then $M(n,0)$ is the colimit of a sequence

$$\cdots \subset M(n-1,q) \subset M(n-1,q+1) \subset \cdots$$

and the map $\rho(n,0)\colon M(n,0) \rightsquigarrow A$ is the induced ho-morphism.

Then the mixed Hodge DGA $M = \bigcup_n M(n,0)$, together with the induced ho-morphism $\rho\colon M \rightsquigarrow A$ will be the required quasi-isomorphism.

Assume that we have constructed a minimal mixed Hodge DGA $\widetilde{M} = M(n,q-1)$ and a ho-morphism $\widetilde{\rho} = \rho(n,q-1) \rightsquigarrow A$ satisfying $(a_{n,q-1})$. Consider the filtered vector spaces of degree $n$ given by

$$V_{\mathbb{Q}} = H^n(C_1(\widetilde{\rho}_{\mathbb{Q}})), \quad V_i = H^n(C_1(\widetilde{\rho}_i)) \quad \text{for } i \in I \quad \text{and} \quad V_{\mathbb{C}} = H^n(C_{1,0}(\widetilde{\rho}_{\mathbb{C}})).$$

By Lemma 3.10 the mixed cone $C(\widetilde{\rho})$ is a mixed Hodge complex. Hence we have filtered isomorphisms

$$(V_{\mathbb{Q}}, W) \otimes \mathbb{C} \cong (V_i, W) \cong (V_{\mathbb{C}}, W)$$

making the triple $(V_{\mathbb{Q}}, \mathrm{Dec}W, F)$ into a mixed Hodge structure. By Lemma 3.12 there are sections $\sigma_{\mathbb{Q}}\colon V_{\mathbb{Q}} \to Z^n(C_1(\widetilde{\rho}_{\mathbb{Q}}))$ and $\sigma_i\colon V_i \to Z^n(C_1(\widetilde{\rho}_i))$ compatible with $W$, together with a section $\sigma_{\mathbb{C}}\colon V_{\mathbb{C}} \to Z^n(C_1(\widetilde{\rho}_{\mathbb{C}}))$, compatible with $W$ and $F$. Define filtered DGAs

$$M_{\mathbb{Q}} = \widetilde{M}_{\mathbb{Q}} \otimes \Lambda(V_{\mathbb{Q}}), \quad M_i = \widetilde{M}_i \otimes \Lambda(V_i) \quad \text{for } i \in I \quad \text{and} \quad M_{\mathbb{C}} = \widetilde{M}_{\mathbb{C}} \otimes \Lambda(V_{\mathbb{C}}).$$

The corresponding filtrations are defined by multiplicative extension. The sections $\sigma_{\mathbb{Q}}$, $\sigma_i$ and $\sigma_{\mathbb{C}}$ allow us to define differentials such that $d(W_p V_{\mathbb{Q}}) \subset W_{p-1} \widetilde{M}^{n+1}_{\mathbb{Q}}$ and $d(F^p V_{\mathbb{C}}) \subset F^q \widetilde{M}^{n+1}_{\mathbb{C}}$, and maps $\rho_{\mathbb{Q}}\colon M_{\mathbb{Q}} \to A_{\mathbb{Q}}$, $\rho_i\colon M_i \to A_i$ and $\rho_{\mathbb{C}}\colon M_{\mathbb{C}} \to A_{\mathbb{C}}$ compatible with the corresponding filtrations. Since, by hypothesis, $\widetilde{M}_{\mathbb{Q}}$ is generated in degrees less than or equal to $n$, it follows that $dV_{\mathbb{Q}} \subset \widetilde{M}^+_{\mathbb{Q}} \cdot \widetilde{M}^+_{\mathbb{Q}}$.

Since $M_i$ is $E_1$–cofibrant, by Theorem 2.12, for every solid diagram

$$\begin{tikzcd} M_i \arrow[r, dotted, "\varphi_u"] \arrow[d, "\rho_i"'] & M_j \arrow[d, "\rho_j"] \\ A_i \arrow[r, "\varphi_u"] & A_j \end{tikzcd}$$

there exists a morphism $\varphi_u\colon M_i \to M_j$ together with a 1–homotopy $R_u$ from $\rho_j\varphi_u$ to $\varphi_u\rho_i$. Since the $M_i$ are minimal, $\varphi_u$ is an isomorphism. Hence we can transport the filtrations $F$ of $M_{\mathbb{C}}$ to $M_{\mathbb{Q}}\otimes\mathbb{C}$. Let $M(n,q)=M_{\mathbb{Q}}$. The morphisms $\rho_{\mathbb{Q}}$ and $\rho_{\mathbb{C}}\varphi_u$, together with the homotopies $R_u$ define a ho-morphism $\rho\colon M(n,q)\rightsquigarrow A$ satisfying $(a_{n,q})$. This ends the inductive step. □

To construct minimal models of morphisms of mixed Hodge diagrams, we adapt the classical construction for morphisms of DGAs (see [10, Section 14]).

**Definition 3.18** A *relative minimal mixed Hodge DGA* is given by an inclusion $M \hookrightarrow \widetilde{M}$ of mixed Hodge DGAs where $\widetilde{M}=M\otimes\Lambda(V)$ is a colimit mixed Hodge extensions of $M$ satisfying

$$d(W_pV)\subset W_{p-1}(M^+\otimes\Lambda(V))\oplus W_{p-1}(M\otimes\Lambda^{\geq 2}V).$$

**Theorem 3.19** *For every morphism $f\colon A\to B$ of 0–connected mixed Hodge diagrams, there exists a relative minimal mixed Hodge DGA $\widetilde{f}\colon M\to\widetilde{M}$, with $M$ minimal, and a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ M \arrow[u, squiggly, "\rho"] \arrow[r, "\widetilde{f}"] & \widetilde{M} \arrow[u, squiggly, "\rho'"'] \end{tikzcd}$$

*where the vertical ho-morphisms are quasi-isomorphisms.*

**Proof** We proceed inductively over $n\geq 1$ and $q\geq 0$ as follows. By Theorem 3.17 there is a minimal mixed Hodge DGA $M$ and a quasi-isomorphism $\rho\colon M\rightsquigarrow A$. As a base case for our induction, we take $\widetilde{M}(1,0)=M$ and $\rho(1,0)=f\rho$. Assume inductively that we have constructed a relative minimal mixed Hodge DGA $f(n,q-1)\colon M\to\widetilde{M}(n,q-1)$ together with a quasi-isomorphism $\widetilde{\rho}(n,q-1)\colon\widetilde{M}(n,q-1)\rightsquigarrow B$. The inductive step follows as in Theorem 3.17, by taking a mixed Hodge extension defined via the mixed cone of the ho-morphism $\widetilde{\rho}(n,q-1)$. The ho-morphism

$$\rho' := \bigcup\widetilde{\rho}(n,0) : \widetilde{M} := \bigcup_n \widetilde{M}(n,0)\rightsquigarrow A,$$

together with the inclusion $\widetilde{f}\colon M \to \widetilde{M}$, give the required commutative diagram. □

### 3.3 Formality of mixed Hodge diagrams

We next show that Deligne's splitting of mixed Hodge structures induces a 1–splitting of $(A, W)\otimes_{\mathbb{Q}}\mathbb{C}$, for any given mixed Hodge DGA $(A, W, F)$ such that $(A, W)$ is $E_1$–cofibrant. By the results of Section 2, this descends to a 1–splitting of $(A, W)$ whenever $(A, W, F)$ is a minimal mixed Hodge DGA of finite type. Together with the existence of minimal models, this proves $E_1$–formality for the rational component of 0–connected mixed Hodge diagrams with finite homotopy type. We actually prove these results for morphisms of such objects.

**Lemma 3.20** (cf [21, Theorem 9.6]) *Let $f\colon A \to B$ be a morphism of mixed Hodge DGAs of finite type and let $f_{\mathbb{C}} := f\otimes_{\mathbb{Q}}\mathbb{C}$. Then $\mathrm{Dec} f_{\mathbb{C}}$ admits a 0–splitting over $\mathbb{C}$. Furthermore, if $A$ and $B$ are $E_1$–cofibrant, then $f_{\mathbb{C}}$ admits a 1–splitting over $\mathbb{C}$.*

**Proof** Since for all $n \geq 0$, the triple $(A^n, \mathrm{Dec}W, F)$ is a mixed Hodge structure, by Lemma 3.11 we have functorial decompositions

$$A^n_{\mathbb{C}} = \bigoplus I_n^{p,q}, \quad \text{with } \mathrm{Dec}W_m A^n_{\mathbb{C}} = \bigoplus_{p+q\leq m} I_n^{p,q}.$$

Since the differentials and products of $A$ are morphisms of mixed Hodge structures, we have $d(I_n^{p,q}) \subset I_{n+1}^{p,q}$ and $I_n^{p,q}\cdot I_{n'}^{p',q'} \subset I_{n+n'}^{p+p',q+q'}$. Then the $A^{p,n-p} := \bigoplus_r I_n^{-p-r,r}$ define a 0–splitting of the filtered DGA $(A_{\mathbb{C}}, \mathrm{Dec}W)$. Apply the same argument to define a 0–splitting for $B_{\mathbb{C}} = \bigoplus B^{p,n-p}$. Since $\mathrm{Dec} f\colon \mathrm{Dec}A \to \mathrm{Dec}B$ is a morphism of graded mixed Hodge structures and Deligne's splittings are functorial, the morphism $\mathrm{Dec} f_{\mathbb{C}}$ is compatible with these 0–splittings.

For $E_1$–cofibrant DGAs, the décalage functor has an inverse defined by shifting the weight filtration. Indeed, if $A$ is $E_1$–cofibrant, by Lemma 2.11 we have $W_p A^n = \mathrm{Dec}W_{p+n}A^n$. Then the $\widetilde{A}^{p,n-p} := A^{p-n,2n-p}$ define a 1–splitting of $A$ with respect to the filtration $W$. The same argument applies to $B$. The map $f_{\mathbb{C}}$ is compatible with these 1–splittings. □

**Lemma 3.21** *Let $f\colon M \hookrightarrow \widetilde{M}$ be a relative minimal mixed Hodge DGA, with $M$ minimal. If $M$ and $\widetilde{M}$ have finite type then $f$ admits a 1–splitting over $\mathbb{Q}$.*

**Proof** Let $t_n M$ denote the subalgebra of $M$ generated by $M^{\leq n}$. Likewise, denote by $t_n\widetilde{M}$ the subalgebra of $\widetilde{M}$ generated by $M^{\leq n+1} + \widetilde{M}^{\leq n}$. The minimality conditions on $M$ and $f$ ensure that both $t_n M$ and $t_n\widetilde{M}$ are stable under the differentials. Hence

$t_n M$ and $t_n \widetilde{M}$ are filtered sub-DGAs of $M$ and $\widetilde{M}$ respectively. Denote the restriction of $f$ by $t_n f\colon t_n M \to t_n \widetilde{M}$. Then $f$ can be written as the inductive limit of $t_n f$ over $n \geq 0$. Since $f$ is a morphism of $E_1$–cofibrant DGAs of finite type, it follows that:

(i) $t_n f\colon t_n M \to t_n \widetilde{M}$ is a morphism of $E_1$–cofibrant finitely generated DGAs.

(ii) $t_n f$ is stable under the automorphisms of $f$: there is a map $\mathrm{Aut}(f, W) \to \mathrm{Aut}(t_n f, W)$.

(iii) There is an inverse system of groups $(\mathrm{Aut}(t_n f, W))_n$ and an isomorphism of groups
$$\mathrm{Aut}(f, W) \to \varprojlim \mathrm{Aut}(t_n f, W).$$

Since $t_n(f \otimes \mathbb{C}) \cong t_n f \otimes \mathbb{C}$, Lemma 3.20 implies that the morphisms $t_n f \otimes \mathbb{C}$ inherit 1–splittings. Hence the morphisms $t_n f$ admit 1–splittings by (i) and Theorem 2.26. It suffices to show that the 1–splittings of $t_n f$ allow us to define a 1–splitting of $f$. This follows as in the proof of Theorem 6.2.1 of [17], using properties (ii) and (iii). □

**Definition 3.22** We say that a mixed Hodge diagram $A$ has *finite homotopy type* if there exists a quasi-isomorphism $M \rightsquigarrow A$ where $M$ is a minimal mixed Hodge DGA of finite type.

**Theorem 3.23** *The rational component of every morphism $f\colon A \to B$ of 0–connected mixed Hodge diagrams with finite homotopy type is $E_1$–formal.*

**Proof** By Theorem 3.19 there exists a minimal model $\tilde{f}\colon M \to \widetilde{M}$ of $f$. By assumption both $M$ and $\widetilde{M}$ have finite type. By Lemma 3.20, $\tilde{f}_{\mathbb{C}}$ admits a 1–splitting. Therefore $\tilde{f}_{\mathbb{Q}}$ admits a 1–splitting by Lemma 3.21. We obtain a commutative diagram

$$\begin{tikzcd}
(A_{\mathbb{Q}}, d, W) \arrow[d, "f_{\mathbb{Q}}"] & (M_{\mathbb{Q}}, d, W) \arrow[l, "\sim"'] \arrow[r, "\cong"] \arrow[d, "\tilde{f}_{\mathbb{Q}}"] & (E_1(M_{\mathbb{Q}}), d_1, W) \arrow[r, "\sim"] \arrow[d, "E_1(\tilde{f}_{\mathbb{Q}})"] & (E_1(A_{\mathbb{Q}}), d_1, W) \arrow[d, "E_1(f_{\mathbb{Q}})"] \\
(B_{\mathbb{Q}}, d, W) & (\widetilde{M}_{\mathbb{Q}}, d, W) \arrow[l, "\sim"'] \arrow[r, "\cong"] & (E_1(\widetilde{M}_{\mathbb{Q}}), d_1, W) \arrow[r, "\sim"] & (E_1(B_{\mathbb{Q}}), d_1, W)
\end{tikzcd}$$

where the horizontal arrows are $E_1$–quasi-isomorphisms. □

The previous result can be restated in terms of a formality property for the forgetful functor
$$U_{\mathbb{Q}}^{\mathrm{ft}}\colon \mathrm{Ho}\big(\mathbf{MHD}^{\mathrm{ft}}\big) \to \mathrm{Ho}_1\big(\mathbf{F}\mathsf{DGA}^{\mathrm{ft}}(\mathbb{Q})\big)$$
defined by sending every 0–connected mixed Hodge diagram with finite homotopy type to its rational component.

**Corollary 3.24** *There is an isomorphism of functors $E_1 \circ U_{\mathbb{Q}}^{\mathrm{ft}} \cong U_{\mathbb{Q}}^{\mathrm{ft}}$.*

# 4 Mixed Hodge theory of complex algebraic varieties

We review Navarro's functorial construction of mixed Hodge diagrams associated with complex algebraic varieties within the context of cohomological descent categories and the extension criterion of functors of Guillén and Navarro [15]. Together with Theorem 3.23, this will lead the main result of this paper.

## 4.1 Mixed Hodge diagrams associated with algebraic varieties

Let us recall the notion of a cubical codiagram in a given category $\mathcal{D}$ [15, Section 1]. The non-empty parts of a non-empty set $S$, ordered by the inclusion, define a category $\square_S$. Every inclusion of sets $u\colon S\to T$ induces a functor $\square_u\colon \square_S\to\square_T$ defined by $\square_u(a)=u(a)$. Denote by $\Pi$ the category whose objects are finite products of categories $\square_S$ and whose morphisms are the functors associated to injective maps in each component. A *cubical codiagram of* $\mathcal{D}$ is a pair $(X,\square)$ where $\square$ is an object of $\Pi$ and $X\colon\square\to\mathcal{D}$ is a functor.

The Thom–Whitney simple functor defined by Navarro in [22] for strict cosimplicial DGAs is easily adapted to cubical codiagrams of DGAs (see [15, 1.7.3]).

Given a non-empty finite set $S$, denote by $L_S$ the DGA over $\boldsymbol{k}$ of smooth differential forms over the hyperplane of the affine space $\mathbb{A}^S_{\boldsymbol{k}}$, defined by the equation $\sum_{s\in S} t_s=1$.

For $r\geq 0$, let $\sigma[r]$ be the increasing filtration of $L_S$ defined by letting $t_s$ be of pure weight 0 and $dt_s$ of pure weight $-r$, for every generator $t_s$ of degree 0 of $L_S$, and extending multiplicatively. For every filtered DGA $(A,W)$, we have a family of filtered DGAs $L^r_S(A)$ indexed by $r\geq 0$:

$$W_pL^r_S(A):=\bigoplus_q(\sigma[r]_qL_S\otimes W_{p-q}A).$$

**Definition 4.1** The *$r$–Thom–Whitney simple* of a cubical codiagram of filtered DGAs $A=((A,W)^\alpha)$ is the filtered DGA $\boldsymbol{s}^r_{TW}(A,W)$ defined by the end

$$W_p\boldsymbol{s}^r_{TW}(A)=\int_\alpha\bigoplus_q\big(\sigma[r]_qL_\alpha\otimes W_{p-q}A^\alpha\big).$$

For a cubical codiagram of bifiltered DGAs $A=((A,W,F)^\alpha)$, the $(r,0)$*–Thom–Whitney simple* is defined analogously:

$$W_pF^q\boldsymbol{s}^{r,0}_{TW}(A)=\int_\alpha\bigoplus_l\big(\sigma[r]_lL_\alpha\otimes W_{p-l}F^qA^\alpha\big).$$

**Definition 4.2** Let $A$ be a cubical codiagram of mixed Hodge diagrams. The *Thom–Whitney simple of* $A$ is the diagram of DGAs

$$s_{TW}(A) = \left(s^1_{TW}(A_{\mathbb{Q}}, W) \overset{s(\varphi)}{\longleftarrow\!\!-\!\!-\!\!\rightarrow} s^{1,0}_{TW}(A_{\mathbb{C}}, W, F)\right).$$

**Theorem 4.3** *The category of mixed Hodge diagrams* **MHD** *with the Thom–Whitney simple functor* $s_{TW}$ *and the class of quasi-isomorphisms is a cohomological descent category.*

**Proof** The Thom–Whitney simple of a cubical codiagram of mixed Hodge diagrams is a mixed Hodge diagram. Indeed, it suffices to prove that the associated functor of strict cosimplicial objects is a mixed Hodge diagram. This follows from [22, 7.11]. Consider the functor $U_{\mathbb{Q}}$: **MHD** $\to \mathsf{DGA}(\mathbb{Q})$ defined by sending every mixed Hodge diagram $A$ to the DGA $A_{\mathbb{Q}}$ over $\mathbb{Q}$. This functor commutes with the Thom–Whitney simple. The class of quasi-isomorphisms of mixed Hodge diagrams is obtained by lifting the class of quasi-isomorphisms of DGAs. By [15, Proposition 1.7.4] the category of DGAs admits a cohomological descent structure. Hence by [15, Proposition 1.5.12], this lifts to a cohomological descent structure on **MHD**. □

Denote by **Sch**$(\mathbb{C})$ the category of complex reduced schemes that are separated and of finite type. In what follows, **MHD** is the category of mixed Hodge diagrams indexed by $I = \{0 \to 1 \leftarrow 2 \to 3\}$.

**Theorem 4.4** [22, Section 9] *There exists a functor* $\mathbb{H}dg$: **Sch**$(\mathbb{C}) \to$ Ho(**MHD**) *satisfying the following conditions:*

(1) *The rational component of* $\mathbb{H}dg(X)$ *is* $A_{\mathbb{Q}}(X) \cong \mathcal{A}_{Su}(X^{\mathrm{an}}; \mathbb{Q})$.

(2) *The cohomology* $H(\mathbb{H}dg(X))$ *is the mixed Hodge structure of the cohomology of* $X$.

**Proof** Denote by $V^2(\mathbb{C})$ the category of pairs $(X, U)$, where $X$ is a smooth projective scheme over $\mathbb{C}$ and $U$ is an open subscheme of $X$ such that $D = X - U$ is a normal crossings divisor. By [22, Theorem 8.15] there is a functor $\mathbb{H}dg$: $V^2(\mathbb{C}) \to$ **MHD** such that:

(1′) The rational component of $\mathbb{H}dg(X, U)$ is $A_{\mathbb{Q}}(U) \cong \mathcal{A}_{Su}(U^{\mathrm{an}}; \mathbb{Q})$.

(2′) The cohomology $H(\mathbb{H}dg(X, U)$ is the mixed Hodge structure of the cohomology of $U$.

By Theorem 4.3 the Thom–Whitney simple endows the category of mixed Hodge diagrams with a cohomological descent structure. For every elementary acyclic diagram

$$\begin{tikzcd}
(\widetilde{Y}, \widetilde{U} \cap \widetilde{Y}) \arrow[r, "j"] \arrow[d, "g"'] & (\widetilde{X}, \widetilde{U}) \arrow[d, "f"] \\
(Y, U \cap Y) \arrow[r, "i"] & (X, U)
\end{tikzcd}$$

of $V^2(\mathbb{C})$, the mixed Hodge diagram $\mathbb{H}dg(X,U)$ is quasi-isomorphic to the Thom–Whitney simple of the mixed Hodge diagrams associated with the remaining components. Therefore the functor

$$V^2_{\mathbb{C}} \xrightarrow{\mathbb{H}dg} \mathbf{MHD} \xrightarrow{\gamma} \mathrm{Ho}(\mathbf{MHD})$$

satisfies the hypothesis of Theorem 2.3.6 of [22] on the extension of functors. □

## 4.2 Formality

**Theorem 4.5** *Let $f\colon Y \to X$ be a morphism of complex algebraic varieties. If $X$ and $Y$ are nilpotent spaces, then the rational $E_1$–homotopy type of $f$ is a formal consequence of the first term of spectral sequence associated with the weight filtration: there exists a diagram*

$$\begin{tikzcd}
(A_{\mathbb{Q}}(X), W) \arrow[d, "f_{\mathbb{Q}}"] & (M_X, W) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\widetilde{f}_{\mathbb{Q}}"] & (E_1(A_{\mathbb{Q}}(X)), W) \arrow[d, "E_1(f_{\mathbb{Q}})"] \\
(A_{\mathbb{Q}}(Y), W) & (\widetilde{M}_Y, W) \arrow[l, "\sim"'] \arrow[r, "\sim"] & (E_1(A_{\mathbb{Q}}(Y)), W)
\end{tikzcd}$$

*which commutes in the homotopy category* $\mathrm{Ho}_1(\mathbf{F}\mathsf{DGA}(\mathbb{Q}))$.

**Proof** By Theorem 4.4 there is a functor $\mathbb{H}dg\colon \mathbf{Sch}(\mathbb{C}) \to \mathrm{Ho}(\mathbf{MHD})$ whose rational component is the Sullivan–de Rham functor $X \mapsto A_{\mathbb{Q}}(X) = \mathcal{A}_{Su}(X^{\mathrm{an}};\mathbb{Q})$. In addition, for a nilpotent space $X$, the minimal model of $A_{\mathbb{Q}}(X)$ has finite type. The result follows from Theorem 3.23. □

The previous result can be restated in terms of a formality property for the composite functor

$$\mathcal{A}^{\mathrm{nil}}_{\mathbb{Q}}\colon \mathbf{Sch}^{\mathrm{nil}}(\mathbb{C}) \xrightarrow{\mathbb{H}dg} \mathrm{Ho}\big(\mathbf{MHD}^{\mathrm{ft}}\big) \xrightarrow{U_{\mathbb{Q}}} \mathrm{Ho}_1\big(\mathbf{F}\mathsf{DGA}^{\mathrm{ft}}(\mathbb{Q})\big)$$

defined by sending nilpotent complex algebraic varieties to their Sullivan–de Rham algebra endowed with the multiplicative weight filtration.

**Corollary 4.6** *There is an isomorphism of functors $E_1 \circ \mathcal{A}_{\mathbb{Q}}^{\mathrm{nil}} \cong \mathcal{A}_{\mathbb{Q}}^{\mathrm{nil}}$.*

**Remark 4.7** The above formality property is concerned only with the weight filtration, and does not provide any formality statement for the mixed Hodge structures involved. In fact, there exist examples for which the mixed Hodge structures on the rational homotopy groups are not a formal consequence of the mixed Hodge structures on cohomology (see Carlson, Clemens and Morgan [5]). Note as well that the nilpotence condition ensures that the minimal models have finite type, and hence one can apply descent from $\mathbb{C}$ to $\mathbb{Q}$.

# 5 An application: The Hopf invariant

The Hopf invariant of algebraic morphisms $f\colon \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1_{\mathbb{C}}$ has been long studied using various techniques. Here we study this discrete invariant in the context of algebraic geometry, using Theorem 4.5 and intersection theory. The results of this section easily generalize to morphisms $f\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n_{\mathbb{C}}$, for $n \geq 1$. However, for the sake of simplicity, we shall only develop the case $n = 1$.

## 5.1 The Hopf invariant

We first recall Whitehead's definition of the Hopf invariant in the context of differential forms and show that it can be computed in the context of rational homotopy, via Sullivan minimal models.

Consider a differentiable map $f\colon S^3 \to S^2$. Denote by $f^*\colon \mathcal{A}_{\mathrm{dR}}(S^2) \to \mathcal{A}_{\mathrm{dR}}(S^3)$ the induced morphism of algebras. Choose fundamental classes $[S^2]$ and $[S^3]$ together with normalized volume forms $w_2$ and $w_3$ of $S^2$ and $S^3$ respectively, satisfying $\int_{S^2} w_2 = \int_{S^3} w_3 = 1$. Let $\theta$ be a one-form in $\mathcal{A}_{\mathrm{dR}}(S^3)$ such that $f^*(w_2) = d\theta$. The *Hopf invariant of $f$* is defined by

$$H(f) := \int_{S^3} \theta \wedge d\theta\,.$$

While the definition of $H(f)$ is independent of the choice of $\theta$ and the orientation of $S^2$, it does depend on the choice of orientation of $S^3$. Homotopic maps have the same Hopf invariant. Geometrically, $H(f)$ is given by the linking number of pre-images of two distinct regular values of $f$. In particular it is always an integer number, and it defines a homomorphism $H\colon \pi_3(S^2) \to \mathbb{Z}$ (see Bott and Tu [3, Section 18]).

**5.1** We define the normalized minimal model of a continuous map $f\colon S^3 \to S^2$ as follows. Let $M(S^2) = \Lambda(\alpha, \beta)$ be the free $\mathbb{Q}$–DGA generated by $\alpha$ in degree 2 and $\beta$ in degree 3 with differentials $d\alpha=0$ and $d\beta=\alpha^2$. The morphism $\rho_2\colon M(S^2)\to\mathcal{A}_{\mathbb{Q}}(S^2)$ defined by sending $\alpha$ to the volume form $w_2$ of $S^2$ is a Sullivan minimal model of $S^2$. Likewise, let $M(S^3)=\Lambda(\gamma)$ be the free $\mathbb{Q}$–DGA generated by $\gamma$ in degree 3 with trivial differential. The morphism $\rho_3\colon M(S^3)\to\mathcal{A}_{\mathbb{Q}}(S^3)$ defined by sending $\gamma$ to the volume form $w_3$ of $S^3$ is a Sullivan minimal model of $S^3$. By dimensional arguments, every possible morphism $\widetilde{f_\lambda}\colon M(S^2)\to M(S^3)$ is of the form $\alpha\mapsto 0$ and $\beta\mapsto\lambda\cdot\gamma$, with $\lambda\in\mathbb{Q}$. Furthermore, any two such homotopic morphisms coincide. The map $\widetilde{f_\lambda}$ is a minimal model of $f^*$ if and only if the diagram

$$\begin{tikzcd}
\mathcal{A}_{\mathbb{Q}}(S^2) \arrow[r, "f^*"] & \mathcal{A}_{\mathbb{Q}}(S^3) \\
M(S^2) \arrow[u, "\rho_2"] \arrow[r, "\widetilde{f_\lambda}"'] \arrow[ur, phantom, "\simeq"] & M(S^3) \arrow[u, "\rho_3"']
\end{tikzcd}$$

commutes up to homotopy. In such case we say that the above diagram is a *normalized minimal model* of $f^*$ with respect to the chosen volume forms. Note that as in the definition of the Hopf invariant, the sign of $\lambda$ depends on the choice of orientation of $S^3$.

**Proposition 5.2** *Let $f\colon S^3\to S^2$ be a differentiable morphism. Then $\widetilde{f_\lambda}$ is a normalized minimal model of $f^*$ if and only if $H(f)=\lambda$.*

**Proof** Let $\theta$ be a one-form of $\mathcal{A}_{\mathbb{Q}}(S^3)$ satisfying $d\theta=f^*(w_2)$. Define a homotopy $h\colon M(S^2)\to\mathcal{A}_{\mathbb{Q}}(S^3)\otimes\Lambda(t,dt)$ by letting $h(\alpha)=d(\theta\cdot t)$ and $h(\beta)=\lambda w_3(1-t^2)$. Then $\delta^0 h=\rho_3\circ\widetilde{f_\lambda}$ and $\delta^1 h=f^*\circ\rho_2$. For $h$ to be a morphism of DGAs, it is necessary and sufficient that $h(\alpha)^2=dh(\beta)$. This is the case only when $d\theta\cdot\theta=\lambda w_3$. We have $H(f)=\int_{S^3}d\theta\cdot\theta=\int_{S^3}\lambda w_3=\lambda$. □

## 5.2 Weight spectral sequence

We study the rational homotopy type and the Hopf invariant of certain algebraic morphisms of complex algebraic varieties, via the weight filtration.

**Definition 5.3** Let $f\colon\mathbb{C}^2\setminus\{0\}\to\mathbb{P}^1_{\mathbb{C}}$ be a morphism of complex algebraic varieties, and $i\colon S^3\hookrightarrow\mathbb{C}^2\setminus\{0\}$ denote the inclusion. We call $H(f\circ i)$ the *Hopf invariant of* $f$.

Consider the smooth compactification $U := \mathbb{C}^2 \setminus \{0\} \hookrightarrow X := \widetilde{\mathbb{P}}^2$ into the blown-up complex projective plane at the origin. We have a diagram

$$\begin{tikzcd} \mathbb{P}^1_\infty \arrow[r, "i"] \arrow[d] & \widetilde{\mathbb{P}}^2 \arrow[d, "\pi"] & \mathbb{P}^1_E \arrow[l, "j"'] \arrow[d] \\ \mathbb{P}^1_\infty \arrow[r] & \mathbb{P}^2 & \{0\} \arrow[l] \end{tikzcd}$$

where $\mathbb{P}^1_\infty$ and $\mathbb{P}^1_E$ are complex projective lines denoting the hyperplane at infinity and the exceptional divisor respectively. The cohomology ring of $\widetilde{\mathbb{P}}^2$ is given by

$$H^*(\widetilde{\mathbb{P}}^2;\mathbb{Q}) = \mathbb{Q}\langle a,b\rangle, \quad \text{with } a\cdot b = 0 \text{ and } a^2 = -b^2,$$

where $a = i_*1_\infty$ and $b = j_*1_E$ denote the classes of $\mathbb{P}^1_\infty$ and $\mathbb{P}^1_E$. The cohomology ring of the complement $D := \widetilde{\mathbb{P}}^2_{\mathbb{C}} - U = \mathbb{P}^1_\infty \sqcup \mathbb{P}^1_E$ can be written as

$$H^*(D;\mathbb{Q}) = \mathbb{Q}\langle x,y\rangle, \quad \text{with } x\cdot y = 0,\ x^2 = 0 \text{ and } y^2 = 0,$$

where $x$ and $y$ denote the classes of a point in $\mathbb{P}^1_\infty$ and $\mathbb{P}^1_E$, respectively.

The differentials and products of the weight spectral sequence can be computed in the Chow rings, using intersection theory. We will use the following result (see Fulton [12, Proposition 2.6]).

**Proposition 5.4** *Let $j\colon D \to X$ denote the inclusion of a Cartier divisor $D$ on a scheme $X$.*

(a) *If $\alpha$ is a cycle on $X$, then $j_*j^*\alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$.*

(b) *If $\alpha$ is a cycle on $D$, then $j^*j_*\alpha = c_1(N_D) \cap \alpha$, where $N_D = j^*(\mathcal{O}_X(D))$.*

The first Chern classes associated with the morphisms $i$ and $j$ above are given by

$$c_1(\mathcal{O}_X(\mathbb{P}^1_\infty)) = a, \quad c_1(\mathcal{O}_X(\mathbb{P}^1_E)) = b, \quad c_1(N_\infty) = x \quad \text{and} \quad c_1(N_E) = -y.$$

Using Proposition 5.4 we obtain the intersection products

$$1_\infty\cdot a := i^*a = i^*i_*1_\infty = c_1(N_\infty) = x \quad \text{and} \quad 1_E\cdot b := j^*b = j^*j_*1_E = c_1(N_E) = -y.$$

Since $\mathbb{P}^1_\infty \cap \mathbb{P}^1_E = \varnothing$, it follows that $1_\infty\cdot b = 0$ and $1_E\cdot a = 0$.

With these results we can write the first term of the weight spectral sequence associated with the compactification $U \hookrightarrow \widetilde{\mathbb{P}}^2$. By definition the only non-trivial terms are

$$E_1^{0,q}(U) = H^q(\widetilde{\mathbb{P}}^2;\mathbb{Q}) \quad \text{and} \quad E_1^{-1,q}(U) = H^{q-2}(D;\mathbb{Q}),$$

and the differentials $d_1\colon H^{q-2}(D) \to H^q(\widetilde{\mathbb{P}}^2)$ are given by the Gysin morphisms $i_*$ and $j_*$. Let $u = 1_\infty$ and $v = 1_E$. Since $x = u \cdot a$ and $y = -v \cdot b$, we can write

$$E_1^{*,*}(U) = \Lambda(u, v, a, b)/R$$

as the quotient of the free bigraded algebra generated by $u$ and $v$ of bidegree $(-1, 2)$, and $a$ and $b$ of bidegree $(0, 2)$, by the ideal of relations

$$R = (uv, ub, va, a^2 + b^2, ab).$$

The differential is defined on the generators by $du = a$, $dv = b$ and $da = db = 0$.

Since $\mathbb{P}^1_{\mathbb{C}}$ is smooth and compact the first term of the associated weight spectral sequence satisfies

$$E_1^{0,q}(\mathbb{P}^1_{\mathbb{C}}) = H^q(\mathbb{P}^1_{\mathbb{C}}; \mathbb{Q}) \quad \text{and} \quad E_1^{-p,q}(\mathbb{P}^1_{\mathbb{C}}) = 0 \quad \text{for } p \neq 0.$$

Therefore we can write

$$E_1^{*,*}(\mathbb{P}^1_{\mathbb{C}}) = \Lambda(\alpha)/\alpha^2$$

as the quotient of the free bigraded algebra generated by $\alpha$ in bidegree $(0, 2)$ by the ideal $(\alpha^2)$, and with trivial differential.

**Proposition 5.5** *Let $f\colon U := \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1_{\mathbb{C}}$ be a morphism of complex algebraic varieties extending to a morphism $g\colon \widetilde{\mathbb{P}}^2_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$.*

(1) *There exists a unique $\varepsilon \in \mathbb{Z}$ such that $E_1(g)\colon E_1(\mathbb{P}^1_{\mathbb{C}}) \to E_1(U)$ satisfies $\alpha \mapsto \varepsilon(a \pm b)$.*

(2) *The map $\widetilde{f}\colon M(S^2) \to M(S^3)$ given by $\beta \mapsto \varepsilon^2\gamma$ defines a normalized minimal model of $f^*$.*

(3) *The Hopf invariant of $f$ is $H(f) = \varepsilon^2$.*

**Proof** By dimensional arguments, any morphism $E_1(g)\colon E_1(\mathbb{P}^1_{\mathbb{C}}) \to E_1(U)$ can be written as $\alpha \mapsto \varepsilon_1 a + \varepsilon_2 b$, where $\varepsilon_i \in \mathbb{Q}$. The compatibility condition for $\alpha^2 = 0$ implies that $\varepsilon_1 = \pm\varepsilon_2$. The weight spectral sequence associated with a compactification is defined over $\mathbb{Z}$. Hence, $E_1(g)$ is defined over $\mathbb{Z}$, and (1) follows. Let us prove (2). Define a quasi-isomorphism $\rho\colon M(S^2) \to E_1(\mathbb{P}^1_{\mathbb{C}})$ by letting $\rho(\alpha) = \alpha$ and $\rho(\beta) = 0$. Likewise, define a quasi-isomorphism $\rho'\colon M(S^3) \to E_1(U)$ by $\rho'(\gamma) = ua + vb$. The map $h\colon M(S^2) \to E_1(U) \otimes \Lambda(t, dt)$ defined by $h(\alpha) = \varepsilon(a \pm b)t - \varepsilon(u \pm v)\, dt$ and $h(\beta) = \varepsilon^2 \cdot (ua + vb)(1 - t^2)$ is a homotopy from $\rho' \circ \widetilde{f}$ to $E_1(g) \circ \rho$. Hence $\widetilde{f}$ is a minimal model of $E_1(g)$. By Theorem 4.5 this defines a normalized minimal model of $f^*$. This proves (2). Assertion (3) follows from Proposition 5.2. □

**Example 5.6** Let $q \geq 1$, and let $f\colon U \to \mathbb{P}^1_{\mathbb{C}}$ be the morphism defined by $(x_0, x_1) \mapsto [x_0^q : x_1^q]$. For $q = 1$, this morphism is the Hopf fibration. Then $f$ extends to a morphism $g\colon \widetilde{\mathbb{P}}^2_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. The induced morphism at the level of spectral sequences $E_1(g)\colon E_1(\mathbb{P}^1_{\mathbb{C}}) \to E_1(U)$ is given by $\alpha \mapsto q(a-b)$. Indeed, the pre-image $g^{-1}(p)$ of a point $p$ is a family of $q$ lines that intersect both $\mathbb{P}^1_E$ and $\mathbb{P}^1_{\infty}$ at a point in $\widetilde{\mathbb{P}}^2_{\mathbb{C}}$. We find that $H(f) = q^2$. In particular we recover the well-known result that $H(f) = 1$ for the Hopf fibration.

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